Analytic function:

Complex variable function.

Any function : $f(z) = u(x, y) + iv(x, y)$ where $u(x,y)$ and $v(x,y)$ are real is known as complex variable function.

Eg:- (i) $f(z) = x^2 + i\,2xy$

(ii) $f(z) = x^3 + y^3 + i(x^2 - y^2)$.

(iii) $f(z) = \sin x \sin y + ie^{x} \cos y$.

All these are complex variable function.

Analytic function.

Any function.

$$f(z) = u(x,y) + iv(x,y)$$ is said to be analytic at a point if it is Differentiable in some neighbouring of z_0.

i.e. If we consider a domain D function is Differentiable at each point of Domain D.

Note:- Analytic functn is also known as holomorphic or regular functn (Entire functn).

If $f(z)$ is Analytic then,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \Big\} \text{ exist at everywhere.}$$

If $f(z)$ is analytic then $\boxed{f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}}$ exist.

Continuous functh.

Any functh $f(z)$ is continuous at $z = z_0$. when limit of funith = value of functh.

$$\boxed{\lim_{z \to z_0} f(z) = f(z_0)}.$$

→ value of functh

every differentiable functh is continuous.

Theorem If $f(z) = u(x,y) + iv(x,y)$, is analytic. (differential value) then four partial derivative $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$, and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$,

and,

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}} \quad \text{OR} \quad \boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$

This eqn is called cauchy Reimann eqn.

and if said examine then proof also $f'(z)$ and it's also give result.

Proof. If $f(z) = u(x,y) + iv(x,y)$ is analytic (i.e. Differentiable) then $f'(z)$ exist.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exist.}$$

$$z = x + iy$$

$$\Delta z = \Delta x + i\Delta y.$$

$$f(z) = u(x,y) + iv(x,y).$$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y).$$

$$\cancel{f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} - \{u(x,y) + iv(x,y)\}}.$$

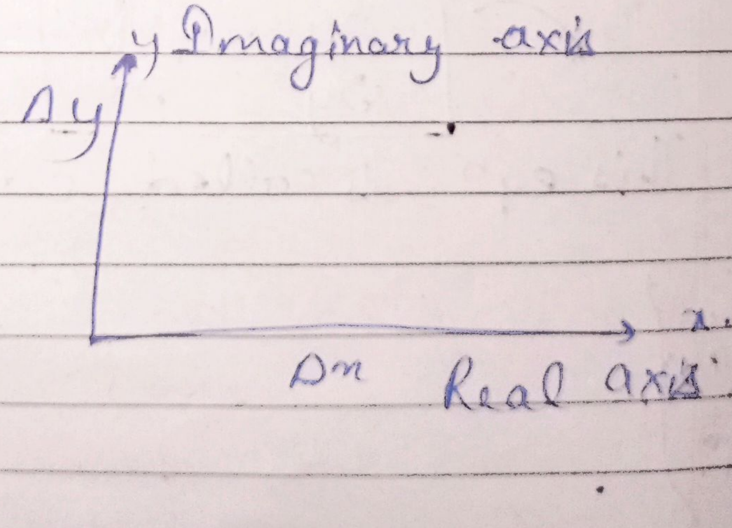
$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - \{u(x,y) + iv(x,y)\}}{\Delta x + i\Delta y}.$$

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y}$$

$$+ \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{i\{v(x+\Delta x, y+\Delta y) - v(x,y)\}}{\Delta x + i\Delta y}.$$

Along Real Axis.

$\Delta y = 0$



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$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$+ i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}.$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \quad \text{—— (A).}$$

Since, $f'(z)$ exist along imaginary axis.

at $\Delta x = 0$,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y}.$$

$$+ \lim_{\Delta y \to 0} \frac{\cancel{i}\{v(x, y+\Delta y) - v(x,y)\}}{\cancel{i}\Delta y}.$$

$$= -i \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}.$$

$$+ \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}.$$

$$\boxed{f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}} \quad \text{—— (B).}$$

Since $f'(z)$ exist, so $-\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ also exists.

from A and B.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{OR} \quad u_x = v_y$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad u_y = -v_x$$

Harmonic funct^n.

Any funct^n $f(z) = u(x,y)$ is said to be Harmonic funct^n when its real and imaginary parts satisfied laplace eq^n.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

i.e. $\Delta^2 u = 0$

$\Delta^2 v = 0$

Every analytic funct^n is harmonic funct^n.

If $f(z) = u(x,y) + iv(x,y)$ is analytic then,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{--- (i)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (ii)}$$

partial differentiate (i) w.r. to x and (ii) w.r.t. y and sum of them.

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$$\cancel{\frac{\partial^2 u}{\partial x^2}} = \cancel{\frac{\partial^2 v}{\partial^2 y}} \cdot A.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{\frac{\partial^2 v}{\partial x \partial y}} - \cancel{\frac{\partial^2 v}{\partial y \partial x}}$$

$$= \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

i.e. u satified laplace eq^n.

partial diff. (I) w.r.to y and (II) w.r.to x. and subtracting of them.

$$\cancel{\frac{\partial^2 u}{\partial y \partial x}} - \cancel{\frac{\partial^2 u}{\partial x \partial y}} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

i.e. v satisfied laplace eq^n.

hence, $u(x,y)$ and $v(x,y)$ are satisfied
i.e. u and v are harmonic.

Qu. Let $f(z)$ is analytic in Domain D. if $f(\bar{z})$ is also Analytic in Domain D. P.T $f(z)$ is constant in Domain D.

Dt.

Pg.

Soln. $f(z) = u(x,y) + iv(x,y)$ — (I).
is analytic in Domain D.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ — (i)}.$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ — (ii)}.$$

$\overline{f(z)} = u(x,y) - iv(x,y)$.
is also Analytic.

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \text{ — (iii)}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \text{ — (iv)}.$$

(i) + (iii).

$$\boxed{\frac{\partial u}{\partial x} = 0}$$

(ii) + (iv)

$$\boxed{\frac{\partial u}{\partial y} = 0}$$

putting $\frac{\partial u}{\partial x}$ in (i) and $\frac{\partial u}{\partial y}$ in (ii).

$$\boxed{\frac{\partial v}{\partial y} = 0}$$

$$\boxed{\frac{\partial u}{\partial y} = 0}$$

$\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$, $u = \text{constant} = C_1$

$$\frac{\partial v}{\partial y} = 0 , \frac{\partial v}{\partial x} = 0$$

$$V = \text{constant} = c_2 .$$

i.e, $f(z) = u(x,y) + i\,v(x,y)$

$$\boxed{f(z) = c_1 + i c_2}.$$

Harmonic Conjugate.

If $f(z) = u(x,y) + i v(x,y)$, is a complex variable funct^n and $f(z)$ is analytic (differentiable) then $v(x,y)$ is harmonic conjugate of $u(x,y)$ and $u(x,y)$ is called harmonic conjugate of $v(x,y)$.

If $u(x,y)$ is given, then $v(x,y)$. is harmonic conjugate of $u(x,y)$ which is to be determine,

$$v = v(x,y).$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

{from using chain rule}.

By C-R eq^n.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\frac{\partial v}{\partial} = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy.$$

$$\int dv = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy$$

y constant   Those Term which donot contain x.

$$v = -\int \frac{\partial u}{\partial y} dx + \int \frac{\partial u}{\partial x} dy + c.$$

y constant   Those term which do not contain x.

Q4. find the harmonic conjugate of the following functn.

(i) $u(x,y) = x^3 - 3xy^2$.

(ii) $u(x,y) = e^x \cos y$.

(iii) $u(x,y) = 2x(1-y)$

(iv) $u(x,y) = 2x - x^3 + 3xy^2$.

(v) $u(x,y) = e^x(x\cos y - y\sin y)$.

(vi) $u(x,y) = \sin hx \sin y$

(vii) $u(x,y) = y/x^2+y^2$.

Soln (i) $u(x,y) = x^3 - 3xy^2$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$.

$\frac{\partial u}{\partial y} = -6xy$

$\partial v = -(-6xy)dx + (3x^2 - 3y^2)dy$.

$$\int dv = \int 6xy \, dx + \int$$

y constant   These term which do not contain x.

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$$V = \frac{6yx^2}{2} - \frac{\cancel{3}y^3}{\cancel{3}} + c.$$

$$\boxed{V = 3x^2y - y^3 + c}$$

II method: $dv = 6xy\,dx + 3x^2dy - 3y^2dy.$
$dv = \{3x^2y\}.$

Q1. P.T $u(x,y) = e^x \cos y$ is harmonic find the harmonic congugate ~~harm~~ of $u(x,y)$ also find $f(z)$.

Soln. $\frac{\partial u}{\partial x} = e^x \cos y$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \quad \text{---} (i).$$

$$\frac{\partial u}{\partial y} = -e^x \sin y.$$

$$\frac{\partial^2 u}{\partial y^2} = -e^x \cos y \quad \text{---} (ii).$$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0.$ u is harmonic.

Let, $v(x,y)$ is the harmonic conjugate of $u(x,y)$.

$$V = v(x,y)$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

using,
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}.$$

$$\partial v = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

$dv = e^x \sin y \, dx + e^x \cos y \, dy$.

$\int dv = \underbrace{\int e^x \sin y \, dx}_{y\ \text{constant}} + \underbrace{\int e^x \cos y \, dy}_{\text{Those term which do not contain } x.} + c$

$\boxed{v = \sin y e^x + c}$ This is harmonic conjugate

$f(z) = u + iv$.
$= e^x \cos y + i(e^x \sin y + c)$.

$f(z) = e^x(\cos y + i \sin y) + ic$.

$f(z) = e^x \cdot e^{iy} + ic$
$= e^{x+iy} + ic$

$\boxed{f(z) = e^z + ic}$

Qu. find the harmonic conjugate of $u(x,y)$.
$u(x,y) = e^x(x \cos y - y \sin y)$.

Solⁿ If $v(x,y)$ is harmonic conjugate of $u(x,y)$.
$v = (x,y)$

$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$.

By CR eqⁿ.
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$,
$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$ — (1).

$u = e^x (x\cos y - y\sin y)$.

$$\frac{\partial u}{\partial x} = e^x \cdot \frac{\partial}{\partial x}(x\cos y - y\sin y) + (x\cos y - y\sin y)\frac{\partial e^x}{\partial x}.$$

$$\frac{\partial u}{\partial x} = e^x \cos y + (x\cos y - y\sin y)e^x.$$

$$\frac{\partial u}{\partial y} = e^x \cdot \frac{\partial}{\partial y}(x\cos y - y\sin y).$$

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y).$$

putting $\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}$ in (1).

$$du = -e^x(-x\sin y - y\cos y - \sin y)\,dx + \{e^x\cos y - y\sin y)e^x\}\,dy.$$

Integrating,

$$\int du = \int (e^x x\sin y + e^x y\cos y + e^x \sin y)\,dx + \int \{e^x\cos y + (x\cos y - y\sin y)e^x\}\,dy.$$

y const.

Those term which not contain x.

$V = \sin y \cdot e^x(x-1) + y\cos y\, e^x + \sin y \cdot e^x + c.$

$V = e^x \sin y\, \cancel{(x-1+1)} + y\cos y e^x + c$

$$\boxed{V = e^x(x\sin y + y\cos y) + c}$$

Dt.
Pg.

Qu. P.T $u = \frac{1}{2}\log(x^2+y^2)$. is harmonic and find its harmonic conjugate.

Soln. $u = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{\cancel{2}x}{\cancel{2}(x^2+y^2)} = \frac{x}{x^2+y^2}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\cdot\frac{x}{x^2+y^2}.$$

$$= \frac{(x^2+y^2)\cdot\frac{\partial}{\partial x}x - x\cdot\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}.$$

$$= \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{———} \quad (A).$$

$$\frac{\partial u}{\partial y} = \frac{1}{\cancel{2}(x^2+y^2)}\,\cancel{2}y$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}.$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\frac{y}{x^2+y^2}$$

$$= \frac{(x^2+y^2)\frac{\partial y}{\partial y} - y \cdot \frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2 - y^2}{(x^2+y^2)^2} \quad \text{——} \; B$$

A + B

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$= \frac{\cancel{y^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

u is harmonic.

Let $v = v(x,y)$ is harmonic conjugate of $u(x,y)$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy, \quad \left\{ \text{using } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right.$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$dv = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dx$$

$$\Rightarrow \int du = \int_{y\ \text{constant}} \frac{y}{x^2+y^2}\,dx + \int \frac{x}{x^2+y^2}\,dy + c.$$

those term which do not contain x.

$$v = -y \cdot \frac{1}{y} \tan^{-1}\frac{x}{y} + c$$

$$v = -\tan^{-1}\frac{x}{y} + c.$$

$$v = -\cot^{-1}\frac{y}{x} + c$$

$$v = \tan^{-1}\frac{y}{x} - \frac{\pi}{2} + c$$

$$v = \tan^{-1}\frac{y}{x} + c_1.$$

By using
$\tan^{-1}\frac{1}{x} = \cot^{-1}x.$
$\tan^{-1}\frac{x}{y} = \cot^{-1}\frac{y}{x}.$
$\cot^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x} = \frac{\pi}{2}$
$\tan^{-1}\frac{y}{x} - \frac{\pi}{2} = -\cot^{-1}\frac{y}{x}$

* Taken $c - \frac{\pi}{2} = c_1.$

Qu. find the harmonic conjugate (i) $u(x,y) = \sinh x \sin y.$

(ii) $u(x,y) = \dfrac{y}{x^2+y^2}.$

Dt.
Pg.

Cauchy's Reimann Equation in polar co-ordinate.

(1).

If $f(z) = u + iv$ is a complex variable function.

in polar co-ordinate. $z = r.e^{i\theta}$.

$f(r.e^{i\theta}) = u + iv$ —— (ii).

partial differentiate (ii) w.r. to r.

$$f'(re^{i\theta}) \frac{\partial}{\partial r} re^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}.$$

$$f'(r.e^{i\theta}).e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \quad \text{—— (iii)}.$$

partial differentiate (ii) w.r. to θ.

$$f'(r.e^{i\theta}) \frac{\partial}{\partial \theta}(re^{i\theta}) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

$$f'(r.e^{i\theta})\, r.e^{i\theta} i = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

$$f'(r.e^{i\theta})\, e^{i\theta} = \frac{1}{ir}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

$$f'(re^{i\theta})e^{i\theta} = \frac{-i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{—— (iv)}$$

from ~~comparing~~ (iii) and (iv). & comparing real and imaginary.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

OR.

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$$

$$r\frac{\partial v}{\partial r} = \cancel{\frac{r}{r}} - \frac{\partial u}{\partial \theta}.$$

This eq^n is called CR- eq^n in polar co-ordi-nate.

Qus. P.T analytic funct^n with constant Modulus is ~~constant~~ constant.

Sol^n. $f(z) = u + iv$; —— (i).

i.e. C-R. eq^n satisfied so,

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \text{—— (ii).}$$

$$|f(z)| = \sqrt{u^2 + v^2}.$$

$$|f(z)| = c \text{ (given).}$$

$$\sqrt{u^2 + v^2} = c.$$

$$u^2 + v^2 = c^2 \text{ —— (iii)}$$

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partiate differentiate (iii) w.r. to x.

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0.$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{——} \quad (iv).$$

partial diff. w.r. to y.

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad \text{—} \quad (v).$$

putting $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ in (iv)

$$u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y} = 0 \quad \text{——} \quad (vi)$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad \text{——} \quad (v)$$

(vi) $\times u$ + (v) v

$$u^2\frac{\partial v}{\partial y} - uv\cancel{\frac{\partial u}{\partial y}} = 0.$$

(vii)

$$vu\cancel{\frac{\partial u}{\partial y}} + v^2\frac{\partial v}{\partial y} = 0$$

$$(u^2+v^2)\frac{\partial v}{\partial y} = 0.$$

i.e. $\frac{\partial v}{\partial y} = 0.$

from (II) i.e. $\frac{\partial u}{\partial x} = 0$

Similarly. $\frac{\partial v}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$.

hence, $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$

u constant.

$\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} = 0$. v constant

$$f(z) = u + iv$$
$$= c_1 + i c_2 \quad \text{(Constant)}.$$

Qu. P.T a Harmonic functn. satisfies. the differential Equation. $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$

Soln. Since, u is harmonic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (I)}$$

using,

$$z = x + iy$$
$$\bar{z} = x - iy.$$

$$z + \bar{z} = 2x$$
$$x = \frac{z + \bar{z}}{2} \quad \text{--- (II)}$$

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$$z - \bar{z} = 2iy$$

$$y = \frac{z - \bar{z}}{2i} \text{ ——— (iii).}$$

$u = (x, y).$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z}.$$

$$= \frac{\partial u}{\partial x}\cdot\frac{1}{2} + \frac{\partial u}{\partial y}\cdot\frac{1}{2i}. \qquad \left\{ \begin{aligned} \frac{\partial x}{\partial z} &= \frac{1}{2} \\ \frac{\partial y}{\partial z} &= \frac{1}{2i} \end{aligned} \right.$$

$$\frac{\partial u}{\partial z} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right] \text{ ——— (4).}$$

$u = u(x, y)$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial}{\partial \bar{z}} y.$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right] \text{ ——— (5)}$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \bar{z}}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)$$

$$= \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)$$

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$$= \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}\right].$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0 \quad \{\text{using (1)}\}.$$

Note:-

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{4\partial^2 u}{\partial z \partial \bar{z}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

#
$$\boxed{4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}$$

Qu. If $f(z)$ is a regular. such that $f'(z) \neq 0$.
P.T

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f(z)| = 0.$$

Soln.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \log|f(z)|.$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}} \log|f(z)|$$

$$= 2 \cdot \frac{\partial^2}{\partial z \partial \bar{z}} \log|f(z)|^2.$$

Dt.

Pg.

$$= 2 \cdot \frac{\partial^2}{\partial z \partial \bar{z}} \log |f(z)|^2 .$$

$$= 2 \frac{\partial^2}{\partial z \partial \bar{z}} \log f(z) \cdot \overline{f(z)} \cdot \left\{ \text{using } z\bar{z} = |z|^2 . \right.$$

$$= 2 \cdot \frac{\partial^2}{\partial z \partial \bar{z}} \left\{ \log f(z) + \log \overline{f(z)} \right\}$$

$$= 2 \cdot \frac{\partial}{\partial z} \left\{ \frac{\partial}{\partial \bar{z}} \log f(z) + \frac{\partial}{\partial \bar{z}} \log \overline{f(z)} \right.$$

$$= 2 \cdot \frac{\partial}{\partial z} \left\{ 0 + \frac{\partial}{\partial \bar{z}} \log \overline{f(z)} \right\}$$

$$= 2 \frac{\partial}{\partial z} \left\{ \frac{1}{\overline{f(z)}} \overline{f'(z)} \right\}$$

$$= 2 \cdot \frac{\partial}{\partial z} \left(\frac{f'(\bar{z})}{\overline{f(z)}} \right)$$

$$= 0 .$$

If $f(z)$ is a regular functⁿ. of z P.T.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2 .$$

Soln.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2.$$

$$= 4\cdot\frac{\partial^2}{\partial z\partial \bar{z}}|f(z)|^2.$$

$$= 4\cdot\frac{\partial^2}{\partial z\partial \bar{z}} f(z)\cdot\overline{f(z)}$$

$$= 4\cdot\frac{\partial}{\partial z}\left\{f(z)\cdot\overline{f'(z)}\right\}.$$

$$= 4\,\overline{f'(z)}\,f'(z).$$

$$= 4|f'(z)|^2.$$

Qu. P.T $f(r.e^{i\theta}) = \log r + i\theta.$ $\quad -\pi < \theta < \pi.$
is analytic except at origin.

Soln.
$z = x + iy$
$x = r\cos\theta$
$y = r\sin\theta.$
$z = r\cos\theta + ir\sin\theta$
$z = r\cdot e^{i\theta}$
Then, $f(z) = \log r + i\theta.$
comparing. $f(z) = u + iv,$
$u = \log r.$
$v = \theta.$

In polar co-ordinate. C-R eqn.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$u = \log r$

$$\frac{\partial u}{\partial r} = \frac{1}{r} \quad \text{—— (I)}$$

$$\frac{\partial v}{\partial \theta} = 1$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r} \quad \text{—— (II)}$$

from (I) and (II)

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \text{—— (III)}$$

$$\frac{\partial v}{\partial r} = 0$$

$$\frac{\partial u}{\partial \theta} = 0$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = 0$$

$$\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}} \quad \text{—— (IV)}$$

here, from (III) and (IV) CR - eq^n satisfy then functi^n is analytic ~~acc~~ except at origin because at $r = 0$, $\frac{\partial u}{\partial r} \neq \frac{1}{r}\frac{\partial v}{\partial \theta}$

Q. P.T the both the real and imaginary part of the following functⁿ satisfy. Cauchy riemann and laplace eqⁿ.

(i) $f(z) = \cos z$.
(ii) $f(z) = \sinh z$.

Sol (i) $f(z) = \cos z$

hint

$= \cos(x+iy)$.

$= \cos x \cos iy - \sin x \sin iy$.

$f(z) = \cos x \cosh y - \sin x \, i \sinh y$.

$u(x,y) = \cos x \cdot \cosh y$,

$v(x,y) = -\sin x \sinh y$.

using.
$\cos iy = \cosh y$
$\sin iy = i \sinh y$

hint

(ii)

$f(z) = \sinh z$

$= \sinh(x+iy)$

$= -i \sin(ix-y)$

$= -i\{\sin ix \cos y - \cos ix \sin y\}$

$= -i \sin ix \cos y + i \cos ix \sin y$

$= -i \cdot i \sinh x \cos y + i \cos ix \sin y$

$= \sinh x \cos y + i \cosh x \cdot \sin y$

using
$\sin iy = i \sinh y$
$\sin i(x+iy) = i \sinh(x+iy)$
$\sin(ix-y) = i \sinh(x+iy)$

$\sinh(x+iy) = \frac{1}{i} \sin(ix-y)$

$= -i \sin(ix-y)$

$u(x,y) = \sinh x \cos y$

$v(x,y) = \cosh x \cdot \sin y$.

Que. If $f(z) = u + iv$ is analytic function of z and $U - V = e^x(\cos y - \sin y)$. find $f(z)$ in Term of z.

Soln. $f(z) = u + iv$ —— (I).
$if(z) = iu + i^2 v$.
$if(z) = iu - v$ —— (II).
(I) + (II)
$(1+i) f(z) = (u-v) + i(u+v)$.
Let $(1+i) f(z) = F(z)$.
$u - v = U$
$u + v = V$
$F(z) = U + iV$. —— (III).
$U = u - v = e^x(\cos y - \sin y)$
Let V is harmonic conjugate of U.

$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy$.

$$\left\{ \begin{array}{l} \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \\ \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \end{array} \right.$$

$dV = -\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy$ —— (IV)

$U = e^x(\cos y - \sin y)$
$\frac{\partial U}{\partial x} = (\cos y - \sin y) \cdot e^x$.
$\frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$,

putting in (IV)
$dV = -e^x(-\sin y - \cos y) dx + e^x(\cos y - \sin y)$

Dt. ____
Pg. ____

$\int dv = -\int e^x(-\sin y - \cos y)\,dx + \int e^x(\cos y - \sin y)\,dy$
y const"

Those term which do not contain x

$V = -(-\sin y - \cos y)e^x + C$
$V = (\sin y + \cos y)e^x + c$

$f(z) = u + iv$
$= e^x(\cos y - \sin y) + i\{(\sin y + \cos y)e^x + c\}$
$= e^x\cos y - e^x\sin y + ie^x\sin y + ie^x\cos y + ic$

$= e^x(\cos y + i\sin y) + ie^x(\cos y + i\sin y) + ic$

$= e^x \cdot e^{iy} + ie^x \cdot e^{iy} + ic$

$f(z) = e^{x+iy} + i \cdot e^{x+iy} + ic$

using $z = x + iy$

$f(z) = e^z + i \cdot e^z + ic$
$f(z) = e^z(1+i) + ic$
$(1+i) f(z) = e^z(1+i) + ic$. using $f(z) = (1+i) f(z)$.

Qu. If $u - v = (x-y)(x^2 + 4xy + y^2)$, and $f(z) = u + iv$ is analytic function of $z = x + iy$. find $f(z)$ in Term of z.

Dt. ______

Pg. ______

Qu P.T the following function are ~~know~~ no where analytic.

(I) $\bar{z}$.

(II) $z\bar{z}$

(III) $z-\bar{z}$

(IV) $e^{x}\cdot e^{-iy}$

(I) $f(z)=\bar{z}$.

$\left\{ \begin{array}{l} z=x+iy, \\ \bar{z}=x-iy. \end{array} \right.$

$f(z)=x-iy$.

$f(z)=u+iv$.

$u=x,\ v=-y$.

$\frac{\partial u}{\partial x}=1,\ \frac{\partial u}{\partial y}=0$

$\frac{\partial v}{\partial x}=0,\ \frac{\partial v}{\partial y}=-1$.

$\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$

i.e. function for every value of x, y.

i.e " is no where analytic.

Qu. Construct the analytic functn. $f(z)$ of is the real part is. $2xy+2x$.

Qu If $f(z)=\frac{x^3y(y-ix)}{x^6+y^2}$ $(z\neq0)$, $f(0)=0$.

P.T the increament ratio, $\frac{f(z)-f(0)}{z}$ tends to 0 as $z\rightarrow0$ along any radius vector. But not as $z\rightarrow0$ as any manner.

Sol^n

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{\frac{x^3 y (y - ix)}{x^6 + y^2} - 0}{x + iy}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 y (y - ix)}{(x^6 - y^2)(x + iy)}$$

along origin we take $y = mx$.

$$= \lim_{x \to 0} \frac{x^3 \cdot mx (mx - ix)}{(x^6 + m^2 x^2)(x + imx)}$$

$$= \lim_{x \to 0} \frac{x^5 m (m - i)}{x^3 (x^4 + m^2)(1 + im)}$$

$$= \lim_{x \to 0} \frac{x^2 m (m - i)}{(x^4 + m^2)(1 + im)}$$

$$= 0$$

Taking, $y = x^3$.

$$\lim_{x \to 0} \frac{x^3 \cdot x^3 (x^3 - ix)}{(x^6 + x^6)(x + ix^3)}$$

$$= \lim_{x \to 0} \frac{x^7 (x^2 - i)}{2x^7 (1 + ix^2)}$$

$$= -i/2 \neq 0$$

Qu. P.T $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

Sol^n. $I = \lim_{z \to 0} \frac{\bar{z}}{z}$

$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x - iy}{x + iy}$

along origin, consider a path $y = mx$.

$= \lim_{x \to 0} \frac{x - imx}{x + imx}$.

$= \frac{1 - im}{1 + im}$

here limit depends upon m, so, limit $z \to 0$.

$\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

Q. Examine the continuity of the funct^n.

$f(z) = \frac{[\operatorname{Im} z]^2}{|z|}$ when $z \neq 0$, $f(0) = 0$.

Sol^n Any funct^n is continuous when (limit of funct^n = value of function)

$z = x + iy$

$|z| = \sqrt{x^2 + y^2}$

$\operatorname{Im}_z = y$.

Taking $x = r\cos\theta$ and

$y = r\sin\theta$.

When, $z \to 0$ Then $r \to 0$,

$$f(z) = \frac{[Im z]^2}{|z|}$$

$$= \frac{y^2}{\sqrt{x^2+y^2}}.$$

lim of funct^n = $\lim_{z \to 0} f(z) = \lim_{r \to 0} \frac{r^{\cancel{2}} \sin^2\theta}{\cancel{r}}$,

$= 0$.

Limit of funct^n of value of funct^n.

Qu. Examine the continuity at the origin.

$$f(z) = \frac{Re z + Im z}{|z|}, \quad z \neq 0 \text{ and } f(0) = 0.$$

Sol^n

$z = x + iy$

$Re\, z = x$

$Img.\, z = y$

$|z| = \sqrt{x^2+y^2}$

$$\therefore f(z) = \frac{Re z + Im z}{|z|}$$

$$f(z) = \frac{x+y}{\sqrt{x^2+y^2}}$$

In Polar Co-ordinate.

Taking $x = r\cos\theta$

$y = r\sin\theta$

$$f(z) = \frac{r\cos\theta + r\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

at $z \to 0$

$$\lim_{z\to 0} f(z) = \lim_{r\to 0} \frac{\cancel{r}(\cos\theta + \sin\theta)}{\cancel{r}}$$

$$= \cos\theta + \sin\theta$$

limit depends upon θ.

So limit does not exist

Function is not continuous at origin.

Q. Examine the continuity at origin for the fun^n $f(z) = \frac{Re\, z \cdot Im\, z}{|z|}$, $z \neq 0$. $f(0) = 0$.

Q. Examine the Continuity at Origin $f(z) = \sin\theta$ for $z = re^{i\theta}$, $r > 0$. $f(z) = 0$ for $z = 0$.

$x = r\cos\theta$

$y = r\sin\theta$

$\frac{y}{r} = \sin\theta$.

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$x^2+y^2=r^2$

$$f(z) = \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{mx}{\sqrt{x^2+m^2x^2}}$$

$$= \frac{m}{\sqrt{1+m^2}}$$

Here, limit depends upon m. double limit does not exist ∴ Function is Discontinuous at Origin.

Milnes Thomson Method for Finding f(z)

If $f(z) = u(x,y) + iv(x,y)$ is an analytic funⁿ. If $u(x,y)$ is given then by Milnes Thomson Method.

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c.$$

Where,

$$\phi_1 = \frac{\partial u}{\partial x}, \quad \phi_2 = \frac{\partial u}{\partial y}$$

Similarly,

If $v(x,y)$ is given then,

$$f(z) = \int \varphi_1(z,0)\,dz + i\int \varphi_2(z,0)\,dz + c$$

Where,

$\psi_1 = \frac{\partial v}{\partial y}$ $\psi_2 = \frac{\partial v}{\partial x}$.

Q. Find $f(z)$ when the given fun^n are real part of $f(z)$

Hint

① $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

② $\sin x \cdot \cosh y + 2\cos x \cdot \sinh y + x^2 - y^2 + 4xy$

Sol^n ② $u(x,y) = \sin x \cdot \cosh y + 2\cos x \cdot \sinh y + x^2 - y^2 + 4xy$

$\phi_1 = \frac{\partial u}{\partial x} = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$\phi_1(z,0) = \cos z \cosh 0 - 2\sin z \sinh 0 + 2z + 4\times 0$

$\phi_1(z,0) = \cos z + 2z$.

$\phi_2 = \frac{\partial u}{\partial y} = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$\phi_2(z,0) = 0 + 2\cos z - 0 + 4z$

$\phi_2(z,0) = 2\cos z + 4z$

By Milnes Thomson Method.

$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + C$.

$$= \int (\cos z + 2z)\,dz - i\int (2\cos z + 4z)\,dz + c$$

$$= \sin z + \frac{\cancel{2}z^2}{\cancel{2}} - 2i\sin z - i\frac{\cancel{4}z^2}{\cancel{2}} + c$$

$$= z^2 + \sin z - 2i\sin z - 2iz^2 + c.$$

$$\boxed{f(z) = z^2 + \sin z - 2i\sin z - 2iz^2 + c}$$

Q. Find $f(z)$ of which the imaginary component is $2x(1-y)$.

Fundamental Theorem of limit :-

If $\omega = f(z) = u(x,y) + iv(x,y)$ & $z_0 = x_0 + iy_0$

$\omega_0 = u_0 + iv_0$. Then $\lim_{z \to z_0} f(z) = \omega_0$ iff.

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \;\&\; \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

If.

$$\lim_{z \to z_0} f(z) = \omega_0$$

By definition of limit

$$|f(z) - \omega_0| < \varepsilon \quad , \quad \varepsilon > 0$$

$$0 < |z - z_0| < \delta \quad , \quad \delta > 0$$

Dt.
Pg.

i.e. $|f(z) - w_0| < \varepsilon$

$|u + iv - (u_0 + iv_0)| < \varepsilon$

$|(u - u_0) + i(v - v_0)| < \varepsilon$ —— ①

Again,

$0 < |z - z_0| < \delta$

$0 < |x + iy - (x_0 + iy_0)| < \delta$

$0 < |x - x_0 + i(y - y_0)| < \delta$

$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$ —— ②

Again,

$|u - u_0| \le |u - u_0 + i(v - v_0)|$ —— ③

& $|v - v_0| \le |(u - u_0) + i(v - v_0)|$ —— ④

from ① & ②

$|u - u_0| < \varepsilon$

$|v - v_0| < \varepsilon$

& $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$

$\lim_{(x,y) \to (x_0, y_0)} u(x, y) = u_0$

$\lim_{(x,y) \to (x_0, y_0)} v(x, y) = v_0$

Dt.

Pg.

Only,

$$|u-u_0| < \frac{\varepsilon}{2}$$

For

$$0 < \{(x-x_0)^2 + (y-y_0)^2\}^{1/2} < \delta_1$$

$$|v-v_0| < \frac{\varepsilon}{2}$$

$$0 < [(x-x_0)^2 + (y-y_0)^2]^{1/2} < \delta_2$$

$$|(u-u_0)+i(v-v_0)| \leq |u-u_0| + |v-v_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$|(u-u_0)+i(v-v_0)| < \varepsilon$$

$$|u+iv-(u_0+iv_0)| < \varepsilon$$

$$|f(z)-w_0| < \varepsilon$$

i.e. $\lim_{z \to z_0} f(z) = w_0$, where,

$$0 < |x+iy-(x_0+iy_0)| < \delta.$$