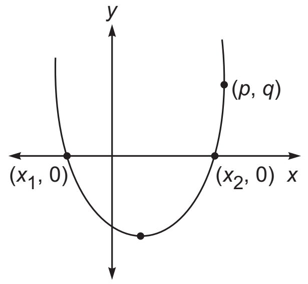
3 Mock Test Paper—3

1. Find the number of solutions for x if $50 \le x \le 120$ such that $\left[\frac{x}{6}\right] + \left[\frac{x}{5}\right] + \left[\frac{x}{10}\right] = \frac{7x}{15}$, where $[y]$ indicates the greatest integer less than or equal to y.
   (1) 0 (2) 1
   (3) 2 (4) 3
2. The number of ordered pairs of positive integers (m, n) such that $\frac{1}{m} + \frac{1}{n} = \frac{1}{15}$ is:
   (1) 10 (2) 2
   (3) 4 (4) 9
3. Let 'f' be a function such that for real x & y, $f(x) \times f(y) = f(x + y)$ & $\frac{f(x)}{f(y)} = f(x - y)$ for all $f(y) \ne 0$.
   Given that $f(0) = 1$ and $f(1) = 3$, then $f(x - 1) + f(-x - 1)$
   (1) cannot be negative.
   (2) has minimum value 2/3.
   (3) lies between 0 and 2/3.
   (4) none of these.
4. Let $a = \frac{1^2}{1} + \frac{2^2}{3} + \frac{3^2}{5} + \ldots + \frac{1001^2}{2001}$ and $b = \frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + \ldots + \frac{1001^2}{2003}$. The integer closest to $a - b$ is
   (1) 500 (2) 501
   (3) 1000 (4) 1001
5. Which of the following statements is false?
   (1) The product of three consecutive even numbers must be divisible by 48.
   (2) x% of y% of z is same as z% of x% of y.
   (3) The factorial of any natural number greater than 1 cannot be a perfect square.
   (4) The numbers $(100)_2$, $(100)_3$, $(100)_4$, $(100)_5$ … and so on, when converted to decimal system are all in an arithmetic progression.
6. Find the length of the string wound on a cylinder of height 48 cm and a base diameter of $5\frac{1}{11}$ cm. The string makes exactly four complete turns round the cylinder while its two ends touch the cylinder's top and bottom.
   (1) 192 cm (2) 80 cm
   (3) 64 cm (4) Cannot be determined
7. In Coorg, the production of tea is three times the production of coffee. If a percent more tea and b percent more coffee were produced, the aggregate amount would be $5c$ percent more. But if b percent more tea and a percent more coffee were produced, the aggregate amount produced would be $3c$ percent more. What is the ratio $a : b$?
   (1) 1 : 3 (2) 1 : 2
   (3) 2 : 1 (4) 3 : 1
8. A girl wished to purchase m roses for n rupees. (m and n are integers). The shopkeeper offered to give the remaining 10 flowers also if she paid him a total of ₹ 2. This would have resulted in a saving of 80 paise per dozen for her. How many flowers did she wish to buy initially?
   (1) 4 (2) 5
   (3) 6 (4) 7
9. Samit went to the market with ₹ 100. If he buys three pens and six pencils he uses up all his money. On the other hand if he buys three pencils and six pens he would fall short by 20%. If he wants to buy equal number of pens & pencils, how many pencils can he buy?
   (1) 4 (2) 5
   (3) 6 (4) 7

Directions for Questions 10 and 11: The graph given below represents a quadratic expression, $f(x) = ax^2 + bx + c$. The minimum value of $f(x)$ is –2 and it occurs at $x = 2$



x_1 and x_2 are such that $x_1 + x_2 + 2x_1x_2 = 0$.

10. What is the nature of the roots of the equation $f(x) = 0$?
(1) Complex conjugates
(2) Rational
(3) Irrational
(4) Cannot be determined

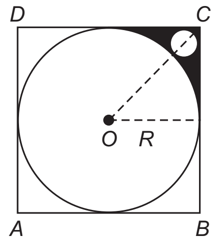
11. Find the value of the function at $x = 3$
(1) 2 (2) – 1
(3) 5 (4) none of these

12. How many numbers are there between 100 and 1000, which have exactly one of their digits as 7?
(1) 300 (2) 1444
(3) 225 (4) 729

13. A group of men was employed to shift 545 crates. Every day after the first, 6 more men than the previous day were put on the job. Also, every day after the first, each man working, shifted 5 fewer crates than the number of crates moved by each man the previous day. The result was that during the latter part of the period, the number of crates shifted per day began to go down. 5 days were required to finish the work.
What was the number of crates shifted on the third day?
(1) 137 (2) 169
(3) 26 (4) 152

14. A group of men was employed to shift 545 crates. Every day after the first, 6 more men than the previous day were put on the job. Also, every day after the first, each man working, shifted 5 fewer crates than the number of crates moved by each man the previous day. The result was that during the latter part of the period, the number of crates shifted per day began to go down. 5 days were required to finish the work.
What was the number of men working on the fifth day?
(1) 13 (2) 25
(3) 19 (4) 75

15. In the figure given, the radius of the smaller circle is $\sqrt{2} - 1$ cm. Then the area of shaded region will be



(1) $3 - \sqrt{2}$ cm^2
(2) $\frac{3}{4}(4 - \pi) + \sqrt{2}(4 - \sqrt{2}\pi)$ cm^2
(3) $\frac{3}{4}(4 - 5\pi) + \frac{1}{\sqrt{2}}(4 - 3\pi)$ cm^2
(4) Data insufficient

16. Two sisters go up 40-step escalators. The older sister rides the up escalator, but can only take 10 steps up during the ride since it is quite crowded. The younger sister runs up the empty down escalator, arriving at the top at the same time as her sister. How many steps does the younger sister take?
(1) 70 (2) 60
(3) 80 (4) 70

17. In how many ways may the numbers {1, 2, 3, 4, 5, 6} be ordered such that no two consecutive terms have a sum which is divisible by 2 or 3?
(1) 6 (2) 12
(3) 15 (4) 14

18. Let $f(n)$ denote the square of the sum of the digits of n. Let $f^2(n)$ denote $f(f(n))$, $f^3(n)$ denote $f(f(f(n)))$ and so on. Then $f^{1998}(11) =$
(1) 49 (2) 56
(3) 169 (4) 16

19. A biologist catches a random sample of 60 fish from a lake, tags them and releases them. Six months later she catches a random sample of 70 fish and finds 3 are tagged. She assumes 25% of the fish in the lake on the earlier date have died or moved away and that 40% of the fish on the later date have arrived (or been born) since. What does she estimate as the number of fish in the lake on the earlier date?
(1) 840 (2) 280
(3) 560 (4) 750

20. Michael Jordan's probability of hitting any basketball shot is three times than mine, which never exceeds a third. To beat him in a game, I need to hit a shot myself and have Jordan miss the same shot. If I pick my shot optimally, what is the maximum probability of winning which I can attain?
(1) $\frac{1}{16}$ (2) $\frac{1}{12}$
(3) $\frac{5}{6}$ (4) 1/14

ANSWER KEY

1. (4)	2. (4)	3. (2)	4. (2)
5. (4)	6. (2)	7. (4)	8. (2)
9. (1)	10. (3)	11. (2)	12. (3)
13. (2)	14. (2)	15. (3)	16. (1)
17. (2)	18. (3)	19. (1)	20. (2)

Solutions and Shortcuts

Solution 1: Level of Difficulty (2)

Since a greatest integer function always delivers an integer value, $\frac{7x}{15}$ must be an integer.

So possible values of x lying in the given domain would be 60, 75, 90, 105 and 120. Out of these 60, 90 and 120 satisfy the given equation.

Solution 2: Level of Difficulty (2)

$$\frac{1}{m}+\frac{1}{n}=\frac{1}{15} \Rightarrow \frac{m+n}{mn}=\frac{1}{15} \Rightarrow mn - 15(m+n) = 0$$

Adding 225 to both sides we get, $mn - 15(m + n) + 225 = 225 \Rightarrow (m - 15)(n - 15) = 225$.

Now we have to consider all the factors of 225 which are 1, 3, 5, 9, 15, 25, 45, 75 and 225. Hence 9 possible ordered pairs of (m, n) satisfy the given condition.

Solution 3: Level of Difficulty (2)

The given properties lead us to an exponential function, i.e. $f(x) = a^x$. Given that $f(1) = 3 \Rightarrow a = 3$. So the given function is $f(x) = a^x$.

$$f(x-1)+f(-x-1) = 3^{x-1} + 3^{-x-1} = \frac{1}{3}(3^x + 3^{-x}) \geq \frac{1}{3}\cdot 2\sqrt{3^x \cdot 3^{-x}}$$

$[\therefore \text{A.M} \geq \text{G.M.}]$

$\Rightarrow 3^{x-1} + 3^{-x-1} \geq 2/3$.

Hence the minimum value of $3^{x-1} + 3^{-x-1} = 2/3$.

Solution 4: Level of Difficulty (2)

$$a-b=\frac{1^2}{1}+\frac{2^2-1^2}{3}+\frac{3^2-2^2}{5}+\frac{4^2-3^2}{7}+\ldots+\frac{1001^2-1000^2}{2001}-\frac{1001^2}{2003}$$

Since = = 1 for all $k \geq 0$, it follows that

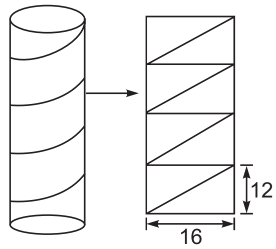
$a - b =$ – $\approx 1001 - 500.25 \approx 500.75$

Hence, $a - b \approx 501$.

Solution 5: Level of Difficulty (2)

The numbers in option (4) can be written as $2^2, 3^2, 4^2, 5^2 \ldots$ so on, which are not in an arithmetic progression. All the other statements always hold true.

Solution 6: Level of Difficulty (2)



The base circumference $= \frac{22}{7}\times\frac{56}{11} = 16$ cm. The length of one complete turn $= \sqrt{16^2+12^2} = 20$ cm. Hence, total length = 80 cm.

Solution 7: Level of Difficulty (2)

Let x be the production of tea and y be the production of coffee. It is given that $x = 3y$.

Also, $x(100 + a) + y(100 + b) = (x + y)(100 + 5c)$

$\rightarrow \quad x(a - 5c) = y(5c - b)$

i.e. $\quad 3a - 15c = 5c - b$

i.e. $\quad 3a + b = 20c \quad$ (i)

Also, $x(100 + b) + y(100 + a) = (x + y)(100 + 3c)$

$\rightarrow \quad x(b - 3c) = y(3c - a)$

i.e. $\quad 3b - 9c = 3c - a$

i.e. $\quad a + 3b = 12c \quad$ (ii)

Eqn. (i) – 3 × Eqn. (ii) given,

$$3a + b = 20c$$
$$a + 3b = 12c$$
$$-8b = -16c$$

i.e. $\quad b = 2c$

$\therefore \quad 3a = 18c \rightarrow a = 6c$

$\therefore \quad \frac{a}{b} = \frac{6c}{2c} = \frac{3}{1}$

Hence (4).

Solution 8: Level of Difficulty (2)

$n = 1$; since n is an integer < 2.

$\therefore \quad \frac{100}{m} - \frac{200}{m+10} = \frac{80}{12}$

$\therefore \quad m = 5$.

Hence [2].

Solution 9: Level of Difficulty (1)

9 pens + 9 pencils would cost him ₹ 225. Hence, he can buy 4 pens and 4 pencils for ₹100.

Solution 10: Level of Difficulty (2)

The differentiation of the function would give:

$2ax + b = 0 \rightarrow x = -b/2a = 2 \rightarrow -b = 4a \rightarrow b = -4a$.

Hence, the expression would become: $x^2 - 4x + c$. At $x = 2$, it becomes $4 - 8 + c = -2 \rightarrow c = 2$. Hence, the equation is $x^2 - 4x + 2$.

10. 3

It can be seen for this equation that the roots are irrational.

Solution 11: Level of Difficulty (1)

$\rightarrow b = -4a$.

Hence, the expression would become: $x^2 - 4x + c$. At $x = 2$, it becomes $4 - 8 + c = -2 \rightarrow c = 2$

Hence, the equation is $x^2 - 4x + 2$.

At $x = 3$, the function becomes $11 - 12 = -1$.

Solution 12: Level of Difficulty (2)

Solution 13: Level of Difficulty (2)

Consider a three digit number in which 7 comes exactly once i.e. from 107 to 997.

If you fix seven at the units place and use the other digits for filling in the other 2 places you will get 72 numbers in which 7 comes at unit place i.e. $\underline{8} \times \underline{9} \times {}_{\underline{7}}$

Similarly, at the ten's place there would be i.e. $\underline{8} \times {}_{\underline{7}} \times 9 = 72$ ways

And also at the hundred's place there would be ${}_{\underline{7}} \times \underline{9} \times 9 = 81$ ways

So, the total numbers from 100 to 1000, there are $72 + 72 + 81 = 225$ numbers in which they have exactly one of their digit is 7.

Solution 13: Level of Difficulty (2)

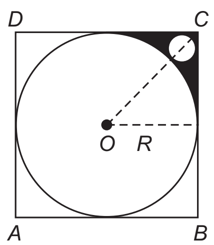
Solve through trial and error (2)

Solution 14: Level of Difficulty (2)

Solve through trial and error (2)

Solution 15: Level of Difficulty (3)

Let radius of bigger circle be R.



$\therefore\ OC = R\sqrt{2}$ cm.

If 'r' be the radius of smaller circle. Then $R\sqrt{2} = R + r + r\sqrt{2}$ (By Fig.)

$$R(\sqrt{2} - 1) = r(\sqrt{2} + 1)$$

$$R(\sqrt{2} - 1) = (\sqrt{2} - 1)(\sqrt{2} + 1) \quad (\because r = \sqrt{2} - 1)$$

$$\Rightarrow \qquad R = (\sqrt{2} + 1)\text{cm.}$$

$$\text{Area of shaded part} = R^2 - \frac{\pi R^2}{4} - \pi r^2$$

$$= (\sqrt{2} + 1)^2 - \frac{\pi}{6}(\sqrt{2} + 1)^2 - \pi(\sqrt{2} - 1)^2$$

$$= 2 + 1 + 2\sqrt{2} - \pi\left(\frac{2+1+2\sqrt{2}}{4} + 2 + 1 - 2\sqrt{2}\right)$$

$$= 3 + 2\sqrt{2} - \frac{\pi}{4}(3 + 2\sqrt{2} + 12 - 8\sqrt{2})$$

$$= 3 + 2\sqrt{2} - \frac{\pi}{4}(15 - 6\sqrt{2})$$

$$= 3\left(1 - 5\frac{\pi}{4}\right) + 2\sqrt{2}\left(1 - 3\frac{\pi}{4}\right)$$

$$= \frac{3}{4}(4 - 5\pi) + \frac{1}{\sqrt{2}}(4 - 3\pi)$$

Solution 16: Level of Difficulty (2)

In the time it takes the older sister to reach the top, the up escalator has carried her forward 30 steps, because the total length of the escalator is 40 steps and she took 10 steps herself. Therefore, in the same time the down escalator pushes the younger sister back 30 steps. To make up this set-back and cover the original 40 steps separating her from the top the younger sister must make a total of 70 steps.

Solution 17: Level of Difficulty (2)

Each of the numbers from 1 to 6 can only have two possible neighbours to avoid sums divisible by 2 or 3. For example, 5 may neighbour only 2 or 6. If we write the numbers 1, 4, 3, 2, 5, and 6 in that order in a ring then each number will be next to its two possible neighbors. Therefore, to order the six numbers successfully we need only start at any point along the ring (6 choices) and list the numbers as they appear around the ring in either direction (2 choices) for a total of 12 possible orderings.

Solution 18: Level of Difficulty (1)

We find $f^4(11) = 169, f^6(11) = 169 \quad \therefore f^{1998}(11) = 169$.

Solution 19: Level of Difficulty (2)

Suppose there are N fishes initially. Then after 6 months, $N \times 75\%/60\% = 5N/4$ fishes, and $60 \times 75\% = 45$ tagged. Estimate $45/(5N/4) = 3/70$, or $N = 840$

Solution 20: Level of Difficulty (2)

If I choose a shot that I will make with probability p (where p is between 0 and 1/3), then Michael Jordan will make the same shot with probability $3p$. Hence, the probability that I make a shot that Jordan subsequently misses is $p(1 - 3p)$. The graph of this function is a parabola which equals zero when $p = 0$ and $p = \frac{1}{3}$. By symmetry the vertex (maximum) is midway between the two, at $p = \frac{1}{6}$.

Hence, the best chance I have of winning the game is $\frac{1}{12}$.