Discrete signal and system.

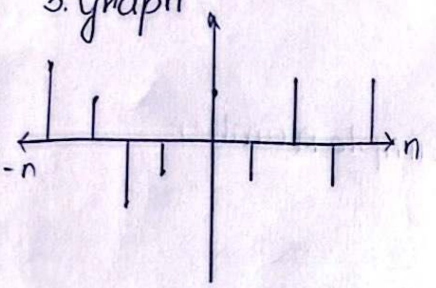
Disbrete time signal

There are those signal in which signal have value only for discrete value of lime.

They can be oblained by sampling of continious time signal or by accumul ating variable over time

It represent as

1. Sequence of numbers : $x(n) = \{0, 1, 8, 0.5, 0.685\}$
2. Function of n: $x(n) = 8.5$ for $n >= 0$, otherwise 0
3. Graph

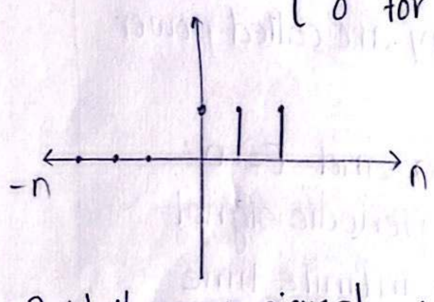


Basic/elementry Signal of discrete time

1. Unit step signal

It models the phenomenon that changes values in steps

i.e $u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$



2. Unit ramp signal

$u_r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$

3. Unit impulse/Sample sequence.

It makes the phenomenat that appear only for short period of time

$$\delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n\neq 0 \end{cases}$$

4 Exponential Function

It may be real or complex

It may be growing or decaying

The exponential function is a sequence of form

$x(n) = a^n$ for all n

5. Sinusodial Signal

$x(n) = A\cos(wn+\theta)$, $-\infty < n < \infty$

where n = is a integer variable called sample number

A = amplitude of sinusodial

W = frequency in radon per second

θ = phase in radian

Energy signal	Power signal.
The signal that have finite total energy and zero average power are called energy signal.	The signal that have finite average power and infinite total energy are called power signal.
$0 < E < \infty$ and $P_{av} = 0$	$0 < P_{av} < \infty$ and $E = \infty$
Most of non periodic signal	Most of periodic signal
Time limited	Exist over infinite time
$E = \lim_{N\to\infty} \sum_{n=-N/2}^{N/2} \lvert x(n)\rvert^2$	$P_{av} = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=-N/2}^{N/2} \lvert x(n)\rvert^2$

Q. Check signal $x(n) = u(n)$ and $x(n) = \delta(n)$ is energy or power signal.

1. For $x(n) = u(n)$

weknow that,

$$x(n) = u(n) = \begin{cases} 1 & \text{for } n >= 1 \\ 0 & \text{for } n < 0 \end{cases}$$

Now,

$$P_{av} = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=-\frac{N}{2}}^{N/2} |x(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N/2} |u(n)|^2$$

$$= \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N/2} 1.$$

$$= \lim_{N \to \infty} \frac{1}{N+1} \times \left(\frac{N}{2} + 1\right)$$

$$= \lim_{N \to \infty} \frac{1}{N(1+\frac{N}{2})} \times N\left(\frac{1}{2} + \frac{1}{N}\right)$$

$$= \frac{1}{2} \text{ finite.}$$

Now,

$$E = \lim_{N \to \infty} \sum^{N/2} |x(n)|^2$$

$\therefore E = \infty$

$$= \lim_{N \to \infty} \sum_{n=-N/2}^{N/2} |u(n)|^2$$

$\therefore$ Hene $x(n) = u(n)$ is power signal.

$$= \lim_{N \to \infty} \sum_{n=0}^{N/2} 1.$$

$$= \lim_{N \to \infty} \frac{N}{2} + 1.$$

Periodicity of discrete time signal.

A discrete time signal is said to be periodic if it repeats itself after every samples

Mathematically, a discrete time signal is said to be perio-dic if and only if given condition is true.

$x(n+N) = x(n)$ for all n

The smalles value on N for which the condition holds is called fundamental period.

Q. Check the signal $x(n) = \cos(2n\pi/5) + \sin(\frac{n\pi}{3})$ is periodic or not.

Given $x(n) = \cos\frac{2n\pi}{5} + \sin\frac{n\pi}{3}$

Here,

$\cos\frac{2\pi n}{5} = \cos 2.n.\frac{1}{5}.\pi = \cos\left(2\pi.\frac{6}{30}.n\right)$

$\sin\frac{n\pi}{3} = \sin n.\frac{1}{3}.\pi = \sin\left(2\pi.\frac{5}{30}.n\right)$

As the frequency of both component can be expressed in form $f_0 = \frac{r}{N}$ with $N=30$

$\therefore$ the given signal is periodic with period $N=30$.

1.4. Transformation of independent variable.

1. Time Shifting

- It is the shifting of signal in time
- It is done by adding or subtracting the current of the shift of independent variable(n) in the function.
- Subtracting will shift the signal to right. It is called time delay.
- Adding will shift the signal to left It is called time advance.

2. Time Scaling

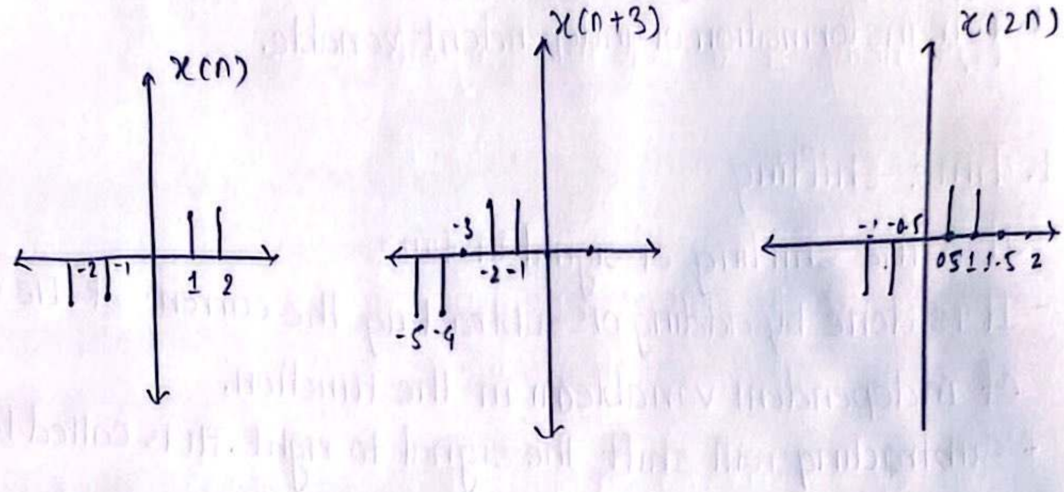
- It compresses and dilates the signal
- It is done by multiplying the independent variable (n) by scale amount.
- If amount >1, the signal become narrower it is called compression.
- If amount is <1, the signal become wider It is called dilation.

3. Time Reversal.

It flips the signal over y-axis

It is done by scaling the signal with negative amount.

let us consider $x(n) = \begin{cases} 3 & n = 1, 2 \\ -2 & n = -1, -2 \\ 0 & \text{otherwise} \end{cases}$



1.5. Discrete time Fourier series and properties.

A periodic discrete time signal $x(n)$ with period N can be represen[ted] as Fourier series.

where,

$$a_k = \frac{1}{N} \sum_{k=N} a_k e^{jk(2\pi/N)n} \quad \text{—(i)}$$

where,

$$a_k = \frac{1}{N} \sum_{n=N} x(n)\, e^{-jk(2\pi/N)n} \quad \text{——(ii)}$$

where N= period of discrete time signal $x(n)$

eqn (i) is called synthetic eqution

eqn (ii) is called analytic equation

The properties of fourier series.

let us consider $x(n)$ and $y(n)$ be a perodic signal with perio[d] N such that.

$$x(n) \xleftrightarrow{DTFS} a_k$$

$$y(n) \xleftrightarrow{DTFS} b_k$$

1. Linearity

$$A x(n) + B y(n) \xleftrightarrow{DTFS} A a_k + B b_k$$

2. Periodic convolution

$$\sum_{l=N} x(l)\, y(n-l) \longleftrightarrow N a_k b_k$$

3. Multiplication

$$x(n)\, y(n) \longleftrightarrow \sum_{i=n} a_i b_{k-i}$$

4. First difference

$$x(n) - x(n-1) \longleftrightarrow (1 - e^{jk(2\pi/n)}) a_k$$

5. Conjugation

$$x^*(n) \longleftrightarrow a^*_{-k}$$

6. Time Reversal

$$x(-n) \longleftrightarrow a_{-k}$$

7. Time shifting

$$x(n-n_0) \longleftrightarrow a_k\, e^{-jk\, 2\pi/N\, n_0}$$

8. Frequency shifting

$$e^{jm\, 2\pi/N n}\, x(n) \longleftrightarrow a_{k-m}.$$

1. Discrete Time Forier Transform and properties.

It is the mathematical tool which is used to convert a discrete time sequence into frequency time domain.

Mathematically if $x(n)$ is a discrete time sequence then DIFT is defined as

$$F[x(n)] = X(w) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

Properties:

let us consider $x(n)$, $x_1(n)$, $x_2(n)$ be signal such that

$$x(n) \xleftrightarrow{DIFT} X(w)$$
$$x_1(n) \longleftrightarrow X_1(w)$$
$$x_2(n) \longleftrightarrow X_2(w)$$

1. Linearity

$$a x_1(n) + b x_2(n) \longleftrightarrow a X_1(w) + b X_2(w)$$

2. Time-shifting

$$x(n-k) \longleftrightarrow e^{-jwk} X(w)$$

3. Frequency shifting

$$x(n)\, e^{jw_0 n} \longleftrightarrow X(w-w_0)$$

4. Time Reversal

$$x(-n) \longleftrightarrow X(-w)$$

5. Frequency Differaction

$$n x(n) \longleftrightarrow j \frac{d}{dw} X(w)$$

6. Time convolution

$$x_1(n) * x_2(n) \longleftrightarrow x_1(w)\, X_2(w)$$

7. Conjugation

$$x^*(n) \longleftrightarrow X(-w)$$

$$x^*(-n) \longleftrightarrow X(w)$$

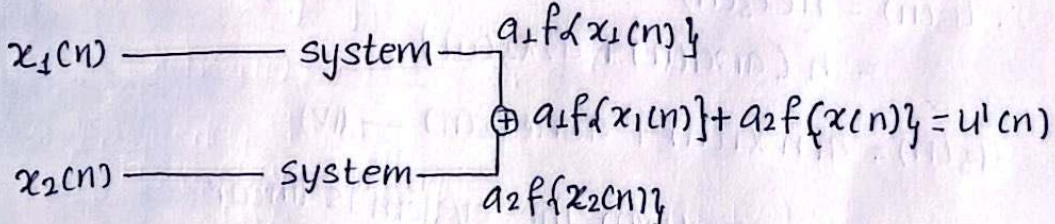
8. Preseval's Relation

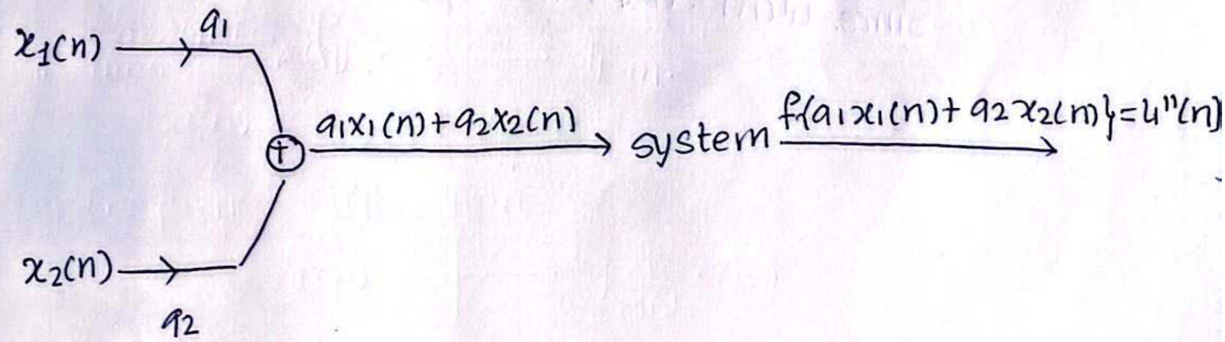
$$\sum_{n=-\infty}^{\infty} |x(n)|^2 \longleftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(w)|^2 dw$$

1.7 Discrete Time system Properties

1. Linearity

A system is called linear if superposition principle applies to that system. This means a linear system may be defined as one which response to the sum of the weighted input is same at the sum of weighted response.





If $u'(n) = u''(n)$: the system is linear otherwise the system is non-linear where a_1 and a_2 are scalar quantities

Q. Determine whether the system is linear or not

$y(n) = n x(n)$ ——— (A)

Let $x_1(n)$ and $x_2(n)$ be the two input and $u_1(n)$ and $u_2(n)$ b respective output.

$$u_1(n) = n x_1(n) \text{ ——— (i)}$$
$$u_2(n) \quad n x_2(n) \text{ ——— (ii)}$$

let $u'(n)$ be the linear combination of output

$$u'(n) = a_1 u_1(n) + a_2 u_2(n)$$
$$u'(n) = a_1 n x_1(n) + a_2 n x_2(n) \text{ ——— (iii)}$$

let $x_3(n)$ be the linear combination of input and $u_3(n)$ be the crossponding output.

$$x_3(n) = a_1 x_1(n) + a_2 x_2(n)$$
$$\therefore u_3(n) = n x_3(n)$$
$$= n(a_1 x_1(n) + a_2 x_2(n))$$
$$u_3(n) = a_1 n x_1(n) + a_2 n x_2(n) \text{ — (iv)}$$

Since $u'(n) = u_3(n)$ the system is linear

2) Time variant and Time invariant system.

→ If the output response of the system change with time, then the system is said to be time variant.

→ To check for time variance, we need to delay the i/p by 'k' unit and represent it by $u(n-k)$

→ Again, we delay the system by 'k' unit where we find independent variable either t or n^2 and represent it by $u(n-k)$

→ If $y(n,k) = u(n-k)$: system = shift invariance.

→ If $u(n,k) \neq u(n-k)$; system = time variant.

Q. Check $y(n) = n x(n)$ is shift variance or not.

Soln.

Delaying only the input by 'k' unit.

$y(n,k) = n x(n-k)$ ——— (i)

delaying throughout the system by k unit

$y(n-k) = (n-k) x(n-k)$ ——— (ii)

Here,

$y(n,k) \neq y(n-k)$

So ~~th~~ system is shift variance.

3) Causal and Non causal system

If the present output of system depends only the present and past input and output then system is called causal system.

If the present op of system depend on the future value of input or output irrespective of present and past response such system are called non causal system.

Q. Check for causability or not

$y(n) = x(n) + x(n-1) + x(n+2)$

for $n=0$

$y(0) = x(0) + x(-1) + x(2)$

$y(0)$	$x(0)$	$x(-1)$	$x(2)$
↑	↑	↑	
Present o/p	present I/p	past i/p	future i/p

Since present o/p depends on future i/p of system is non-causal

4) Memory and memory less system.

→ If o/p at $x=x_0$ depends only on i/p at $x=0$ then the system is called memoryless or static

→ If o/p at $x=x_0$ depends on i/p or o/p at $x \neq x_0$, the system is called memory or dynamic system.

5. Stable and non stable system.

If op of a system is bounded (finite) for input such system is called stable system.

If input is bounded then $(x(n)) \leq m_x < \infty$ for stability

$(y(n)) \leq m_y < \infty$, where m_x and m_y are finite quantities.

Q. Check wheather system is stable or not.

$y(n) = e^{x(n)}$

let the input be

$|x(n)| \leq m_x < \infty$

where m_x is finite quantity

now, $|y(n)| \leq e^{|x(n)|}$

$|y(n)| \leq e^{m_x}$

Since e^{m_x} always produced finite value Hence the system is stable.

1.8. Linear time Invariant (LTI) system, convolution, sum, properties of LTI system.

If the system is both linear and ~~finite~~ time invariant, then the system is called LIT system, They are of two type continous time and discrete time. They are characterized by their impulse response. The complete characterization of any LIT system in term of its impulse response is performed by convolution integeral in continious time and by convolution sum in discrete time.

let us consider a relaxed LIT system, A relaxed system means if input $x(n)$ is zero then output $y(n)$ is also zero.

let an input impulse function $\delta(n)$ is applied to this system then its output is denoted by $h(n)$ and is known as impulse response of the system.

Since system is time invarient if we delay input by k-samples then output should also be delayed by k-samples.

i.e.

i) $\delta(n) \xrightarrow{T} y(n) = h(n)$

ii) $\delta(n-k) \xrightarrow{T} y(n) = h(n-k)$

Now multiplying both side $x(k)$ we get.

iii) $x(k)\,\delta(n-k) \xrightarrow{T} y(n) = x(k)\,h(n-k)$

Since the system is linear, we can appily supposition theorem as.

iv) $$\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \xrightarrow{T} y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

Thus we have

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

The summation is called convolution sum which is obtained by taking convolution of $x(n)$ and $h(n)$

$$y(n) = x(n) \underset{+}{*} h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

where

$*$ Convolution sum (Mathematically)

$+$ Circular convolution

Q. Find the convolution sum of given input sequence.

$x(n) = \{1, 2, 1, 2\}$

$h(n) = \{1, 1, 1\}$

→ Here, $x(n) = \{1, 2, 1, 2\}$

i.e $x(0) = 1.$

$x(1) = 2$

$x(2) = 1.$ $\quad \therefore$ length of $x(n) = 4$

$x(3) = 2$

$x_l = 0$ [lowest value] $\quad \left\{ \begin{array}{l} \text{left } x_1 \\ \text{Right } x_2 \end{array} \right.$

$x_n = 3$ [highest value]

$h(n) = (1, 1, 1)$

i.e $h(0) = 1.$

$h(1) = 1.$

$h(2) = 1.$

length of $h(n) = 3$

$h_l = 0$

$h_n = 2$

$\therefore$ The total length of convolution of $x(n)$ and $h(n)$ is given by

length of $y(n)$ = length of $x(n)$ + length of $h(n) - 1.$

$= 4 + 3 - 1.$

$= 6.$

Now,

$y_l = x_l + h_l = 0 + 3 = 3$

$y_n = x_n + h_n = 3 + 2 = 5$

$$\therefore y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\ h(n-k)$$

$$\therefore y(n) = \sum_{k=0}^{5} x(k)\ h(n-k)$$

now,

at $n=0$, $y(0) = \sum_{k=0}^{5} x(k)\ h(-k)$

$= x(0)\ h(-0) + x(1)\ h(-1) + x(2)\ h(-2) + x(3)\ h(-3) + x(4) \cdot h(-4) + x(5)\ h(5)$

$= 1\times1 + 2\times0 + 1\times0 + 2\times0 +$

$= 1.$

$\therefore y(0) = 1.$

at $n=1$ $\sum_{k=0}^{5} x(k)\ h(1-k)$

$= x(0)h(1) + x(1)h(0) + x(2)h(-1) + x(3)h(-2) + x(4)h(-3) + x(5)h(4)$

$= 1\times1 + 2\times1 + 1\times0 + \dots$

$= 1+2$

$= 3.$

at $n=2$, $y(2) = \sum_{k=0}^{5} x(k).h(2-k)$

$= x(0)\ h(2) + x(1)\ h(1) + x(2)h(0) + x(3)h(-1) + x(4)h(-2) + x(5)\ h(-3)$

$= 1\times1 + 2\times1 + 1\times1 + 0 + 0 + \dots$

$= 1+2+1$

$\therefore y(2) = 4$

At $n=3, y(3) = \sum_{k=0}^{5} x(k)h(3-k)$

$= x(0)h(3) + x(1)h(2) + x(2)h(1) + x(3)h(0) + x(4)h(-1) + x(5)h(-2)$

$= 1\times0 + 2\times1 + 1\times1 + 2\times1 + 0 + 0$

$= 5$

At $n=4$, $y(4) = \sum_{k=0}^{5} x(k).h(4-k)$

$= x(0).h(4) + x(1)h(3) + x(2)h(2) + x(3)h(1) + x(4)h(0) + x(5)h(-1)$

$= 1\times0 + 2\times0 + 1\times1 + 2\times1 + 0\times1 + 0\times0$

$y(4) = 3$

At $n=5$, $y(5) = \sum_{k=0}^{5} x(k).h(5-k)$

$= x(0)h(5) + x(1)h(4) + x(2)h(3) + x(3)h(2) + x(4)h(1)$
$+ x(5)h(0)$

$= 2\times1 + 0\times1 + 0\times1$

$y(5) = 2.$

Thus, $y(n) = \{1,3,4,5,3,2\} \Rightarrow$ linear convolution.

Q. Find the convolution sum using linear convolution and check the result using tabular convolution.

$$x_1(n) = \{2, \underset{\uparrow}{4}, 3\}, \quad x_2(n) = \{1, 3, \underset{\uparrow}{5}, 7\}$$

Soln:

$x_1(n) = \{2, \underset{\uparrow}{4}, 3\}$

$x_1(-1) = 2$ $\quad x_l = -1$

$x_1(0) = 4$ $\quad x_h = 1$

$x_1(1) = 3$

length of $x_1(n) = 3$

$x_2(n) = \{1, 3, \underset{\uparrow}{5}, 7\}$

$x_2(-2) = 1$

$x_2(-1) = 3$

$x_2(0) = 5$

$x_2(1) = 7$

length of $x_2(n) = 4$ $\quad x_l = -2$

$x_h = 1.$

$y_l = x_{1L} + x_{2L} = -1 + (-2) = -3$

$y_h = x_{1h} + x_{2h} = 1 + 1 = 2$

$y(n) = 3 + 4 - 1 = 6$

$$y(n) = x_1(n) * x_2(n) = \sum_{k=-3}^{2} x_1(k)\, x_2(n-k)$$

At $n = -3$,

$$y(-3) = \sum_{k=-3}^{2} x_1(k)\, x_2(-3-k)$$

$= x_1(-3)x_2(-6) + x_1(-2)x_2(-1) + x_1(-1)\,x_2(-2) + x_1(0)x_2(-3)$

$+ x_1(1)x_2(-4) + x_1(2)\,x_2(-5)$

$= 2 \times 1$

$= 2$

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At $n=-2$

$$y(-2) = \sum_{k=3}^{2} x_1(k)\, x_2(-2-k)$$

$$= x_1(-3)\,x_2(1) + x_1(-2)\,x_2(0) + x_1(-1)\,x_2(-1) + x_1(0)\,x_2(-2) + x_1(1)\,x_2(-3) + x_1(2)\,x_2(-4)$$

$$= 2\times 3 + 4\times 1$$

$$y(-2) = 10$$

At $n=-1$

$$y(-1) = \sum_{k=-3}^{2} x_1(k)\, x_2(-1-k)$$

$$= x_1(-3)\,x_2(2) + x_1(-2)\,x_2(1) + x_1(-1)\,x_2(0) + x_1(0)\,x_2(-1) + x_1(1)\,x_2(-2) + x_1(2)\,x_2(-3)$$

$$= \cancel{2\times 2}\ \ 2\times 5 + 4\times 5 + 3\times 1$$

$$y(-1) = 25.$$

At $n=0$

$$y(0) = \sum_{k=-3}^{2} x_1(k)\, x_2(-k)$$

$$= x_1(-3)\,x_2(3) + x_1(-2)\,x_2(2) + x_1(-1)\,x_2(1) + x_1(0)\,x_2(0) + x_1(1)\,x_2(-1) + x_1(2)\,x_2(-2)$$

$$y(0) = 2\times 7 + 4\times 5 + 3\times 3 = 43$$

at $n=1$

$$y(1) = \sum_{k=-3}^{2} x_1(k)\, x_2(1-k)$$

$$= x_1(-3)\,x_2(4) + x_1(-2)\,x_2(3) + x_1(-1)\,x_2(2) + x_1(0)\,x_2(1) + x_1(1)\,x_2(0) + x_1(2)\,x_2(-1)$$

$$= 4\times 7 + 3\times 5 = 43$$

At $n=2$

$$y(2) = \sum_{k=-3}^{2} x_1(k)\, x_2(2-k)$$

$$= 3 \times 7 = 21.$$

$\therefore y(n) = \{2, 10, 25, 43, 43, 21.$

Now, Tabular convolution

$$x_1(n) = \{2, 4, 3\}$$

$$x_2(n) = \{1, 3, 5, 7\}$$

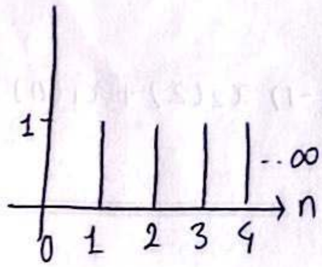
	2	4	3
1	2	4	3
3	6	12	9
5	10	20	15
7	14	28	21

Q. Given $x_1(n) = u(n) - u(n-5)$

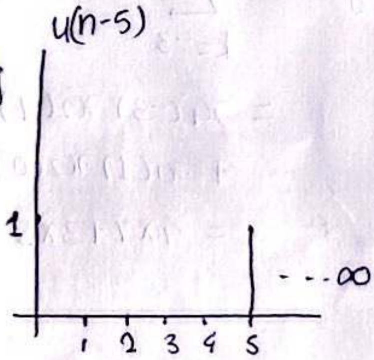
$x_2(n) = 2\,[u(n) - 4(n-3)]$

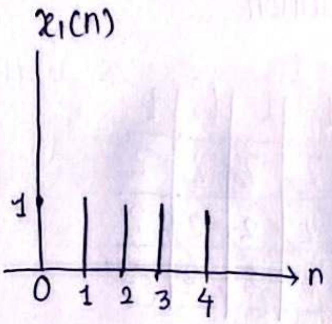
1. Plot $x_1(n)$ and $x_2(n)$
2. Calculate and plot $y(n) = x_1(n) * x_2(n)$

Soln: $u(n)$ is a unit step ie $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$



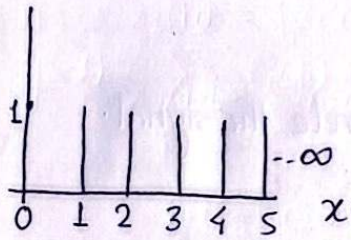
$\begin{Bmatrix} n-5=0 \\ n=5 \end{Bmatrix}$



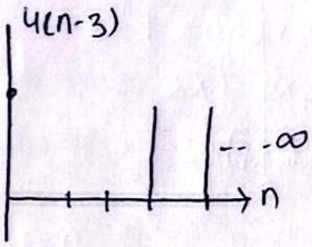


$x_1(n) = \{1, 1, 1, 1, 1\}$

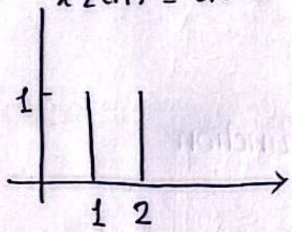
$x_2(n) = 2[u(n) - u(n-3)]$



$x_1(n) = \{1, 1, 1, 1, 1\}$



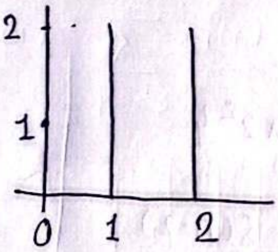
$x_2(n) = u(n) - u(n-3)$



$x_2(n) = \{1, 1, 1\}$

$= 2\{1, 1, 1\}$

$= \{2, 2, 2\}$



$\therefore y(n) = x_1(n) * x_2(n)$

length of $y(n) = 5+3-1$

$= 7$

$y_l = 0+0 = 0$

$y_n = y+2 = 6$

$\therefore y_n = \sum_{n=0}^{6} x_1(k)\, x_2(n-k)$

Tabular convolution

At $n=0$, $y(0)=2$
At $n=1$, $y(1)=4$
At $n=2$, $y(2)=6$
At $n=3$, $y(3)=6$
At $n=4$, $y(4)=6$
At $n=5$, $y(5)=4$
At $n=6$, $y(6)=2$

$\therefore y(n) = \{2, 4, 6, 6, 6, 4, 2\}$

	1	1	1	1	1
2	2	2	2	2	2
2	2	2	2	2	2
2	2	2	2	2	2

Imp. Obtain the linear convolution of two discrete the signal.

$$x(n) = \sum_{k=0}^{2} \delta(n-k)$$

$$h(n) = 2^n \{u(n) - u(n-3)\}$$

Sol^n

$$x(n) = \sum_{k=0}^{2} \delta(n-k)$$

$$x(n) = \delta(n-0) + \delta(n-1) + \delta(n-2)$$

$$= \delta(n) + \delta(n-1) + \delta(n-2)$$

$$\delta(n) = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$$ i.e unit impulse function

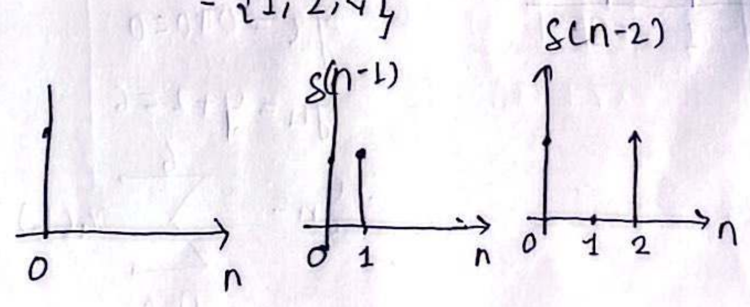
$$h(n) = u(n) - u(n-3) = \{1, 1, 1\}$$

$$h(n) = 2^n \{u(n) - u(n-3)\} = 2^n \{1, 1, 1\}$$

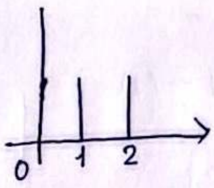
$$= \{1 \times 2^0, 1 \times 2^1, 1 \times 2^2\}$$

$$= \{1, 2, 4\}$$

$\therefore$ $\delta(n)$



$\therefore x(n) = \delta(n) + \delta(n-1) + \delta(n-2)$



$x(n) = \{1, 1, 1\}$
$h(n) = \{1, 2, 4\}$

$y(n) = 3 + 3 - 1 = 5$
$y_l = 0 + 0 = 0$
$y_n = 2 + 2 = 4$

$$y(n) = \sum_{k=0}^{4} x(k)\, h(n-k)$$

At $n=0$, $y(0) = 1$
At $n=1$, $y(1) = 3$
At $n=2$, $y(2) = 7$
At $n=3$, $y(3) = 6$
At $n=4$, $y(4) = 4$.

Tabular convolution

	1	1	1
1	1	1	1
2	2	2	2
4	4	4	4

$\therefore y(n) = \{1, 3, \cancel{4}7, 6, 4\}$

Q2. Obtain the linear convolution of two discrete time signal given as

$x(n) = u(n)$

$h(n) = a^n u(n)$, $a<1$

Solⁿ: Here $x(n) = u(n) = \{1, 1, 1, \ldots \infty\}$ — (1)

{from mathematical expression of u n}

$h(n) = \begin{cases} a^n & n \geq 0 \\ 0 & \text{otherwise } (n<0) \end{cases}$

$h(n) = \{a^0,$

$h(n) = \{a^0, a^1, a^2, \ldots \infty\}$ — ②

$h(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

$= \sum_{K=\infty}^{\infty} x(k)\, h(k)$

{-2, ... 2} range taken

$y(n) = x(-2)h(n+2) + x(-1)h(n+1) + x(0).h(n) + x(1)h(n-1) + x(2)h(n-2) +$

Note when both the sequence are infinite then in this case input $x(k)$ is kept as it is and $h(k)$ is delayed.

$y(n) = \cdots h(-2)\,u(n+2) + h(-1)\,u(n+1) + h(0)\,u(n) + h(1)\,u(n-1) +$

$h(2)\,u(n-2)$

Substituting $h(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{fo } n<0 \end{cases}$

for $n=0$, $h(0)=1$

$n=1$, $h(1) = a^1$

$n=2$, $h(2) = a^2$

$n=3$, $h(3) = a^3$

Scanned with CamScanner

$y(n) = 1.u(n) + au(n-1) + a^2 u(n-2) + a^3 u(n-3) + \cdots$ Ⓐ

at $n=0$, $y(0)=1$

at $n=1$, $y(1) = 1+a$

at $n=2$, $y(2) = 1+a+a^2$

at $n=3$, $y(3) = 1+a+a^2+a^3$

at $n=n$, $y(n) = \sum_{k=0}^{n} a^k$ —— Ⓑ

We know standard equation of summation is given by

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{for } a \neq 1 \\ N_2 - N_1 + 1 & \text{for } a = 1 \end{cases}$$

using the formula for eqn B

$$y(n) = \frac{a^0 - a^{n+1}}{1-a}$$

$$y(n) = \frac{1-a^{n+1}}{1-a}$$

$x(n) = u(n)$

$h(n) = a^n u(n) \quad a<1$

Graphical method for convolution sum the linear convolution of two sequence is given by

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad \text{———} (1)$$

we have to calculate

$y(n)$ at $n = 0, 1, 2, \ldots\ldots$

For $n=0$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) \quad \text{——} (ii)$$

Here, $h(-k)$ represent the folding of $h(k)$

for $n=1$

$$y(1) = \sum_{-\infty}^{\infty} x(k)\, h(1-k)$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k+1)$$

The term $h(1-k)$ can be written as $h(-k+1)$ which can be obtained by $h(-k)$ delayed by '1' sample. Similarly, for other 'n' output $y(n)$ is calculated.

Thus, different operation involved in graphical convolution are

1) Folding operation : It indicate folding of sequence $h(k)$

2) shifting operation: It indicate time shifting of $h(-k)$ i.e

$h(-k+1)$

3) Multiplication: It indicate the multiplication of $x(k)$ and $h(n-k)$

4) Summation: It indicates addition of all product term obtained from multiplication of $x(k)$ and $h(n-k)$

Q. Obtained the convolution sum using graphical method

$$x(k) = \{1, 2, 1, 2\}$$
$$h(k) = \{1, 1, 1\}$$

Here,

$x(k) = 0$ to 3; 4

$h(k) = 0$ to 2; 3

$y(n) = 0$ to 5; $4+3-1 = 6$

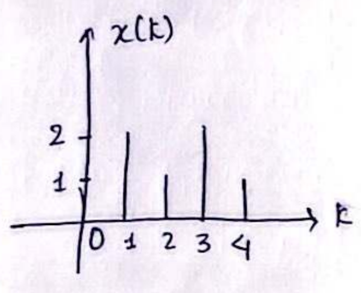
Since $y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

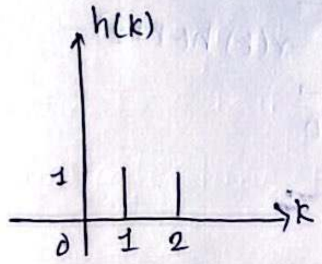
$$y(n) = \sum_{k=0}^{5} x(k)\, h(n-k) \quad \text{——— (1)}$$

Now at $n=0$,

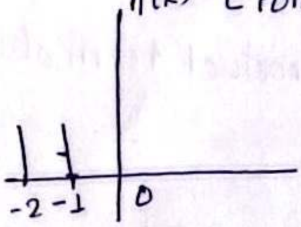
$$y(0) = \sum_{k=0}^{5} x(k)\, h(-k)$$

Graphical representation of $x(k)$ and $h(-k)$ is given by



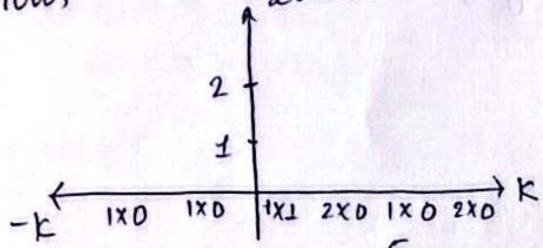


h(k) [Folding of h(k)]

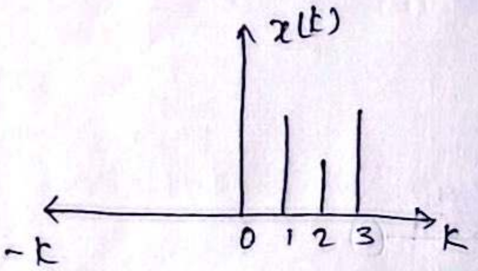


now, x(k)(h-k) [Multiplication] = {0,0,1,0,0,0}

y(0) = 1

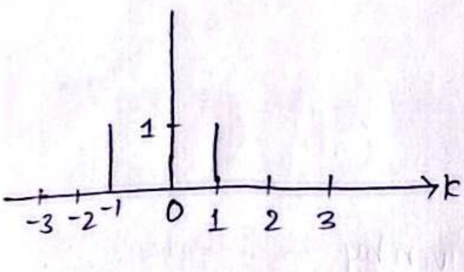


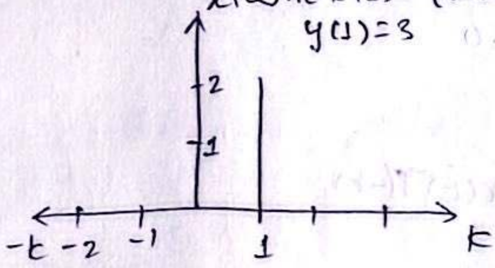
$$\text{At } n=1,\ y(1) = \sum_{k=0}^{5} x(k)h(-k) = \sum_{k=0}^{5} x(k)h(-k+1)$$



x(k)h(-k+1) = {0,1,2,0,0}

y(1) = 3





$$\text{at } n=2,\ y(2) = \sum_{k=0}^{5} x(k)h(-k+2)$$

Properties of LTI system.

1. Commutative property

According to this property, output of LTI system having system response h(n) and input x(n) is same as that of system response x(n) and input h(n) i.e

$$x(n) * h(n) = h(n) * x(n)$$

$$\sum_{k=\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

2. Discriptive property

According to this property if the same input is two system in parallel then the input convolve with- the combination as well as each system individually i.e

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

3. Associative property

According to this property the convolution of input with more then one system follows the same rule as multiplication

$$x(n) * [h_1(n) * h_2(n)] = [x(n) * h_1(n)] * h_2(n)$$

4. Cascaded connection

If a system is given as

$$x(n) \longrightarrow \boxed{h_1(n)} \longrightarrow \boxed{h_2(n)} \longrightarrow \boxed{h_3(n)} \longrightarrow \cdots \boxed{h_n(n)} \longrightarrow y(n)$$

The equivalent of the above system can be drawn as

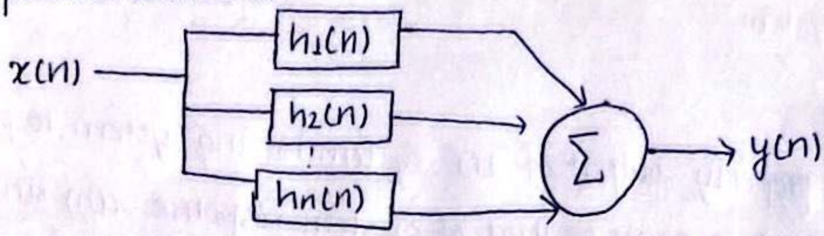
$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

where

$$h(n) = h_1(n) * h_2(n) * h_3(n) - - - - * h_n(n)$$

Thus for a cascaded combination of system it convolved into single system.

5. parallel connection



It can be represented as follows

$$x(n) \longrightarrow h(n) \longrightarrow y(n)$$

where $h(n) = h_1(n) + h_2(n) + \ldots\ldots h_n(n)$

Thus for a parallel connection of individual system that combined system will be the sum of individual system.

6. Stability of LIT system

Q. A LIT system is BIBO stable if its impulse response is absolutely summable. Justify

⇒ A system is said to be stable if and only if its output sequence $y(n)$ is bounded for every bounded input $x(n)$

If $x(n)$ is bounded, there exist a constant 'm_x' such that

$$|x(n)| \leq m_x < \infty$$

Similarly if the output is bounded, there exist a constant M_y such that for all n

now,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

If we ~~ka~~ take the absolute value of both side of this equation, we get

$$|y(n)| = \left| \sum_{k=\infty}^{\infty} h(k) * x(n-k) \right|$$

The absolute value of sum of term is always less than or equal to the sum of the absolute value of the terms.

Hence,

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)|$$

If the input is bounded, there exist a finite number of 'm_x' such that $|x(n)| \le m_x$ substituting the upper bound of $x(n)$ in the equation above we get,

$$|y_n(n)| \le m_x \sum_{k=-\infty}^{\infty} |h(k)|$$

From these equation we observe that the o/p is bounded if the impulse response of the system satisfies the condition.

$$S = \sum_{k=-\infty}^{\infty} |h(k)| < \infty \quad (k = -\infty)$$

i.e a LTI system is stable if its impulse response is

$$|AB| \le |A|\,|B|$$

absolulely, summable, This condition is not only sufficient but it is also necessary to ensure the stability of system.

7. Causality of LIT system

Q. Prove that a LIT system is a causal if and only if its impulse response is zero for negative value of n, i.e $h(n)=0$ for $n<0$

Q. State and prove paley weiner criteria for the causality of LIT system.

Ans: let as consider a LIT system having an o/p at time $n=n_0$ given by the convolution sum.

$$y(n_0)= \sum_{k=-\infty}^{\infty} h(k)\, x(n_0-k)$$

or,

$$h(n_0)= \sum_{k=0}^{\infty} h(k)\, x(n_0-k) + \sum_{k=-\infty}^{-1} x(n_0-k).h(k)$$

$$= \left[h(0)\,x(n_0) + h(1)\,x(n_0-1) + h(2)\,x(n_0-2) \ldots\right]$$

$$+\left[h(-1)\,x(n_0+1) + h(-2)\,x(n_0+2) + \ldots\right]$$

We observe that the term in the first sum involve $x(n_0)$, $x(n_0-1)\ldots$ which are present and past value of the input signal.

On the other hand, the term on the second sum, involves future value of input signal i.e $x(n_0+1)$, $x(n_0+2)\ldots$ now if if the output at time $n=n_0$ is to depend onlyon the present and past input then clearly the impulse response of the system must satisfy the condition $h(x)=0$ fon $n<0$

Since $h(x)$ is the response of the relaxed LIT system to a unit impulse appilied to $n=0$ if follows that $h(n)=0$ for $n<0$ is both a necessary and sufficient condition for causality.

1.9. Frequency response of LIT system

The frequency Response $H(e^{jw})$ of LTI system is define as complex gain that the system applies to complex exponential input (e^{jw})

The fourier trans-form of system input and output are

$$Y(e^{jw}) = H(e^{jw})X(e^{jw})$$

When frequency response is expressed in polar form then magnetude and phase of fourier transform of a system input and output are related by

$$|Y(e^{jw})| = |H(e^{jw})||X(e^{jw})|$$

$$\angle Y e^{jw} = \angle H(e^{jw}) + \cancel{\angle e^{jw}} \angle X(e^{jw})$$

where $|H(e^{jw})|$ is the magnitude response or gain of system and $\angle H(e^{jw})$ is phase respone or phase ~~H~~ shift.

1.10. Sampling of continious time signal, spectral properties of samples signal.

Sampling of continious time signal

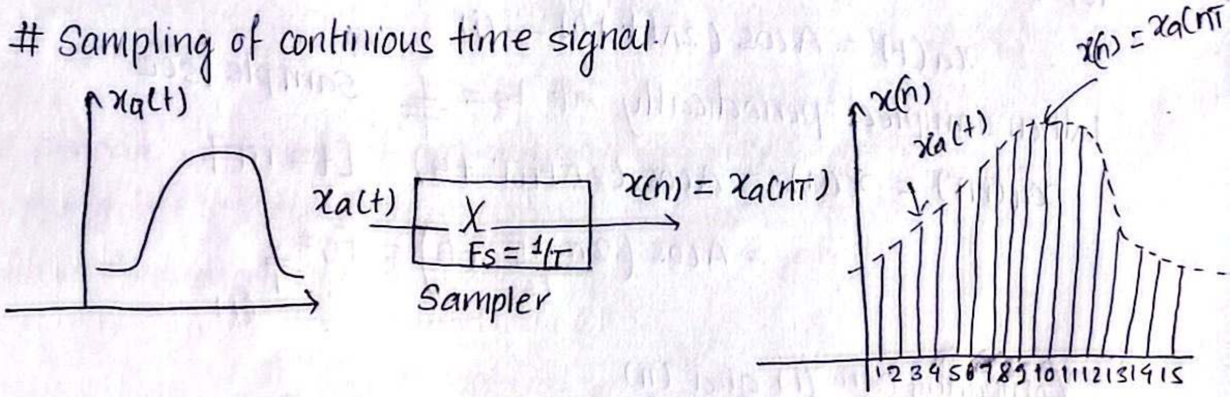


Fig: Periodic sampling of analog signal.

Sampling is the process of conversion of continious time signal, $x_a(t)$ into discrete time signal i.e

$$x(n) = x_a(nT) \; ; \; -\infty < n < \infty$$

Where $x(n)$ is a discrete time signal obtained by taking sample of an analog signal $x_a(T)$ in every t sec. The time interval 'T' between the successive sample is known as sampling period of sampling interval and its reciprocal is known as sampling rate or sampling frequency

i.e $F_s = \frac{1}{T}$

F_s = sample per second or Hertz (Hz)

Periodic sampling established a relationship between the time variable 't' of continious time signal and 'n' of discrete time signal as $t = nT$

$$t = \frac{n}{F_s}$$

As a result a relationship between frequency variable F or Ω for analog signal and Frequency variable f or w for discrete signal.

let us consider an analog sinusodial signal in the form

$$x_a(t) = A\cos(2\pi Ft + \theta) \text{ — (i)}$$

When sampled periodically at $F_s = \frac{1}{T}$ sample/sec

$$x_a(nT) = x(n) = A\cos(2\pi FnT + \theta) \quad [t = nT]$$

$$= A\cos\left(2\pi \frac{nF}{F_s} + \theta\right) = F_s = \frac{1}{T} \text{ (ii)}$$

Comparing eqn (i) and (ii)

$$f = \frac{F}{F_s} \quad \text{or,} \quad w = \Omega T = \frac{\Omega}{F_s}$$

We know

$-\infty < F < \infty$ and $\begin{bmatrix} f \rightarrow \text{Frequency of digital} \\ F \rightarrow \text{Frequency of analog} \\ F_s \rightarrow \text{sampled Frequency} \end{bmatrix}$

$-\infty < \Omega < \infty$

Also, the maximum rang of discrete time signal is

$-\frac{1}{2} \leq f \leq \frac{1}{2}$ and $-\pi < w < \pi$

Substituting these values

$$-\frac{1}{2T} = -\frac{F_s}{2} \leq F \leq \frac{F_s}{2} = \frac{1}{2T}$$

$$-\frac{\pi}{T} = -\pi F_s \leq \Omega \leq \pi F_s = \frac{\pi}{T}$$

Since the highest frequency is a discrete time signal is $w = \pi$ or $f = \frac{1}{2}$ it follows that with a sampling rate F_s, the crossponding heighest value of f and Ω are.

$$F_{max} = \frac{F_s}{2} = \frac{1}{2T} \text{ and } \Omega_{max} = \pi F_s = \frac{\pi}{T}$$

We observed that a fundamental difference between continuous and discrete time signal in their range of value of the re frequeney variable F (analog Frequency) and f (digital frequency) or Ω and w

$\Omega = 2\pi f$

$w = 2\pi f$

Thus periodic sampling of a continous time signal implies a mapping the infinite frequency rate for the variable F or Ω into a finite frequency rate f or w.

Sampling theorem

It states that a continious time signal can be completely represented in its samples and recovered back if the sampling frequency is greater than or equal to the maximum frequency of the signal

i.e $F_s \geq 2 F_{max}$

$F_s \longrightarrow$ sampling frequency

$F_{max} \longrightarrow$ Maximum frequency of signal

Note:- Nyquist rate $= 2F_{max}$

Nyquist interval $= \frac{1}{2F_{max}}$

$$\text{Resolution}(\Delta) = \frac{X_{max} - X_{min}}{L-1}$$

where $X_{max} = A_1 + A_2$

$X_{min} = -X_{max}$

$L =$ no. of quantization level.

$$F_s(\text{sample/sec}) = \frac{\text{Bit rate}}{\text{bit/sample}}$$

$b = \log_2(L) \longrightarrow$ bits / sample

$b \rightarrow$ no. of bit per sample

Continious Time signal	Discrete Time signal.
1) $x_a(t) = A\cos(2\pi F t + \theta)$	$x(n) = A\cos(2\pi f n + \theta)$
$= A\cos(\Omega t + \theta)$; $-\infty < t < \infty$	$= A\cos(2\pi n T + \theta)$
$\Omega = 2\pi F$	i.e $x(n) = x_a(nT)$
$\Omega =$ radian/sec	$w = 2\pi f$
$F =$ Frequency in Hz	$w \rightarrow$ radian/samples
	$F \rightarrow$ cycles/samples
ii) $W = \pi$	
iii) $-\infty < \Omega < \infty$	$-\frac{\pi}{T} < \Omega \leq$
$-\infty < F < \infty$	

Q. A digital communication link carries binary coded words representating the sample of an input sample $x_a(t) = 3\cos 600\pi t + 2\cos 800\pi t$. The link is operated as 10,000 bits/sec each input sample is quantized into 1024 different voltage records, determined.

1) What is the sampling frequency and folding frequency
2) What is the nyquist rate for the signal $x_a(t)$
3) What are the frequency in resulting discrete time signal $x(n)$.
4) What is the resolution?

Here,

$x_a(t) = 3\cos 600\pi t + 2\cos 800\pi t$

bitrate = 10,000 bit/sec

no. of quantization level $(L) = 1024$

since

$$b = \log_2(L) = \log_2(1024)$$
$$= 10 \text{ bit/sample}$$

Sampling frequency is given by

$$F_s = \frac{\text{Bit rate}}{\text{bit/sample}} = \frac{10000}{10}$$
$$= 1000\,Hz$$

Folding frequency $= \frac{F_s}{2} = \frac{1000}{2} = 500\,Hz$

Comparing given signal with standard form

$x(t) = A_1 \cos \pi F_1 t + A_2 \cos 2\pi F_2 t$

we get,

$A_1 = 3$ $\quad F_1 = 300$

$A_2 = 2$ $\quad F_2 = 400 = F_{max}$

$\therefore$ Nyquist rate $= 2F_{max}$
$= 2 \times 400$
$= 800\,Hz$

Nyquist interval $= \frac{1}{2F_{max}} = \frac{1}{800} = 0.00125$ sec.

3. Discrete time signal is obtained by

$t \to \frac{n}{F_s}$ (nT)

$t = \frac{n}{1000}$

$x(n) = 3\cos 600\pi \frac{n}{1000} + 2\cos 800\pi \frac{n}{1000}$

$= 3\cos 2\pi \frac{300n}{1000} + 2\cos 2\pi \frac{400n}{1000}$

$\therefore$ $x(n) = 3\cos 2\pi\, 0.3n + 2\cos 2\pi\, 0.4n$ which is the required discret time signal

with $f_1 = 0.3Hz$ and $f_2 = 0.4Hz$

u) Resolution $= \frac{X_{max} - X_{min}}{L-1}$

where $X_{max} = A_1 + A_2 = 3+2 = 5$

$X_{min} = -X_{max} = -5$

$L = 1024$

$\therefore$ Resolution $= \frac{5+5}{1024-1} = \frac{10}{1023} = 0.009775.$

Q. Consider the analog signal $x_a(t) = 3\cos 100\pi t$

1) Determine the minimum sampling rate required to avoid aliasing effect (Nyquist rate)

2) Suppose that a signal is sampled at sampling frequency of 200Hz What is the discrete time signal.

Here, $x_a(t) = 3\cos 100\pi t$

Comparing this signal with $x(t) = A\cos 2\pi f t$

$\therefore A = 3$

$F = 50 = F_{max}$

Now sampling rate i.e Nyquist rate

$= 2 F_{max}$

$= 2 \times 50$

$= 100 Hz$

2) $F_S = 200 Hz$

Now, discrete time signal is

$t \rightarrow \frac{n}{200}$

$\therefore x(n) = 3\cos 100\pi \frac{n}{200}$

$= 3\cos 2\pi\, 0.25n$

which is required signal with $f = 0.25 Hz$

Q. Consider a signal $x_a(t) = 10\cos 2\pi(1000)t + 5\cos 2\pi(5000)t$

calculate I) Nyquist rate

II) If signal is sampled at 4kHz at will the signal be recovered from its sample?

Here, $x_a(t) = 10\cos 2\pi(1000)t + 5\cos 2\pi(5000)t$

Comparing these equation with standard form

$A_1 = 10 \quad F_1 = 1000$

$A_2 = 5 \quad F_2 = 5000 = F_{max}$

1) Nyquist rate $= 2 \times F_{max}$

$= 2 \times 5000$

$= 10,000 Hz$

2) Here $F_S = 4KHz = 4000 Hz$

and $t = \frac{n}{4000}$

$x(n) = 10\cos 2\pi\, 1000 \frac{n}{4000} + 5\cos 2\pi\, 5000 \frac{n}{4000}$

$= 10\cos 2\pi\left(\frac{1}{4}\right)n + 5\cos 2\pi\left(1 + \frac{1}{4}\right)n$

Scanned with CamScanner

$$= 10\cos 2\pi\left(\frac{1}{4}\right)n + 5\cos\left(2\pi + \frac{2\pi}{4}\right)n$$

$$= 10\cos 2\pi\left(\frac{1}{4}\right)n + 5\cos 2\pi\left(\frac{1}{4}\right)n$$

$$= 15\cos 2\pi\left(\frac{1}{4}\right)n$$

Reconstruction of signal from discrete time signal as

$$n = tF_s$$
$$t \times 4000$$

$$x_a(t) = 15\cos 2\pi \frac{1}{4} \times t \times 4000$$

$$= 15\cos 2\pi \cancel{\pi} 1000\, t$$

Here we observe that the reconstructed signal contain only one frequency term of 1000 Hz and amplitude is 15

Thus the effect of frequency $F_2 = 5000$ is completely lost and the amplitude of F_1 is increased this show thus with the sampling of 4000 Hz the signal is not recovered from its simplicity

This is because the sampling term is not satisfied.

The nyquist rate calculate for given signal is 10,000 Hz ~~since~~ Hance minimum sampling frequency must be 10,000 Hz to avoid aliasing effect.

Q. The discrete time signal $x(n) = 6.5\cos(0.1\pi n)$ is quantized of $\Delta = 0.01$. How many bit are required in ADC. If the maximum frequency that can be reconstructed from above signal is 500Hz Determine the recognized continuous time signal.

Here, $x(n) = 6.5\cos 0.1\pi n$

$\Delta = 0.01$

$$\Delta = \frac{X_{max} - X_{min}}{L-1}$$

$X_{max} = A = 6.5$

$X_{min} = -X_{max} = 6.5$

Now,

$$0.01 = \frac{6.5+6.5}{L-1}$$

$L = 1301$

Now, No. of bit (b) $= \log_2(L) = 10.35$

max frequence $= 500Hz = F_{max}$

$F_s = 2F_{max}$

$2 \times 500 = 1000Hz$

Now,

$x(n) = 6.5\cos(0.1\pi n)$

For continious time signal, $n = tF_s = t \times 1000$

$X_a(t) = 6.5\cos(0.1\pi t \times 1000)$

$= 6.5\cos 100t$. #

End
of
Chapter

Ayustush Pant

Z-transform (6-hr)

Convert the time domain to Frequency domain
Analysis of discrete time signals
Used to determine region of convergence

i.e $X(z) = \frac{(z+a)(z+b)}{(z-5)(z+4)}$

If z= finite, then region of convergence is determined when $z \leq 5$

Direct z-transform.

i.e $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

where z is continious complex variable where as,
two side analysis

i.e $z[x(n)] = \sum_{n=-\infty}^{0} x(n) z^{-n} + \sum_{n=0}^{\infty} x(n) z^{-n}$

→ The z-transform of discrete time signal $x[n]$ is defined as finite or infinite sum or infinite power series.

$x[n]$ is denoted by, $X(z) = z[x[n]]$

$x[n]$ and $x[z]_z$ is ~~denoted~~ indicated by

$$x[n] \xrightarrow{z} x[z] \xrightarrow{z^{-1}} x[n]$$

$$x[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[z] = z\{x[n]\}$$

Q. Find z-transform

(i) $x(n) = \{1, 2, 1, 3, 4\}$ [Causal signal]

Soln:

$x(0) = 1$
$x(1) = 2$
$x(2) = 1$
$x(3) = 3$

$$\therefore Z[x(n)] = X[z] = \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n}$$

$$= x(0)\cdot z^{-0} + x(1)\cdot z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

$$= 1 + 2z^{-1} + z^{-2} + 3z^{-3} + 4z^{-4}$$

Z- transform of some elementry signals.

1. $x[n] = \delta[n]$

$$\delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n\neq 0 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n} = z^{0} = 1$$

i.e $\sum_{n=0} \delta(n) z^{-n} = 1\cdot z^{0} = 1.$

2. $x[n] = \delta[n-k]$

i.e

$$X[z] = \sum_{n=-\infty}^{\infty} \delta[n-k]\, z^{-1} = z^{-k}$$

ROC: All z except at $z=0$

3. $x[n] = u[n]$

$$u(n) = \begin{cases} 1 & \text{for } n\geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$x(z) = \sum_{n=-\infty}^{\infty} u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}\cdot u(n)$$

$$= u(0)z^{0} + u(1)z^{-1} + u(2)z^{-2} + u(3)z^{-3} + \cdots$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

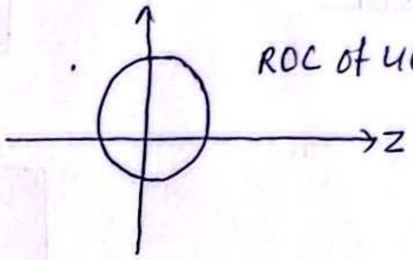
$$= \frac{1}{1-z^{-1}} \quad \text{if } |z^{-1}| < 1$$

$$= \frac{1}{1-\frac{1}{z}}$$

$$= \frac{z}{z-1}.$$

let $z-1=0$

$z=1$ so ROC $|z|>1$



4. $x[n] = u[-n-1]$

$\Rightarrow$ $$x[z] = \sum_{n=-\infty}^{\infty} [-n-1]\, z^{-n}$$

i.e $-n-1=0$

$-n=1$

$n=-1$

$$= \sum_{n=-\infty}^{-1} z^{-n}$$

$$= \sum_{n=1}^{\infty} z^{n}$$

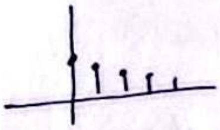
$\Rightarrow$ $$= \frac{z}{1-z} \quad \text{if } |z|<1$$

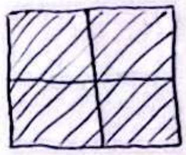
$$= \frac{1}{1-z^{-1}}, \text{ ROC } |z|<1.$$

Analysis of ROC

Signal | ROC

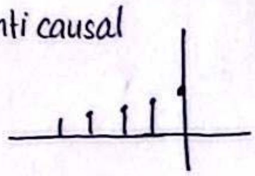
i) Causal

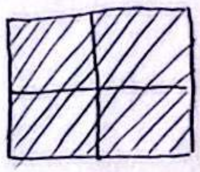




entire
z-plane
expect $z=0$

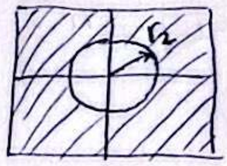
ii Anti causal





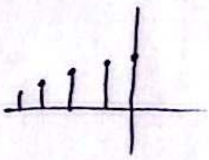
entire z-plane
except $z=\infty$

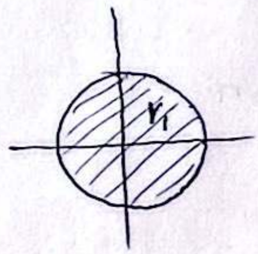
iii) Two sided



$|z| > r_2$

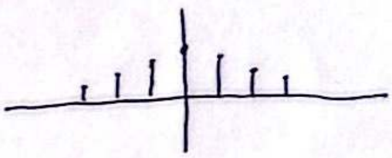
iv) Anti causal

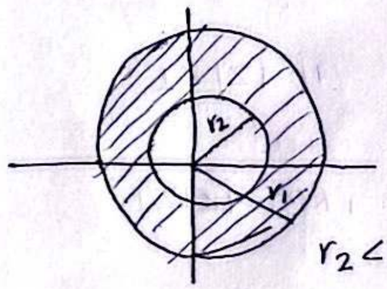




$|z| < r_1$

v) Two seided





$r_2 < |z| < r_1$

Analysis of ROC:

We can express variable z in polar form

$$z = re^{i\theta}$$

then,

$$X[z]\Big|_{z=re^{i\theta}}$$

$$= \sum_{n=-\infty}^{\infty} x[n](re^{j\theta})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\} e^{-j\theta n}$$

The region of convergence (ROC) of X[z] is the set of all values of z for the set of all values of z for which X(z) attains a finite value

In ROC of X(z)

$|X(z)| < \infty$ But

$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\} e^{-j\theta n}\right| \leq \sum_{n=-\infty}^{\infty} |\{x[n]r^{-n}\} e^{-j\theta n}|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]r^{-n}|$$

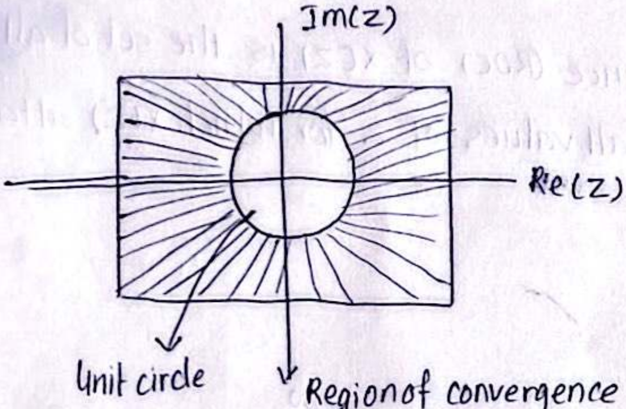
Advantages of z-transform

- Discrete time signal and linear time invariant system can be completely characterized by z-transform.
- The stability of LTI system can be determined using z-transform
- By calculating z-transform of given system discrete fourier transform (DFT) can be determined.

- The sol^n of differential equations can be simplified using z-transform.
- Mathematical calculations are reduced by using z-transform for eg: Convolution operation is transformed into simple multiple operation.
- Entire family of digital filters can be obtained from one prototype design using z-transform.

Region of convergence (ROC)

The set of values of z in the z-plane ~~fro~~ for which the magnetude of $X(z)$ is finite is called convergence.



zero at $z=0$
pole at $z=1$.

Properties of ROC:

1. The ROC of $X(z)$ consists of a ring circle in the z-plane centered about the origin.
2. The ROC doesnot contain any pole.
3. The z-transform $X(z)$ converges uniformly if and only if the ROC of $X(z)$ include the unit circle i.e ROC of $X(z)$ has value of z for which

$$\sum_{n=-\infty}^{\infty} |x(n)\, r^{-n}| < \infty$$

4. If $x(n)$ is of finite duration then the will be the entire z-plane except at $z=0$ and $z=\infty$

5. If $x(n)$ is right seided sequence then ROC will not include infinity.

6. If $x(n)$ is left seided sequence, then ROC will not include $z=0$ but incase of, if $x(n)=0$ for all $n>0$ then ROC will include $z=0$.

7. If $x(n)$ is a two sided sequence, then ROC will consist of a ring then ROC will consist of a ring in z-plane, that includes the circle $|z| = r_0$, i.e ROC will include the intersection of region of convergence of components of the given signals.

8. If $X(z)$ is rational, then the ROC will extend to ∞

9. if $x(n)$ is causal, then ROC will include $z=\infty$

10. If $x(n)$ is anti causal, then ROC will include $z=0$.

Properties of z-transform

1. Linearity

$$x_1(n) \xleftrightarrow{Z} X_1(z)$$
$$x_2(n) \longleftrightarrow X_2(z)$$

then, $x(n) = a_1x_1(n) + a_2x_2(n) \xleftrightarrow{Z} X(z) = a_1X_1(z) + a_2X_2(z)$

where a_1 and a_2 are constant.

2. Time shifting

If $x(n) \xleftrightarrow{Z} X(z)$

then, $x(n-k) \xleftrightarrow{Z} z^{-k}X(z)$

ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z=0$ if $k>0$ and $z=\infty$ if $k<0$

3. Scaling in z-domain

if $x(n) \xleftrightarrow{Z} X(z)$

ROC: $r_1 < |z| < r_2$

Then, $a^n x(n) \longleftrightarrow X(a^{-1}z)$

ROC: $|a| r_1 < |z| < |a| r_2$

4. Time Reversal

If $x(n) \xleftrightarrow{Z} X(z)$

ROC: $r_1 < |z| < r_2$

then, $x(n) \xleftrightarrow{Z} X(z^{-1})$

ROC: $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

5. Differentiation in z-domain.

If $x(n) \xleftrightarrow{Z} X(z)$, then

$n\,x(n) \xleftrightarrow{Z} -z \frac{d}{dz} X(z)$

or, $nx(n) \xleftrightarrow{Z} z^{-1} \frac{d}{dz^{-1}} X(z)$

6. Convolution of two sequence

If $x_1(n) \xleftrightarrow{Z} X_1(z)$

& $x_2(n) \xleftrightarrow{Z} X_2(z)$

If $x_1(n)$ and $x_2(n)$ are z-transformable then

$x(n) = x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(z) * X_2(z)$

7. Correlation property

It states that if $x_1(n) \xleftrightarrow{Z} X_1(z)$

$x_2(n) \xleftrightarrow{Z} X_2(z)$

then correlation of two properties is given by

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-l)$$

$$r_{x_1x_2}(l) \xleftrightarrow{Z} R_{x_1x_2}(z) = X_1(z)\, X_2(z^{-1})$$

Q. Determine the correlation of given sequence

$x_1(n) = (1, 2, 3, 4)$
$x_2(n) = (4, 3, 2, 1)$

Soln:
Here

$x_1(n) = (1, 2, 3, 4)$

$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$ —(i)

$x_2(n) = (4, 3, 2, 1)$

$X_2(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}$

$X_2(z^{-1}) = 4 + 3z + 2z^2 + z^3$ —(ii)

$$R_{x_1x_2}(z) = (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(4 + 3z + 2z^2 + z^3)$$
$$= z^3 + 4z^2 + 10z + 20 + 25z^{-1} + 24z^{-2} + 16z^{-3}$$

∴ Required correlation of $x_1(n)$ and $x_2(n)$ is,

$$r_{x_1x_2}(l) = Z^{-1}[R_{x_1}R_{x_2}(z)]$$
$$= \{1, 4, 10, 20, 25, 24, 16\}$$

//only coefficient of $R_{x_1x_2}$

8. Conjugate property

If $x(n) \xleftrightarrow{Z} X(z)$

then, $x^*(n) \xleftrightarrow{Z} X^*(z^*)$

9. Initial value theorem

It states that if $x(n)$ is a causal discrete time signal with z-transform $X(z)$, then the initial value may be determined by using following expression:

$$x(0) = \lim_{n \to 0} x(n) = \lim_{|z| \to \infty} X(z)$$

10. Final value Theorem

It states that for a discrete time signal $x(n)$ if $X(z)$ and the poles of $X(z)$ are all inside the unit circle then the final value of discrete time signal $x(\infty)$ may be determined. by

$$x(\infty) = \lim_{|z| \to 1} (1 - z^{-1})\, x(z)$$

Q. Determine the initial and final values of the given signal

$$x(z) = 2 + 3z^{-1} + 2z^{-2}$$

Soln:

$$x(0) = \lim_{z \to \infty} x(z)$$

or, $$x(0) = \lim_{z \to \infty} (2 + 3z^{-1} + 4z^{-2})$$

or, $$x(0) = \lim_{z \to \infty} \left(2 + \frac{3}{z} + \frac{4}{z^2}\right)$$

or, $$x(0) = 2 + \frac{3}{\infty} + \frac{4}{\infty}$$

$\therefore$ $x(0) = 2$ is required initial values

For final value we have

$$x(\infty) = \lim_{|z| \to 1} [(1 - z^{-1})\, x(z)]$$

$$= \lim_{|z| \to 1} (1 - z^{-1})(2 + 3z^{-1} + 4z^{-2})$$

$$= \lim_{z \to 1} [2 + 3z^{-1} + 4z^{-2} - 2z^{-1} - 3z^{-2} - 4z^{-3}]$$

$$= \lim_{z \to 1} (2 + z^{-1} + z^{-2} - 4z^{-3})$$

$$= 2 + 1 + 1 - 4$$

$$= 0$$

$\therefore$ $x(\infty) = 0$ is the final value

Some important formula and ROC

SN	x(n)	X(Z)	ROC
1	$\delta(n)$	1	All z-plane
2	$\delta(n-k)$	z^{-k}	$\|z\|>0,\ k>0$ & $\|z\|<\infty,\ k<0$
3	$u(n)$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\|z\|>1$
4.	$u(-n-1)$	$\frac{1}{1-z^{1}} = \frac{z}{z-1}$	$\|z\|<1$
5.	$n\,u(n)$	$\frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$	$\|z\|>1$
6.	$a^n u(n)$	$\frac{z}{z-a}$	$\|z\|<a$
7.	e^{-an}	$\frac{z}{z-e^{-a}}$	$\|z\|>\|e^{-a}\|$
8.	$\sin w_0 n$	$\frac{z\sin w_0}{z^2-2z\cos w_0+1}$	$\|z\|>1$
9	$\cos w_0 n$	$\frac{z(z-\cos w_0)}{z^2-2z\cos w_0+1}$	$\|z\|>1$
10	$e^{-an}\sin w_0 n$	$\frac{z e^{-a}\sin w_0}{z^2-2ze^{-a}\cos w_0+e^{-2a}}$	$\|z\|>\|e^{-a}\|$

Inverse z-transform.

Mathematically the inverse z-transform is expressed as

$x(n) = z^{-1}\, X(z)$ — ①

To perform inverse z-transform following methods are used

i) Long division method

ii) PEE (partial Function Expansion) Method.

iii) Residue Method.

i) Long Division method.

since, $X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$

Z-transform of discrete time signal is expressed as

$$X(z) = \frac{N(z)}{D(z)}$$

$$= \sum_{n=0}^{\infty} a_n z^{-n}$$

$$= a_0 z^0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$$

The coefficient 'a_n' are required inverse z-transform.

Q. Using long division method determine the ~~z~~ inverse z-transform.

$$X(z) = \frac{1}{1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}}$$

when ROC: $|z| > 1$

when ROC: $|z| < \frac{1}{2}$

a) When ROC $|z|>1$.

It is exterior of circle, therefore $x(n)$ is a causal signal. Therefore by dividing the numerator of $x(z)$

$$X(z) = \frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{N(s)}{D(s)}$$

$$= 1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} \overline{\big) \; 1 \;} \quad 1+\frac{3}{2}z^{-1}+7/4\, z^{-2}$$

$$\frac{\begin{array}{r} 1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} \\ - \quad + \quad - \end{array}}{\frac{3}{2}z^{-1}-\frac{1}{2}z^{-2}}$$

$$\frac{\begin{array}{r} -\frac{3}{2}z^{-1}-\frac{9}{4}z^{-2}+\frac{3}{4}z^{-3} \\ + \quad - \end{array}}{7/4\, z^{-2}-\frac{3}{4}z^{-3}}$$

$$\frac{\begin{array}{r} -\frac{7}{4}z^{-2}-\frac{21}{8}z^{-3}+\frac{7}{8}z^{-4} \\ + \quad - \end{array}}{\frac{15}{8}z^{-3}-\frac{7}{8}z^{-4}}$$

$$\therefore \; X(z) = \frac{N(s)}{D(s)} = 1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}+\cdots$$

Inverse transform of $x(z)$ is given by

$$x(n) = \left\{1, \frac{3}{2}, 7/4, \frac{15}{8}, \cdots\right\}$$

b) When $|z|<\frac{1}{2}$

It is the interior of circle.

Hence $x(n)$ is an anti causal so to obtain power series expansion in positive power ~~oz~~ of z we perform the long divison as following

$$X(z) = \frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{1}{\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1} = \frac{N(s)}{D(s)}$$

$$\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 \overline{\Big) \; 1 \;} \Big(2z^2+6z^3+14z^4+30z^5$$

$$\begin{array}{l} 1 \\ \underline{1-3z+2z^2} \\ \quad - \; + \; - \\ 3z-2z^2 \\ \underline{3z-9z^2+6z^3} \\ \quad - \; + \; - \\ 7z^2-6z^3 \\ \underline{7z^2-21z^3+14z^4} \\ 15z^3-14z^4 \\ \underline{15z^3-45z^4+30z^5} \\ \quad - \; + \; - \\ 31z^4-30z^5 \end{array}$$

$$X(z) = 2z^2+6z^3+14z^4+30z^5+ - - -$$

$$x(n) = \{0, 0, 2, 6, 14, 30, - - - \}$$

Q. Using LDM determine LTI of $X(z) = \dfrac{1+z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-3}}$

ROC $|z| < \frac{1}{2}$

We know $|z| < \frac{1}{2}$ is the interior of circle so the $x(n)$ is anti Causal signal By LDM

$$\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 \overline{\Big) \; 1+z^{-1} \;} \Big(2z+8z^2+20z^3$$

$$\begin{array}{l} 1+z^{-1} \\ \underline{z^{-1}-3+2z} \\ \quad - \; \mp \\ 4-2z \\ \underline{4-12z+8z^2} \\ \quad - \; + \\ 10z-8z^2 \end{array}$$

$$\therefore X(z) = 2z+8z^2+20z^3+ - - -$$

$$x(n) = \{0, 2, 8, 20 - - - - \}$$

Partial fraction expansion method.

With the help of PFE, we can convert a system transfer function into a sum of standard function for which firstly denumenator of the transform function are factorized into prime factor and inverse z-transform is obtained to get required soln.

1) $F(z) = \dfrac{H(z)}{(az+b)(cz+d)} = \dfrac{A}{(az+b)} + \dfrac{B}{(cz+d)}$

2) $F(z) = \dfrac{N(z)}{(az+b)(cz^2+d)} = \dfrac{A}{(az+b)} + \dfrac{Bz+C}{cz^2+d}.$

Question Use pfe to find the inverse of z-transform of

$$H(z) = \frac{-4+8z^{-1}}{1+6z^{-1}+8z^{-2}}$$

Here, $H(z) = \dfrac{-4+8z^{-1}}{1+6z^{-1}+8z^{-2}}$

$= \dfrac{-4+8z^{-1}}{8z^{-2}+6z^{-1}+1}$

$= \dfrac{-4+8z^{-1}}{(4z^{-1}+1)(2z^{-1}+1)}$

let $H(z) = \dfrac{-4+8z^{-1}}{(4z^{-1}+1)(2z^{-1}+1)} = \dfrac{A}{4z^{-1}+1} + \dfrac{B}{2z^{-1}+1}$

Or, $-4+8z^{-1} = A(2z^{-1}+1) + B(4z^{-1}+1)$

Comparing the terms

$-4 = A+B \quad — (i)$

$8 = 2A+4B \quad — (ii)$

Solving ① and ⑪ , $A=-12,\ B=8$

$$H(z) = \frac{-12}{4z^{-1}+1} + \frac{8}{2z^{-1}+1}$$

$$= -12 \times \left(\frac{1}{1+4z^{-1}}\right) + 8 \times \frac{1}{1+2z^{-1}}$$

$$= -12 \times (-4)^n u(n) + 8(-2)^n u(n)$$

Q. Determine $x(n)$ using inverse z-transform.

$$X(z) = \frac{z}{3z^2-4z+1}$$

if ROC ① $|z| > 1$

② $|z| < \frac{1}{3}$

③ $\frac{1}{3} < |z| < 1$.

Here, $X(z) = \frac{A}{(z-1)} + \frac{B}{(3z-1)}$

Solving Ⓐ and Ⓑ

$A = \frac{1}{2}$, $B = -\frac{3}{2}$

$$X(z) = \frac{1}{2}\left(\frac{z}{z-1}\right) - \frac{3}{2}\left(\frac{z}{3z-1}\right)$$

i) $x(n) = \frac{1}{2}(1)^n u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$

for $|z|>1$.

2) At ROC $|z| < \frac{1}{3}$

$x(n) = \frac{1}{2}(1)^n u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(-n-1)$

↳ anti causal , non causal.

3) At ROC : $\frac{1}{3} < |z| < 1$

$x(n) = \frac{1}{2}(1)^n u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$

non causal ↑ ↓ causal.

3) Residue method

In this method we obtain inverse z- transform of $X(z)$ i.e $x(n)$ as,

$$x(n) = \sum_{\substack{\text{all poles} \\ \text{of } X(z)}} \text{residue of } (X(z)\, z^{n-1})$$

Hence the residue for any pole of oder 'm' at $z=\beta$ is given by

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z\to\beta} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-\beta)^m \times X(z)\; z^{n-1} \right] \right\}$$

Q. Use residue method to find the inverse z- transform. $x(n)$ for

$$X(z) = \frac{z}{(z-1)(z-2)}$$

Here $X(z) = \dfrac{z}{(z-1)(z-2)}$

$X(z)$ has two poles of order 1 i.e $m=1$ at $z=1$ and $z=2$

i) For pole at $z=1$ $(\beta=1)$

$$\text{Residue} = \frac{1}{(1-1)!} \lim_{z\to 1} \left\{ \frac{d^{1-1}}{dz^{1-1}} \left[(z-1)^1 \frac{z}{(z-1)(z-2)} z^{n-1} \right] \right\}$$

$$= \lim_{z\to 0} \left[\frac{z}{z-2} \; z^{n-1} \right.$$

$$= \frac{1}{1-2} = 1^{n-1} \cdot$$

$$= (-1)\, 1^{n-1}$$

Residue. $= -1.$

ii) Similarly at pole $z=2$ $(\beta=2)$

$$\text{Residue} = \frac{1}{(1-1)!} \lim_{z\to 2} \left\{ \frac{d^0}{dz^0} \left[(z-2)^1 \frac{z}{(z-1)(z-2)} z^{n-1} \right] \right\}$$

$$= \lim_{z\to 2} \frac{z}{z-1} \; z^{n-1}$$

$$= 2 \cdot 2^{n-1} = 2^n$$

Residue $= 2^n$

Here, $x(n) = \sum_{\text{all pole}}$ residue of $(X(z) z^{n-1})$

$= \{-1+2^n\} u(n)$

Transfer function and impulse response

The transfer function is defined as the ratio of z- transform of output of the z-transform of input.

i.e $H(z) = \frac{Y(z)}{X(z)}$ — (1)

$H(z) \rightarrow$ Transfer function

$Y(z) \rightarrow$ z- transform of output $y(n)$

$X(z) \rightarrow$ z-transform of input $x(n)$

Thus the impulse response $h(n)$ is obtained by taking IZT of eqn (1)

i.e $h(n) = z^{-1}\left[\frac{Y(z)}{X(z)}\right]$

Q. Find the transfer function and impulse response of given LCCD equation

$y(n) = \frac{1}{2} y(n-1) + x(n) + \frac{1}{3} x(n-1)$

Taking z-transform on both side

$Y(z) = \frac{1}{2} Y(z) z^{-1} + X(z) + \frac{1}{3} X(z) z^{-1}$

$Y(z) - \frac{1}{2} Y(z) z^{-1} = X(z) \frac{1}{3} X(z) z^{-1}$

$Y(z)\left[1 - \frac{1}{2} z^{-1}\right] = X(z)\left[1 + \frac{1}{3} z^{-1}\right]$

$\cancel{X(z)}$ $\frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} = H(z)$

which is required transfer function

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Now taking IZT of transfer function i.e $H(z)$

$$z^{-1}(H(z)) = z^{-1}\left[\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\right]$$

$$= z^{-1}\left[\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\right]$$

$$= \left(\frac{1}{2}\right)^{n}u(n) + \frac{1}{3}\left(\frac{1}{2}\right)^{n}u(n-1)$$

$\therefore$ ROC $|z| > 1$.

Q. Find the z- transform of

① $x(n) = \left(\frac{1}{2}\right)^{n}u(n)$

② $\alpha(n) = \alpha^{n}u(n)$

Here, $x(n) = \left(\frac{1}{2}\right)^{n}u(n)$

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = \begin{cases} \left(\frac{1}{2}\right)^{n} & n > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{-\infty}^{0} x(n) z^{-n} + \sum_{0}^{\infty} x(n) z^{-n}$$

$$= 0 + \sum_{0}^{\infty}\left(\frac{1}{2}\right)^{n} z^{-n}$$

$$= \sum_{0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^{n}$$

$$\therefore X(z) = 1 + \frac{1}{2}z^{-1} + \left(\frac{1}{2}z^{-1}\right)^{2} + \left(\frac{1}{2}z^{-1}\right)^{3} - - - - - -$$

2) Here,

$$x(n) = \alpha u(n) = \begin{cases} 2^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$x(2) = \sum_{-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{-\infty}^{0} \cancel{x(n)} z^{-n} + \sum_{0}^{\infty} x(n) z^{-n}$$

$$= \sum_{0}^{\infty} \alpha^n z^{-n}$$

$$= \sum_{0}^{\infty} (\alpha z^{-1})^n$$

$$= 1 + (\alpha z^{-1})^1 + (\alpha z^{-1})^2 + (\alpha z^{-1})^3 + \cdots$$

$$= \frac{1}{1-\alpha z^{-1}}$$

$$= \frac{z}{z-\alpha}$$

Remember

(1) $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ (2) $\sum_{n=1}^{\infty} a^n = \frac{a}{1-a}$ (3) $\sum_{a=2}^{\infty} a^n = \frac{a^2}{1-a}$ (4) $\sum_{n=b}^{\infty} a^n = \frac{a^b}{1-a}$

Q. Find the z- transform of given signal.

1) $x(n) = 2^n u(n) + 3^n u(-n-1)$

2) $x(n) = \beta^n u(-n-1)$

1. $x(n) = 2^n u(n) + 3^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} + \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^n z^{-n} + \sum_{n=-1}^{-\infty} 3^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (2z^{-1})^n + \sum_{n=-1}^{\infty} (3z^{-1})^n$$

$$= \frac{1}{1-2z^{-1}} + z^{-1}\left[x_2(n)\right]$$

$$z^{-1}[x_2(n)] = \sum_{n=-1}^{-\infty} (3z^{-1})^n$$

$$z^{-1}[x_2(n)] = \sum_{n=-\infty}^{-1} (3z^{-1})^n$$

let $l=-n$ then

when $n=-\infty$, $l=\infty$

$n=-1$, $l=1$

$$z^{-1}[x_2(n)] = \sum_{l=1}^{\infty} (3z^{-1})^{-l}$$

$$= \sum_{l=1}^{\infty} \left(\frac{2}{3}\right)^l$$

$$= \frac{\frac{z}{3}}{1-\frac{z}{3}}$$

$$\therefore \quad x(n) = \frac{1}{1-2z} + \frac{z/3}{1-z/3}$$

2) $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \beta^n z^{-n}$$

$$= \sum_{n=-1}^{\infty} (\beta z^{-1})^n$$

$$= z^{-1}[x_2(n)]$$

Now, $Z^{-1}[x_2(n)] = \sum_{n=-1}^{-\infty} (\beta Z^{-1})^n$

$$Z^{-1}[x_2(n)] = \sum_{n=-\infty}^{-1} (\beta Z^{-1})^n$$

let $l = -n$ then

when, $n = -\infty$, $l = \infty$

$n = -1$ $\quad l = 1$

$$Z^{-1}[x_n(n)] = \sum_{l=1}^{\infty} (\beta Z^{-1})^{-l}$$

$$= \sum_{l=1}^{\infty} \left(\frac{Z}{\beta}\right)^l$$

$$= \frac{\frac{Z}{\beta}}{1 - \frac{Z}{\beta}}$$

$$x(n) = \frac{Z/\beta}{1 - Z/\beta} \quad \#$$

Unit 4: Discrete filter structure

Linear time invariant system (LIT) can be classified according to length of impulse response. Thus discrete time LIT system can be finite impulse response and infinite impulse response. If the impulse response is the finite duration then the system is known as finite duration impulse response (FIR) system.

If the impulse response is of infinite duration then the system is known as infinite duration impulse response (IIR) system.

FIR system	IIR system
Finite duration impulse response	Infinite duration impulse response.
For casual FIR system, $h(n)=0$ for $n<0$ and $n \geq m-1$ then convolution o/p become: $y(n) = \sum_{k=0}^{m-1} h(k)\, x(n-k)$	For IIR system $h(n)=0$, for $n<0$, then convolution o/p become $y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$
It has finite memory length.	It has infinite memory length.
It can be realized with the finite no of additional, multiplication and memory location	IIR can't be realized with convolution sum because it required infinite response so it is described by difference equation (LCCD)

Moving average system

$$y(n) = \frac{1}{m}\sum_{k=0}^{m} x(n-k)$$

Cumulative average system

$$y(n) = \frac{1}{n+1}\sum_{k=0}^{n} x(k)$$

Structure of FIR system.

FIR system is described by difference equation

$$y(n) = \sum_{k=0}^{m-1} b_k\, x(n-k)$$

and the system function is

$$H(z) = \sum_{k=0}^{m-1} b_k\, z^{-k}$$

where $m \rightarrow$ length of FIR system.

FIR system can be realized in following forms:

1) Direct Form Structure (Cononical form)

For a causal FIR system, $y(n) = \sum_{k=0}^{m} h(k)\, x(n-k)$

$\therefore y(n) = h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + h(3)\,x(n-3) + \ldots$

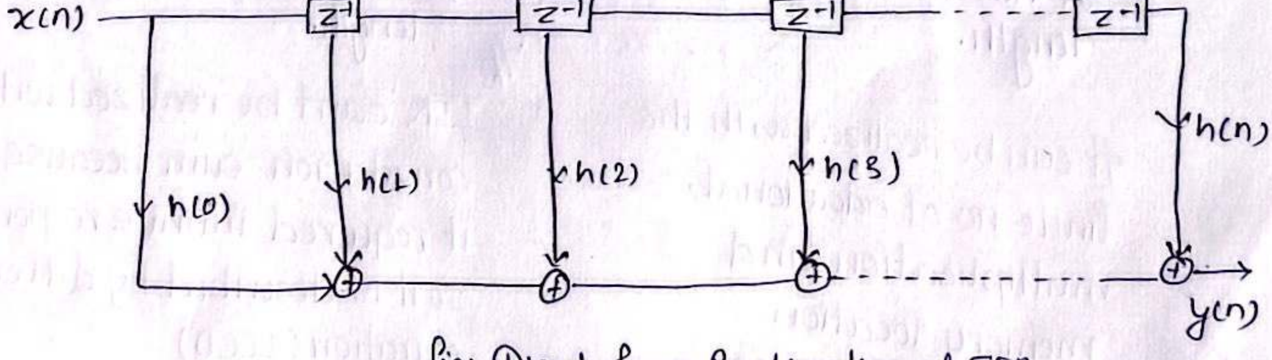

fig: Direct form Realization of FIR.

2) Cascade structure.

Here, $H(z) = \prod_{k=0}^{K} H_k(z) = H_0(z) \cdot H_1(z) . H_2(z) \ldots$

where
$H_k(z) = b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2} + \ldots$

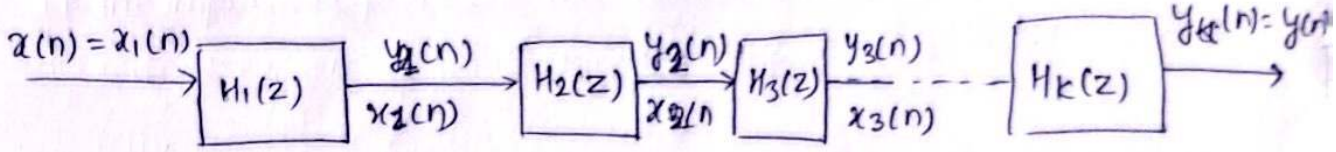

fig: Cascade realization of FIR system.

Structure of IIR system.

The casual IIR system are characterized by constant coefficient difference equation.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{m} b_k x(n-k)$$

or,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{m} b_k . z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

From this equation we observe that realisation of IIR system involves a recursive computational algorithm system. The IIR system realized as

1) Direct form I and II system
2) Cascade realization structure
3) Parallel realization structure.

1) Direct form

Since we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{m} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = H_1(z) \cdot H_2(z)$$

where $H_1(z) = \sum_{k=0}^{m} b_k z^{-k} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}$

and $H_2(z) = \dfrac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$

$$= \frac{1}{1+a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots + a_N z^{-N}}$$

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Structure:-

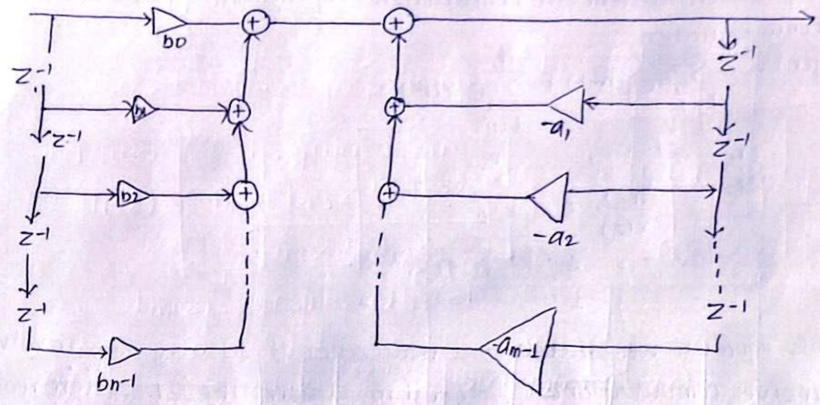
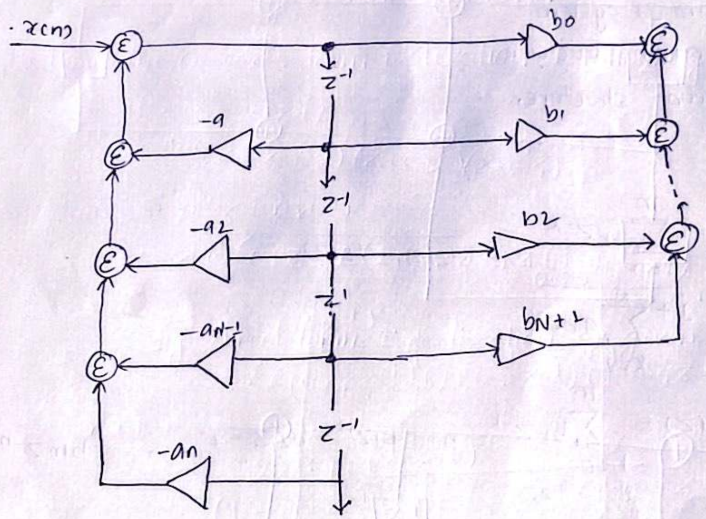


fig: Structure I direct form.



Obtain the direct form realization of the system $y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1)$ ★★

$\therefore y(n) - 0.5y(n-1) + 0.25y(n-2) = x(n) + 0.4x(n-1)$

taking z transform on both side.

$$y(z) = 0.5y(z)z^{-1} + 0.25y(z)z^{-2} = x(z) + 0.4x(z)z^{-1}$$

$$y(z) = x(z) \frac{(1+0.4z^{-1})}{(1-0.5z^{-1}+0.25z^{-2})}$$

$$H(z) = \frac{y(z)}{x(z)} = \frac{1+0.4z^{-1}}{1-0.5z^{-1}+0.25z^{-2}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \dots}$$

Comparing we get

$b_0 = 1$

$b_1 = 0.4$

$a_1 = -0.5 \Rightarrow -a_1 = 0.5$

$a_2 = 0.25 \Rightarrow -a_2 = -0.25$

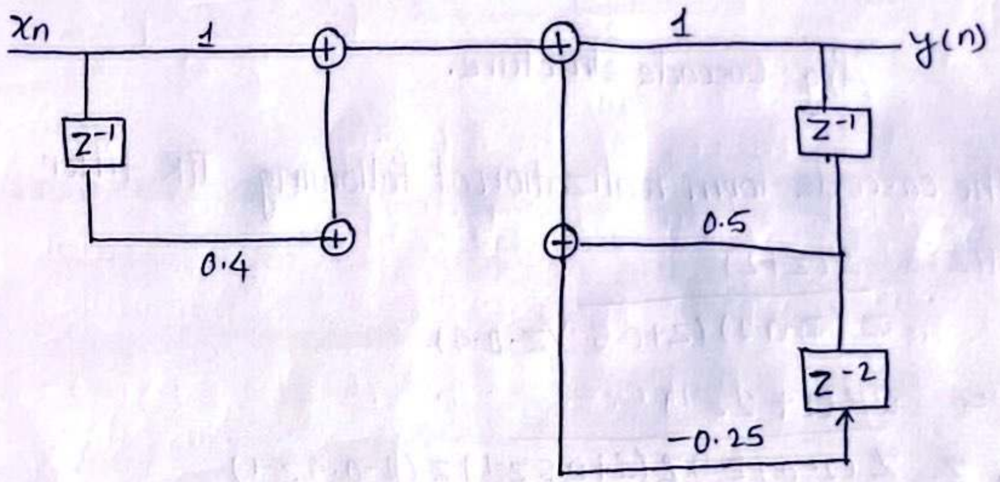


fig: direct form I realization at given system

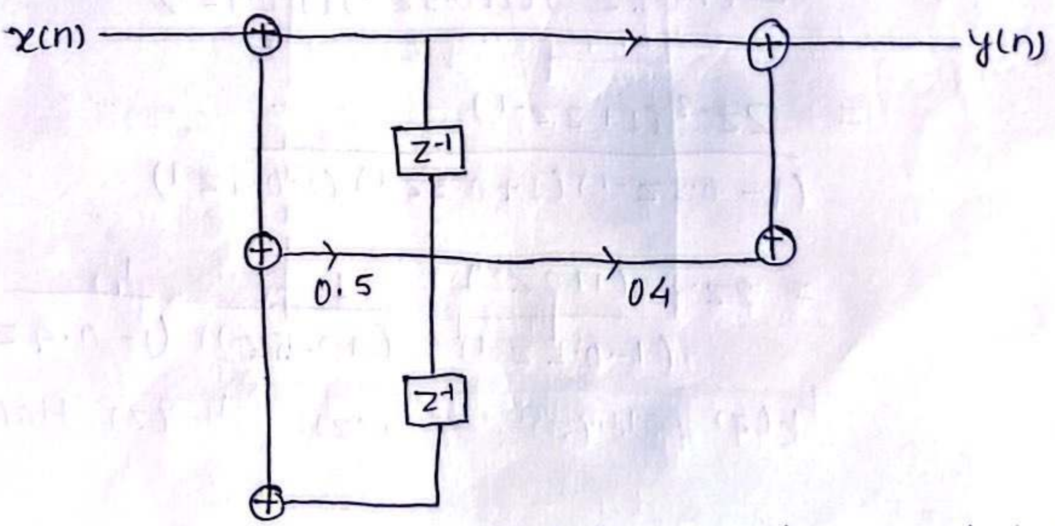


fig: direct form II realization at given system.

2) Cascade structure.

let $H(z) = \prod_{k=1}^{k} H_k(z) = H_1(z).H_2(z).H_3(z)...$

Where k is the integer part $\frac{N+1}{2}$

and H(z) has the general form

$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$$

$$H_k(z) = H_1(z)\,H_2(z)\,H_3(z)....$$

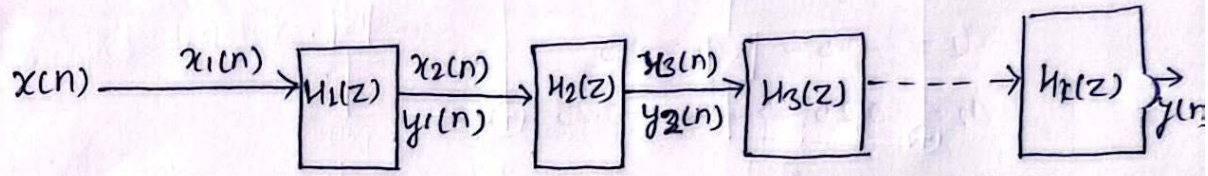


fig: Cascade structure.

Q. Obtain the cascade form realization of following IIR filter

$$H(z) = \frac{2(z+2)}{z(z-0.1)(z+0.5)(z-0.4)}$$

$$= \frac{2\cancel{z}(1+2z^{-1})}{z.\cancel{z}(1-0.1z^{-1})z(1+0.5z^{-1})z(1-0.4z^{-1})}$$

$$= \frac{2(1+2z^{-1})}{z^3(1-0.1z^{-1})(1+0.5z^{-1})(1-0.4z^{-1})}$$

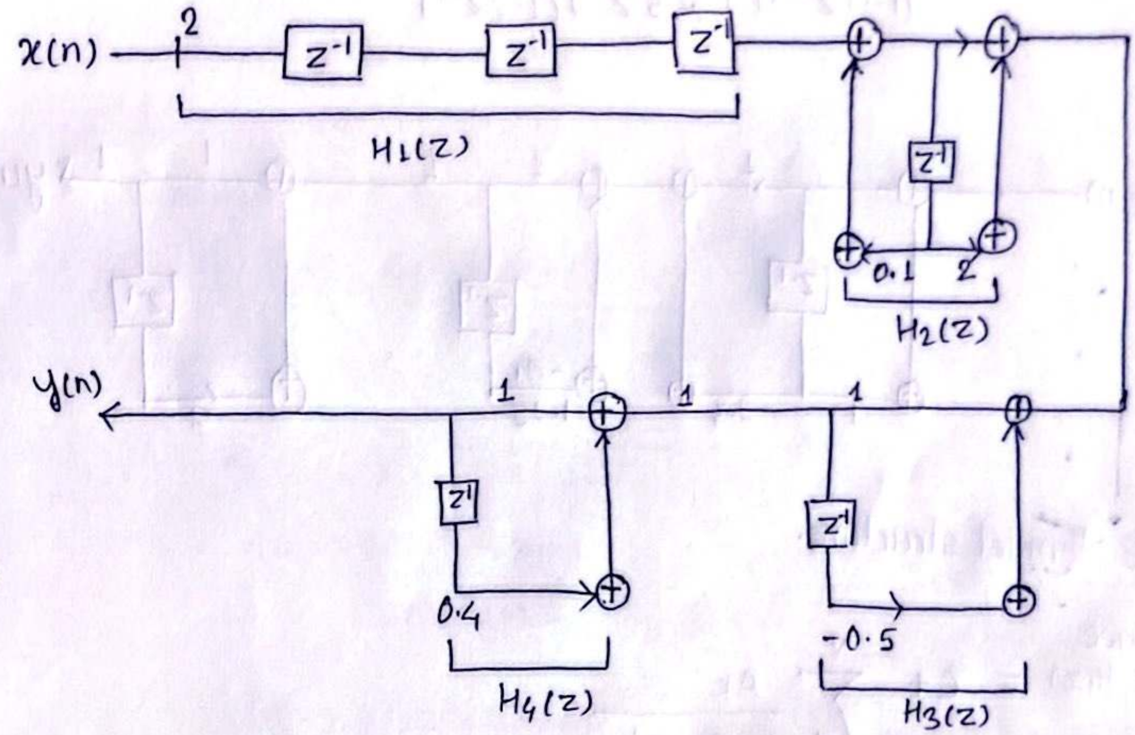
$$= \frac{2z^{-3}(1+2z^{-1})}{(1-0.1z^{-1})(1+0.5z^{-1})(1-0.4z^{-1})}$$

$$= 2z^{-3}\,\frac{(1+2z^{-1})}{(1-0.1z^{-1})}\;\frac{1}{(1+0.5z^{-1})}\cdot\frac{1}{(1-0.4z^{-1})}$$

$$H_k(z) = H_1(z) \quad H_2(z) \quad H_3(z) \quad H_4(z)$$

$x(n) \longrightarrow H_1(z) \longrightarrow H_2(z) \longrightarrow H_3(z) \rightarrow H_4(z) \rightarrow y(n)$

The cascade structure of above function is given by



Q. Draw the cascade structure for the following

$$y(n) = 2.3y(n-1) - 1.7y(n-2) + 0.4y(n-3) + x(n) - 2.3x(n-1)$$

Taking Z-transform

$$Y(z) = 2.3Y(z)z^{-1} - 1.7Y(z)z^{-2} + 0.4Y(z)z^{-3} + X(z) - 2.3X(z)z^{-1}$$

$$Y(z) = -2.3Y(z)z^{-1} + 1.7Y(z)z^{-2} - 0.4Y(z)z^{-3} = XZ - 2.3X(z)z^{-1}$$

$$\frac{Y(z)}{X(z)} = \frac{1 - 2.3z^{-1}}{1 - 2.3z^{-1} + 1.7z^{-2} - 0.4z^{-3}}$$

$$\therefore H(z) = \frac{1 - 2.3z^{-1}}{1 - 2.3z^{-1} + 1.7z^{-2} - 0.4z^{-3}}$$

Comparing with

$$\frac{b_0 + b_1z^{-1} + b_2z^{-2} + \cdots}{1 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3}}$$

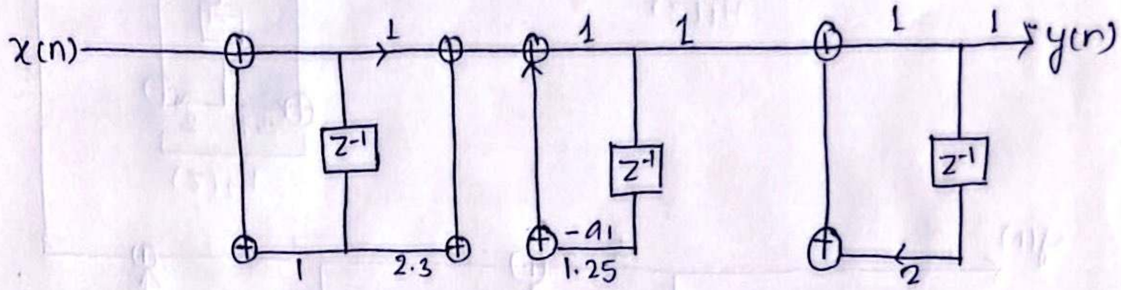
we get

$b_0 = 1$ $\quad a_1 = -2.3$ $\quad a_2 = 1.7$ $\quad a_3 = -0.4$

$b_1 = -2.3$ $\quad -a_1 = 2.3$ $\quad -a_2 = -1.7$ $\quad -a_3 = 0.4$

Now,

$$H(z) = \frac{(1-2.3z^{-1})}{(1-1z^{-1})(1-1.25z^{-1})(1-2z^{-1})}$$



3. Parallel structure.

Since

$$H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1-P_k z^{-1}}$$

or,

$$H(z) = C + H_1(z) + H_2(z) + H_3(z) + \cdots$$

$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2} + \cdots}$$

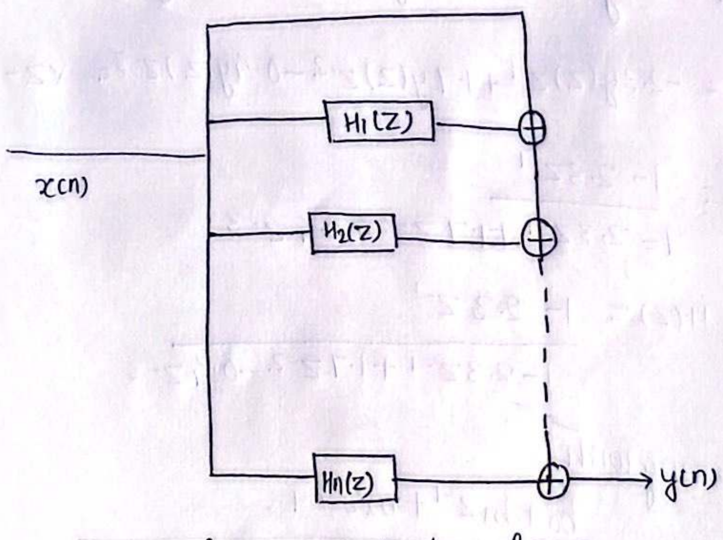


fig: Parallel structure of IIR.

Q. Obtain the parallel structure of given IIR filter

$$H(z) = \frac{3(2z^2+5z+4)}{(z+2)(2z+1)}$$

Dividing both side by 'z'

$$\frac{H(z)}{z} = \frac{\cancel{3}\,1.5(2z^2+5z+4)}{z(z+2)\cancel{2}(z+\frac{1}{2})}$$

$$\frac{H(z)}{z} = \frac{A}{z} + \frac{B}{z+2} + \frac{C}{(z+\frac{1}{2})}$$

$$A = \left.\frac{1.5(2z^2+5z+4)}{(z+2)(z+0.5)}\right|_{z=0} = \frac{1.5\times 4}{2\times 0.5} = 6$$

$$B = \left.\frac{1.5(2z^2+5z+4)}{z(z+\frac{1}{2})}\right|_{z=-2} = 1$$

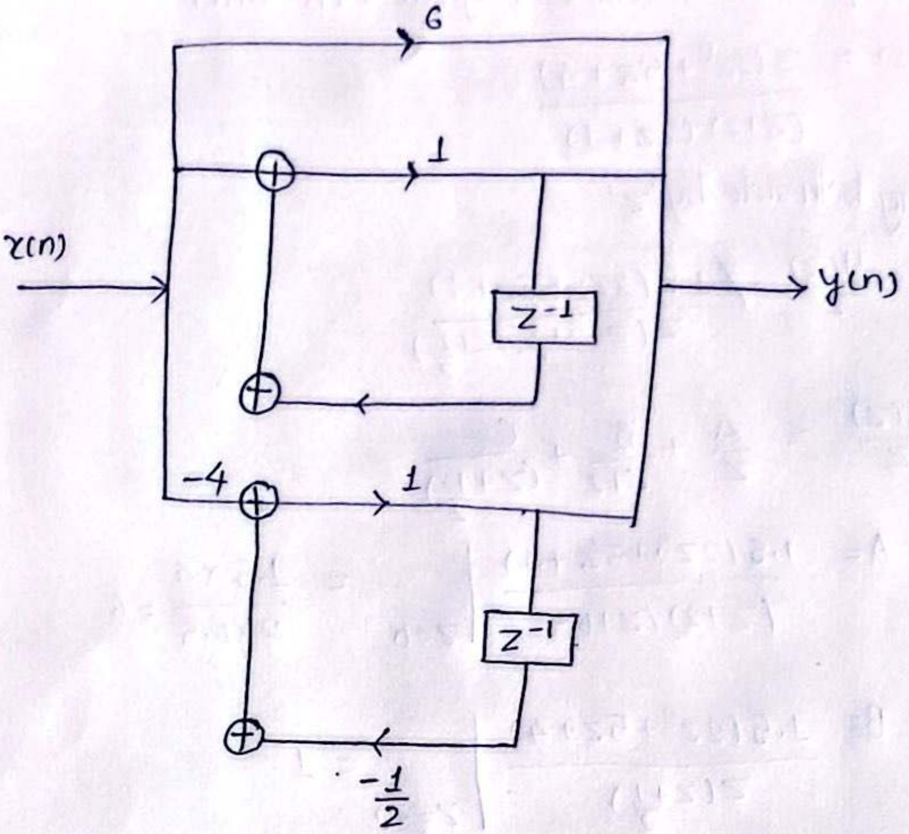
$$C = \left.\frac{1.5(2z^2+5z+4)}{z(z+2)}\right|_{z=-\frac{1}{2}} = -4$$

$$\frac{H(z)}{z} = \frac{6}{z} + \frac{1}{(z+2)} - \frac{4}{(z+\frac{1}{2})}$$

$$H(z) = 6 + \frac{z}{(z+2)} - \frac{4}{(z+\frac{1}{2})}$$

$$= 6 + \frac{1}{(1+2z^{-1})} - \frac{4}{(1+\frac{1}{2}z^{-1})}$$

$$H(z) = H_1(z) + H_2(z) + H_3(z)$$



Q. Obtain IIR filter using direct - II form.

$$H(z) = \frac{1}{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}}$$

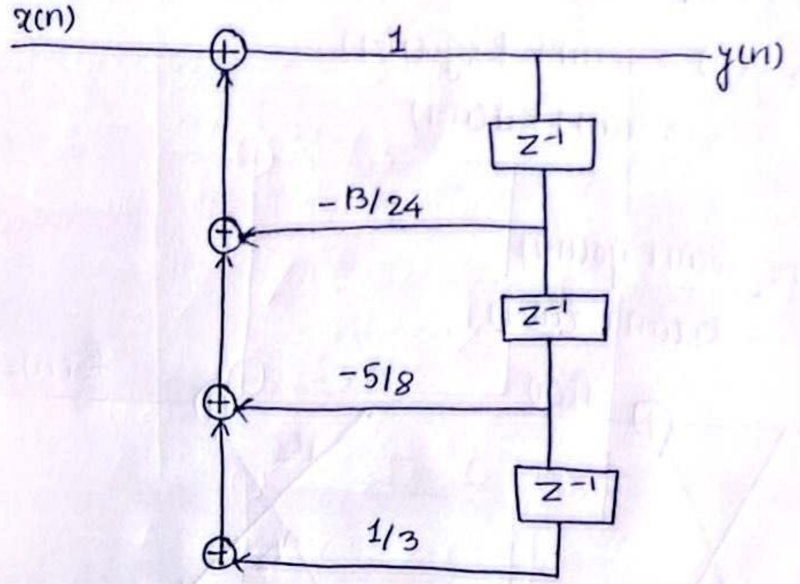
Comparing with $\dfrac{b_0 + b_1 z^{-1} + \cdots}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots}$

$b_0 = 1$

$a_1 = \frac{13}{24} \Rightarrow -a_1 = -\frac{13}{24}$

$a_2 = \frac{5}{8} \Rightarrow -a_2 = -\frac{5}{8}$

$a_3 = \frac{1}{3} \Rightarrow -a_3 = 1/3$



Lattice structure for IIR filter

The lattice filter are used in digital speech processing an in the implementation of adaptive flerting

let us begin the development by considering the sequence of FIR filter with the system function.

$H_m(z) = A_m(z)$ where $m = 0, 1, 2, --- m-1$

and

$$A_m(z) = 1 + \sum_{k=1}^{m} a_m(k) z^{-k}$$ is polynomial

for which $A_0(z) = 1$ $\alpha_m(k) \longrightarrow$ coefficient of filter

if $x(n)$ is the input sequence to the filter $A_m(z)$ and $y(n)$ is output sequence we have

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)\, x(n-k)$$

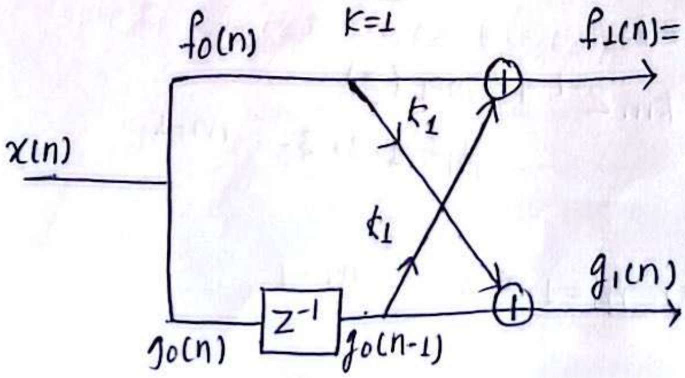


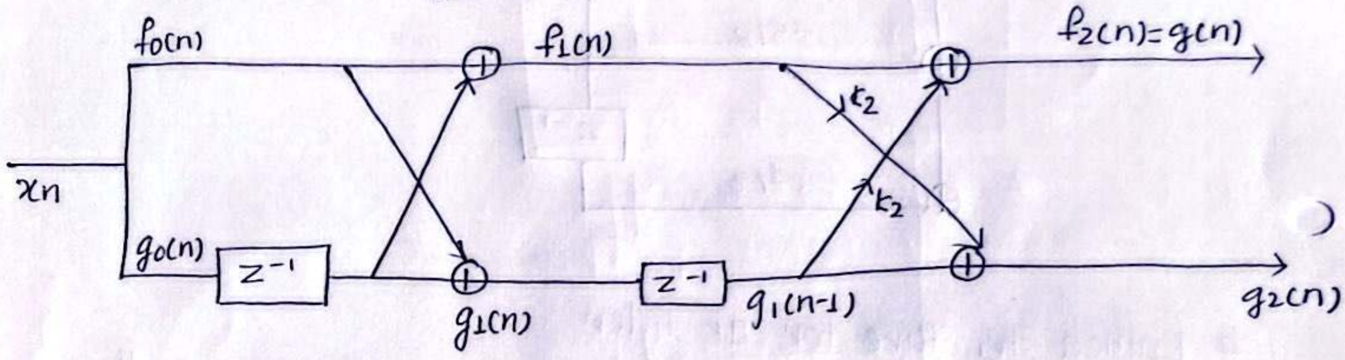
fig: Single stage lattice filter

From figure, $f_0(n) = g_0(n) = x(n)$

$$f_1(n) = y(n) = f_0(n) + k_1 g_0(n-1)$$
$$= x(n) + k_1 x(n-1)$$

and

$$g_1(n) = f_0(n) + g_0(n-1)$$
$$= k_1 x(n) + x(n-1)$$



$$y(n) = f_1(n) + k_2\, g_1(n-1)$$

$g_2(n)$

Conversion of Lattice coefficient to direct form filter coefficient

The direct form FIR filters coefficient $\alpha_m(k)$ can be obtained from Lattice coefficient 'k_1' by using following relation.

$$\Rightarrow A_0(z) = B_0(z) = 1.$$

$$A_m(z) = A_{m-1}(z) + k_m z^{-1} + B_{m-1}(z)$$

inverse of $A_m(z)$ $\qquad m = 1, 2, 3 \ldots m-1$

and

$$B_m(z) = z^{-m} A_m(z^{-1}), \quad m = 1, 2, 3, \ldots m-1$$

The solution is obtained recursively with m

Q. Given a theee stage lattice filter with coefficient $k_1 = \frac{1}{4}$, $k_2 = \frac{1}{4}$, $k_3 = \frac{1}{3}$
Determine FIR filter coefficient for direct form structure

Soln

$k_1 = \frac{1}{4}$, $k_2 = \frac{1}{4}$, $k_3 = \frac{1}{3}$

Since we have, $A_m(z) = A_{m-1} + k_m z^{-1} \times B_{m-1}(z)$

Now, at m=1, $A_1(z) = A_0(z) + k_1 z^{-1} \times B_0(z)$

$A_0(z) = B_0(z) = 1$ and $k_1 = \frac{1}{4}$

$\therefore A_1(z) = 1 + \frac{1}{4} z^{-1} \times 1$

$A_1(z) = 1 + 0.25 z^{-1}$

$B_1(z) = 0.25 + z^{-1}$

at m=2, $A_2(z) = A_1(z) + k_2 z^{-1} B(z)$

$= 1 + 0.25 z^{-1} + \frac{1}{4} z^{-1} (0.25 + z^{-1})$

$A_2(z) = 1 + 0.3125 z^{-1} + 0.25 z^{-2}$

$B_2(z) = 0.25 + 0.3125 z^{-1} + z^{-2}$

at m=3, $A_3(z) = A_2(z) + k_3 z^{-1} B_2(z)$

$= 1 + 0.3125 z^{-1} + 0.25 z^{-2} + \frac{1}{3} z^{-1} (0.25 + 0.3125 z^{-1} + z^{-2})$

$= 1 + 0.3125 z^{-1} + 0.25 z^{-2} + 0.083 z^{-1} + 0.104 z^{-2} + 0.33 z^{-3}$

$A_3 = 1 + 0.3955 z^{-1} + 0.354 z^{-2} + 0.33 z^{-3}$ #

$\alpha_3(0) = 1$

$\alpha_3(1) = 0.395$

$\alpha_3(2) = 0.354$

$\alpha_3(3) = 0.33 = k_3$

Conversion of direct form FIR filter coefficient to lattice coefficient

Since we have,

$$A_m(z) = A_{m-1}(z) + k_m z^{-1} B_{m-1}(z)$$

$$= A_{m-1}(z) + k_m [B_m(z) - k_m - A_{m-1}(z)]$$

Solving we get,

$$A_{m-1}(z) = \frac{A_m(z) - k_m B_m(z)}{1 - k_m^2} \quad \text{—} (1)$$

where $m = m-1, m-2, \text{-----}, 1$

starting with $m = m-1$ in eqn (1)

$$k_m = \alpha_m(m), \quad \alpha_{m-1}(0) = 1.$$

Thus we compute all- lower degree polynomial $A_m(z)$ begining with $A_{m-1}(z)$ and obtain the desired lattice coefficient from the relation $k_m = \alpha_m(m)$ if $|k_m| \neq 1$

$$\therefore \alpha_{m-1}(k) = \frac{\alpha_m(k) - k_m B_m(k)}{1 - k_m^2}$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\, \alpha_m(m-k)}{1 - \alpha_m^2(m)}$$

$$1 \leq k \leq m-1$$

Q. Determine the lattice coefficient crroesponding to the FIR firler with the system function

$H(z) = A_m(z) = 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}, \; m=3$

⇒ Here

$A_m(z) = 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}$

$\alpha_3(0) = 1$

$\alpha_3(1) = \frac{13}{24}$

$\alpha_3(2) = \frac{5}{8}$

$\alpha_3(3) = \frac{1}{3} = k_3$

$B_m(z) = \frac{1}{3} + \frac{5}{8} z^{-1} + \frac{13}{24} z^{-2} + z^{-3}$

Since,

$A_{m-1}(z) = \dfrac{A_m(z) - B_m(z).k_m}{1 - k_m^2}$

Now, m=3,

$A_2(z) = \dfrac{A_3(z) - B_3(z)k_3}{1 - k_3^2}$

$= \dfrac{1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3} - \frac{1}{3}\left\{\frac{1}{3} - \frac{5}{8} z^{-1} + \frac{13}{24} z^{-2} + z^{-3}\right\}}{1 - \left(\frac{1}{3}\right)^2}$

$A_2(z) = \dfrac{1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3} - \frac{1}{9} - \frac{5}{24} z^{-1} - \frac{13}{72} z^{-2} - \frac{1}{3} z^{-3}}{\frac{8}{9}}$

$= \dfrac{8/9 + \frac{1}{3} z^{-1} + \frac{60-13}{72} z^{2}}{}$

$A_2(z) = 1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2}$

$B_2(z) = \frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2}$

$\alpha_2 = k_2 = \frac{1}{2}$

at $m = 2$,

$$A_1(z) = \frac{A_2(z) - B_2(z) k_2}{1 - k_2^2}$$

$$= \frac{1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2} - \frac{1}{2}\left\{\frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2}\right\}}{1 - \left(\frac{1}{2}\right)^2}$$

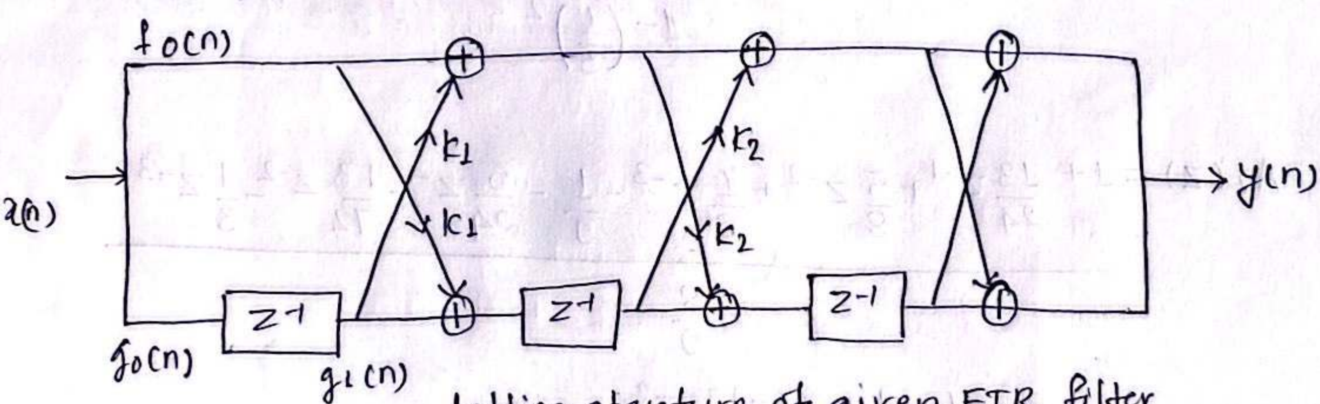
$$= \frac{\frac{3}{4} + \frac{3}{10} z^{-1}}{3/4}$$

$A_1(z) = 1 + \frac{1}{4} z^{-1}$

$\alpha_1(0) = 1, \ \alpha_1(1) = \frac{1}{4} = k_1$

$B_1(z) = \frac{1}{4} + z^{-1}$

$k_1 = \frac{1}{4}, \ k_2 = \frac{1}{2}, \ k_3 = \frac{1}{3}$



Lattice structure of given FIR filter $A_m(z)$

Q. Draw lattice structure of IIR system.

$$H(z) = \frac{1}{2 + 1.4z^{-1} + 1.8z^{-2}}$$

$$H(z) = \frac{1}{A_m(z)}$$

$$A_m(z) = 2 + 1.4z^{-1} + 1.8z^{-2}$$

$$= 2\left\{1 + \frac{1.4}{2} z^{-1} + \frac{1.8}{2} z^{-2}\right\}$$

$$= 2\{1 + 0.7z^{-1} + 0.9z^{-2}\}$$

$\alpha_2(0) = 1$

$\alpha_2(1) = 0.7$

$\alpha_2(2) = 0.9 = k_2$

$$B_2(z) = 0.9 + 0.7z^{-1} + z^{-2}$$

Again

$$A_m(z) = \frac{A_m(z) - k_m B_m(z)}{1 - k_m^2}$$

At $m=2$

$$A_1 = \frac{A_2(z) - k_2 B_2(z)}{1 - (0.9)^2}$$

$$= \frac{(1 + 0.7z^{-1} + 0.9z^{-2}) - 0.9(0.9 + 0.7z^{-1} + z^{-2})}{1 - 0.81}$$

$$= \frac{0.19 + 0.07z^{-1}}{0.19}$$

$$A_1(z) = 1 + 0.368z^{-1}$$

$\alpha_1(0) = 1$

$\alpha_1(1) = 0.368 = k_1$

$$B_1(z) = 0.368 + z^{-1}$$

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At $m=1$

$$A_0 = \frac{A_1(z) - k_1 B_1(z)}{1-(k_1)^2}$$

$$= \frac{1+0.368z^{-1} - 0.368(0.368+z^{-1})}{1-(0.368)^2}$$

$$= \frac{1-0.135}{1-0.135}$$

$$= \frac{0.86}{0.86}$$

$$= 1.$$

$\alpha_0(0) = 1 = k_0.$

$k_0 = 1$
$H_1 = 0.368$
$H_2 = 0.9$

~~Take lat~~ The lattice structure of given function is

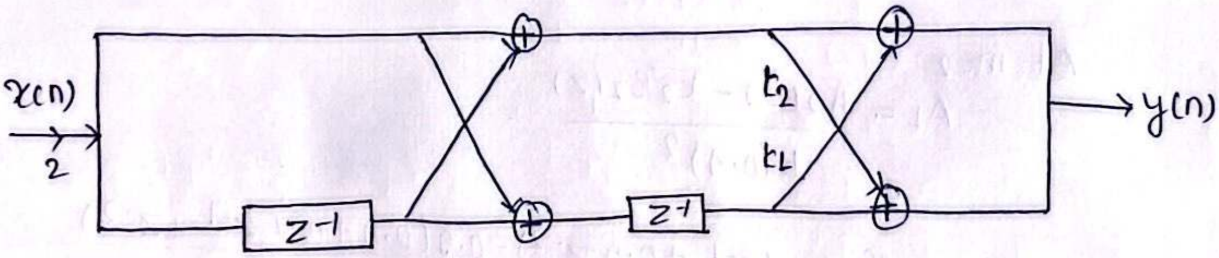


fig: Lattice structure of H(z)

Q. Draw lattice structure for given difference

$$y(n) = 2x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$$

Taking z transform

$$y(z) = 2X(z) + \frac{4}{5}X(z)z^{-1} + \frac{3}{2}X(z)z^{-2} + \frac{2}{3}X(z)z^{-3}$$

$$\frac{Y(z)}{X(z)} = \left(2 + \frac{4}{5}z^{-1} + \frac{3}{2}z^{-2} + \frac{2}{3}z^{-3}\right) = H(z)$$

$$A_m(z) = 2\left\{1 + \frac{2}{5}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{3}z^{-3}\right\}$$

$$A_3(z) = k_0\left\{1 + \frac{2}{5}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{3}z^{-3}\right\}$$

$$k_3 = 1/3$$

$$B_3(z) = \frac{1}{3} + \frac{3}{4}z^{-1} + \frac{2}{5}z^{-2} + z^{-3}$$

Since $A_{m-1}(z) = \dfrac{A_m(z) - k_m B_m(z)}{1 - k_m^2}$

At $m=3$, $A_2(z) = \dfrac{A_3(z) - k_3 B_3(z)}{1 - k_3^2}$

$$= \frac{1 + \frac{2}{5}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{3}z^{-3} - \frac{1}{3}\left\{\frac{1}{3} + \frac{3}{4}z^{-1} + \frac{2}{5}z^{-2} + z^{-3}\right\}}{1 - \left(\frac{1}{3}\right)^2}$$

$$= \frac{8/9 + \frac{3}{20}z^{-1} + \frac{37}{60}z^{-2}}{1 - 1/9}$$

$$= \frac{\frac{8}{9} + \frac{3}{20}z^{-1} + \frac{37}{60}z^{-2}}{8/9}$$

$$A_2(z) = 1 + \frac{81}{100} z^{-1} + \frac{111}{160} z^{-2}$$

$$k_2 = \frac{111}{160}$$

$$B_2(z) = \frac{111}{160} + \frac{81}{160} z^{-1} + z^{-2}$$

at $m=2$

$$A_1(z) = \frac{A_2(z) - k_2 B_2(z)}{1-(k_2)^2}$$

$$= \frac{1 + \frac{81}{160} z^{-1} + \frac{111}{160} z^{-2} - \frac{111}{160}\left(\frac{111}{160} + \frac{81}{160} z^{-1} + z^{-2}\right)}{1 - \left(\frac{111}{160}\right)^2}$$

$$= \frac{\frac{13279}{25600} + \frac{3969}{25600} z^{-1}}{\frac{13279}{25600}}$$

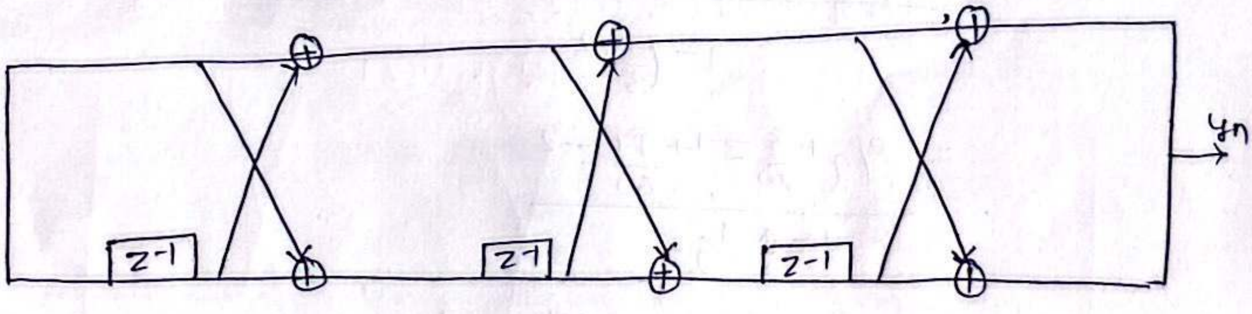
$$A_1(z) = 1 + \frac{81}{271} z^{-1}$$

$$k_1 = \frac{81}{271} \quad \therefore B_1(z) = \frac{81}{271} + z^{-1}$$

at $m=1$

$$A_0(z) = \frac{A_1(z) - k_1 B_1(z)}{1-(k_1)^2}$$

$$A_0(z) = 1$$



Chapter 7 Discrete fourier transform

The discrete fourier transform of discrete time signal $x(n)$ is denoted by $x(k)$ and given by $x(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$

where, $k = 0, 1, 2, --- N-1$.

Since summation is taken for N-points so it is known as N-point ~~DDF~~ DFT

Inverse DFT or process of conversion of $x(K)$ to $x(n)$ is known is as inverse discrete fourier transform (IDFT) and given by,

$$X(n) = \frac{1}{N}\sum_{k=0}^{N-1} x(k) e^{(j2\pi kn)/4N}$$

where $n = 0, 1, 2, 3, --- N-1$.

Discrete fourier transform is very powerful tool for frequency analysis of discrete time signals. DFT is itself a sequence rather then a function of a continious variable and it crossponds to samples which are equally spaced in frequency. so DFT is speeefic kind of Discrete transform used is fourier analysis. It is used for transforming $x(n)$ of finite length into $X(K)$ of finite length and viceversa.

i.e $x(n) \longleftrightarrow x(k)$

Using twiddle function

$$W_N^{kn} = e^{-j2\pi/N}$$

$e^{-j2\pi/N}$ = twiddle factor.

$$\text{DFT} \Rightarrow x(k) = \sum_{n=0}^{N-1} x(n). W_N^{kn}$$

$$\text{IDFT} \Rightarrow X(n) = \frac{1}{N}\sum_{K=0}^{N-1} x(n) W_n^{-kn}$$

DFT As a linear transform.

$$DFT \Rightarrow X_N = W_N \cdot x(n)$$

$$IDFT \Rightarrow x_n = W_N^{-1} X_N$$

$$W_N^{-1} = \frac{1}{N} W_N^{*}$$

Where $x(n) = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$ and $X(N) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}$

and

$$W_N = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_N & W_N^2 & W_N^3 & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & W_N^6 & W_N^{2(N-2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & W_N^{3(N-1)} & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Compute the DFT of 4 points sequence using both methods $x(n) = \{0,1,2,3\}$

Soln:- Here $x(n) = \{0,1,2,3\}$

The DFT of sequence $x(n)$ is given by

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

Since the sequence is 4 point i.e $N=4$

$$\therefore X(K) = \sum_{n=0}^{N-1} x(n)\, e^{(-j2\pi kn)/4}$$

$$= \sum_{n=0}^{3} x(n)\, e^{(-j2\pi kn)/4}$$

$$\therefore X(K) = x(0)\, e^{\frac{-j 2\pi K \times 0}{4}} + x(1)\, e^{\frac{-j 2\pi K}{4}} + x(2)\, e^{\frac{-j 2\pi K 2}{4}} + x(3)\, e^{-j\frac{2\pi K 3}{4}} \quad \text{——(1)}$$

Again $K = 0, 1, 2 \ldots N-1$.

$\therefore K = 0, 1, 2, 3.$

at $K=0$. from eqn (1)

$$X(0) = x(0)e^0 + x(1)\cdot e^0 + x(2)\, e^0 + x(3) + e^0$$

$$= 0+1+2+3$$

$$= 6.$$

$$X(K) = \sum_{n=0}^{3} x(n)\, e^{(-j 2\pi K n)/4}$$

$$X(0) = \sum_{n=0}^{3} x(n)\, e^0$$

$$= 6.$$

at $K=1$.

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j 2\pi . n/4}$$

$$= x(0)\, e^{-j 2\pi . 0/4} + x(1) e^{-j 2\pi\, 1/4} + x(2) e^{-j 2\pi 2/4}$$

$$+ x(3)\, e^{(-j 2\pi\, 3)/4}$$

$$= x(0) e^0 + x(1)\, e^{-j\pi/2} + x(2)\, e^{-j\pi} + x(3)\, e^{-j 3\pi/2}$$

$$= 0 + e^{-j\pi/2} + 2e^{-j\pi} + 3e^{-j 3\pi/2}$$

$$= (\cos\pi/2 - i\sin\pi/2) + 2(\cos\pi - i\sin\pi) + 3(\cos 3\pi/2 - i\sin 3\pi/2)$$

$$= -j + 2(-1) + 3(0+j)$$

$$= -j - 2 + j3$$

$$= -2 + j2$$

at $k=2$

$$x(2) = \sum_{n=0}^{3} x(n)\, e^{-j 2\pi 2n/4}$$

$$= \sum_{n=0}^{3} x(n)\, e^{-j\pi n}$$

$$= x(0)e^{-j\pi\times 0} + x(1)e^{-j\pi} + \cancel{e}\, x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 0 + 1e^{-j\pi} + 2e^{-j2\pi} + 3e^{-j3\pi}$$

$$= \cancel{0}\ \cancel{1}(\cos\pi - j\sin\pi) + 2(\cos 2\pi - j\sin 2\pi) + 3(\cos 3\pi - j\sin 3\pi)$$

$$= (-1-0) + 2(1-0) + 3(-1-0)$$

$$= -1+2-3$$

$$= -4+2 = -2$$

at $k=3$

$$x(3) = \sum_{n=0}^{3} x(n)\, e^{-j 2\pi 3n/4}$$

$$= \sum_{n=0}^{3} x(n)\, e^{-j\frac{3n\pi}{2}}$$

$$= x(0)\, e^{-j\frac{3\times 0\pi}{2}} + x(1)\, e^{-j\frac{3\times 1\times\pi}{2}} + x(2)\, e^{-j\frac{3\times 2\times\pi}{2}}$$

$$+ x(3)\, e^{-j\frac{3\times 3\times\pi}{2}}$$

$$= 0 + 1\times e^{-j3\pi/2} + 2e^{-j3\pi/\cancel{2}} + 3e^{-j9\pi/2}$$

$$= \left(\cos\frac{3\pi}{2} - j\sin 3\pi/2\right) + 2(\cos 3\pi/\cancel{2} - j\sin 3\pi/\cancel{2}) + 3(\cos 9\pi/2 - j\sin$$

$$= (0+1j) + 2(-1-0) + 3(0-1j)$$

$$= 1j\cancel{*} - 2 - 3j$$

$$= -2-2j$$

$x(K) = \{6, -2+j2, -2, -2-j2\}$

Using twiddle factor

$x(K) = W_N \cdot x_n$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_N & W_N^2 & W_N^3 \\ 1 & W_N^2 & W_N^4 & W_N^6 \\ 1 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

Since,

$W_N = e^{-j2\pi/N}$

$W_4 = e^{-j2\pi/4} = -j$

$W_4^2 = -j \cdot -j = -1$

$W_4^3 = j \; W_4^1 \cdot W_4^2 = -j \cdot 1 = j$

$W_4^4 = (-1)(-1) = 1$

$W_4^6 = -1.$

$W_4^9 = -j$

Subtituting we get

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$= \{6, -2+j2, -2, -2-j2\}$

Q. Find the dft of the given sequence using linear transformation and twiddle factor method.

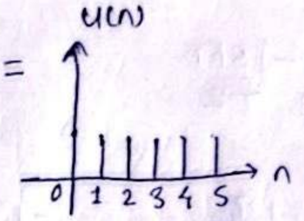
$x(n) = u(n) - u(n-4)$

Soln:- Since DFT of a sequence is given by

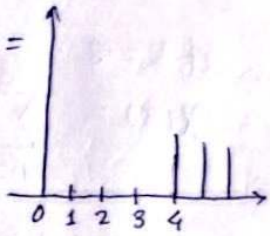
$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Given function

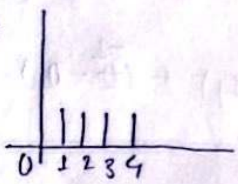
$x(n) = u(n) - u(n-4)$ $\quad u(n) =$



$u(n-4) =$



$x(n) =$



$\therefore x(n) = \{1, 1, 1, 1\}$

Since the sequence is 4 points i.e $N = 4$.

$$\therefore X(k) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi kn}{4}}$$

at $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n)\, e^{0}$$

$= x(0) + x(1) + x(2) + x(3)$

$= 1 + 1 + 1 + 1 = 4.$

at $k = 1$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi n}{2}}$$

$= x(0)e^{0} + x(1)\, e^{\frac{-j\pi}{2}} + x(2)\, e^{-j\pi} + x(3)\, e^{\frac{-j\pi 3}{2}}$

$= 1 + \{0 - j1\} + (-1 - 0j) + (0 - j(-1))$

$= 0$

at $k=2$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\,2\cdot 4\pi n}{4}}$$

$$= x(0)e^{0} + x(1)e^{-j\pi} + e^{-2j\pi} + e^{-3j\pi}$$

$$= 1 + (-1-0) + (1-0) + (-1-0)$$

$$= 0$$

at $k=3$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\,3\pi n}{2}}$$

$$= 1 + x(1)e^{\frac{-j3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{\frac{-j9\pi}{2}}$$

$$= 1 + (0-1j) + (-1-0j) + (0-1j)$$

$$= 0$$

$\therefore\ x(k) = \{4, 0, 0, 0\}$

Using twiddle e

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_N & W_N^2 & W_N^3 \\ 1 & W_N^2 & W_N^4 & W_N^6 \\ 1 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1+1 \\ 1-j-1+j \\ 1-1+1-1 \\ 1+j-1-j \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \therefore\ X(k) = \{1, 0, 0, 0\}$$

Determine the DFT of $x(n) = \{1,1,0,0\}$ and check the validity of DFT Using IDFT

Here,

DFT of $x(n)$ is given by

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi kn}{N}}$$

Given $x(n) = \{1,1,0,0\}$

$$\text{so } X(K) = \sum_{n=0}^{3} x(n) e^{\frac{-j2\pi kn}{N}}$$

for $k=0$

$$X(K) = \sum_{n=0}^{3} x(n) e^{0}$$

$$= x(0)+x(1)+x(2)+x(3)$$

$$= 1+1+0+0$$

$$= 2$$

for $k=1$.

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi n}{2}}$$

$$= x(0)e^{0} + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$$

$$= 1-j$$

for $k=2$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi n 4}{4}}$$

$$= \sum_{n=0}^{3} x(n) e^{-j\pi n}$$

$$= x(0)e^{0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 0$$

for, $k=3$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j\frac{3\pi}{2}}$$

$$= x(0)e^{0} + x(1)\, e^{-j\frac{3\pi}{2}} + x(2)\, e^{-j3\pi} + x(3)\, e^{-j9\pi/2}$$

$$= 1 + j \qquad X(K) = \{2, 1-j, 0, 1+j\}$$

Inverse DFT of a function is given by

$$x(n) = \frac{1}{N} W_N^{*} X(K)$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_N^{1*} & W_N^{2*} & W_N^{3*} \\ 1 & W_N^{2*} & W_N^{4*} & W_N^{6*} \\ 1 & W_N^{3*} & W_N^{6*} & W_N^{9*} \end{bmatrix} \begin{bmatrix} X(1) \\ X(2) \\ X(3) \\ X(4) \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 2 + 1 - j + 0 + 1 + j \\ 2 + j + 1 + 0 - j + 1 \\ 2 - 1 + j + 0 - 1 - j \\ 2 - j - 1 + 0 + j - 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore x(n) = \{1, 1, 0, 0\}$$

$$x(n) \xrightarrow{DFT} X(K) \xrightarrow{IDFT} x(n)$$

Circular convolution

SN	Input sequence	Expression	Explaination
1.	Input sequence	$x(n)$	Plot $x(n)$ in ~~article~~ anticlockwise direction i.e +ve direction
2.	Circular folding	$x(-n)$	Plot $x(n)$ in clockwise direction i.e negative direction.
3.	Circular delay	$x(n-k)$	shift $x(n)$ in anticlockwise direction by 'k' samples.
4.	Circular advance	$x(n+k)$	Shift $x(n)$ in clockwise direction by k samples.

The circular convolution using linear convolution is given by

$$y(n) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N$$

Where, $m = 0, 1, 2, 3 \ldots N-1$

or,

$$y(m) = \sum_{n=0}^{N-1} x_2(n)\, x_1((m-n))_N$$

Q Given the two sequence of length 4 as under $x(n) = \{0, 1, 2, 3\}$ $h(n) = \{2, 1, 1, 2\}$ Compute the convolution.

Soln According the defination of circular convolution

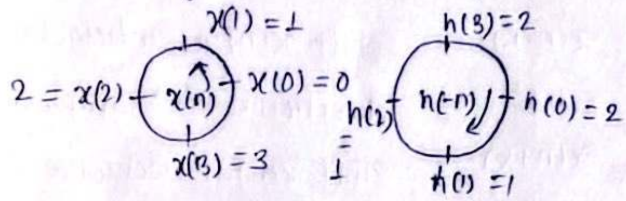
$$y(m) = \sum_{n=0}^{N-1} x_1(m)\, x_2((m-n))_N$$

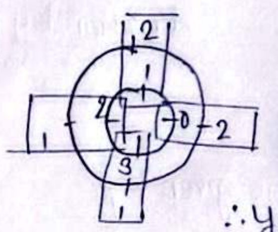
$$\therefore\ y(m) = \sum_{n=0}^{3} x(n)\, h((m-n))_4$$

Now at $m=0$

$$y(0) = \sum_{n=0}^{3} x(n)\, h(-n)_4$$

Plotting $x(n)$ and $h(-n)$ we get





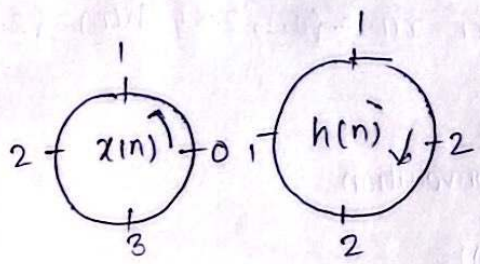
$$\therefore y(0) = 0\times2 + 3\times1 + 1\times2 + 2\times1$$
$$= 0+3+2+2$$
$$= 7$$

at $m=1$

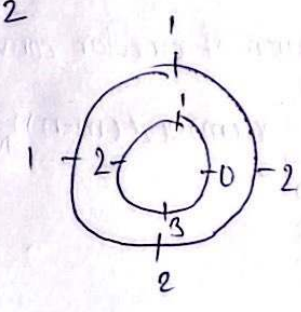
$$y(1) = \sum_{0}^{3} x(n)\; h((1-n))_4$$
$$= \sum_{0}^{3} x(n)\; h((-n)+1))_4$$

Plotting $h(1-n)$ is equivalent to $h((-n)+1) \rightarrow$ Circular advance of $h(-n)$

[Clockwise '1' shift of $h(-n) = h((-n)+1)$ "



$\Rightarrow$



$$y(1) = 2\times0 + 3\times2 + 1\times2 + 1\times1$$
$$= 0+6+2+1 = 9$$

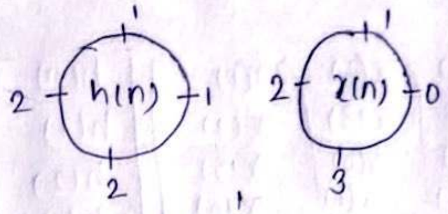
Now at m=2

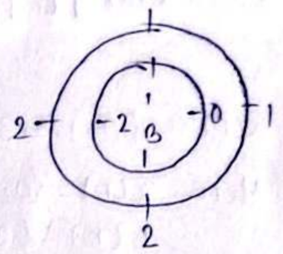
$$y(2) = \sum_{n=0}^{3} x(n)\, h((2-n))_N$$

$$= \sum_{n=0}^{3} x(n)\, h((-n)+2))_N$$

plotting $h((2-n))$ is equivalent to $h((-n)+2)$

i.e





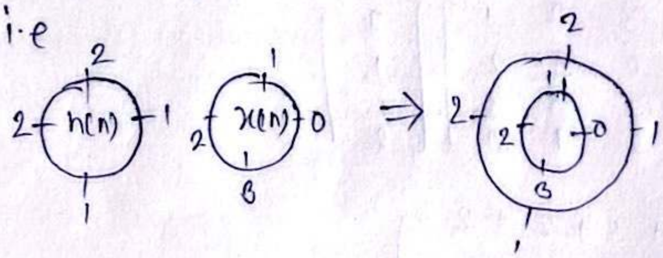
$$\therefore y(2) = 1\times0 + 3\times2 + 2\times2 + 1\times1$$
$$= 0+6+4+1$$
$$= \underline{11}.$$

Now at m=3.

$$y(3) = \sum_{n=0}^{3} x(n)\, h((3-n))_4$$

plotting $h(3-n)$ is equivalent to $h((-n)+3)$

i.e



$$y(3) = 0\times1 + 3\times1 + 2\times2 + 2\times1$$
$$= 0+3+4+2$$
$$= 9.$$

$$\therefore y(m) = \{7, 9, 11, 9\}$$

Circular convolution using matrix method.

The graphical method for calculating the circular convolution is quite difficult when many no of sample points are given so matrix method is used to replace such difficulty as

$$y(m) = x(n) \, \textcircled{N} \, h(n) \text{ or } h(n) \, \textcircled{N} \, x(n)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} x(0) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(3) & x(2) \\ x(2) & x(1) & x(0) & x(3) \\ x(3) & x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix}$$

Q. Find the circular convolution of $x(n) = \{0,1,2,3\}$ and $x(n) = \{2,1,1,2\}$ using matrix method.

Soln:- Here $x(n) = \{0,1,2,3\}$
$h(n) = \{2,1,1,2\}$

$$y(m) = x(n) \, \textcircled{N} \, h(n)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} x(0) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(3) & x(2) \\ x(2) & x(1) & x(0) & x(3) \\ x(3) & x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 & 2 & 1 \\ 1 & 0 & 3 & 2 \\ 2 & 1 & 0 & 3 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 3 + 2 + 2 \\ 2 + 0 + 3 + 4 \\ 4 + 1 + 0 + 6 \\ 6 + 2 + 1 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 9 \\ 11 \\ 9 \end{bmatrix}$$

Q. Find the circular convolution of $x(n) = \{1, 2, 3, 1\}$ and $h(n) = \{4, 3, 2, 2\}$

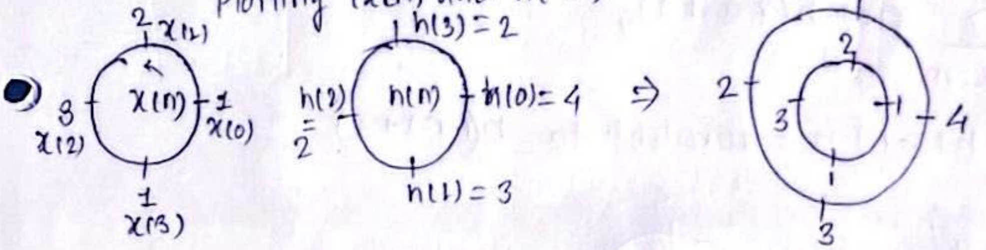
Soln:- According to the defn of circular convolution

$$y(m) = \sum_{n=0}^{N-1} x(n)\, h((m-n))_4$$

at $m=0$

$$y(0) = \sum_{n=0}^{3} x(n)\, h(-n)_4$$

Plotting $(x(n))$ and $h(-n)$

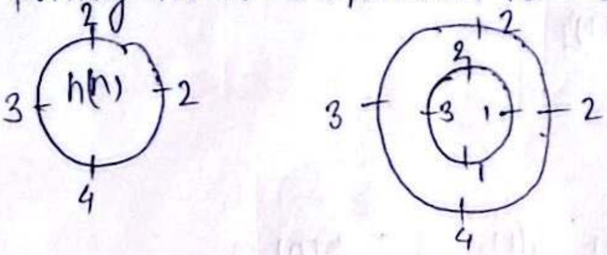


$$\therefore y(0) = 1\times4 + 1\times3 + 2\times3 + 2\times2$$
$$= 4+3+6+4$$
$$= 17$$

at $m=1$

$$y(1) = \sum_{n=0}^{3} x(n)\, h((1-n))_4$$

ii) plotting $h(1-n)$ is equivalent to $h((-n)+1)$

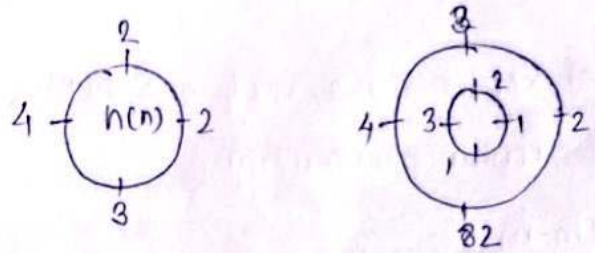


$$y(1) = 1\times2 + 1\times4 + 3\times3 + 2\times2$$
$$= 2+4+9+4$$
$$= 19.$$

at $m=2$

$$y(2) = \sum_{n=0}^{3} x(n)\, h((2-n))_4$$

Plotting $h(2-n)$ is equivalent to $h((-n)+2)$

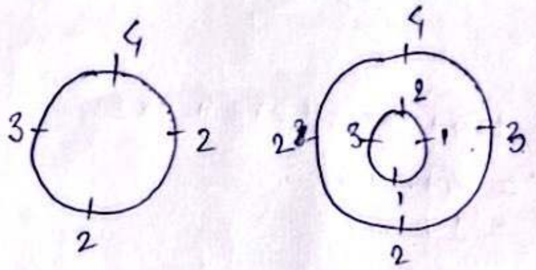


$$y(2) = 1\times2 + 1\times2 + 4\times3 + 3\times2$$
$$= 2 + 2 + 12 + 6$$
$$= 22.$$

at y $m=3$.

$$y(3) = \sum_{n=0}^{3} x(n)\, h((3-n))_4$$

Plotting $h(3-n)$ is equivalent to $h((-n)+3)$



$$y(3) = 1\times3 + 1\times2 + 3\times2 + 4\times2$$
$$= 3 + 2 + 6 + 8$$
$$= \cancel{2}\,19$$

$$y(m) = \{17, 19, 22, \cancel{2}19\}$$

Using matrix mathod.

$$y(m) = x(n) \; N \; h(n)$$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \begin{bmatrix} x(0) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(3) & x(2) \\ x(2) & x(1) & x(0) & x(3) \\ x(3) & x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times4 + 1\times3 + 3\times2 + 2\times2 \\ 2\times4 + 1\times3 + 1\times2 + 3\times2 \\ 3\times4 + 2\times3 + 1\times2 + 1\times2 \\ 1\times4 + 3\times3 + 2\times2 + 2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3+6+4 \\ 8+3+2+6 \\ 12+6+2+2 \\ 4+9+4+2 \end{bmatrix}$$

$$= \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

17.

Q. Compute the 8-point circular convolution for the given sequence

$x_1(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

$x_2(n) = \sin\left(\frac{3\pi n}{8}\right) \quad 0 \le n \le 7$

for $x_2(n)$

at $n=0 \quad x_2(0) = \sin 0 = \cancel{0.92}\ 0$

at $n=1 \quad x_2(1) = \sin\left(\frac{3\pi \times 1}{8}\right) = 0.92$

$x_2(2) = \sin\left(\frac{3\pi \times 2}{8}\right) = 0.70$

$x_2(3) = \sin\left(\frac{3\pi \times 3}{8}\right) = -0.38$

$x_2(4) = \sin\left(\frac{3\pi \times 4}{8}\right) = -1$

$x_2(5) = \sin\left(\frac{15\pi}{8}\right) = \cancel{1} -0.38$

$x_2(6) = \sin\left(\frac{18\pi}{8}\right) = 0.70$

$x_2(7) = \sin \frac{21\pi}{8} = 0.92$

Convolution using matrix method is given by

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} x_1(0) & x_1(7) & x_1(6) & x_1(5) & x_1(4) & x_1(3) & x_1(2) & x_1(1) \\ x_1(1) & x_1(0) & x_1(7) & x_1(6) & x_1(5) & x_1(4) & x_1(3) & x_1(2) \\ x_1(2) & x_1(1) & x_1(0) & x_1(7) & x_1(6) & x_1(5) & x_1(4) & x_1(3) \\ x_1(3) & x_1(2) & x_1(1) & x_1(8) & x_1(7) & x_1(6) & x_1(5) & x_1(4) \\ x_1(4) & x_1(3) & x_1(2) & x_1(1) & x_1(0) & x_1(7) & x_1(6) & x_1(5) \\ x_1(5) & x_1(4) & x_1(3) & x_1(2) & x_1(1) & x_1(0) & x_1(7) & x_1(6) \\ x_1(6) & x_1(5) & x_1(4) & x_1(3) & x_1(2) & x_1(1) & x_1(0) & x_1(7) \\ x_1(7) & x_1(6) & x_1(5) & x_1(4) & x_1(3) & x_1(2) & x_1(1) & x_1(0) \end{bmatrix} \times \begin{bmatrix} x_2(0) \\ x_2(1) \\ x_2(2) \\ x_2(3) \\ x_2(4) \\ x_2(5) \\ x_2(6) \\ x_2(7) \end{bmatrix}$$

y

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.92 \\ 0.70 \\ -0.38 \\ -1 \\ -0.38 \\ 0.70 \\ 0.92 \end{bmatrix}$$

$$= \begin{bmatrix} -0.38 + 0.70 + 0.92 \\ 0 + 0.92 + 0.70 + 0.92 \\ 0.92 + 0.70 + 0.92 \\ 0.92 + 0.70 - 0.38 \\ 0.92 + 0.70 - 0.38 - 1 \\ 0.70 - 0.38 - 1 - 0.38 \\ -0.38 - 1 - 0.38 + 0.70 \\ -1 - 0.38 + 0.70 + 0.92 \end{bmatrix} = \begin{bmatrix} 1.24 \\ 2.54 \\ 2.54 \\ 1.24 \\ 0.24 \\ -1.06 \\ -1.06 \\ 0.24 \end{bmatrix}$$

required circular Convolution

$y(m) = \{1.24, 2.54, 2.54, 1.24, 0.24, -1.06, -1.06, 0.24\}$

Q. Use 4 point DFT and IDFT to determine circular convolution of given sequence.

$x_1(n) = \{1, 2, 3, 1\}$

$x_2(n) = \{4, 3, 2, 2\}$

Here

$$x_1(n) \; N \; x_2(n) \longrightarrow x_1(k)\, x_2(k) \longrightarrow X_3(k) \xrightarrow{IDFT} x_3(n) \text{ or } y(m)$$

We know,

$x_1(k) = W_N \, x_1(n)$

$$\begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+3+1 \\ 1-j2-3+j \\ 1-2+3-1 \\ 1+j2-3-j \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ -2-j \\ 1 \\ -2+j \end{bmatrix} \quad \therefore \; x_1(k) = \{7, -2-j, 1, -2+j\}$$

again $x_2(k) = W_N \, x_2(n)$

$$\begin{bmatrix} x_2(0) \\ x_2(1) \\ x_2(2) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 \\ 2-j \\ 1 \\ 2+j \end{bmatrix}$$

$x_3(k) = x_1(k) \cdot x_2(k)$

$= \{7, -2-j, 1, -2+j\}\{11, 2-j, 1, 2+j\}$

$= \{77, -5, 1, -5\}$

$x_3(n) = y(m) = x_1(n) \circledN x_2(n) = \frac{1}{N} W_N^* x_3(k)$

$$\begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 77 \\ -5 \\ 1 \\ -5 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 68 \\ 77 - 5j - 1 + 5j \\ 77 + 5 + 1 + 5 \\ 77 + 5j - 1 - 5j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 68 \\ 76 \\ 88 \\ 76 \end{bmatrix} = \begin{bmatrix} 17 \\ 19 \\ 22 \\ 19 \end{bmatrix}$$

$x_3(n) = \{17, 19, 22, 19\}$ which is circular convolution

linear convolution

	4	3	2	2
1	4	3	2	2
2	8	6	4	4
3	12	9	6	6
1	4	3	2	2

$x_1(n) \circledast x_2(n) = \{4, 11, 20, 19, 13, 8, 2\}$

Circular convolution and linear convolution is not same.

Q. Determine the circular convolution of

$x(n) = \{1, 0.5, 1, 0.5, 1, 0.5, 1, 0.5\}$

$h(n) = \{0, 1, 2, 3\}$

Soln: To find the circular convolution of length of x(n) and h(n) must be equal, so we have to add 4-zeros in the signal h(n) to make length of x(n) and h(n) equal, this process of adding zeros in the sequence in signal is known as zero padding

Here,

$x(n) = \{1, 0.5, 1, 0.5, 1, 0.5, 1, 0.5, 1, 0.5\}$

$h(n) = \{0, 1, 2, 3\}$

adding 4 zeros to make length of x(n) and h(n) equal

$h(n) = \{0, 1, 2, 3, 0, 0, 0, 0\}$

Convolution using matrix method is given by

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix} = \begin{bmatrix} x(0) & x(7) & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) \\ x(1) & x(0) & x(7) & x(6) & x(5) & x(4) & x(3) & x(2) \\ x(2) & x(1) & x(0) & x(7) & x(6) & x(5) & x(4) & x(3) \\ x(3) & x(2) & x(1) & x(0) & x(7) & x(6) & x(5) & x(4) \\ x(4) & x(3) & x(2) & x(1) & x(0) & x(7) & x(6) & x(5) \\ x(5) & x(4) & x(3) & x(2) & x(1) & x(0) & x(7) & x(6) \\ x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & x(0) & x(7) \\ x(7) & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & x(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \\ h(4) \\ h(5) \\ h(6) \\ h(7) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 \\ 0.5 & 1 & 0.5 & 1 & 0.5 & 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0.5+2+0.5\times 3 \\ 0+1+0.5\times 2+1\times 3 \\ 0+0.5+2+0.5\times 3 \\ 0+1+0.5\times 2+0.5\times 3 \\ 0+0.5+2\overset{+}{\cancel{\times}}0.5\times 3 \\ 0+1+0.5\times 2+1\times 3 \\ 0+0.5+2\times 0.5\times 3 \\ 0+1+0.5\times 2+1\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 5 \\ 4 \\ 5 \\ 4 \\ 5 \\ 4 \\ 5 \end{bmatrix}$$

required circular convolution

$y(m) = \{4, 5, 4, 5, 4, 5, 4, 5\}$