Q. No. 1 (a)

Acontinuity of function

A function $f(x)$ is said to be continuous at $x=a$ if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

or, $LHL = RHL = f(a)$

Differentiability of a function

A function $f(x)$ is said to be differentiable at $x=a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = f'(a)$

i.e. $RHD = LHD =$ finite number

Second Part:

Here $f(x)$ is differentiable at $x=a$

i.e. $RHD = LHD =$ finit number $= f'(a)$

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = f'(a)$$

We can write

$$f(a+h) - f(a) = \frac{f(a+h) - f(a)}{h} \times h$$

$$\lim_{h \to 0} [f(a+h) - f(a)] = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \times h$$

$$= \left\{ \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right\} \times \lim_{h \to 0} h$$

$$= f'(a) \times 0$$

$$= 0$$

$\therefore \lim_{h\to 0} f(a+h) - f(a) = 0$

$\lim_{h\to 0} f(a+h) = f(a)$

$RHL = f(a)$

Similarly

$\lim_{h\to a} f(a-h) = f(a)$

$LHL = f(a)$

$\therefore RHL = LHL = f(a)$

For converse part

Take an example

$f(x) = |x|$ at $x = 0$

Test for continuity

$LHL = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0$

$RHL = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = \lim_{x\to 0^+} (x) = 0$

and

$f(0) = |0| = 0$

$\therefore LHL = RHL = f(0) = 0$

$\therefore f(x)$ is continuous at $x = 0$

Test for differentiability

$RHD = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

$= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - 0}{h} = \lim_{h\to 0} \frac{|h|}{h} = 1$

$$LHD = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{|-h| - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$

$$= -1$$

$\therefore$ LHD $\neq$ RHD at $x = 0$

$f(x)$ is not differentiable at $x = 0$.

This show that every differentiable function is continous but every continuous function may not be differentiable at give point.

Q. No. 1 (b)

Rolle's Theorem

Statement: If $f(x)$ is a function such that

① $f(x)$ is continuous on closed interval $[a,b]$

② $f(x)$ is differentiable on open interval (a,b)

③ $f(a) = f(b)$

Then there exist at least one point $c \in (a,b)$ such that

$$f'(c) = 0$$

Proof:

If $f(x)$ is continuous in $[a,b]$ then there is a maximum value and minimum value

Let $f(c) = M$ is the maximum value

$f(d) = m$ is the minimum value

Case I: If $M = m$ i.e. max and mini. value is same then, the function is constant

i.e. $f(x) = k$

$f'(x) = 0$ for all value of x

Hence theorem is verified

Case II: If $M \neq m$ then either max and mini value or both are different from $f(a)$ and $f(b)$

Let $f(c) = m$ is the different from $f(a)$ and $f(b)$
Then at $x = c$

$$RHD = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h\to 0} \frac{f(c+h) - m}{h} \qquad [m = f(c)]$$

$$= \frac{-ve}{+ve} \leq 0$$

$\therefore$ $RHD \leq 0$

and

$$LHD = \lim_{h\to 0} \frac{f(c-h) - f(c)}{-h}$$

$$= \lim_{h\to 0} \frac{f(c-h) - m}{-h}$$

$$= \cancel{\lim} \frac{-ve}{-ve} \geq 0$$

$LHD \geq 0$

Since $f(x)$ is differentiable in (a,b) we must have

$$RHD = LHD = 0$$

$$\therefore f'(c) = 0$$

Similarly we can show $f'(d) = 0$.

Hence the theorem is verifed.

Geometrical Interpretation

If a function ~~sti~~ satisfy all the conditio[n] of rolle's theorem then $f'(c) = 0$.

there exist a point $x = c$ at which the tange[nt] is parallel to x-axis.

Q. No. 2(a)

Give ~~the~~ curve be

$$x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy - 5y + 6 = 0$$

Here the given curve is third degree. So it may have at most 3-asympotes.

Here x^3 and y^3 are present. So there is no vertical and horizontal asympotes.

For oblique asympotes, Let $y = mx + c$ be required asympotes

For this, let $y = m$ and $x = 1$

$$\phi_3 = 1 + 2m - m^2 \mp 2m^3$$
$$= 1(1 + 2m) - m^2(1 + 2m)$$
$$= (1 + 2m)(1 - m^2)$$
$$= (1 + 2m)(1 + m)(1 - m)$$

$$\phi_2 = 4m^2 + 2m$$
$$\phi_1 = -5m$$

For value of m, $\phi_3(m) = 0$

$$(1 + 2m)(1 + m)(1 - m) = 0$$

$\therefore$ $1 + 2m = 0$ or, $1 + m = 0$ or, $1 - m = 0$

$m = -\frac{1}{2}$ $\quad m = -1$ $\quad m = 1$

Thus m has non-repeated value. So for value of c

$$c = \frac{\Phi_2(m)}{\Phi_3'(m)} = \frac{4m^2+2m}{2-2m-6m^2} = \frac{\cancel{2}m(2m+1)}{\cancel{2}(1-m-3m^2)}$$

$$= \frac{2m^2+m}{1-m-3m^2}$$

At ~~For~~ $m = -\frac{1}{2}$

$$c = \frac{2\cdot\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right) - 3\left(-\frac{1}{2}\right)^2} = \frac{\cancel{2}\cdot\frac{1}{\cancel{4}} - \frac{1}{2}}{1+\frac{1}{2}-\frac{3}{4}} = \frac{\frac{1}{2}-\frac{1}{2}}{\frac{3}{4}} =$$

At $m = 1$

$$c = \frac{2\cdot(1)^2+1}{\cancel{1}-\cancel{1}-3} = \frac{2+1}{-3} = \frac{3}{-3} = -1$$

At $m = -1$

$$c = \frac{2(-1)^2-1}{1-(-1)-3(-1)^2} = \frac{2-1}{1+1-3} = \frac{1}{-1} = -1$$

Therefore $y = mx + c$ become

when $m = -\frac{1}{2}$ and $c = 0$, when $m = +1$ and $c = -1$

$y = \frac{-1}{2}x + 0$

$y = +1x + (-1)$

$y = x - 1$

$x - y - 1 = 0$

$2y = -x$

$x + 2y = 0$

and when $m = -1$ and $c = -1$

$y = -1x + (-1)$

$y = -x - 1$

$x + y + 1 = 0$

Hence $x + 2y = 0$, $x - y - 1 = 0$ and $x + y + 1 = 0$ are required asympotes to give curve

(5) Given $x^{2/3} + y^{2/3} = a^{2/3}$

The parmetric eqⁿ of asteroid is given by

$x = a\cos^3\theta$ — (1) and $y = a\sin^3\theta$ — (2)

Diff. both side w.r. to θ in eq (1) and (2)

$\therefore \frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos^3\theta)$ $\qquad$ $\frac{dy}{d\theta} = \frac{d}{d\theta}(a\sin^3\theta)$

$= 3a\cos^2\theta(-\sin\theta)$ $\qquad$ $= 3a\sin^2\theta(\cos\theta)$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [3a\cos^2\theta(-\sin\theta)]^2 + [3a\sin^2\theta(\cos\theta)]^2$$

$$= 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$= 9a^2\sin^2\theta\cos^2\theta\,[\cos^2\theta + \sin^2\theta]$$

$$= 9a^2\sin^2\theta\cos^2\theta$$

The asteriod is symmetrical about both axis and the arc in the 1st quadrent lies between $\theta = 0$ and $\theta = \pi/2$

$\therefore$ Perimeter of asteriod = 4 × its length in the 1st quadr

$$= 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 4 \times \int_0^{\pi/2} \sqrt{9a^2 \sin^2\theta \cos^2\theta} \, d\theta$$

$$= 4 \times \int_0^{\pi/2} 3a \sin\theta \cos\theta \, d\theta$$

$$= 4 \times 3a \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta$$

$$= 12a \int_0^{\pi/2} \frac{2 \sin\theta \cos\theta}{2} \, d\theta$$

$$= \frac{12a}{2} \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= \frac{12a}{2} \left| \frac{-\cos 2\theta}{2} \right|_0^{\pi/2}$$

$$= 6a \left[\frac{-\cos 2(\pi/2) - (-\cos 2 \times 0)}{2} \right]$$

$$= 3a \left[-\cos\pi + \cos 0 \right]$$

$$= 3a \left[-(-1) - (-1) \right]$$

$$= 3a \left[1 + 1 \right]$$

$$= 6a$$

Q. No. 3

(d) $\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx$

$$= \frac{\Gamma\left(\frac{3+1}{2}\right) \Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\left(\frac{3+4+2}{2}\right)}$$

$$= \frac{\Gamma\frac{4}{2} \quad \Gamma\frac{5}{2}}{2\,\Gamma\frac{9}{2}}$$

$$= \frac{\Gamma 2 \quad \Gamma\frac{5}{2}}{2\,\Gamma\frac{9}{2}}$$

$$= \frac{1! \quad \cancel{\frac{3}{2}} \; \cancel{\frac{1}{2}} \; \cancel{\Gamma\frac{1}{2}}}{2 \times \frac{7}{2} \times \frac{5}{2} \times \cancel{\frac{3}{2}} \times \cancel{\frac{1}{2}} \; \cancel{\Gamma\frac{1}{2}}}$$

$$= \frac{1}{2 \times \frac{7}{2} \times \frac{5}{2}} = \frac{2}{35}$$

(b) $I = \int \frac{dx}{3\sin x + 4\cos x}$

Put $3 = r\cos\theta$, $4 = r\sin\theta$ Then $r^2 = 3^2 + 4^2 = 25$
and $\tan\theta = \frac{4}{3}$

Now,

$$I = \int \frac{dx}{r(\sin x\cos\theta + \cos x\sin\theta)}$$

$$= \frac{1}{5}\int \frac{dx}{\sin(x+\theta)} = \frac{1}{5}\int \operatorname{cosec}(x+\theta)\,dx$$

put $x+\theta = y$
$dx = dy$

$$I = \frac{1}{5}\int \operatorname{cosec} y\,dy$$

$$= \frac{1}{5}\log\left|\tan\frac{y}{2}\right| + c$$

$$= \frac{1}{5}\log\left|\tan\left(\frac{x+\theta}{2}\right)\right| + c$$

$$= \frac{1}{5}\log\left|\tan\left[\frac{x}{2} + \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right]\right| + c$$

(c) $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2$

$I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$ for $(1+\tan\theta) > 0$

$= \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) d\theta$ $\left[\because \int_0^a f(x) = \int_0^a f(a-x\right.$

$= \int_0^{\pi/4} \log\left(1 + \frac{\tan\pi/4 - \tan\theta}{\tan\frac{\pi}{4} + \tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \log\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \log\left(\frac{1+\tan\theta + 1-\tan\theta}{1+\tan\theta}\right) d\theta$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$

$$I = \int_0^{\pi/4} \log(2)\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$I = \log(2) \int_0^{\pi/4} d\theta - I$$

$$2I = \log 2 \int_0^{\pi/4} d\theta = \log 2 \; |\theta|_0^{\pi/4} = \frac{\pi}{4} \cdot \log 2$$

$$2I = \frac{\pi}{4} \log 2$$

$$I = \frac{\pi}{8} \log 2$$

Q. No. 4 (a)

The given equation of curve $y = x^2 + 1$
The point on its locus are

x	-2	-1	0	1	2
y	5	2	1	2	5

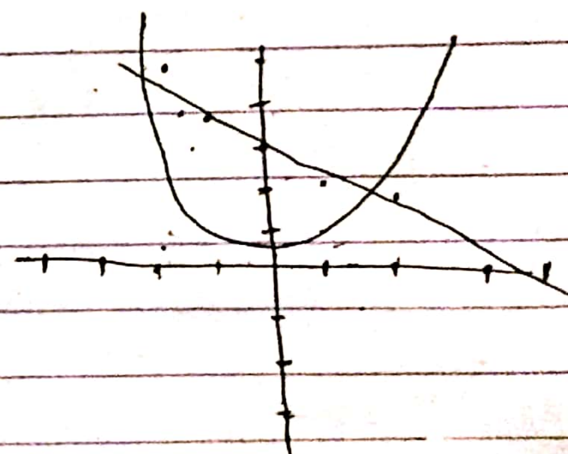
and

line is $y = -x + 3$

x	-2	-1	0	1	2	3
y	5	4	3	2	1	0

∴ the point of intersection are $(-2, 5)$ and $(1, 2)$

Now, the outer radius is $R(x) = -x + 3$
and inner radius is $r(x) = x^2 + 1$



$$V = \pi \int_{-2}^{1} \left[\{R(x)\}^2 - \{r(x)\}^2\right] dx$$

$$= \pi \int_{-2}^{1} \left[(-x+3)^2 - (x^2+1)^2\right] dx$$

$$= \pi \int_{-2}^{1} (x^2 - 6x + 9 - x^4 - 2x^2 - 1)\, dx$$

$$= \pi \int_{-2}^{1} -x^4 - x^2 - 6x + 8 \; dx$$

$$\pi\left[-\frac{x^5}{5}-\frac{x^3}{3}-\cancel{}\,3x^2+8x\right]_{-2}^{1}$$

$$=\pi\left[-\frac{1^5-(-2)^5}{5}-\frac{1-(-2)^3}{3}-\left[3(1)^2-3(-2)\right]+8-8(-2)\right]$$

$$=\pi\left[-\frac{1\times 32}{5}-\frac{1+8}{3}-(3+6)+(8+16)\right]$$

$$=\pi\left[\frac{28}{5}\right]$$

$$=\frac{28}{5}\pi \text{ cubic unit}$$

4(b)

let $u = f(x, y, z)$ be a homogeneous function of three independent variable x, y and z of degree of n

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Proof:

if $u = f(x, y, z)$ be a homogeneous function of x, y and z of degree n.

$$u = x^n g\left(\frac{y}{x}, \frac{z}{x}\right)$$

Let $p = y/x$ and $q = \frac{z}{x}$ then $u = x^n g(p, q)$ —(1

$$\frac{\partial p}{\partial x} = \frac{-y}{x}, \quad \frac{\partial p}{\partial y} = \frac{1}{x}, \quad \frac{\partial p}{\partial z} = 0$$

$$\frac{\partial q}{\partial x} = \frac{-z}{x^2}, \quad \frac{\partial q}{\partial y} = 0, \quad \frac{\partial q}{\partial z} = \frac{1}{x}$$

Diff. (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = nx^{n-1} g(p, q) + x^n\left[\frac{\partial g}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial x}\right]$$

$$= nx^{n-1} g(p, q) + x^n\left[\frac{\partial g}{\partial p}\left(-\frac{y}{x^2}\right) + \frac{\partial g}{\partial q}\left(\frac{-z}{x^2}\right)\right]$$

Similarly diff (1) partially w.r.t y

$$\frac{\partial u}{\partial y} = x^n \left[\frac{\partial g}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial g}{\partial q} \frac{\partial q}{\partial y} \right]$$

$$= x^n \left[\frac{\partial g}{\partial p} \left(\frac{1}{x} \right) + \frac{\partial g}{\partial q} (0) \right]$$

$$= x^n \frac{\partial g}{\partial p} \left(\frac{1}{x} \right)$$

$$= x^{n-1} \frac{\partial g}{\partial p}$$

Again diff. (1) partially w.r.t z

$$\frac{\partial u}{\partial z} = x^n \left[\frac{\partial g}{\partial p} \frac{\partial p}{\partial z} + \frac{\partial g}{\partial q} \frac{\partial q}{\partial z} \right]$$

$$= x^n \left[\frac{\partial g}{\partial p} 0 + \frac{\partial g}{\partial q} \times \frac{1}{x} \right]$$

$$= x^{n-1} \frac{\partial g}{\partial q}$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= x\left[nx^{n-1}g(p,q) - x^{n-2}y\frac{\partial g}{\partial p} - x^{n-2}z\frac{\partial g}{\partial p}\right]$$

$$+ x^{n-1}y\frac{\partial g}{\partial p} + x^{n-1}z\frac{\partial g}{\partial q}$$

$$= n\,x^{n}g(p,q)$$

5 ⓐ

let $f = x^2+y^2+z^2$ and $\phi = ax+by+cz-k$

By using Lagrange method

$$F = f + \lambda \phi$$
$$= (x^2+y^2+z^2) + \lambda(ax+by+cz-k)$$

Differentiating partial

$F_x = 2x + \lambda a$ $\quad$ $F_y = 2y + \lambda b$

$F_z = 2z + \lambda c$ $\quad$ $F_\lambda = ax+by+cz-k$

For extreme value

$F_x = 0$ $\quad$ $F_y = 0$ $\quad$ $F_z = 0$ $\quad$ $F_\lambda = 0$

$2x+\lambda a = 0$ $\quad$ $2y+\lambda b = 0$ $\quad$ $2z+\lambda c = 0$ $\quad$ $ax+by+cz-p=0$

$x = \frac{-\lambda a}{2}$ $\quad$ $y = \frac{-\lambda b}{2}$ $\quad$ $z = \frac{-\lambda c}{2}$

Now,

$ax+by+cz-k=0$

$$a\left(\frac{-\lambda a}{2}\right) + b\left(\frac{-\lambda b}{2}\right) + c\left(\frac{-\lambda c}{2}\right) = k$$

$$\frac{-\lambda(a^2+b^2+c^2)}{2} = k$$

$$\lambda = \frac{-2k}{a^2+b^2+c^2}$$

$\therefore x = \frac{ak}{a^2+b^2+c^2}$, $y = \frac{bk}{a^2+b^2+c^2}$, $z = \frac{ck}{a^2+b^2+c^2}$

$\therefore$ Stationary point $= \left(\frac{ak}{a^2+b^2+c^2}, \frac{bk}{a^2+b^2+c^2} + \frac{ck}{a^2+b^2+c^2}\right)$

$F_{xx} = 2 = F_{yy} = F_{zz} = 2$

$F_{xy} = F_{zx} = F_{yz} = 0$ $\quad \phi_x = a$, $\phi_y = b$, $\phi_z = c$

$$H_{B1} \begin{vmatrix} 0 & a & b \\ a & 2 & 0 \\ b & 0 & 2 \end{vmatrix} = (-2a^2+2b^2) < 0$$

$$H_{B2} = \begin{vmatrix} 0 & a & b & c \\ a & 2 & 0 & 0 \\ b & 0 & 2 & 0 \\ c & 0 & 0 & 2 \end{vmatrix} = -(4a^2+4b^2+4c^2) < 0$$

$\therefore$ f (has minimum value. The minimum is

$$f_{min} \left(\frac{ak}{a^2+b^2+c^2}\right)^2 + \left(\frac{bk}{a^2+b^2+c^2}\right)^2 + \left(\frac{ck}{a^2+b^2+c^2}\right)$$

$$= \frac{k^2}{a^2+b^2+c^2}$$

5(b)

The Riccati's equation of the form

$$\frac{dy}{dx} + Py + Qy^2 = R \quad \text{———} (1)$$

Put $y = y_1 + u$ ——— (2)

where y_1 = particular solution of differential equation

Differentiating (2) w.r.t x

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$$

Substituting value of y and $\frac{dy}{dx}$ by in eq (1)

$$\frac{dy_1}{dx} + \frac{du}{dx} + P(y_1+u) + Q(y_1+u)^2 = R$$

$$\frac{dy_1}{dx} + \frac{du}{dx} + Py_1 + Pu + Qy_1^2 + 2Qy_1u + Qu^2 = R$$

$$\frac{du}{dx} + (P+2Qy_1)u + Qu^2 + \left(\frac{dy_1}{dx} + Py_1 + Qy_1^2\right) = R \quad \text{———} (3)$$

Since

$y = y_1$ be the particular solution of equation (1)

$$\frac{dy_1}{dx} + Py_1 + Qy_1^2 = R$$

Equation (8) becomes

$$\frac{du}{dx} + (P + 2Qy_1)u + Qu^2 + R = R$$

$$\frac{du}{dx} + (P + 2Qy_1)u + Qu^2 = 0$$

$$\frac{du}{dx} + (P + 2Qy_1)u = -Qu^2$$

which is Bernoulli's equation in u and x

6(a) The given differential equation

$$1 \frac{d^2 V_c}{dt^2} + 40 \frac{d V_c}{dt} + 625 V_C = 0$$

Its AE is

$$r^2 + 40r + 625 = 0$$

$$r = \frac{-40 \pm \sqrt{-900}}{2}$$

$$= -20 \pm 15i$$

Then general solution is $V_c = e^{-20t}(c_1 \cos 15t + c_2 \sin 15t)$

Also using $V_c(0) = 6V$

$$6 = C_1 \cos 0 + C_2 (\sin 0)$$

$$6 = C_1$$

$$c_1 = 6$$

Also $V_C' = -20e^{20t}(c_1 \cos 15t + c_2 \sin 15t) + e^{-20t}(-15c_1 \sin 15t + 15c_2 \cos 15t)$

Using $V_c'(0) = 6A$

$$6 = -20c_1 + 15C_2$$

$$6 = -20 \times 6 + 15C_2$$

$$15C_2 = 126$$

$$C_2 = \frac{126}{15}$$

Thus the required solution is $V_c = e^{-200t}\left(6\cos 15t + \frac{126}{15}\sin 15t\right)$

6 (b) $x^2y'' - 2xy' + 2y = 0$

Comparing this equation with

$x^2y'' + axy' + by$

$a = -2, \quad b = 2$

Then AE is

$m^2 + (a-1)m + b = 0$

$m^2 + (-2-1)m + 2 = 0$

$m^2 - 3m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$m = 2$ and $m = 1$

$\therefore y_1 = x^2$ and $y_1 = x$

The general solution of given differential eqn is

$y = C_1 x^1 + C_2 x^2$ —— ①

$y(1) = 3/2$ and $y'(1) = 1$

Since $y(1) = 3/2 \Rightarrow x = 1$ and $y = 3/2$

$3/2 = C_1 \times 1 + C_2 \times 1^2$

$3/2 = C_1 + C_2$

$3 = 2C_1 + 2C_2$ —— ②

Diff (i) w.r.t x

$$y' = c_1 + 2c_2 x$$

Since $y'(1) = 1$

$$1 = c_1 + 2c_2$$

$$c_1 + 2c_2 = 1 \quad \text{—— (iii)}$$

Solving eqn (ii) and (iii)

$c_1 = 2$ and $c_2 = -\frac{1}{2}$

eqn (i) become

$$y = 2x - \frac{1}{2}x^2$$

which is the general solution.

7(a) Given

$$y = x^n \quad \text{——— (1)}$$

Differentating (1) successively with respect to x

$$y_1 = n x^{n-1}$$
$$y_2 = n(n-1) x^{n-2}$$
$$y_3 = n(n-1)(n-2) x^{n-3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n = n(n-1)(n-2)\ldots(n-(n-1)) x^{n-n}$$
$$= n(n-1)(n-2)\ldots(1) x^0$$
$$= n(n-1)(n-2)\ldots 1\, x^0$$
$$= n!\, x^0$$
$$= n!$$

7 (b)

We know,
$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{——— (1)}$$

Here
$$y^2 = 4ax$$
$$\therefore \quad 2y\frac{dy}{dx} = 4a$$
$$\frac{dy}{dx} = \frac{2a}{y}$$
$$\therefore \quad y_1 = \frac{2a}{y}$$

Differentiating w.r.t x
$$y_2 = -\frac{2a}{y^2} \cdot y_1$$
$$= -\frac{2a}{y^2} \cdot \frac{2a}{y}$$
$$y_2 = -\frac{4a^2}{y^3} \qquad \left[y_1 = \frac{2a}{y} \right]$$

From (1)
$$\rho = \frac{\left(1 + \frac{4a^2}{y^2}\right)^{3/2}}{\frac{4a^2}{y^3}}$$

$$= \left(\frac{y^2+4a^2}{4a^2}\right)^{3/2}$$

$$= \frac{(4ax+4a^2)^{3/2}}{4a^2}$$

$$= \frac{(4a)^{3/2}\,(a+x)^{3/2}}{4a^2}$$

$$= (2^2)^{3/2} \quad \frac{a^{3/2}\,(a+x)^{3/2}}{4a^2}$$

$$= \frac{8a^{3/2}}{4a^2}\,(a+x)^{3/2}$$

$$= +2\cdot a^{-1/2}\,(a+x)^{3/2}$$

7 (c)

Given $f(x,y) = x^3 + y^3 - 3xy$

$f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3 = f_{yx}$

For stationary point

$f_x = 0$ and $f_y = 0$

$3x^2 - 3y = 0$ $\qquad$ $3y^2 - 3x = 0$

$x^2 - y = 0$ — (1) $\qquad$ $y^2 - x = 0$

$y^2 = x$ — (2)

Put the value of x from (2) to (1)

$\therefore (y^2)^2 - y = 0$

$y^4 - y = 0$

$y(y^3 - 1) = 0$

$\therefore y = 0, 1$

$\therefore x = y^2 \Rightarrow (0, 1)$

$\therefore$ the point are $(0,0)$ $(1,1)$

Now,

At $(0,0)$

$f_{xx} \cdot f_{yy} - (f_{xy})^2$

$= 6x \cdot 6y - (-3)^2$

$= 12xy - 9$

$= 12 \times 0 - 9 = -9 < 0$ at $(0,0)$

At (1,1)

$$f_{xx} \cdot f_{yy} - (f_{xy})^2$$
$$= 6 \cdot 6 - (-3)^2$$
$$= 36 - 9$$
$$= 27 > 0 \quad \text{at } (1,1)$$

Thus, At (0,0) there is a saddle point ie. -9.

7(d)

Given,

$$\frac{dy}{dx} + \frac{1-\cos 2y}{1-\cos 2x} = 0$$

or, $$\frac{dy}{dx} + \frac{\cos^2 y}{\sin^2 x} = 0$$

$$\frac{dy}{dx} = \frac{-\cos^2 y}{\sin^2 x}$$

$$\int \frac{dy}{dx} = -\int \frac{\cos^2 y}{\sin^2 x}$$

$$\int \frac{dy}{\cos^2 y} = -\int \frac{dx}{\sin^2 x}$$

$$\int \sec^2 y \, dy = -\int \operatorname{cosec}^2 x \, dx$$

$$\tan y + C_1 = \cot x + C_2$$

$$\tan y - \cot x = C_2 - C_1$$

$$\tan y - \cot x = C$$

where $C = C_2 - C_1$