Chapter-3 (4 hrs)

Review of z-Transform

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3.1 Definition of z-transform

3.2 Convergence of z-transform, Region of convergence

3.3 Properties of z-transform (linearity, time shift, multiplication by exponential sequence, differentiation, time reversal, convolution, multiplication) ~~conjugation & conjugate symmetry~~

3.4 Inverse z-transform - by long division, by partial fraction expansion

Introduction of z-transform:-

The z-transform is the discrete time analysis of the Laplace transform. If $x(n)$ be a discrete time signal, then the z-transform of $x(n)$ is given by

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{--- (1)}$$

where z is a complex variable, sometimes the equation (1) is also known as direct z-transform since it transforms $x(n)$ into a complex plane representation of $X(z)$.

The process of conversion of $X(z)$ into $x(n)$ is known as inverse z-transform ie

$$x(n) = z^{-1}[X(z)] \quad \text{--- (2)}$$

The relationship between $x(n)$ and $X(z)$ is represented as

$$x(n) \xrightarrow{z} X(z)$$

$$X(z) \xrightarrow{z^{-1}} x(n)$$

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Since the z-transform is an infinite power series, it exists only for those values of z for which the series converges. The region of convergence (ROC) of $X(z)$ is the set of all the values of z for which $X(z)$ attains a finite value. Thus while calculating the z-transform, we should indicate its region of convergence.

iv) Types of z-transform :-

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1) One sided z-transform :-

A single side or one sided z-transform of discrete time signal $x(n)$ is defined as

$$X(z)^{+} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

or

$$X(z)^{-} = \sum_{n=-\infty}^{0} x(n) z^{-n}$$

which limits the summation from 0 to ∞, while expanding we put only a substitute only positive values of n or in case of negative value, it range from $-\infty$ to 0 where we have to substitute negative values of n, hence due to this design, the one sided z-transform has following characteristics :-

i) It does not contain any information about the signal $x(n)$ for negative values of time ie for $n<0$.

ii) It is unique only for causal signals ~~because~~ because only these signals are zero for $n<0$

$x(n) = 0$, $n < 0 \rightarrow$ causal system

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iii) The one sided z-transform $X^{+}(z)$ of $x(n)$ is identical to two sided z-transform of the signal $x(n) \cdot u(n)$, the Roc of $X^{+}(z)$ is always the exterior of the circle.

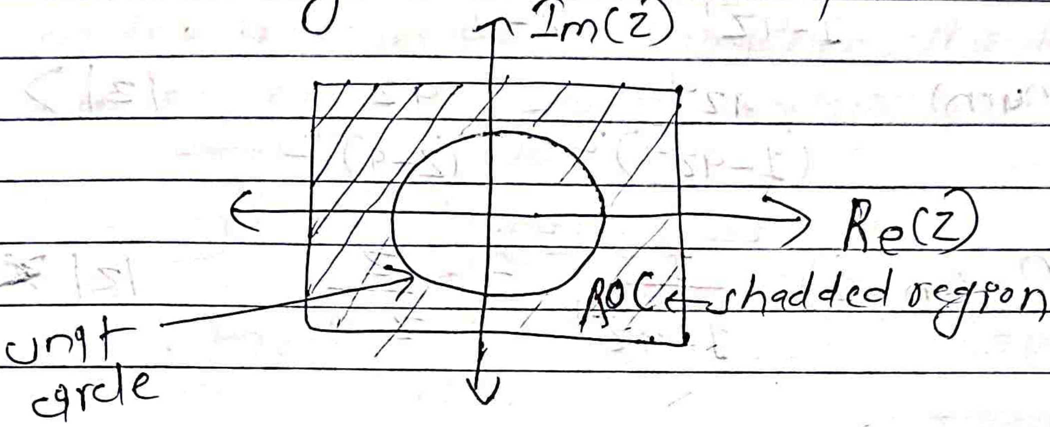

fig :- Roc of $u(n)$

2) Two sided / double sided z-transform

A double sided z-transform of discrete time signal $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

while expanding the summation we should put both positive and negative values of n.

formulas

	Discrete time signal $x(n)$	z-transform $X(z)$	ROC
1)	$\delta(n)$	1	Entire z-plane
2)	$\delta(n-k)$	z^{-k}	Entire z-plane except $z=0$
3)	$\delta(n+k)$	z^{k}	Entire z-plane except $z=\infty$
4)	$u(n)$	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$	$\|z\|>1$
5)	$u(-n)$	$\frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z}$	$\|z\|>1$

$u(-n-1) = \frac{z}{1-z}$

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6) $n u(n)$ $\quad \frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2}$ $\quad |z| > 1$

7) $a^n u(n)$ $\quad \frac{1}{1-az^{-1}} = \frac{z}{z-a}$ $\quad |z| > |a|$

8) $n a^n u(n)$ $\quad \frac{az^{-1}}{(1-az^{-1})^2} = \frac{az}{(z-a)^2}$ $\quad |z| > |a|$

9) $-a^n u(-n-1)$ $\quad \frac{1}{1-az^{-1}} = \frac{z}{z-a}$ $\quad |z| \cancel{>} |a|$

10) $-n a^n u(-n-1)$ $\quad \frac{az^{-1}}{(1-az^{-1})^2} = \frac{az}{(z-a)^2}$ $\quad |z| < |a|$

11) $\sin \omega_0 n$ $\quad \frac{z \sin \omega_0}{z^2 - 2z\cos\omega_0 + 1}$ $\quad |z| > 1$

12) $\cos \omega_0 n$ $\quad \frac{z(z-\cos\omega_0)}{z^2 - 2z\cos\omega_0 + 1}$ $\quad |z| > 1$

13) $(\sin \omega_0 n)\, u(n)$ $\quad \frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}}$ $\quad |z| > 1$

14) $(\cos \omega_0 n)\, u(n)$ $\quad \frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}}$ $\quad |z| > 1$

15) $\sinh \omega_0 n$ $\quad \frac{z \sinh \omega_0}{z^2 - 2z\cosh\omega_0 + 1}$ $\quad |z| > 1$

16) $\cosh \omega_0 n$ $\quad \frac{z(z-\cosh\omega_0)}{z^2 - 2z\cosh\omega_0 + 1}$ $\quad |z| > 1$

17) $(a^n \cos \omega_0 n)\, u(n) = \frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0 + a^2 z^{-2}}$ $\quad |z| > |a|$

18) $(a^n \sin\omega_0 n)u(n)$ $\qquad \dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$ $\qquad |z|>|a|$

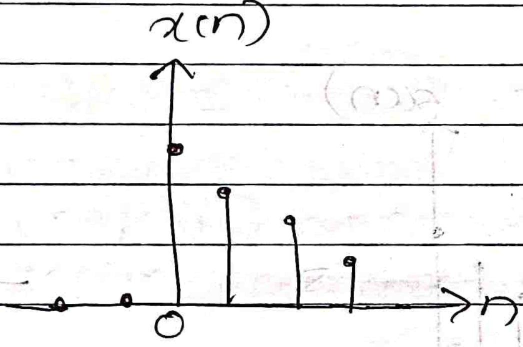
Region of convergence (ROC)

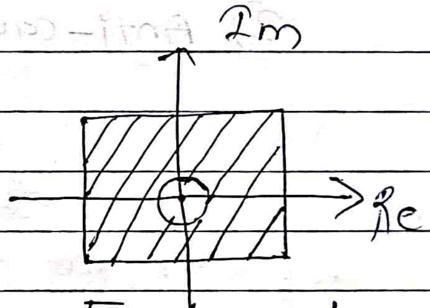
The set of values of 'z' in a z-plane (X → Real part, y → imaginary) for which the magnitude of X(z) is finite is known as the region of convergence. The ROC of some signal is plotted below:–

Finite Duration signals

signals | ROC's

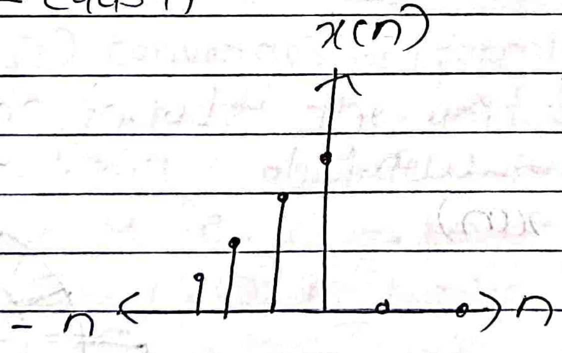
1) causal

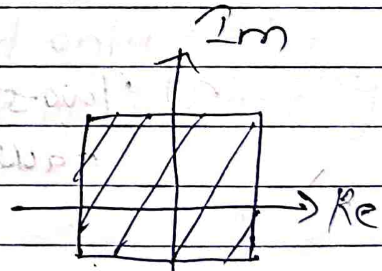




[entire z-plane except z = 0]

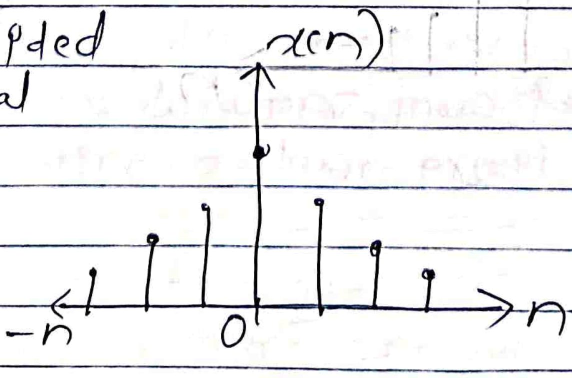
2) Anti-causal

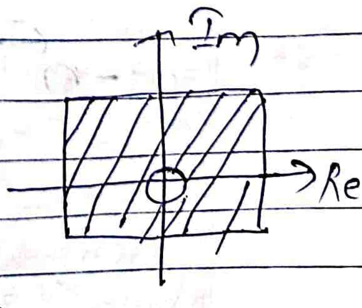




[entire z-plane except z = ∞]

3) Two-sided causal



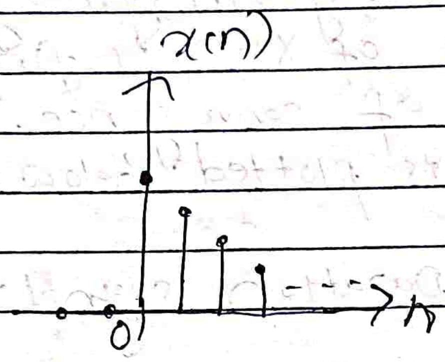


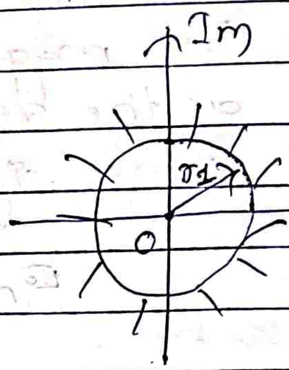
[Entire z-plane except z = 0 and z = ∞]

Infinite Duration signals

signals | **Roc's**

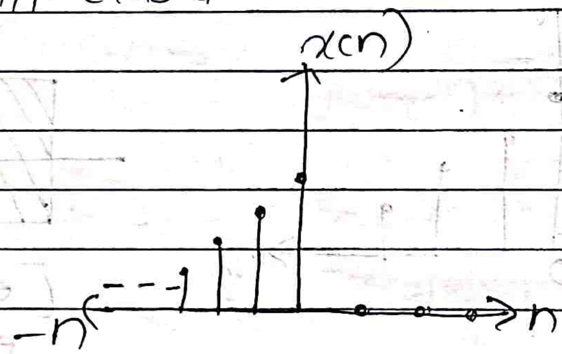
1) Causal

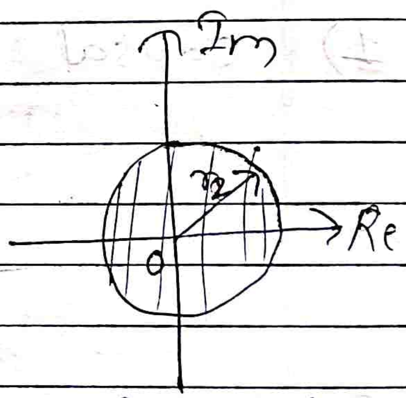




ROC: $|z| > r_1$

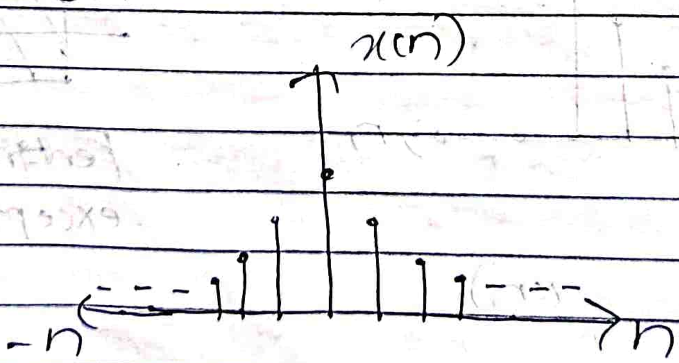
2) Anti-causal

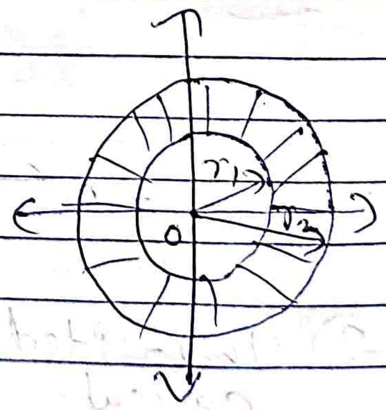




ROC: $|z| < r_2$

3) Two-sided causal





ROC: $r_1 < |z| < r_2$

Q) Write down properties of ROC with examples.

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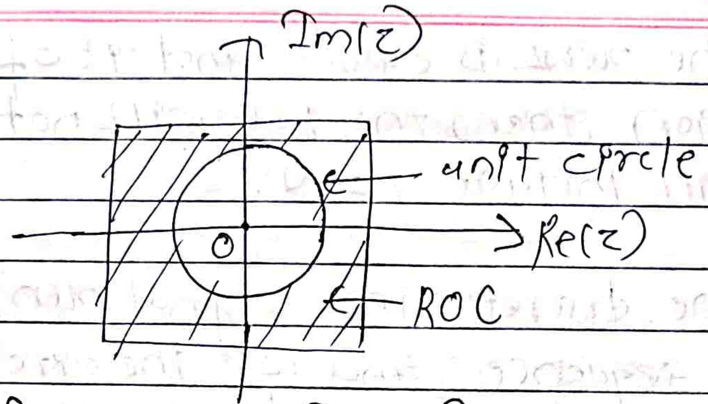

fig:- ROC of u(n)

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In other words, ROC (Region of convergence) is specified for the z-transform, where is converges.

Imp

Properties of ROC:-

Z-transform → $X(z)$
discrete time signal → $x(n)$

1) The ROC of $X(z)$ consists of a ring or circle in the z-plane centered about the origin.

2) ROC does not contain any pole.

3) $X(z)$ converges uniformly if and only if the ROC include the unit circle or if $\{x(n) r^{-n}\}$ is always absolutely summable.
ie

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

4) If the discrete time signal $x(n)$ is of finite duration, then ROC will be the entire z-plane except $z=0$ and $z=\infty$.

5) If $x(n)$ is anti-causal and is of finite duration then the ROC will not include infinity, will include $z=0$.

6) If the x(n) is causal and is of finite duration, then the ROC will not include $z=0$, will include $z=\infty$.

7) If the discrete time signal x(n) is a two-sided sequence, and if the circle $|z|=r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z|=r_0$. This means that the ROC will include the intersection of the ROC's of the components.

(anti-causal)
8) If discrete-time signal x(n) is a right-sided sequence, then the ROC will not include ∞.

9) If the discrete-time signal x(n) is a left-sided sequence, then the ROC will not include $z=0$. But in a case of $x(n)=0$ for all $n>0$, then ROC will include $z=0$.

10) If the z-transform X(z) is rational, then the ROC will extend to infinity. This means that the ROC will be bounded by poles.

Q) Find the z-transform of discrete time unit step signal u(n).

Soln

Given, $x(n) = u(n)$

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$u(n) = 1$, for $n \geq 0$
0 , otherwise

It is causal signal.

Z-transform of $u(n)$ is given by,

$$Z[u(n)] = Z[x(n)] = \cancel{X}(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=\infty}^{0} u(n) z^{-n} + \sum_{n=0}^{\infty} u(n) z^{-n}$$

$$= 0 + \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= z^0 + z^{-1} + z^{-2} + z^{-3}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \frac{1}{1-\frac{1}{z}} , \left|\frac{1}{z}\right| < 1$$

$$X(z) = \frac{z}{z-1} \quad ; \text{ROC}: |z| > 1$$

$\because \; 1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}$; ROC: $|a| < 1$

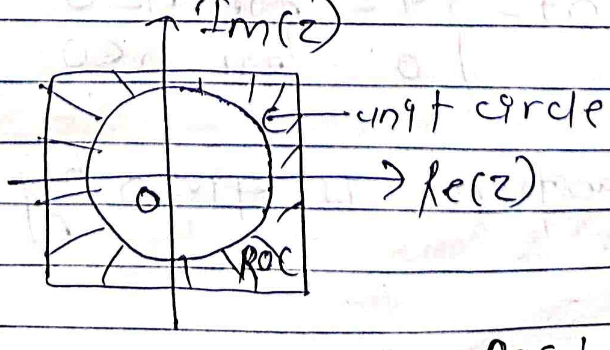

fig:- ROC of $u(n)$ ROC: $|z| > 1$

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Again, $x(n) = u(-n)$

$$Z[u(-n)] = \sum_{n=-\infty}^{\infty} u(-n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} u(-n) z^{-n} + \sum_{n=0}^{\infty} u(-n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} u(-n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} 1 \cdot z^{-n}$$

$$= \cdots + z^3 + z^2 + 1$$

$$= \frac{1}{1-z}$$

$$= \frac{1/z}{\frac{1}{z} - 1}$$

$$= \frac{z^{-1}}{z^{-1} - 1} \qquad |z| > 1$$

Q.) Find the z-transform and the ROC the discrete time signal $x(n)$ given as :-

$$x(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

Soln

Given, $x(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$

The z-transform is given by

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{0} x(n) z^{-n} + \sum_{n=0}^{\infty} x(n) z^{-n}$$

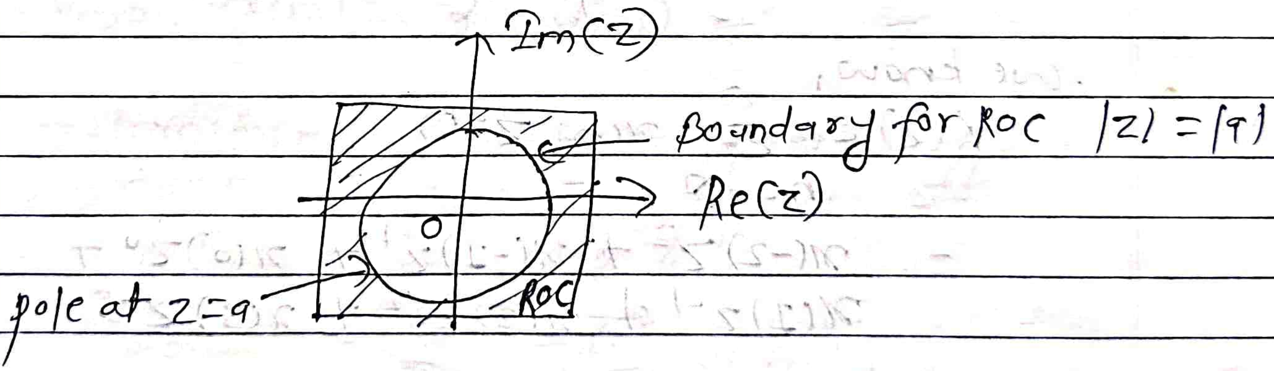
$$= 0 + \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= a^0 z^0 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{1}{1-\frac{a}{z}} \qquad \left|\frac{a}{z}\right| < 1$$

$$= \frac{z}{z-a} \qquad |z| > |a|$$



Q3) Determine the z-transform of the following signal:

a) $x(n) = \{1, 2, 5, 7, 0, 1\}$

soln

We have : $X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$

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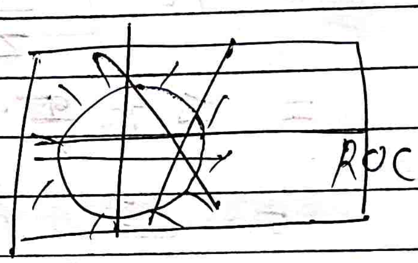
$$= x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5}$$

$$= 1 + \frac{2}{z} + \frac{5}{z^2} + \frac{7}{z^3} + \frac{0}{z^4} + \frac{1}{z^5}$$

$$= 1 + \frac{2}{z} + \frac{5}{z^2} + \frac{7}{z^3} + \frac{1}{z^5}$$

Roc is entire z-plane except $z=0$.



(b) $x(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

Soln

Given, $x(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

We know,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= x(-2)z^2 + x(-1)z^1 + x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

$$= 1 \cdot z^2 + 2 \cdot z + 5 \cdot 1 + \frac{7}{z} + \frac{0}{z^2} + \frac{1}{z^3}$$

$$= z^2 + 2z + 5 + \frac{7}{z} + \frac{1}{z^3}$$

Roc is entire z-plane except $z=0$ and $z=\infty$

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c) $x(n) = \delta(n)$

soln

we know that

$$\delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{otherwise} \end{cases}$$

Now,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \cdots + x(0) z^{0} + \cdots$$

$$= 1 \cdot 1$$

$$= 1$$

ROC is entire z-plane

Q4) Determine ROC of : $x(n) = a^n . u(n)$

soln

given, $x(n) = a^n . u(n)$

z-transform of $x(n)$ is given by,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n . u(n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} a^n . u(n) . z^{-n} + \sum_{n=0}^{\infty} a^n . u(n) . z^{-n}$$

$$\because u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore X(z) = \sum_{n=0}^{\infty} \alpha^n . z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$

$$= \left(\frac{\alpha}{z}\right)^0 + \left(\frac{\alpha}{z}\right)^1 + \left(\frac{\alpha}{z}\right)^2 + \left(\frac{\alpha}{z}\right)^3 + \cdots$$

$$= \frac{1}{1-\frac{\alpha}{z}}$$

$$= \frac{z}{z-\alpha}$$

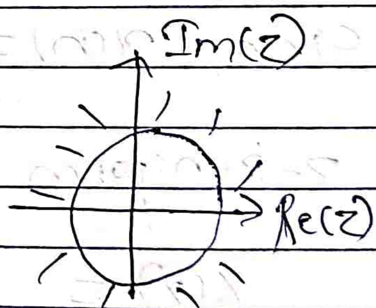
To find RoC, we must have

$\left|\frac{\alpha}{z}\right| < 1$, for which $X(z)$ is +ve

or $\frac{|\alpha|}{|z|} < 1$

or, $|\alpha| < |z|$

$\therefore z > \alpha$



Thus, ROC is a plane at $z > \alpha$.

Q5) Determine the z-transform of the signal

$x(n) = \left(\frac{1}{2}\right)^n u(n)$,

$$u(n) = \begin{cases} 1 & , n \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

Also determine Roc.

Soln

We know,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{n} u(n) z^{-n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{n} u(n) z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} u(n) z^{-n}$$

$$= 0 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} . z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^{n}$$

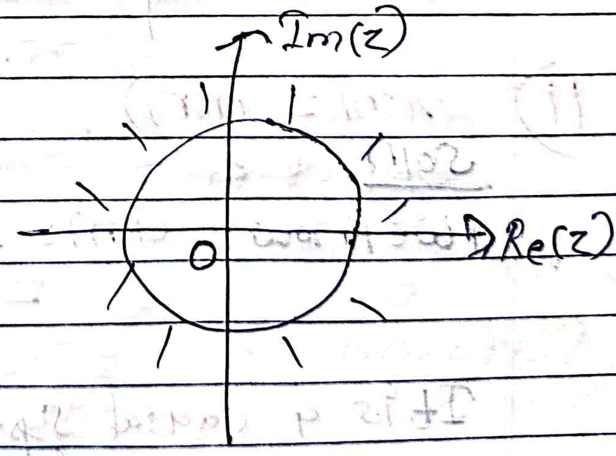
$$X(z) = \frac{1}{1-\frac{1}{2z}}$$

For region of convergence, X(z) is +ve

$$\frac{1}{2z} < 1$$

or, $\frac{1}{2} < z$

$\therefore z > \frac{1}{2}$



ROC is the plane at $z > 1/2$.

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Q6) Determine Z-transform of $x(n) = \delta(n)$ and
i) $x(n) = u(n)$.

Soln

We know, $\delta(n) = 1$ for $n = 0$
0, otherwise

Z-transform of $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$\delta(n)$ is causal system as $\delta(n) = 1$ at $n = 0$

$$= \sum_{n=0}^{\infty} \delta(n) . z^{-n}$$

$$= \delta(0) z^{0} + \delta(1) z^{-1} + \dots$$
$$= 1.1 + 0 + 0 + \dots$$
$$= 1$$

$X(z)$ is 1 for all z.

ii) $x(n) = u(n)$
Soln
We know $u(n) = 1$, for $n \geq 0$
$= 0$, otherwise

It is a causal system
ie

$$X(z) = \sum_{n=0}^{\infty} u(n) . z^{-n}$$

$$= u(0) . z^{0} + u(1) . z^{-1} + u(2) z^{-2} + \dots$$

$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$

$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$

$= \frac{1}{1 - \frac{1}{z}}$

$= \frac{z}{z-1}$

ROC is at $|z| > 1$.

(i) obtain z-transform of the signal:
$x(n) = 2^n . u(n) + 3^n . u(-n-1)$.
Also plot ROC.

soln

$x(n) = 2^n . u(n) + 3^n . u(-n-1)$

z-transform of $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[2^n . u(n) + 3^n . u(-n-1)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 2^n . u(n) z^{-n} + \sum_{n=-\infty}^{\infty} 3^n . u(-n-1) z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^n . z^{-n} + \sum_{n=-\infty}^{-1} 3^n z^{-n}$$

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$$= \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n + \sum_{n=-\infty}^{-1} \left(\frac{z}{3}\right)^{-n}$$

Let, $d = -n$

then,

when $n = -\infty$, $d = \infty$

when $n = -1$, $d = 1$

$$= \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n + \sum_{d=1}^{\infty} \left(\frac{z}{3}\right)^d$$

$$= \frac{1}{1-\frac{2}{z}} + \frac{z/3}{1-\frac{z}{3}}$$

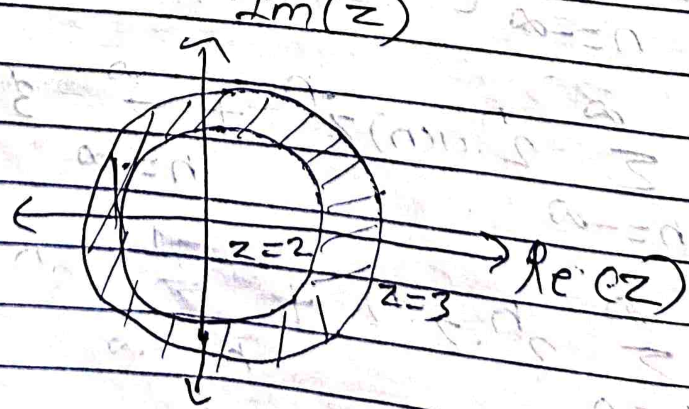
$$X(z) = \frac{z}{z-2} - \frac{z}{3-z}$$

for RoC, we must have:

$$\left|\frac{2}{z}\right| < 1 \text{ and } \left|\frac{z}{3}\right| < 1$$

$\Rightarrow 2 < z$ $\Rightarrow z < 3$

ie



Z-Transform in polar form :-

Soln

we have :

$$z = re^{j\theta}$$

where r = magnitude

θ = phase

Now,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot \left(re^{j\theta}\right)^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot r^{-n} e^{-j\theta n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, r^{-n}$$

$$= \sum_{n=-\infty}^{-1} x(n) \cdot r^{-n} + \sum_{n=0}^{\infty} x(n) \cdot r^{-n}$$

let, $l = -n$ or $n = -l$ for 1st summation

$$X(z) = \sum_{l=1}^{\infty} x(-l) \cdot r^{l} + \sum_{n=0}^{\infty} x(n)\, r^{-n}$$

Note :-

- r should be a small value such that the product $x(-l)\, r^{l}$ is finite.

- r should be large enough such that the product $x(-l) \cdot r^{-n}$ is finite.

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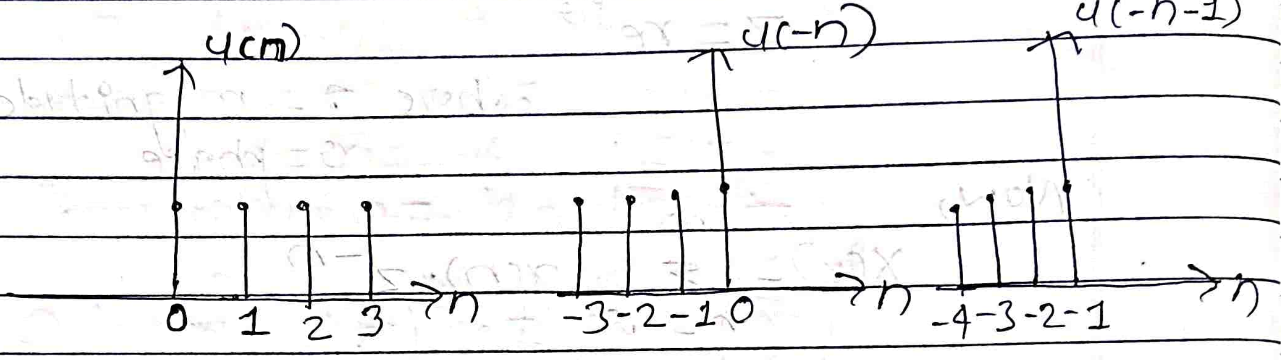
Q) Obtain the z-transform of signal:
$x(n) = \beta^n u(-n-1)$.

soln



$$x(n) = \beta^n u(-n-1)$$

Z-transform of x(n) is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \beta^n . u(-n-1) . z^{-n}$$

$$= \sum_{n=-\infty}^{-1} \beta^n . z^{-n}$$

let, $n = -d$, then

$$X(z) = \sum_{d=1}^{\infty} \beta^{-d} . z^{d}$$

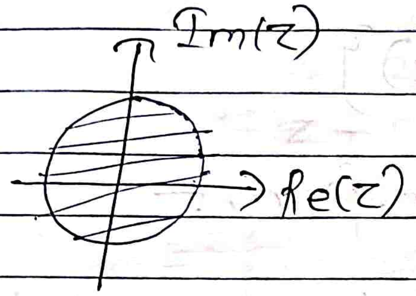
$$= \sum_{d=1}^{\infty} \left(\frac{z}{\beta}\right)^d$$

$$= \frac{z/\beta}{1 - z/\beta}$$

$$= \frac{z}{\beta - z}$$

ROC is given at $\left|\frac{z}{\beta}\right| < 1$

ie. $z < \beta$



Note:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-a}$$

$$\sum_{n=2}^{\infty} a^n = \frac{a^2}{1-a}$$

$$\sum_{n=6}^{\infty} a^n = \frac{a^6}{1-a}$$

Note :-

$$Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n).z^{-n}$$

⇒ If $x(n)$ is causal,
ie $x(n) = 0$ for $n < 0$, then the z-transform is called "one sided z-transform".

for causal $x(n)$,

$$\boxed{X(z) = \sum_{n=0}^{\infty} x(n).z^{-n}}$$

⇒ If $x(n)$ is anti-causal,
ie $x(n) = 0$ for $n \geq 0$, then

$$\boxed{X(z) = \sum_{n=-\infty}^{-1} x(n).z^{-n}}$$

It is also called as one sided z-transform

⇒ The inverse z-transform is used to obtain original time domain discrete signal $x(n)$ from complex domain $X(z)$.

$$x(n) = z^{-1}\{X(z)\}$$

Properties of z-transform :-

1) Linearity :- The z-transform is linear. This property states that the z-transform of a linear combination of discrete-time signal is equal to the same linear combination of their z-transform.
Mathematically,
If $x_1(n) \xleftrightarrow{z} X_1(z)$
and $x_2(n) \xleftrightarrow{z} X_2(z)$

Then, $x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{z} X(z)$

$$X(z) = a_1 X_1(z) + a_2 X_2(z)$$
where a_1 and a_2 are constant

Q) Find z-transform of $x(n) = \delta(n+1) + 2\delta(n) + 2\delta(n-4)$.

soln

Here, $X(z) = a_1 \cdot z\{x_1(n)\} + a_2 \cdot z\{x_2(n)\} + a_3 \cdot z\{x_3(n)\}$
$= z(\delta(n+1)) + 2z(\delta(n)) + 2z(\delta(n-4))$

we have
$z(\delta(n)) = 1$
$z(\delta(n-k)) = z^{-k}$
$z(\delta(n+k)) = z^{k}$

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Now, $Z(\delta(n+1)) = z^{1}$
$Z(\delta(n)) = 1$
$Z(\delta(n-4)) = z^{-4}$

Hence, $X(z) = z + 2\times 1 + 2\times z^{-4}$
$= z + 2 - \frac{2}{z^{4}}$ Ans

2) Time Reversal :-
Time reversal property states that
If $x(n) \xleftrightarrow{Z} X(z)$ ROC: $r_1 < |z| < r_2$
Then
$x(-n) \xleftrightarrow{Z} X(z^{-1})$ ROC: $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

Q) obtain z-transform of $x(n) = u(-n)$.
soln
we have,
$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\cdot z^{-n}$$

If $x(n) = u(n)$. Then

$$X(z) = \sum_{n=-\infty}^{\infty} u(n)\cdot z^{-n}$$
$$= \sum_{n=0}^{\infty} 1\cdot z^{-n}$$
$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$
$$= \frac{1}{1-\frac{1}{z}}$$
$$= \frac{z}{z-1} \quad \text{ROC: } \left|\frac{1}{z}\right| < 1$$
ie $|z| > 1$

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If $x(n) = u(-n)$
By using Time Reversal property,

$$Z\{u(-n)\} = X(z^{-1})$$

$$= \frac{z^{-1}}{z^{-1}-1}$$

$$= \frac{\frac{1}{z}}{\frac{1}{z}-1}$$

$$= \frac{\frac{1}{z}}{\frac{1-z}{z}}$$

$$= \frac{1}{1-z}$$

$\therefore$ ROC is $|z| < 1$

3) Time shifting property:-
Time shifting property states that
If $x(n) \xleftrightarrow{z} X(z)$
Then $x(n-n_0) \xleftrightarrow{z} z^{-n_0} X(z)$

The region of convergence (ROC) of $z^{-n_0}X(z)$ will be the same as that of $X(z)$ except for $z=0$ if $n_0>0$ and $z=\infty$ if $n_0<0$.

proof:
we know, $Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n).z^{-n}$

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for $x(n-n_0)$:

$$Z\{x(n-n_0)\} = \sum_{n=-\infty}^{\infty} x(n-n_0) \cdot z^{-n}$$

let, $n-n_0 = m$

$\Rightarrow n = m+n_0$

Then,

$$Z\{x(n-n_0)\} = \sum_{m=-\infty}^{\infty} x(m)\, z^{-(m+n_0)}$$

$$Z\{x(m)\} = \sum_{m=-\infty}^{\infty} x(m)\, z^{-m} \cdot z^{-n_0}$$

The z^{-n_0} is constant.

$$Z\{x(n-n_0)\} = z^{-n_0} \sum_{m=-\infty}^{\infty} x(m)\, z^{-m}$$

$$\boxed{\therefore Z\{x(n-n_0)\} = z^{-n_0}\, X(z)}$$ proved #

4) Scaling property :-

The scaling property states that

If $x(n) \xleftrightarrow{Z} X(z)$ ROC: $r_1 < |z| < r_2$

then

$a^n x(n) \xleftrightarrow{Z} X(a^{-1}z)$ ROC: $|a| r_1 < |z| < |a| r_2$

Here 'a' is any constant which may be real or complex quantity.

Q) A discrete time signal is given by:

$x(n) = 2^n \cdot u(n-2)$.

find its z-transform and ROC.

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Soln

Given, $x(n) = 2^n \cdot u(n-2)$

We have,

$$Z\{u(n)\} = \frac{1}{1-z^{-1}}$$

Using time-shifting property, we have

$$Z\{u(n-2)\} = z^{-2} \frac{1}{1-z^{-1}}$$

$$= \frac{z^{-2}}{1-z^{-1}}$$

Using scaling property

$$Z[2^n u(n-2)] = Z\{u(n-2)\}\Big|_{z \to z/2}$$

$$= \frac{\left(\frac{z}{2}\right)^{-2}}{1-\left(\frac{z}{2}\right)^{-1}}$$

$$= \frac{\frac{z^{-2}}{2^{-2}}}{1-\frac{z^{-1}}{2^{-1}}}$$

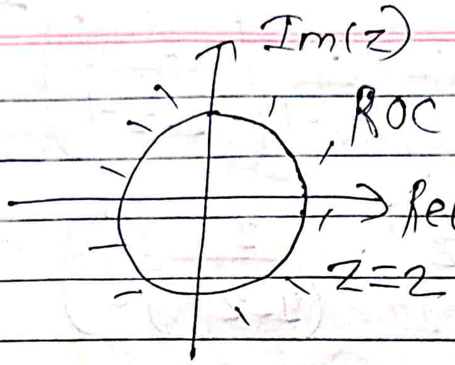
$$= \frac{2^2 z^{-2}}{1-2z^{-1}}$$

$$= \frac{4z^{-2}}{1-2z^{-1}}$$

ROC is given by: $2z^{-1} < 1$

$$\frac{2}{z} < 1$$

ie $|z| > 2$



⇒ Differentiation property:-
It states that
If $x(n) \overset{z}{\longleftrightarrow} X(z)$
Then

$$n\,x(n) \overset{z}{\longleftrightarrow} -z \frac{d}{dz} X(z)$$

or,

$$n\,x(n) \overset{z}{\longleftrightarrow} z^{-1} \frac{d}{dz^{-1}} X(z)$$

(1) A discrete time signal is expressed as $x(n) = n^2 . u(n)$. Determine its z-transform.

Soln

given, $x(n) = n^2 . u(n)$

$$X(z) = z[x(n)] = z[n^2 . u(n)]$$
$$= z[n[n . u(n)]]$$

Now, using differentiation property,

$$X(z) = z^{-1} \frac{d}{dz^{-1}} [z[n . u(n)]]$$

Again using differentiation property for the factor in bracket, we get

$$X(z) = z^{-1} \frac{d}{dz^{-1}} \left[z^{-1} \frac{d}{dz^{-1}} z[u(n)] \right]$$

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$$= z^{-1}\frac{d}{dz^{-1}}\left[z^{-1}\frac{d}{dz^{-1}}\left[\frac{1}{1-z^{-1}}\right]\right]$$

$$= z^{-1}\frac{d}{dz^{-1}}\left[z^{-1}\cdot\frac{1}{(1-z^{-1})^2}\right]$$

$$= z^{-1}\frac{d}{dz^{-1}}\left[\frac{z^{-1}}{(1-z^{-1})^2}\right]$$

$$= z^{-1}\left[\frac{(1-z^{-1})^2\cdot 1 - z^{-1}\cdot 2(1-z^{-1})\times(-1)}{(1-z^{-1})^4}\right]$$

$$= z^{-1}\left[\frac{(1-z^{-1})^2 + 2z^{-1}(1-z^{-1})}{(1-z^{-1})^4}\right]$$

$$= z^{-1}\left[\frac{1-z^{-1}+2z^{-1}}{(1-z^{-1})^3}\right]$$

$$= z^{-1}\left[\frac{1+z^{-1}}{(1-z^{-1})^3}\right]$$ Ans

6) Convolution property :-

We know that convolution of two discrete-time signals $x_1(n)$ and $x_2(n)$ is expressed as

$$x(n) = x_1(n) \circledast x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Now, the convolution property of z-transform states that

If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$

and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$

Then

$$x(n) = x_1(n) \circledast x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)\cdot X_2(z)$$

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4) Determine cross-correlation sequence $r_{x_1 x_2}(l)$ of the sequences:

$x_1(n) = (\underset{\uparrow}{1}, 2, 3, 4)$

$x_2(n) = (\underset{\uparrow}{4}, 3, 2, 1)$

Soln

The cross-correlation sequence can be obtained using the correlation property of z-transform. Therefore, for the given $x_1(n)$ and $x_2(n)$, we have

$$X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

$$X_2(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}$$

and

$$X_2(z^{-1}) = 4 + 3z + 2z^2 + z^3$$

Now,

$$R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

$$= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3}) \cdot (4 + 3z + 2z^2 + z^3)$$

$$= 4 + 3z + 2z^2 + z^3 + 8z^{-1} + 6 + 4z + 2z^2 + 12z^{-2} + 9z^{-1} + 6 + 3z + 16z^{-3} + 12z^{-2} + 8z^{-1} + 4$$

$$= 20 + 25z^{-1} + 24z^{-2} + 16z^{-3} + 10z + 4z^2 + z^3$$

~~Applying~~

$$= z^3 + 4z^2 + 10z + 20 + 25z^{-1} + 24z^{-2} + 16z^{-3}$$

Applying inverse z-transform:

$$r_{x_1 x_2}(l) = \{1, 4, 10, \underset{\uparrow}{20}, 25, 24, 16\}$$

Ans

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8) Conjugation Property :-
Conjugation property states that

If $x(n) \xleftrightarrow{z} X(z)$
Then
$$x^*(n) \xleftrightarrow{z} X^*(z^*)$$

Now, if $x(n)$ is real then
$$X(z) = X^*(z^*)$$
This means that if z-transform $X(z)$ has a pole or zero at $z = z_0$, then it must also have a pole or zero at the complex conjugate point $z = z_0^*$.

9) Initial value Theorem :-
Initial value theorem states that if $x(n)$ is a causal discrete-time signal with z-transform $X(z)$, then the initial value may be determined by using the following expression:

$$x(0) = \lim_{n \to 0} x(n) = \lim_{|z| \to \infty} X(z)$$

10) Final Value Theorem :-
The final value theorem states that for a discrete time signal $x(n)$, if $X(z)$ and the poles of $X(z)$ are all inside the unit circle, then the final value of the discrete-time signal, $x(\infty)$ may be determined by using the following expression.

$$x(\infty) = \lim_{n \to \infty} x(n) = \lim_{|z| \to 1} \left[(1 - z^{-1}) X(z)\right]$$

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11) Time Delay (for one sided z-transform)

This property states that,

If $x(n) \xleftrightarrow{z} X(z)$, then

$$x(n-k) \xleftrightarrow{z} z^{-k}\left[X(z) + \sum_{n=1}^{k} x(-n) z^{n}\right] \quad k>0$$

12) Time Advance Property :-

This property states that

If $x(n) \xleftrightarrow{z} X(z)$

Then

$$x(n+k) \xleftrightarrow{z} z^{k}\left[X(z) - \sum_{n=0}^{k-1} x(n) z^{-n}\right]$$

Q) If $X(z) = 2 + 3z^{-1} + 4z^{-2}$. find initial and final value of $x(n)$

soln

The given expression is

$$X(z) = 2 + 3z^{-1} + 4z^{-2}$$

For initial value,

$$x(0) = \lim_{n\to 0} x(n)$$

$$= \lim_{|z|\to\infty} X(z)$$

$$= \lim_{z\to\infty} (2 + 3z^{-1} + 4z^{-2})$$

$$= 2 + \frac{3}{\infty} + \frac{4}{\infty}$$

$$= 2 + 0 + 0$$

$$= 2 \quad \text{Ans}$$

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For final value,

$$x(\infty) = \lim_{n\to\infty} x(n)$$

$$= \lim_{z\to 1} \left[(1-z^{-1})\, X(z)\right]$$

$$= \lim_{z\to 1} \left[(1-z^{-1})(2+3z^{-1}+4z^{-2})\right]$$

$$= \lim_{z\to 1} \left[2+3z^{-1}+4z^{-2}-2z^{-1}-3z^{-2}-4z^{-3}\right]$$

$$= \lim_{z\to 1} \left[2+z^{-1}+z^{-2}-4z^{-3}\right]$$

$$= \lim_{z\to 1} \left[2+\frac{1}{z}+\frac{1}{z^2}-\frac{4}{z^3}\right]$$

$$= 2+\frac{1}{1}+\frac{1}{1}-\frac{4}{1}$$

$$= 4-4$$

$$= 0 \quad \underline{\text{Ans}}$$

Q) Determine $x(n)$ by using convolution for:

$$X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

Soln

Given, $X(z) = \dfrac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$

~~Fe~~ ~~X(z)~~=

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ie $X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$ & $X_2(z) = \frac{1}{1+\frac{1}{4}z^{-1}}$

Taking inverse z-transform:

$x_1(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$ and $x_2(n) = \left(-\frac{1}{4}\right)^n \cdot u(n)$

We have,

$$x(n) = x_1(n) \circledast x_2(n)$$

$$= \sum_{k=0}^{n} x_1(n-k)\, x_2(k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{n-k} u(n-k) \cdot \left(\frac{-1}{4}\right)^k u(k)$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \frac{(-1/4)^k}{(1/2)^k}$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{-1}{2}\right)^k$$

$$= \left(\frac{1}{2}\right)^n \frac{1-(-1/2)^{n+1}}{1-(-1/2)}$$

$$= \left(\frac{1}{2}\right)^n \frac{1+\frac{1}{2}\left(-\frac{1}{2}\right)^n}{3/2}$$

$$= \left(\frac{1}{2}\right)^n \frac{2}{3}\left[1+\frac{1}{2}\left(-\frac{1}{2}\right)^n\right]$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n+1} \cdot \left(-\frac{1}{2}\right)^n \cdot \frac{2}{3}$$

$$= \frac{2}{3}\left[\left(\frac{1}{2}\right)^n + (-1)^n \left(\frac{1}{2}\right)^{2n+1}\right]$$

Ans

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(i) Consider the analog signal: $x_a(t) = 3\cos 100\pi t$
i) Determine the minimum sampling rate required to avoid aliasing.
ii) Suppose that the signal is sampled at the rate $F_s = 200Hz$. What is the discrete-time signal obtained after sampling?
iii) What is the frequency $0 < F < F_s/2$ of a sinusoid that yields samples identical to those obtained in part (II)?

Soln
Here, $x_a(t) = 3\cos 100\pi t$

(i-ans) $x_a(t) = 3\cos 100\pi t$
Replace 't' by 'nT'
$x_a(nT) = 3\cos 100\pi nT$
$= 3\cos 100\pi \frac{n}{F_s}$
$x(n) = 3\cos \frac{100}{F_s}\pi n$

To avoid aliasing effect we must have
$\frac{100}{F_s} \leq 1$
$100 \leq F_s$

OR
$f = 50Hz$
$f_s \geq 2 f_{max}$
$f_s \geq 2\times 50$
$f_s \geq 100Hz$
$\therefore f_s = 100Hz$

So, the minimum sampling rate to avoid aliasing is 100Hz Ans

(ii-ans) $F_s = 200Hz$
$x(n) = 3\cos \frac{100}{F_s}\pi n$
or, $x(n) = 3\cos \frac{100}{200}\pi n$

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$x(n) = 3\cos\frac{1}{2}\pi n$

Here, $\frac{1}{2} \leq 1$, so there is no aliasing effect

$\therefore$ Required discrete time signal obtained after sampling is

$$x(n) = 3\cos\frac{1}{2}\pi n$$

Ans

iii-ans) Here, $x(n) = 3\cos\frac{1}{2}\pi n$

Replace n by $\frac{t}{T} = t \cdot F_s = 100t$

$$x_a(t) = 3\cos\frac{1}{2}\pi\, 100t$$

$$= 3\cos 50\pi t$$

$$= 3\cos 2\pi \cdot 25 \cdot t$$

$\therefore F = 25\,Hz$

Ans

1) Examine whether following systems are stable or not:—

i) $y(n) = x(n) + 1$

The system is stable

If $|x(n)| < A$ for all n, then we easily get $|y(n)| < A+1$

ii-ans) $y(n) - y(n-1) = x(n) + x(n-1)$

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a) $y(t) = x^2(t)$
If $x(t)$ is bounded, say $|x(t)| < A$ for all t,
we easily get
$|y(t)| < A^2$
Hence the system is stable.

b) $y(n) = x(n) + x(n+2)$
$|y(n)| = |x(n) + x(n+2)| \leq$
$|x(n)| + |x(n+2)| < 2A$

for a bounded input with $|x(n)| < A$ for all n.

c) $y(n) = \frac{1}{x(n)}$

The system is not stable. It is because for a bounded input, namely, $x(n) = 0$, the output is unbounded.

Inverse z-transform :-

The procedure for transforming the z-domain to the time domain function is known as inverse z-transform. Mathematically, the inverse z-transform is expressed as: $x(n) = z^{-1}[X(z)]$

To perform the inverse z-transform of the given expression, basically there are three methods :

1) Long Division method
2) partial fraction expansion method
3) Residue ~~fraction~~ method.

1) Long Division method :-

Consider a signal defined as :

$$X(z) = \frac{N(z)}{D(z)}$$

where; $N(z)$ = Numerator polynomial
$D(z)$ = Denominator polynomial

→ $X(z)$ is obtained by dividing $N(z)$ by $D(z)$.
→ Inverse z-transform is taken to obtain $x(n)$.

$$\text{ie } X(z) = \frac{N(z)}{D(z)} = \sum_{n=0}^{\infty} a_n z^{-n}$$

$$= a_0 z^0 + a_1 z^{-1} + a_2 z^{-2} + \dots$$

where, a_n are the values of $x(n)$.

→ The ROC determine whether the series has +ve or -ve exponents.

Causal sequence will have -ve exponents and non-causal sequence have +ve exponents.

Q) If $h(n) = \{1, 2, 3\}$ and $y(n) = \{1, 1, 2, -1, 3\}$ find $x(n)$.

Soln

Here; $h(n) = \{1, 2, 3\}$

$\Rightarrow H(z) = 1 + 2z^{-1} + 3z^{-2}$

$y(n) = \{1, 1, 2, -1, 3\}$

$\Rightarrow Y(z) = 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}$

Now, $X(z) = \dfrac{Y(z)}{H(z)}$

$= \dfrac{1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}}{1 + 2z^{-1} + 3z^{-2}}$

$$
\begin{array}{r|l}
 & 1 - z^{-1} + z^{-2} \\
1 + 2z^{-1} + 3z^{-2} & 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4} \\
 & \underline{-1 \mp 2z^{-1} \mp 3z^{-2}} \\
 & -z^{-1} - z^{-2} - z^{-3} \\
 & \underline{-z^{-1} - 2z^{-2} - 3z^{-3}} \\
 & \quad + \quad + \quad + \\
 & z^{-2} + 2z^{-3} + 3z^{-4} \\
 & \underline{z^{-2} + 2z^{-3} + 3z^{-4}} \\
 & - \quad - \quad - \\
 & 0
\end{array}
$$

$\therefore X(z) = \dfrac{Y(z)}{H(z)} = 1 - z^{-1} + z^{-2}$ Ans

Now, $x(n) = Z^{-1}\{X(z)\}$

$= Z^{-1}\{1 - z^{-1} + z^{-2}\}$

$= \{1, -1, 1\}$ Ans

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Q) Using LDM, determine IZT of:

$$X(z) = \frac{1+z^{-1}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

(a) ROC at $|z|>1$
(b) ROC $|z|<1/2$

Soln.

a) $|z|>1$, ie ROC is outside the unit circle signal is causal.

Now,

$$\begin{array}{r|l} & 1+\frac{5}{2}z^{-1}+\frac{13}{4}z^{-2}+\dots \\ \hline 1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} & 1+z^{-1} \\ & 1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} \\ & \quad - \quad + \quad - \\ \hline & \frac{5}{2}z^{-1}-\frac{1}{2}z^{-2} \\ & \frac{5}{2}z^{-1}-\frac{15}{4}z^{-2}+\frac{5}{4}z^{-3} \\ & - \quad + \quad - \\ \hline & \frac{13}{4}z^{-2}-\frac{5}{4}z^{-3} \\ & \frac{13}{4}z^{-2}-\frac{39}{8}z^{-3}+\frac{13}{8}z^{-4} \\ & - \quad + \quad - \\ \hline & \frac{29}{8}z^{-3}-\frac{13}{8}z^{-4} \end{array}$$

$$X(z) = 1+\frac{5}{2}z^{-1}+\frac{13}{4}z^{-2}+\dots$$

$$x(n) = z^{-1}\{X(z)\}$$
$$= z^{-1}\left\{1+\frac{5}{2}z^{-1}+\frac{13}{4}z^{-2}+\dots\right\}$$
$$= \left\{1, \frac{5}{2}, \frac{13}{4}, \dots\right\}$$ Ans

$\frac{z^{-1}}{\frac{1}{2}z^{-2}} = 2z$

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b) $|z| < \frac{1}{2}$, ROC lies within the unit circle so, the sequence is anticausal.

To obtain the power series expansion in +ve powers of z we shall perform long division as:

$$
\begin{array}{r|l}
 & 2z + 8z^2 + 20z^3 + \ldots \\
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-1} + 1 \\
 & \underline{z^{-1} - 3 + 2z} \\
 & 4 - 2z \\
 & \underline{4 - 12z + 8z^2} \\
 & 10z - 8z^2 \\
 & \underline{10z - 30z^2 + 20z^3} \\
 & 22z^2 - 20z^3
\end{array}
$$

$\therefore X(z) = 2z + 8z^2 + 20z^3 + \ldots$

$= \ldots + 20z^3 + 8z^2 + 2z$

$x(n) = z^{-1}\{X(z)\}$

$= z^{-1}\{\ldots + 20z^3 + 8z^2 + 2z\}$

$\therefore x(n) = \{\ldots 20, 8, 2\}$ Ans #

2) Partial fraction Expansion method :-

with the help of partial fraction expansion method, we can convert a system transfer function into a sum of standard functions.

To do this, firstly the denominator of the transfer function H(z) are factorized into prime factors and inverse z-transform is taken to obtain the required solution / simple poles of H(z) is obtained.

i) $$F(z) = \frac{H(z)}{(az+b)(cz+d)}$$

$$= \frac{A}{az+b} + \frac{B}{cz+d}$$

ii) $$F(z) = \frac{H(z)}{(az+b)(cz^2+d)}$$

$$= \frac{A}{az+b} + \frac{Bz+C}{cz+d}$$

Q) use partial fraction method to find inverse z-transform of H(z).

$$H(z) = \frac{-4+8z^{-1}}{1+6z^{-1}+8z^{-2}}$$

soln

Given, $$H(z) = \frac{-4+8z^{-1}}{1+6z^{-1}+8z^{-2}}$$

$$= \frac{-4+8z^{-1}}{1+4z^{-1}+2z^{-1}+8z^{-2}}$$

$$= \frac{-4+8z^{-1}}{1(1+4z^{-1})+2z^{-1}(1+4z^{-1})}$$

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$$= \frac{-4+8z^{-1}}{(1+4z^{-1})(1+2z^{-1})}$$

It can be expressed as :

$$H(z) = \frac{A}{(1+4z^{-1})} + \frac{B}{(1+2z^{-1})}$$

$$= \frac{A(1+2z^{-1})+B(1+4z^{-1})}{(1+4z^{-1})(1+2z^{-1})}$$

Comparing the like terms

$$-4+8z^{-1} = A(1+2z^{-1})+B(1+4z^{-1})$$

or, $-4+8z^{-1} = (A+B)+(2A+4B)z^{-1}$

Comparing coefficients

$$A+B = -4 \;]\times 2$$
$$2A+4B = 8$$
$$\underline{-\quad -\quad\quad -}$$
$$-2B = -16$$

$\therefore B = 8$, $A = -4-B = -4-8$

$A = -12$

$\therefore A = -12,\ B = 8$

Hence, $H(z) = \frac{-12}{1+4z^{-1}} + \frac{8}{1+2z^{-1}}$

$$h(n) = z^{-1}[H(z)]$$

$$= z^{-1}\left[\frac{-12}{1+4z^{-1}} + \frac{8}{1+2z^{-1}}\right]$$

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$$= -12z^{-1}\left(\frac{1}{1+4z^{-1}}\right) + 8z^{-1}\left(\frac{1}{1+2z^{-1}}\right)$$

$$h(n) = -12(-4)^n \cdot u(n) + 8(-2)^n \cdot u(n)$$

Ans

Q) Find the causal signal $x(n)$, which is having z-transform as:

$$X(z) = \frac{z^3}{(1+z)(z-1)^2}$$

Soln

Here, $X(z) = \frac{z^3}{(1+z)(z-1)^2}$

The degree of polynomial in numerator and denominator is same so,
Divide both sides by z, we get:

$$\frac{X(z)}{z} = \frac{z^2}{(1+z)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{A}{1+z} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{A(z-1)^2 + B(1+z)(z-1) + C(1+z)}{(1+z)(z-1)^2}$$

$$z^2 = A(z-1)^2 + B(z^2-1) + C(z+1)$$

put $z = 1$, then

$$1 = A \times 0 + B \times 0 + C \times 2$$

$$C = \frac{1}{2}$$

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put $z = -1$, then

$$1 = A(-1-1)^2 + B(1-1) + C(-1+1)$$

or, $1 = A \times 4 + 0 + 0$

or, $A = \frac{1}{4}$

put $z = 0$, then

$$0 = A(0-1)^2 + B(0-1) + C(0+1)$$

or, $0 = A \times 1 + B + C$

or, $0 = \frac{1}{4} + B + \frac{1}{2}$

or, $B = \frac{3}{4}$

$\therefore A = \frac{1}{4}, B = \frac{3}{4}, C = \frac{1}{2}$

substituting above values in above equation

$$\frac{X(z)}{z} = \frac{1}{4}\left(\frac{1}{z+1}\right) + \frac{3}{4}\left(\frac{1}{z-1}\right) + \frac{1}{2}\left(\frac{1}{(z-1)^2}\right)$$

$$X(z) = \frac{1}{4}\frac{z}{z+1} + \frac{3}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$$

Taking inverse z-transform

$$x(n) = \frac{1}{4}(-1)^n \cdot u(n) + \frac{3}{4}(1)^n \cdot u(n) + \frac{1}{2} n \cdot u(n)$$

OR

$$A = \frac{X(z)}{z} \cdot (z+1) \Big|_{z=-1}$$

$$= \frac{z^2}{\cancel{(1+z)}(z-1)^2} \cdot \cancel{(z+1)} \Big|_{z=-1}$$

$$= \frac{z^2}{(z-1)^2} \Big|_{z=-1}$$

$$= \frac{1}{4}$$

$$C = \frac{X(z)}{z} \cdot (z-1)^2 \Big|_{z=1}$$

$$= \frac{z^2}{(z+1)(z-1)^2} \cdot (z-1)^2 \Big|_{z=1}$$

$$= \frac{z^2}{z+1} \Big|_{z=1}$$

$$= \frac{1}{2}$$

$$B = \frac{X(z)}{z} \cdot (z-1) \Big|_{z=1}$$

$$= \frac{z^2}{(z+1)(z-1)^2} \cdot (z-1) \Big|_{z=1}$$

$$= \frac{z^2}{(z+1)(z-1)} \Big|_{z=1}$$

$$= \frac{z^2}{z^2-1} \Big|_{z=1} = \frac{1}{}$$

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$$B = \frac{d}{dz} \frac{X(z)}{z} \cdot (z-1)^2 \Big|_{z=1}$$

$$= \frac{d}{dz} \frac{z^2}{(1+z)(z-1)^2} (z-1)^2 \Big|_{z=1}$$

$$= \frac{d}{dz} \frac{z^2}{1+z} \Big|_{z=1}$$

$$= \frac{(z+1) \cdot 2z - z^2 \cdot 1}{(z+1)^2} \Big|_{z=1}$$

$$= \frac{2z^2 + 2z - z^2}{(z+1)^2} \Big|_{z=1}$$

$$= \frac{2 + 2 - 1}{4}$$

$$= \frac{3}{4}$$

$\therefore A = \frac{1}{4}, \; B = \frac{3}{4}, \; C = \frac{1}{2}$

3) Residue method :-

In this method, we obtain inverse z-transform $x(n)$, by summing residues of $[X(z) \cdot z^{n-1}]$ at all poles.

Mathematically,

$$x(n) = \sum_{\substack{\text{all poles} \\ X(z)}} \text{residues of } [X(z) \cdot z^{n-1}]$$

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Here, the residue for any pole of order 'm' at $z=\beta$ is:

$$\text{Residue} = \frac{1}{(m-1)!} \lim_{z \to \beta} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[(z-\beta)^m X(z) \cdot z^{n-1} \right] \right\}$$

(i) Use the Residue method to find the inverse z-transform, x(n) for

$$X(z) = \frac{z}{(z-1)(z-2)}$$

soln.

given, The transform is

$$X(z) = \frac{z}{(z-1)(z-2)}$$

X(z) has two poles of order m=1 at z=1 and at z=2

We can obtain the corresponding residues as ahead:

for poles at z=1

$$\text{Residue} = \frac{1}{0!} \lim_{z \to 1} \left\{ \frac{d^0}{dz^0} \left[\cancel{(z-1)}^1 \cdot \frac{z}{\cancel{(z-1)}(z-2)} \cdot z^{n-1} \right] \right\}$$

$$= \lim_{z \to 1} \left[\frac{z}{z-2} \cdot z^{n-1} \right]$$

$$= \lim_{z \to 1} \left[\frac{z^n}{z-2} \right]$$

$$= \frac{1}{-1}$$

$$= -1$$

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Similarly, for poles at $z=2$
$\beta = 2$

$$\text{Residue} = \frac{1}{0!} \lim_{z\to 2} \left\{ \frac{d^0}{dz^0} \left[(z-2)^1 \frac{z}{(z-1)(z-2)} \cdot z^{n-1} \right] \right\}$$

$$= \lim_{z\to 2} \left[\cancel{(z-2)} \cdot \frac{z}{(z-1)\cancel{(z-2)}} \cdot z^{n-1} \right]$$

$$= \lim_{z\to 2} \left[\frac{z^n}{z-1} \right]$$

$$= \frac{2^n}{1}$$

$$= 2^n$$

Hence, $x(n) = \sum_{\text{all poles}}$ residues of $[X(z) z^{n-1}]$

$$= \{-1 + 2^n\} \cdot u(n)$$

Causality of Discrete time LTI system :-

A discrete time LTI system which has a rational transfer function $H(z)$ will be causal if and only if:

i) The ROC is the exterior of the circle outside the outermost pole.

ii) with $H(z)$ expressed as a ratio of polynomial in z, the order of numerator should be smaller than order of denominator.

Stability criteria :

A discrete time LTI system is stable if and only if the ROC of its transfer function $H(z)$ includes the unit circle $|z| = 1$.

Pole and zero Relationship :-

zeros of z-transform $X(z)$ are the values of z for which $X(z)=0$.

And, poles of the z-transform $X(z)$ are the values of z for which $X(z)=\infty$.

Here, $X(z) = \frac{N(z)}{D(z)}$

$= \frac{\text{Representation of zero}}{\text{Representation of pole}}$

Q) Find the pole & zeros of the system and plot in the z-plane.

$X(z) = \frac{1}{1-az^{-1}}$, ROC $a>z$.

Soln

Here, $X(z) = \frac{1}{1-az^{-1}}$

$= \frac{1}{1-\frac{a}{z}}$

$= \frac{z}{z-a}$

For zeros, $X(z)=0$

or, $\frac{z}{z-a}=0$

$\therefore z=0$

For poles, $X(z)=\infty$

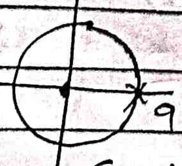
or, $\frac{z}{z-a}=\infty$

or, $\frac{z-a}{z}=0$

or, $z-a=0$

or, $z=a$

$\therefore z=a$



(pole-zero plot)

Magnitude and phase Response :-

The system response is given by :

$$H(j\omega) = 1 - re^{j\theta} - e^{-j\omega}$$
$$= 1 - re^{j(\theta-\omega)}$$
$$= 1 - r[\cos(\theta-\omega) - j\sin(\theta-\omega)]$$
$$= (1 - r\cos(\theta-\omega)) + jr\sin(\theta-\omega)$$

magnitude is given by :

$$|H(j\omega)| = \sqrt{(1-r\cos(\theta-\omega))^2 + (r\sin(\theta-\omega))^2}$$

phase is,

$$\angle H(j\omega) = \tan^{-1}\left[\frac{r\sin(\theta-\omega)}{1-r\cos(\theta-\omega)}\right]$$

$$= \tan^{-1}\left(\frac{Img}{Real}\right)$$

Q) Find the magnitude and phase response of the following system having zero at :

(i) $Z_1 = \{0.25 \pm 0.5j\}$

(ii) $Z_2 = \{0.9 \pm 0.35j\}$

and phase at $P_1 = 0.85 \pm 0.3j$.

Soln

For zeros,

magnitude $|Z_1| = \sqrt{(0.25)^2 + (0.5)^2}$

$$= 0.559$$

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phase $\measuredangle z_1 = \tan^{-1}\left(\frac{0.5}{0.25}\right)$

$= 63.43°$

Similarly, $|z_2| = \sqrt{(0.35)^2 + (0.9)^2}$

$= 0.97$

$\measuredangle z_2 = \tan^{-1}\left(\frac{0.35}{0.9}\right)$

$= 21.25°$

For poles,

$|P_1| = \sqrt{(0.85)^2 + (0.3)^2} = 0.9$

$\measuredangle P_1 = \tan^{-1}\left(\frac{0.3}{0.85}\right) = 19.44$

Now, the transfer function can be expressed as:

$H(j\omega) = 1 - re^{j(\theta - \omega)}$

$$H(j\omega) = \frac{\left\{1 - 0.559\, e^{j(63.43 - \omega)}\right\}\left\{1 - 0.97\, e^{j(21.25 - \omega)}\right\}}{1 - 0.9\, e^{j(19.44 - \theta)}}$$

For magnitude plot in dB (deciBel)

$$H(j\omega) = 10\log\left[1 - 2\times 0.559 \cos(63.43 - \omega) + (0.559)^2\right]$$
$$+ 10\log\left[1 - 2\times 0.97 \cos(21.25 - \omega) + (0.97)^2\right]$$
$$- 10\log\left[1 - 2\times 0.9 \cos(19.44 + \omega) + (0.9)^2\right]$$

For plot,

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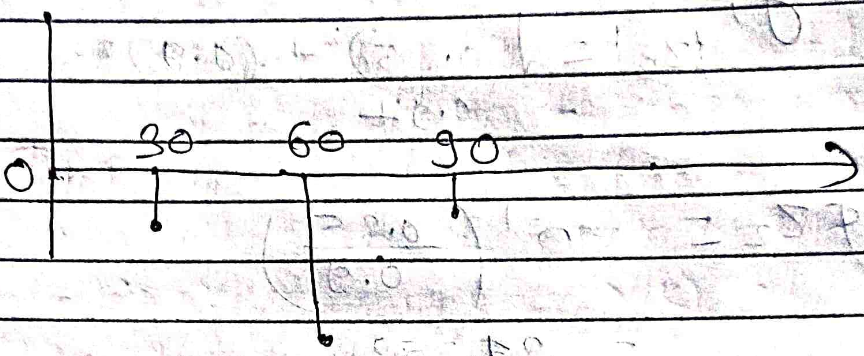
ω	0	30°	60°	90°
H(jω)	-0.18	-6.37	-7.212	-4.96



(fig:- magnitude plot)