Name: Mahesh Rawat

Roll No.: 18 (eighteen)

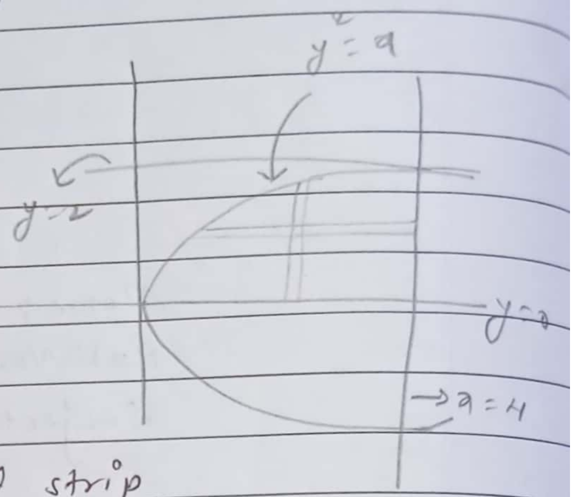
Subject: Calculus II

BE Computer...

Model Paper Solution

1 > a > Evaluate the integral : $\int_0^2 \int_{y^2}^4 y \cos(x^2)\, dx\, dy$

Soln



The curve is bounded by $x = y^2$ to $x = 4$ and $y = 0$ to $y = 2$. Here, the region is first integrated w.r.t. x along horizontal strip which extends from $x = y^2$ to $x = 4$ and then w.r.t. y along vertical strip which extends from $y = 0$ to $y = 2$

By changing the order of integration, we first integrate w.r.t. y along the vertical strip which extends from $y = 0$ to $y = \sqrt{x}$ and then w.r.t. x along horizontal strip which extends from $x = 0$ to $x = 4$.

Hence,

$$\int_0^4 \int_0^{\sqrt{x}} y \cos x^2\, dx\, dy$$

$$= \int_0^4 \cos x^2 \left[\frac{y^2}{2}\right]_0^{\sqrt{x}} dx$$

$$= \int_0^4 \frac{x}{2} \cos x^2\, dx$$

let, $x^2 = t$ when $x = 0$ then $t = 0$

$\therefore 2x\, dx = dt$ $x = 4$ then $t = 16$

$x\, dx = \frac{dt}{2}$

So,

$$\int_0^4 \frac{x}{2} \cos x^2 \, dx = \int_0^{16} \frac{\cos t}{4} \, dt$$

$$= \frac{1}{4} \left[\sin t \right]_0^{16}$$

$$= \frac{1}{4} \sin 16 \quad \underline{\text{Ans.}}$$

1>b> Evaluate the integral: $\int_0^1 \int_0^{(1-x)} \int_0^{(x+y)} e^z \, dz \, dy \, dx$.

Soln given,

$$\int_0^1 \int_0^{(1-x)} \int_0^{(x+y)} e^z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{(1-x)} \left[e^z \right]_0^{(x+y)} dy \, dx$$

$$= \int_0^1 \int_0^{(1-x)} \left(e^{(x+y)} - 1 \right) dy \, dx.$$

$$= \int_0^1 e^x \cdot \left[e^y - y \right]_0^{1-x} dx.$$

$$= \int_0^1 \left\{ e^x \left[e^{1-x} - 1 \right] - 1 + x \right\} dx.$$

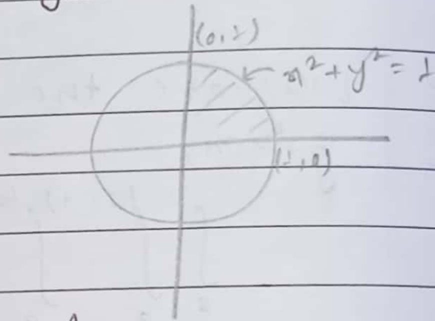
$$= \int_0^1 \left(e - e^x - 1 + x \right) dx.$$

$$= \left[ex - e^x - x + \frac{x^2}{2} \right]_0^1$$

$$= (\cancel{\ell} - \cancel{\ell} - \cancel{1} + \frac{1}{2} + \cancel{1})$$

$$= \frac{1}{2} \text{ Ans.}$$

1>c> Find the volume in the first octant bounded by the coordinate planes, the cylinder $x^2+y^2=1$ and the plane $z+y=3$.

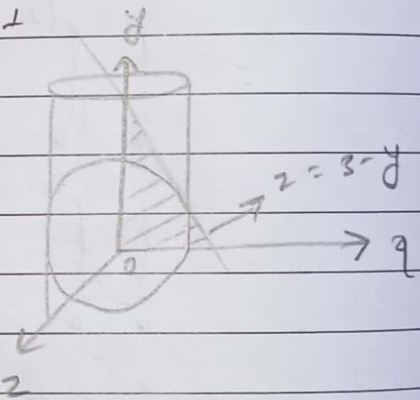


Soln Here, plane is,

$$z+y=3$$

$$\therefore z = 3-y$$

We have integrate $z=3-y$ over the first quadrant base of cylinder of the cycle $x^2+y^2=1$



For the base, limits for y,

$y=0$ to $y=\sqrt{1-x^2}$

and range of x is $0 \le x \le 1$

then,

$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} z \, dy \, dx \quad \text{— (i)}$$

For polar form, let,

$$y = r\sin\theta, \quad x = r\cos\theta$$

$$dy\,dx = r\,dr\,d\theta$$

$$z = 3-y = 3 - r\sin\theta$$

In first octant, range of θ is,

$$0 \le \theta \le \pi/2$$

range of r $0 \le r \le 1$

then eqn (i) becomes,

$$V = \int_0^{\pi/2}\int_0^1 (3 - r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{3r^2}{2} - \sin\theta \frac{r^3}{3}\right]_0^1 d\theta$$

$$= \int_0^{\pi/2} \left[\frac{3}{2} - \frac{\sin\theta}{3}\right] d\theta$$

$$= \left[\frac{3\theta}{2} + \frac{\cos\theta}{3}\right]_0^{\pi/2}$$

$$= \frac{3}{2}\cdot\frac{\pi}{2} + \frac{1}{3}\left(\cos\frac{\pi}{2} - \cos 0\right)$$

$$= \frac{3\pi}{4} - \frac{1}{3} \quad \text{Ans.}$$

2>a> Solve by using power series: $y'' - 4xy' + (4x^2 - 2)y = 0$

Sol^n given,

The differential eq^n is,

$$y'' - 4xy' + (4x^2 - 2)y = 0 \quad \text{— (i)}$$

let, $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$ —(ii)

be the solution of eq^n (i)

then, $y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots$ —(iii)

Also, $y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3$ — (iv)

Now Replacing the values of (ii), (iii) & (iv) in eq^n (i) we get,

$$(2a_2 + 6a_3 x + 12 a_4 x^2 + \cdots) - 4x(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots)$$
$$+ (4x^2 - 2)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots) = 0$$

$$\Rightarrow (2a_2 - 2a_0) + (6a_3 - \cancel{8}4a_1 - 2a_1)x + (12a_4 - 8a_2 + 4a_0 - 2a_2)$$
$$x^2 + (20a_5 - 12a_3 + 4a_1 - 2a_3)x^3 = 0$$

Now, comparing the coefficient of like terms,

$2a_2 - 2a_0 = 0$ $\quad$ $6a_3 - 4a_1 - 2a_1 = 0$ $\quad$ $12a_4 - 8a_2 + 4a_0 - 2a_2 = 0$

$[\therefore a_2 = a_0]$ $\quad$ $\Rightarrow [a_3 = a_1]$ $\quad$ $\Rightarrow 12a_4 - 6a_0 = 0$

$[\therefore a_4 = \frac{1}{2} a_0]$

$20a_5 - 12a_3 + 4a_1 - 2a_3 = 0$

$\Rightarrow 20a_5 = 14a_1 - 4a_1$

$[\therefore a_5 = \frac{1}{2} a_1]$

Now, Replacing the values of $a_0, a_1, a_2, a_3, a_4, a_5 \ldots$,

$$y = a_0 + a_1 x + a_0 x^2 + a_1 x^3 + \frac{1}{2} a_0 x^4 + \frac{1}{2} a_1 x^5 + \cdots$$

$$\Rightarrow y = a_0 \left(1 + x^2 + \frac{1}{2} x^4 + \cdots\right) + a_1 \left(1 + x^3 + \frac{1}{2} x^5 + \cdots\right)$$

which is the required solⁿ.

2> b> i> Express, $2x^2 - 4x + 2$ as Legendre polynomial

Solⁿ Here,

$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \dfrac{3x^2 - 1}{2}$

$\Rightarrow x^2 = \dfrac{2P_2(x) + 1}{3}$

$= \dfrac{2P_2(x) + P_0(x)}{3}$

$$\therefore 2x^2 - 4x + 2 = 2\left(\frac{2P_2(x) + P_0(x)}{3}\right) - 4P_1(x) + 2P_0(x)$$

$$= \frac{4}{3}P_2(x) + \frac{2}{3}P_0(x) - 4P_1(x) + 2P_0(x)$$

$$= \frac{4}{3}P_2(x) - 4P_1(x) + \frac{8}{3}P_0(x)$$

$$= \frac{4}{3}\left[P_2(x) - 3P_1(x) + 2P_0(x)\right]$$

2>b>ii> Show that: $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

Solⁿ Here, the Bessel function is,

$$J_v(x) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{2}\right)^{v+2m} \frac{1}{m!\,\Gamma(v+m+1)} \quad \text{--- (1)}$$

Taking $v = \frac{1}{2}$, we have,

$$J_{1/2}(x) = \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{2}\right)^{2m+\frac{1}{2}} \frac{1}{m!\,\Gamma(m+\frac{1}{2}+1)}$$

$$= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} (-1)^m \left(\frac{x}{2}\right)^{2m+1} \frac{1}{m!\,\Gamma(m+\frac{3}{2})}$$

$$\therefore J_{1/2}(x) = \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{2^{2m+1}\, m!\,\Gamma(m+3/2)} \quad \text{--- (4)}$$

Now, $2^{2m}\, m! = 2m(2m-2)(2m-4) \cdots 4 \cdot 2$

and $2^{2m+1}\,\Gamma(m+3/2) = 2^{m+1}(m+1/2)(m-1/2) \cdots 3/2 \cdot 1/2\,\Gamma(1/2)$

$$= (2m+1)(2m-1) \cdot 3 \cdot 1 \cdot \sqrt{\pi}$$

from (4)

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = \sqrt{\frac{2}{\pi x}}\sin x$$

$$\therefore J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \ \#$$

3> a> i> State first shifting theorem of Laplace Transform and find the Laplace transform of $t\cos at$.

Ans:

If the laplace transform of $f(t)$ is $F(s)$, then Laplace transform of $[e^{at} f(t)]$ is $F(s-a)$ i.e.

if $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

This is the 1st shifting theorem of laplace.

Solution: given,

$t\cos at$

$$\mathcal{L}\{t\cos at\} = -\frac{d}{ds}\mathcal{L}\{\cos at\}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

3> a>

ii> Find the inverse laplace transform of the function

$$\frac{s+2}{s^2(s+3)}$$

Sol^n Here,

$$F(s) = \frac{s+1}{s^2(s+3)}$$

let,

$$F(s) = \frac{s+1}{s^2(s+3)} = \frac{As+B}{s^2} + \frac{C}{s+3}$$

then,

$$(As+B)(s+3) + Cs^2 = s+1$$

$$\Rightarrow As^2 + 3As + Cs^2 + Bs + 3B = s+1$$

$$\Rightarrow (A+C)s^2 + (3A+B)s + 3B = s+1$$

Equating the like terms,

$$A+C=0, \quad 3A+B=1 \quad 3B=1$$

$$\Rightarrow \frac{2}{9}+C=0 \quad \Rightarrow 3A = 1-\frac{1}{3} \quad [\because B = 1/3]$$

$$[\because C = -2/9] \quad [\because A = \frac{2}{9}]$$

$$\therefore F(s) = \frac{2/9\, s + 1/3}{s^2} - \frac{2/9}{s+3}$$

$$= \frac{2s+3}{9s^2} - \frac{2}{9(s+3)}$$

$$= \frac{2s}{9s^2} + \frac{3}{9s^2} - \frac{2}{9(s+3)}$$

$$= \frac{2}{9}\frac{1}{s} + \frac{3}{9}\frac{1}{s^2} - \frac{2}{9}\frac{1}{(s+3)}$$

taking inverse laplace,

$$f(t) = \mathcal{L}^{-1}\left(\frac{2}{9}\right)\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left\{\frac{3}{9}\frac{1}{s^2}\right\} - \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{2}{9}\cdot 1 + \frac{1}{3}t - \frac{2}{9}e^{-3t}$$

$$\therefore F(s) = \frac{2}{9} + \frac{1}{3}t - \frac{2}{9}e^{-3t} \quad \underline{Ans}$$

3>b> Apply laplace transform to solve the initial value problem : $y'' + 4y' + 3y = e^{-t}$, $y(0) = y'(0) = 1$.

$\underline{Sol^n}$ Here,

$$y'' + 4y' + 3y = e^{-t}$$

Taking laplace transform,

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$\Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4[s\mathcal{L}\{y\} - y(0)] + 3\mathcal{L}\{y\} = \frac{1}{s+1}$$

$$\Rightarrow \mathcal{L}\{y\}[s^2 + 4s + 3] - s(1) - 1 - 4(1) = \frac{1}{s+1}$$

$$\Rightarrow \mathcal{L}\{y\} = \frac{1 + (s+5)(s+1)}{(s+1)(s^2+4s+3)}$$

$$= \frac{1 + s^2 + s + 5s + 5}{(s+1)(s^2 + 3s + s + 3)}$$

$$= \frac{s^2 + 6s + 6}{(s+1)(s+1)(s+3)}$$

$$= \frac{s^2 + 6s + 6}{(s+1)^2(s+3)}$$

Using partial fraction,

$$\frac{A}{(s+3)} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2} = \frac{s^2+6s+6}{(s+3)(s+1)^2}$$

$$\Rightarrow A(s+1)^2 + B(s+3)(s+1) + C(s+3) = s^2+6s+6$$

Equating,

$$A+B=1, \quad 2A+4B+C=6, \quad A+3B+3C=6$$

Solving,

$$A=-3/4, \quad B=7/4, \quad C=1/2$$

Putting these values,

$$-\frac{3}{4}\frac{1}{(s+3)} + \frac{7}{4(s+1)} + \frac{1}{2(s+1)^2} = \mathcal{L}\{y\}$$

$$\Rightarrow \cancel{\mathcal{L}\{y\} = -\frac{3}{4}\mathcal{L}}$$

$$\Rightarrow y = -\frac{3}{4}\mathcal{L}\left\{\frac{1}{s+3}\right\} + \frac{7}{4}\mathcal{L}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= -\frac{3}{4}\cdot e^{-3t} + \frac{7}{4}e^{-t} + \frac{1}{2}t\cdot e^{-t}$$

$$\therefore y = \frac{7e^{-t} - 3e^{-3t} + 2te^{-t}}{4} \quad \#$$

4> a> A particle moves along the curve $(t^3+1, t^2, 2t+5)$ Find the component of the velocity and acceleration at $t=1$ along $\vec{i}+\vec{j}+3\vec{k}$.

Soln Here,

$$\vec{r} = (t^3+1, t^2, 2t+5)$$

$$\vec{r} = (t^3+1)\vec{i} + t^2\vec{j} + (2t+5)\vec{k} \quad -(i)$$

Diff w.r.t t,

$$\text{velocity } (\vec{v}) = \frac{dr}{dt} = 3t^2\vec{i} + 2t\vec{j} + 2\vec{k} \quad -(ii)$$

again, diff. w.r.t. t,

$$\text{acceleration } (\vec{a}) = \frac{d^2r}{dt^2} = 6t\vec{i} + 2\vec{j} \quad -(iii)$$

given,

$$\vec{n} = \vec{i} + \vec{j} + 3\vec{k}$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{i}+\vec{j}+3\vec{k}}{\sqrt{1+1+9}} = \frac{1}{\sqrt{11}}(\vec{i}+\vec{j}+3\vec{k})$$

Component of velocity at $t=1$ then,

$$(3t^2\vec{i} + 2t\vec{j} + 2\vec{k}) \cdot \frac{\vec{i}+\vec{j}+3\vec{k}}{\sqrt{11}}$$

$$= \frac{3+2+6}{\sqrt{11}} = \sqrt{11}$$

Component of acceleration at $t=1$ then,

$$(6t\vec{i} + 2\vec{j}) \cdot \frac{(\vec{i}+\vec{j}+3\vec{k})}{\sqrt{11}}$$

$$= \frac{6+2}{\sqrt{11}} = 8/\sqrt{11} \quad \#$$

4 > b > If $\phi = \ln(x^2+y^2+z^2)$, then, find grad ϕ and div (grad ϕ).

Soln Here,

$$\phi = \ln(x^2+y^2+z^2)$$

We have to find,

grad ϕ = ?

div(grad ϕ) = ?

Now

$$\text{grad}\,\phi = \nabla\cdot\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\ln(x^2+y^2+z^2)\right)$$

$$= \frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{k}$$

Again,

$$\text{div}(\text{grad}\,\phi) = \nabla\cdot\text{grad}\,\phi$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left\{\frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{k}\right\}$$

$$= \frac{(x^2+y^2+z^2)\cdot 2 - 2x\cdot 2x}{(x^2+y^2+z^2)^2} + \frac{(x^2+y^2+z^2)\cdot 2 - 2y\cdot 2y}{(x^2+y^2+z^2)^2}$$

$$+ \frac{(x^2+y^2+z^2)\cdot 2 - 2z\cdot 2z}{(x^2+y^2+z^2)^2}$$

$$= \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2}$$

$$\therefore \operatorname{div}(\operatorname{grad}\phi) = \frac{2}{(x^2+y^2+z^2)} \quad \#$$

4> c> Evaluate $\oint_c (x^2-3y)\,dx + (x+\sin y)\,dy$ where c is the boundary of the triangle with vertices $(0,0)$, $(1,0)$ and $(0,2)$.

Soln Here,

$$I = \oint_c (x^2-3y)\,dx + (x+\sin y)\,dy$$ where c is the boundary of the triangle.

Here,

$$I = \oint_c (x^2-3y)\,dx + (x+\sin y)\,dy \quad \text{--- (i)}$$

Comparing (i) with $\oint F_1 dx + F_2 dy$

Where,

$$F_1 = (x^2-3y) \qquad F_2 = (x+\sin y)$$

$$\frac{\partial F_1}{\partial y} = -3 \qquad \frac{\partial F_2}{\partial x} = 1$$

From Green's theorem,

$$\oint F_1 dx + F_2 dy = \iint_S \left(\frac{\partial F_2}{\partial y} - \frac{\partial F_1}{\partial x}\right) dA$$

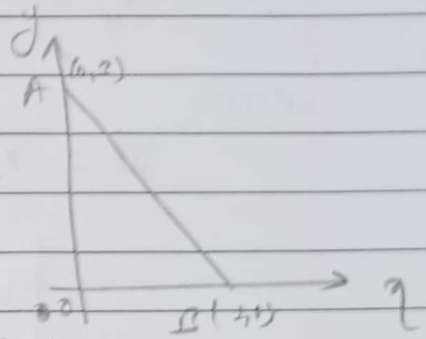
$$= \iint (1+3)\, dx\, dy$$

$$= \iint 4\, dx\, dy$$

Eq^n of line AB,

$y = -2x+2$

$\therefore 2x + y = 2$



Hence, the limit of y is 0 to $2-2x$

the limit of x is 0 to 1.

So,

$$\int_0^1 \int_0^{(2-2x)} 4\, dy\, dx$$

$$= 4 \int_0^1 (2-2x)\, dx.$$

$$= 4 \left[2x - \cancel{2} \frac{x^2}{\cancel{2}} \right]_0^1$$

$$= 4(2-1) = 4 \#$$

5) a) Find $\iint_S (\vec{F} \cdot \vec{n})\, ds$, for $\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$,

$\vec{r} = (u\cos v, u\sin v, 3v)$; $0 \le u \le 1$, $0 \le v \le 2\pi$

Sol^n Here,

$$\vec{F} = x^2 \vec{i} + y^2 \vec{j} + z^2 \vec{k}$$

$\vec{r} = (u\cos v,\ u\sin v,\ 3v)$

Now,

$\vec{r}_u = \cos v\,\vec{i} + \sin v\,\vec{j}$

$\vec{r}_v = -u\sin v\,\vec{i} + u\cos v\,\vec{j} + 3\vec{k}$

and,

$\vec{N} = \vec{r}_u \times \vec{r}_v$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 3 \end{vmatrix}$$

$= 3\sin v\,\vec{i} - 3\cos v\,\vec{j} + \vec{k}\,u$

Now,

$\vec{F} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$

$= u^2\cos^2 v\,\vec{i} + u^2\sin^2 v\,\vec{j} + 9v^2\,\vec{k}$

$\vec{F}\cdot\vec{N} = 3u^2\sin v\cos^2 v - 3u^2\cos v\sin^2 v + 9uv^2$

So,

$$\iint (\vec{F}\cdot\vec{n})\,ds = \iint \vec{F}\cdot\vec{N}\,du\,dv$$

$$= \int_0^{2\pi}\int_0^1 (3u^2\sin v\cos^2 v - 3u^2\cos v\sin^2 v + 9uv^2)\,du\,dv$$

$$= \int_0^{2\pi}\left(u^3\sin v\cos^2 v - \cos v\sin^2 v\,u^3 + \frac{9v^2u^2}{2}\right)_0^1 dv$$

$$= \int_0^{2\pi} \left(\sin v \cos^2 v - \cos v \sin^2 v + \frac{9v^2}{2} \right) dv$$

$$= \left(-\frac{\cos^3 v}{3} \right)_0^{2\pi} - \left(\frac{\sin^3 v}{3} \right)_0^{2\pi} + \frac{9}{2} \left[\frac{v^3}{3} \right]_0^{2\pi}$$

$$= 12\pi^3 \quad \#$$

5>b> Evaluate $\oint_c (\vec{F} \cdot d\vec{r})$ by using stoke's theorem, where $\vec{F} = (y, xz^3, -zy^3)$ & $c: x^2+y^2=4, z=3$

Sol^n Here,

$$\vec{F} = y\vec{i} + xz^3\vec{j} - 3y^3\vec{k}$$

Now,

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz^3 & -3y^3 \end{vmatrix}$$

$$= \vec{i}(-9y^2 - 0) - 0 + \vec{k}(27 - 1)$$

$$= -9y^2\vec{i} + 26\vec{k}.$$

let,

$$\vec{r} = x\vec{i} + y\vec{j} + 3\vec{k}$$

so,

$$\vec{r}_x = \vec{i}$$

$$\vec{r}_y = \vec{j}$$

classmate

Date

Page

$$\vec{N} = \vec{r}_x \times \vec{r}_y = \vec{k}$$

$\therefore$ curl $\vec{F} \cdot \vec{N} = 26$

Now,

$$\oint_c \vec{F} \cdot d\vec{r} = \iint \text{curl}(\vec{F} \cdot \vec{N})\, dx\, dy$$

$$= \iint 26\, dx\, dy$$

$= 26 \cdot$ area of circle

$$= 26 \cdot (2)^2 \pi$$

$$= 104\pi \quad \#$$

6>a> Find the fourier series of $f(x) = \frac{x^2}{2}$ for $-\pi \le x \le \pi$ and deduce that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \pi^2/6$.

Soln Here,

$$f(x) = \frac{x^2}{2} \quad \text{for } -\pi \le x \le \pi$$

Fourier series is,

$$F(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

Where,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^2}{2}\, dx.$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2} \cdot \frac{1}{3} \left[x^3 \right]_{-\pi}^{\pi}$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx.$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x^2}{2} \cos nx \, dx$$

$$= \frac{1}{2\pi} \cdot 2 \int_{0}^{\pi} x^2 \cos nx \, dx$$

$$= \frac{1}{\pi} \left[x^2 \frac{\sin nx}{n} + 2x \frac{\cos nx}{n^2} + 2 \frac{\sin nx}{n^3} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\left(0 + \frac{2\pi \cos n\pi}{n^2} - 0 \right) - (0) \right]$$

$$= \frac{2 \cos n\pi}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

$$= 0$$

Now eq^n (i) becomes,

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left(\frac{2 \cos n\pi}{n^2} \cos nx \right)$$

classmate
Date
Page

$$\Rightarrow \frac{x^2}{2} = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty}\left(\frac{\cos n\pi \cos nx}{n^2}\right)$$

put $x = \pi$

$$\frac{\pi^2}{2} - \frac{\pi^2}{6} = 2\left[\frac{\cos\pi\cos\pi}{1^2} + \frac{\cos^2\pi\cos^2\pi}{2^2} + \cdots\right]$$

$$\Rightarrow \frac{2\pi^2}{6} = 2\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right.$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$ #

6) b) Find the fourier half range cosine and sine of series of $f(x) = e^x$ in $(0, L)$

Sol^n Here,

$$f(x) = e^x$$

Half Range cosine,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L}\right) \quad \text{— (i)}$$

where,

$$a_0 = \frac{1}{L}\int_0^L e^x\,dx$$

$$= \frac{1}{L}\left[e^x\right]_0^L$$

$$= \frac{1}{L}\left(e^L - 1\right)$$

$$a_n = \frac{2}{L}\int_0^L e^x \cos\left(\frac{n\pi x}{L}\right) dx.$$

$$= \frac{2}{L}\left[\frac{e^x}{1^2+\left(\frac{n\pi}{L}\right)^2}\left\{1.\cos\left(\frac{n\pi x}{L}\right) + \frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)\right\}\right]_0^L$$

$$= \frac{2}{L}\left(\frac{L^2}{L^2+n^2\pi^2}\right)\left[e^L\left(\cos n\pi + \frac{n\pi}{L}\sin n\pi\right) - e^0\left(\cos 0^\circ + \frac{n\pi}{L}\sin 0\right)\right.$$

$$= \frac{2L}{L^2+n^2\pi^2}\left(e^L(-1)^n + 0 - 1\right)$$

$$= \frac{2L}{L^2+n^2\pi^2}\left(e^L(-1)^n - 1\right)$$

eq^n (i) becomes,

$$e^x = \frac{1}{L}(e^L - 1) + \sum_{n=1}^{\infty}\left(\frac{2L}{L^2+n^2\pi^2}\right)\left[e^L(-1)^n - 1\right]\cos\left(\frac{n\pi x}{L}\right).$$

Half Range sine,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{--- (ii)}$$

where,

$$b_n = \frac{2}{L}\int_0^L e^x \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left[\frac{e^x}{1+\left(\frac{n\pi}{L}\right)^2}\left(1.\sin\left(\frac{n\pi x}{L}\right) - \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)\right)\right]_0^L$$

$$= \frac{2}{L}\left(\frac{L^2}{L^2+n^2\pi^2}\right)\left[e^L\left(\sin n\pi - \frac{n\pi}{L}\cos n\pi\right)\right] - e^0\left(\sin 0 - \frac{n\pi}{L}\cos 0\right)$$

$$= \frac{2L}{L^2+n^2\pi^2}\left(\frac{n\pi}{L}\left(-(-1)^n e^L + 1\right)\right)$$

$$= \frac{2n\pi}{L^2+n^2\pi^2}\left[1-(-1)^n e^L\right]$$

eq^n (iii) becomes,

$$e^x = \sum_{n=1}^{\infty} \frac{2\pi n}{L^2+n^2\pi^2}\left[1-e^L(-1)^n\right]\sin\left(\frac{n\pi x}{L}\right)$$

7> Attempt any two questions

b> find the breaking time for $u_t + 2u \cdot u_x = 0$, $u(x,0) = e^{-x^2}$

Sol^n Here,

$$f(\lambda) = c(u_0(\lambda)) = 2e^{-\lambda^2}$$

$$\therefore f'(\lambda) = \frac{d\,2e^{-\lambda^2}}{d\lambda}$$

$$= -4\lambda e^{-\lambda^2}$$

$$t = \frac{-1}{f(\lambda)} = \frac{-1}{-4\lambda e^{-\lambda^2}} = \frac{e^{\lambda^2}}{4\lambda}$$

We need to find the max. value of $t = \frac{e^{\lambda^2}}{4\lambda}$

To find the min. set $\frac{dt}{d\lambda} = 0$

$$\therefore \frac{4\lambda \cdot e^{\lambda^2} \cdot 2\lambda - e^{\lambda^2} \cdot 4}{16\lambda^2} = 0$$

$$e^{\lambda^2} 4(2\lambda^2 - 1) = 0$$

But,

$4e^{\lambda^2} \neq 0$, we have,

$$2\lambda^2 - 1 = 0$$

$$\lambda^2 = 1/2$$

$$\therefore \lambda = 1/\sqrt{2}$$

Here, $\frac{\partial^2 t}{d\lambda^2} > 0$ at $\lambda = 1/\sqrt{2}$. So, min. occurs at

$\lambda = 1/\sqrt{2}$.

Hence, breaking time is $t_b = t(\lambda = 1/\sqrt{2})$

$$= \frac{e^{1/2}}{4 \cdot \sqrt{1/2}} = \frac{\sqrt{e}}{2\sqrt{2}} \#$$

7>c> Evaluate $\int_{(1,0,2)}^{(-2,1,3)} [(6xy^3 + 2z^2)dx + 9x^2y^2dy + (4xz+1)dz]$

Soln Here,

$$I = \int_{(1,0,2)}^{(-2,1,3)} [(6xy^3 + 2z^2)dx + 9x^2y^2dy + (4xz+1)dz]$$

Where,

$F_1 = 6xy^3 + 2z^2$ $\quad F_2 = 9y^2x^2$ $\quad F_3 = 4xz+1$

Here,

$$\frac{\partial F_1}{\partial y} = 18xy^2, \; \& \; \frac{\partial F_2}{\partial x} = 18xy^2 \Rightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

$$\frac{\partial F_1}{\partial z} = 4z \; \& \; \frac{\partial F_3}{\partial x} = 4z \Rightarrow \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

Date ____
Page ____

$$\frac{\partial F_2}{\partial z} = 0 \quad \& \quad \frac{\partial F_3}{\partial y} = 0 \Rightarrow \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

So, given integral is exact and solution is,

$$I = \int_{(1,0,2)}^{(-2,1,3)} d\left[\int F_1 dx + \int (\text{term free from } x \text{ in } F_2)\, dy + \int (\text{term free from } x \,\&\, y \text{ in } F_3)\, dz\right]$$

$$= \int_{(1,0,2)}^{(-2,1,3)} d\left[\int (6xy^3 + 2z^2)\, dx + 0 + \int 1\, dz\right]$$

$$= \left[6 \cdot \frac{x^2}{2} y^3 + 2z^2 x + z\right]_{(1,0,2)}^{(-2,1,3)}$$

$$= (12 - 36 + 3) - (8 + 2)$$

$$= -31 \quad \text{Ans.}$$