practice Linear programming + Infinite series (23 marks)

Gandak EE Pokhara)

Q.N 2 b) maximize $z = 20x_1 + 20x_2$ subject to

$x_1 \ge 0, x_2 \ge 0, -x_1 + x_2 \le 1, x_1 + 3x_2 \le 15$

$3x_1 + x_2 \le 21$ (8)

Solution:

Let x_3, x_4, x_5 be slack variable

Then,

Maximize: $z - 20x_1 - 20x_2 = 0$ — (i)

$-x_1 + x_2 + x_3 = 1$ — (ii)

$x_1 + 3x_2 + x_4 = 15$ — (iii)

$3x_1 + x_2 + x_5 = 21$ — (iv)

To find optimal solution augmented matrix be,

	z	x_1	x_2	x_3	x_4	x_5	constant	ratio (c/p.c)
R_1	1	-20	-20	0	0	0	0	0
R_2	0	-1	1	1	0	0	1	1/1 = 1
R_3	0	1	3	0	1	0	15	15/3 = 5
R_4	0	3	1	0	0	1	21	21/1 = 21

- Since x_2 colum has greatest non negative entry is -20, x_2 choosen as pivot column
- Since ratio of R_2 Row has least positive ratio, so choose pivot point 1 and 1 = 1 & Row

- To Eliminate value of pivot column rather than pivot element.

Apply $R_1 \rightarrow R_1 + 20R_2$
$R_3 \rightarrow R_3 - 3R_2$
$R_4 \rightarrow R_4 - R_2$

	z	x_1	x_2	x_3	x_4	x_5	constant	ratio.
R_1	1	-40	0	20	0	0	20	0
R_2	0	-1	1	1	0	0	1	-1
R_3	0	4	0	-3	1	0	12	$4/15 = 0.2$
R_4	0	4	0	-1	0	1	20	$4/21 = 0.19$

Similarly,
x_1 is pivot column 4 is pivot element of R_4

x_3 x_4 x_5 are basic variables and x_1, x_2 non basic variable

Apply, $R_1 \rightarrow R_1 + 10 \times R_4$
$R_2 \rightarrow 4R_2 + R_4$
$R_3 \rightarrow R_3 - R_4$

	z	x_1	x_2	x_3	x_4	x_5	constant	ratio
R_1	1	0	0	10	0	10	220	
R_2	0	0	4	3	0	1	24	
R_3	0	0	0	-4	-1	1	-8	
R_4	0	4	0	-1	0	1	21	

Here, x_1, x_2, x_4 are basic variable

Other x_3 and x_5 are non basic variable

$4x_1 = 21$ | $x_1 = 21/4$

$4x_2 = 24 \Rightarrow x_2 = 6$

$-x_4 = -8$ | $x_4 = 8$

$Z = 220$

Again (8)

maximize,

$Z - 20x_1 - 20x_2 = 0$ — (I)

$-x_1 + x_2 + x_3 = 1$ — (II)

$x_1 + 3x_2 + x_4 = 15$ — (III)

$3x_1 + x_2 + x_5 = 21$ — (IV)

To find optimal solution: Agumented matrix be

	Z	x_1	x_2	x_3	x_4	x_5	constant	Ratio C/P
R_1	1	-20	-20	0	0	0	0	0
R_2	0	-1	1	1	0	0	1	-1
R_3	0	1	3	0	1	0	15	15
R_4	0	3	1	0	0	1	21	7

pivot column $= x_1$
pivot element $= 3$
pivot Row $= R_4$
— being -20 is most negative c...

Apply, $R_1 \rightarrow R_1 + \frac{20}{3} \times R_4$ $R_2 \rightarrow 3R_2 + R_4$

$R_3 \rightarrow 3R_3 - R_4$

" Fixed " ; Mandaki Engineering college

Solution: (Q.N)

The given standard form of minimization problem is.

Step1 conversion to dual.

x_1	x_2	x_3	Constant
1	4	2	5 (y_1)
3	1	2	4 (y_2)
2	9	1	

Let y_1 and y_2 be dual variable so,

$$\text{Max} : W = 5y_1 + 4y_2$$

Subrected to,

$$y_1 + 3y_2 \leq 2$$

$$4y_1 + y_2 \leq 9$$

$$2y_1 + 2y_2 \leq 1$$

$$y_1, y_2 \geq 0$$

Step2:

Introducing slack variable y_3, y_4, y_5

Now, writing in eqn form:

Solving by simplex methop

$$W - 6y_1 + 4y_2 = 0 \quad \text{— (I)}$$

$$y_1 + 3y_2 + y_3 = 2 \quad \text{— (II)}$$

$$4y_1 + y_2 + y_4 = 9 \quad \text{— (III)}$$

$$2y_1 + 2y_2 + y_5 = 1 \quad \text{— (IV)}$$

making simplex table for Augument matrics

	w	y_1	y_2	y_3	y_4	y_5	constant	ratio
R_1	1	-5	-4	0	0	0	0	
R_2	0	1	3	1	0	0	2	2/1
R_3	0	4	1	0	1	0	9	9/4
R_4	0	2	2	0	0	1	1	1/2 = 0.5

Since, -5 is most negative entry,

y_1 = pivot column,

And, 2 is pivot element

R_4 is pivot row.

Apply,
$$R_1 \rightarrow R_1 + \frac{5}{2} R_4$$
$$R_2 \rightarrow 2R_2 - R_4$$
$$R_3 \rightarrow R_3 - 2R_4$$

Again making simplex table

	w	y_1	y_2	y_3	y_4	y_5	constant	ratio
R_1	1	0	1	0	0	5/2	5/2	
R_2	0	0	4	2	0	-1	3	
R_3	0	0	-3	0	1	-2	7	
R_4	0	2	2	0	0	1	1	

since there is non-negative entry at Row 1 (top row) so this gives optimize solution.

Basica variables are y_1 y_3 y_4

and non basic variable are y_2, y_5

so,

maximum $W = 5/2$

$2y_1 = 1$

$\therefore y_1 = 1/2$

$2y_3 = 3$

$\therefore y_3 = 3/2$

$y_4 = 7$

W min at $(5/2)$ ~~y~~ $(x_1\ x_2\ x_3) = (0, 0\ 5/2)$

Fixed Q. N (2)

Soln:

The given problem is in standard form of minimization so:

x_1	x_2	constant
2	3	1 (y_1)
3	1	4 (y_2)
4	3	

Maximize, $W = 1y_1 + 4y_2$

subjected to

$2y_1 + 3y_2 \le 4$

$3y_1 + y_2 \le 3$

$y_1, y_2 \ge 0$

Introducing slack variable, y_3 and y_4 so,

$W - y_1 - 4y_2 = 0$ — (I)

$2y_2 + 3y_2 + y_3 = 4$ — (II)

$3y_1 + y_2 + y_4 = 3$ — (III)

By using simplex table.

	W	y_1	y_2	y_3	y_4	Constant	ratio
R_1	1	-1	-4	0	0	0	
R_2	0	2	3	1	0	4	$4/3 = 1.33$
R_3	0	3	1	0	1	3	$3/1 = 3$

since, -4 is most negative entry at top row, so,

y_2 = pivot column.

R_2 has least ratio. (+ve) so

3 is pivot element

Apply. $R_1 \rightarrow 3R_1 + 4R_2$

$R_3 \rightarrow 3R_3 - R_2$

	W	y_1	y_2	y_3	y_4	Constant	ratio
R_1	3	[illegible]	0	4	0	16	
R_2	0	2	3	1	0	4	
R_3	0	7	0	-1	3	5	

This table gives optimal solution as there is non -ve element in top co-

$\therefore$ y_2 y_4 are basis variable.

y_3 $y_1 = 0$ non basic variable

Now,

$$3y_4 = 5$$

$$y_4 = 5/3$$

$$3y_2 = 4$$

$$y_2 = 4/3$$

Hence,

$$\max(W) = 16/3 \text{ at } (y_2, y_4) = (4/3, 5/3)$$

$$\min(W) = 16/3 \text{ at } (x_1, x_2) = (4/3, 0)$$

Infinite series

2b) Define absolute convergence, Find centre radius and interval of Convergence of infinite series.

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n 5^n}$$

→ Ans

A series $\sum_{n=1}^{\infty} u_n$ converges absolutely if the corresponding series $\sum_{n=1}^{\infty} |u_n|$ converges. This is called absolute convergence.

Solution:

The general term of the given series is $u_n = \frac{(x-5)^n}{n 5^n}$

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} =$$

$$= \lim_{n \to \infty} \frac{(x-5)^{n+1}}{(n+1) 5^{n+1}} \cdot \frac{n 5^n}{(x-5)^n}$$

$$= \lim_{n \to \infty} \frac{(x-5)^n (x-5)'}{n 5^{n+1} + 5^{n+1}} \times \frac{n 5^n}{(x-5)^n}$$

$$= \lim_{n \to \infty} \quad n 5^n \left(5 + \frac{5}{n}\right)$$

$$\lim_{n \to \infty} \frac{(x-5)}{\left(5+\frac{5}{n}\right)}$$

$$= \frac{x-5}{5}$$

Then by ratio test the given series is convergent for $\left|\frac{x-5}{5}\right| < 1$ and divergent for $\left|\frac{x-5}{5}\right| > 1$

further test is necessary for $\left|\frac{x-5}{5}\right| = 1$

At $\left|\frac{x-5}{5}\right| = 1$ implies $\frac{x-5}{5} = 1$ and -1

At $\left(\frac{x-5}{5}\right) = 1$, given series reduce to $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ with general term $u_n = \frac{1}{n}$ which is divergent by p-test

At $\frac{x-5}{5} = -1$ given series reduced to $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \dots$ which is alternative series and each term is less than preceding term

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

So it is convergent at $\frac{x-5}{5} = -1$

$$-1 \le \frac{x-5}{5} < 1$$

$$-5 \le x-5 < 5.$$

$$0 \le x \le 10$$

Required interval of convergences is $[0,10]$

and the radius of the series is,

$$r = \frac{10-0}{2} = \frac{10}{2} = 5$$

Also centre of convergence.

$$C = \frac{0+10}{2} = 5$$

Matrix

(similar to this → fixed)

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(7)

Q.N 1b. Check consistency and solve the equation

$x+2y+3z=1$ — (I)

$2x+3y+2z=2$ — (II)

$2x+3y+4z=1$ — (III)

The augumented matrix of A : B is

$$A:B = \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 2 & 3 & 2 & : & 2 \\ 2 & 3 & 4 & : & 1 \end{bmatrix}$$

Applying: $R_2 \rightarrow R_2 - 2R_1$

and $R_3 \rightarrow R_3 - 2R_1$

$$A:B = \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 0 & -1 & -4 & : & 0 \\ 0 & -1 & -2 & : & -1 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$

$$A:B = \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 0 & -1 & -4 & : & 0 \\ 0 & 0 & 2 & : & -1 \end{bmatrix}$$

since Rank of matrix A = Rank of matrix A : B,

so, This is consitent

solving value.

$~~2z=~~$

$x + 2y + 3z = 1$ —(I)

$-y - 4z = 0$ —(II)

$2z = -1$ —(III)

Hence

$z = -1/2$

$-y = 4z$

$= -4 \times \frac{1}{2}$

$-y = -2$

$y = 2$

$x + 2 \times 2 + 3 \times (\frac{-1}{2}) = 1$

$x + 4 - \frac{3}{2} = 1$

$x + 4 = 1 + \frac{3}{2}$

$x + \cancel{4} = \frac{2+3}{2} + 4$

$x = \frac{5}{2} + 4$

$= \frac{13}{2}$

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5a) Solution:

b) Define consistence and inconsistence of linear equations check consistence of given system of linear equation

$$5x+5y-10z=0$$
$$2w+3x+3y+6z=2$$
$$4w+x+y-2z=4,$$

Ans: ~~The matrix~~

The system of linear equation in which rank of augumented matrix is equal to given matrix Then it is called consistence if not it is called inconsistence

$$5x+5y-10z=0 \quad -(1)$$
$$2w-3x-3y+6z=2 \quad -(11)$$
$$4w+x+y-2z=4 \quad -(111)$$

The augumented matrix of given equation is

$$A:B = \begin{bmatrix} 5 & 5 & -10 & 0 & : & 0 \\ -3 & -3 & 6 & 2 & : & 2 \\ 1 & 1 & -2 & 4 & : & 4 \end{bmatrix}$$

Applying $R_1 \to \frac{1}{5}R_1$ in

$$A:B \begin{bmatrix} 1 & 1 & -2 & 0 & : & 0 \\ -3 & -3 & 6 & 2 & : & 2 \\ 1 & 1 & -2 & 4 & : & 4 \end{bmatrix}$$

Applying
$R_2 \rightarrow R_2 + 3R_1$
$R_3 \rightarrow R_3 - R_1$

$$A:B = \begin{bmatrix} 1 & 1 & -2 & 0 & : & 0 \\ 0 & 0 & 0 & 2 & : & 2 \\ 0 & 0 & 0 & 4 & : & 4 \end{bmatrix}$$

$w = 1$ consistent

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1. check the consistency of given equation and solve it.

Given eqn:

$$5x+3y+7z=4 \quad \text{(I)}$$
$$3x+26y+2z=9 \quad \text{(II)}$$
$$7x+2y+10z=5 \quad \text{(III)}$$

The augumented matrix of given equation is

$$A:B = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

Applying: $R_2 \to 5R_2 - 3R_1$

$R_3 \to 5R_3 - 7R_1$

$$A:B = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 121 & -11 & : & 33 \\ 0 & -11 & 1 & : & -3 \end{bmatrix}$$

$11R_3 + R_2$

$-33+0$

Applying: $R_3 \to R_3 + \frac{R_2}{11}$

$\to R_3 + \frac{1}{11} R_2$

-11

$$A:B = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 0 & 121 & -11 & : & 33 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$121y - 112 = 33$$

$$\cancel{y = 33 + 112}$$

$$5x + 80y =$$

$$121y - 112 = 33$$

$$11y - 2 = 3$$

$$5x + 3y + 7z = 4$$

$$5x + 3y + 7 \times 11y - 3 = 4$$

$$5x + 80y - 21 = 4$$

$$5x + 80y = 25$$

26) Gandaki Engineering college (Fixed)

Define absolute convergence, and find the centre and radius of convergence of infinite series. C.N

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n5^n}$$

→ A series $\sum_{n=1}^{\infty} u_n$ converges absolutly if the corresponding series $\sum_{n=1}^{\infty} |u_n|$ converges.

Solution.

$$u_n = \left(\frac{x-5}{n5^n}\right)^n$$

$$u_{n+1} = \left(\frac{x-5}{(n+1)(5^{n+1})}\right)^{n+1}$$

$$= \frac{(x-5)^n (x-5)}{(n+1)\,5^{n+1}}$$

Now, using ratio test:

$$\lim_{x \to \infty} \frac{u_{n+1}}{u_n}$$

$$= \lim_{x \to \infty} \frac{(x-5)^n (x-5)}{(n+1)(5^{n+1})} \times \frac{n5^n}{(x-5)^n}$$

$$= \lim_{x \to \infty} \frac{(x-5)\, \cancel{n5^n}}{n5^n\left(\frac{1}{n}+1\right)5} \times \cancel{n5^n} \Rightarrow \frac{x-5}{5}$$

By the ratio test

if $\left|\frac{x-5}{5}\right| < 1 \longrightarrow$ convergent

if $\left|\frac{x-5}{5}\right| > 1 \longrightarrow$ Divergent

if $\left(\frac{x-5}{5}\right| = 1$ further test to be needed).

$\left|\frac{x-5}{5}\right| = 1$ implies $\rightarrow \frac{x-5}{5}\Big| = -1$ and $\frac{x-5}{5} = 1$

At $\frac{x-5}{5} = 1$ The eqn reduce to $\frac{1}{n}$ and

Seres become $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + 1 \cdot$ (divergent by p-test)

At $\frac{x-5}{5} = -1$ eqn reduce to $\frac{(-1)^n}{\frac{1}{n}}$ which

make alternate series, $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}$

$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = \frac{1}{\infty} = 0$

so, convergent at $\frac{x-5}{5} = -1$

So given series is convergent at

$$-1 \leq \frac{x-5}{5} < 1$$

$$-5 \leq x-5 < 5$$

$$5 + -5 \leq x \leq 5+5$$

$$0 \leq x \leq 10$$

Required convergency interval is $[0, 10]$

Radius of series. $r = \frac{10-0}{2} = 5$

Centre of convergence $= c = \frac{0+10}{2} = 5$

(Everest E. College)

4a) state p-series test. Test the convergence and divergence of the infinite series.

$\sum |\sqrt{n^3+1} - \sqrt{n^3-1}|$

→ A series of the form $\sum_{n=1}^{\infty} \left(\frac{1}{n^p}\right)$ is called p-series

(p = power which is real & constant)

IMP

→ It state: "The series $\sum_{n=1}^{\infty} \left(\frac{1}{n^p}\right)$ is convergent for $p > 1$ and is diverges for $p \leq 1$

Soln: Given series

$$\sum |\sqrt{n^3+1} - \sqrt{n^3-1}|$$

multiply by numerator and by $\sqrt{n^3+1} + (\sqrt{n^3-1})$

$$u_n = (\sqrt{n^3+1} - \sqrt{n^3-1}) \times \frac{\sqrt{n^3+1} + \sqrt{n^3-1}}{\sqrt{n^3+1} + \sqrt{n^3-1}}$$

$$= \frac{(n^3+1) - (n^3-1)}{\sqrt{n^3+1} + \sqrt{n^3-1}}$$

$$u_n = \frac{2}{\sqrt{n^3+1} + \sqrt{n^3-1}}$$

$$= \frac{2}{n^3\left(\sqrt{1+\frac{1}{n^3}} + \sqrt{1-\frac{1}{n^3}}\right)}$$

Choose, $v_n = \frac{1}{n^3}$

Clearly $\frac{1}{n^3}$ is convergent by p-series test. ($\because p = 3 > 1$)

Limit Here Comparision test $\lim_{n\to\infty} \frac{u_n}{v_n}$

$$= \lim_{n\to\infty} \frac{2}{\left(\sqrt{1+\frac{1}{n^3}} + \sqrt{1-\frac{1}{n^3}}\right)}$$

$$= \frac{2}{1+1}$$

$$= \frac{2}{2}$$

$= 1$ which is finite non zero value.

So the given Σv_n is ~~divergent~~ convergent

so Σu_n is also convergent by limit comparison test.

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Test the convergence of $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{5}+1} + \cdots$

Solution: given series.

$$\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{5}+1}$$

$$= \frac{1}{\sqrt{n+1}}(-1)^{n+1}$$

Comparing the given series with $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$

$$u_n = \frac{1}{\sqrt{n}+1}$$

condition for convergency

$u_n \geq 0$

$u_n \geq u_{n+1}$

$\lim_{n\to\infty} u_n = 0$

a) $u_n = \frac{1}{\sqrt{n}+1} \geq 0$ for all $n \geq 1$

b) $u_n \geq u_{n+1}$

$$\frac{u_{n+1}}{u_n} = \left(\frac{1}{\sqrt{n+1}+1} \times \frac{\sqrt{n}+1}{1}\right)$$

$$= \frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}\right)+1} \cdot \sqrt{n}+1$$

$$= \frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\frac{1}{\sqrt{n}}\right)} \times \sqrt{n}+1$$

$$= \frac{\cancel{\sqrt{n}}\left(1+\frac{1}{\sqrt{n}}\right)}{\cancel{\sqrt{n}}\left(\sqrt{1+\frac{1}{n}}+\frac{1}{\sqrt{n}}\right)}$$

$$= \frac{1 + \frac{1}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}}+\frac{1}{\sqrt{n}}} < 1$$

$$= < 1$$

c) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{n}\left(1+\frac{1}{n}\right)}$

$= 0$

So, series is convergent.

(United technical College UTC)
pre board

7b. State and prove the D'Alembert ratio test. Test the series is absolutely Convergent or not. (May be asked)

$$1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}}$$

Let $\sum u_n$ be a series with positive term and suppose that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$$

IMP
rule

Then

a. if $\lambda < 1$ then series is convergen

b. If $\lambda > 1$ the series is divergeny

c. If $\lambda = 1$ then test "Should further test".

Solution.

Given series, $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n\sqrt{n}}$

Comparing the given equation with

$\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ then we get

$$u_n = \frac{1}{n\sqrt{n}}$$

Here

a) $u_n = \frac{1}{n\sqrt{n}} \geq 0$ For all $n \geq 1$

b) $\frac{u_{n+1}}{u_n}$ must be < 1

$$= \frac{1}{(n+1)(\sqrt{n+1})} \times n\sqrt{n}$$

$$= \frac{1}{n(1+\frac{1}{n})(\sqrt{1+\frac{1}{n}})\sqrt{n}} \times n\sqrt{n}$$

$$= \frac{1}{(1+\frac{1}{n})(\sqrt{1+\frac{1}{n}})} < 1$$

$$\Rightarrow u_{n+1} < u_n$$

c) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0$

Here it is convergence by alternative series test so it is absolutely convergence.

c) a Test the convergence or divergence

Soln:

Given series,

$$\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \dots$$

Here $u_n = \dfrac{2n-1}{n(n+1)(n+2)}$

$$= \frac{n\left(2-\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\times n\left(1+\frac{2}{n}\right)}$$

$$= \frac{1}{n^2}\,\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\times\left(1+\frac{2}{n}\right)}$$

$v_n = \frac{1}{n^2}$ is $p > 1$ so it is convergent by p-series test.

Now using limit comparision test

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \frac{\frac{1}{n^2}\,\frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\times\left(1+\frac{2}{n}\right)}}{\frac{1}{n^2}}$$

$$= \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 2 \text{ finite}$$

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2b) Solution:

Q.N State cayley Hamilton theorem and use it to find inverse of matrix. (8)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

→ cayley Hamilton theorem state "Every square matrix statisfy its own characteristics equation $(A - \lambda I) = 0$ if we put value of A in λ of eqn $|A - \lambda I|$ it it satisfy then it is verified.

Trick: $|A - \lambda I| = 0$ — step1:

(it is eqn: must satisfy A) — step2:

(Find inverse)

Solution:

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A - \lambda I =$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} (1-\lambda) & 0 & 1 \\ 1 & (1-\lambda) & 0 \\ 1 & 0 & (2-\lambda) \end{bmatrix}$$

We know,

characteristic eqn:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} (1-\lambda) & 0 & 1 \\ 1 & (1-\lambda) & 0 \\ 1 & 0 & (2-\lambda) \end{vmatrix} = 0$$

$$(1-\lambda)\{(1-\lambda)(2-\lambda) - 0\} + 1(-(1-\lambda) - 0) = 0$$

$$(1-\lambda)\{(1-\lambda)(2-\lambda)\} + (-(1-\lambda)) = 0$$

$$(1-\lambda)\{(1-\lambda)(2-\lambda)\} - (1-\lambda) = 0$$

$$(1-\lambda)\{1(2-\lambda) - \lambda(2-\lambda)\} - 1 + \lambda = 0$$

$$(1-\lambda)\{2 - \lambda - 2\lambda + \lambda^2\} - 1 + \lambda = 0$$

$$(1-\lambda)(2 - 3\lambda + \lambda^2) - 1 + \lambda$$

$$\text{or } 2 - 3\lambda + \lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda = 0$$

$$\text{or, } \cancel{2 - 5}$$

$$\text{or, } 1 - 4\lambda + 4\lambda^2 - \lambda^3 = 0$$

To varify ~~this eqn:~~ cayley hamilton theorem

we have to show,

$$I - 4A + 4A^2 - A^3 = 0$$

$$= I - 4\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}^2 - \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}^3$$

$$= I - \begin{bmatrix} 4 & 0 & 4 \\ 4 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix} + 4\begin{bmatrix} 2 & 0 & 3 \\ 2 & 1 & 1 \\ 3 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 8 \\ 4 & 1 & 4 \\ 8 & 0 & 13 \end{bmatrix}$$

$$= I - \begin{bmatrix} 4 & 0 & 4 \\ 4 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 12 \\ 8 & 4 & 4 \\ 12 & 0 & 20 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 8 \\ 4 & 1 & 4 \\ 8 & 0 & 13 \end{bmatrix}$$

$$=$$

$$= I - \begin{bmatrix} 4 & 0 & 4 \\ 4 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 8-5 & 0-0 & 12-8 \\ 8-4 & 4-1 & 4-4 \\ 12-8 & 0-0 & 20-13 \end{bmatrix}$$

$$= I - \begin{bmatrix} 4+8-5 & 0-0-0 & 4+12-8 \\ 4+8-4 & 4+4-1 & 0+4-4 \\ 4+12-8 & 0-0-0 & 8+20-13 \end{bmatrix}$$

$$= I -$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 0$$

This verify caylehamollton theorem

Multiply bothside by A^{-1}

$$(I - 4A + 4A^2 - A^3) A^{-1} = A^{-1}(0)$$

or $A^{-1} - 4I + 4A - A^2 = 0$

$$A^{-1} = 4I - 4A + A^2$$

$$= 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}^2$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} - \begin{bmatrix} 4 & 0 & 4 \\ 4 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 3 \\ 2 & 1 & 1 \\ 3 & 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ -2 & 1 & 4 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ -2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

2 a) Soln:

Ans: Define linear dependent and independent

- In matrix form, if the number of variables (vectors) is same as rank of the matrix developed by vector then vector are called linear independent..

Let $V = R^3 = \{(x, y, z) : x, y, z \in R\}$ be the vector space over the field F

Then we wish to show $(1,1,0), (1,0,1)$ $(3,1,1)$ form a basis of $V = R^3$

Then,

$$\left|\begin{matrix} 1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}\right|$$

Apply $R_2 \to R_2 - R_1$

$$\sim \left|\begin{matrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{matrix}\right|$$

Applying $R_3 \to R_3 + R_2$

$$\sim \left|\begin{matrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{matrix}\right|$$

where,

Given, vector are 3 and the echelon ha 3 pivot (1 in first and 1 in second and 1 in 3rd) so it is linearly independent

or Rank of matrix = vector no.

So, dimension of vector space = 3

so it

Hence,
since, no of vector = dimension of vector space

so it is basis for R^3

Q.N) Verify cayley hamilton theorem of given matrix and find inverse of

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}$$

Solution:

Given

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}$$

$|A - \lambda I| = 0$ is characteristic equation.

$$A - \lambda I = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1 & 1 \\ 0 & -2-\lambda & 1 \\ -2 & -3 & -\lambda \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$(1-\lambda)\{(-2-\lambda)(-\lambda) - (-3\times 1)\} - 2\,(-1\times 1 - (1\times$$

$$(1-\lambda)\{(-2\lambda+\lambda^2)+3\}-2\{-1-(-2-\lambda)\}$$

$$0=(1-\lambda)\{(2\lambda+\lambda^2)+3\}-2(-1+2+\lambda)$$

$$0=(1-\lambda)\{(2\lambda+\lambda^2)+3\}-2(1+\lambda)$$

$0=$ 1

$$0=\cancel{2\lambda}+\lambda^2+3-2\lambda^2-\lambda^3-\underline{3\lambda}-2-\cancel{2\lambda}$$

or, $\cancel{\lambda^2 - 3\lambda -}$

or, $\lambda^2 \; 3\lambda$.

or $$1-\lambda^2-\lambda^3-3\lambda=0$$

$$\lambda^2+\lambda^3+3\lambda-1=0$$

To verify calye hamilton theorm

put A int λ

$$A^2+A^3+3A+I=0$$

$$\begin{bmatrix}1 & -1 & 1\\ 0 & -2 & 1\\ -2 & -3 & 0\end{bmatrix}^2+\begin{bmatrix}1 & -1 & 1\\ 0 & -2 & 1\\ -2 & -3 & 0\end{bmatrix}^3+3\begin{bmatrix}1 & -1 & 1\\ 0 & -2 & 1\\ -2 & -3 & 0\end{bmatrix}$$

$$-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & -2 \\ -2 & 8 & -5 \end{bmatrix} + \begin{bmatrix} -1 & 5 & -3 \\ 2 & 6 & -1 \\ 8 & 1 & 6 \end{bmatrix}$$

$$+ \begin{bmatrix} 3 & -3 & 3 \\ 0 & -6 & 3 \\ -6 & -9 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 0$$

To find inverse of matrix multiply both side by A^{-1}

$$A^{-1}(A^2 + A^3 + 3A - I) = A^{-1}(0)$$

$$A + A^2 + 3I - A^{-1} = 0$$

$$A^{-1} = A + A^2 + 3I$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix} + \begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}^2 + 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & -2 \\ -2 & 8 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix}$$

2a) Define Eigen value and Eigen vector of the square matrix A. Find the Eigen value and corresponding Eigen vector of the square matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

→ Let A be a n×n square matrix then. For any non zero column vector X if $(AX = \lambda X)$ then λ is called Eigen value of A and X is called corresponding Eigen vector.

Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

The characteristics equation is

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$= 2-\lambda \{(2-\lambda)(1-\lambda)\} - 1((1-\lambda))$

$= 2-\lambda \{(2-\lambda)(1-\lambda)\} - (1-\lambda)$

$= 2-\lambda (2-2\lambda-\lambda+\lambda^2) - (1-\lambda)$

$= 2-\lambda (2-3\lambda+\lambda^2) - (1-\lambda)$

$= 4-6\lambda+2\lambda^2-2\lambda+3\lambda^2-\lambda^3-1+\lambda$

$0 = 3 - 7\lambda + 5\lambda^2 - \lambda^3$

$\lambda = 1, \text{ and } 3$

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be eigen vector column

~~At $\lambda = 1$~~

$AX = \lambda X$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 2x+y+z \\ x+2y+z \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (1)$$

At ~~x~~ $\lambda = 1$

From $-①$

$$\begin{bmatrix} 2x+y+z \\ x+2y+z \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$z = z$ | $x+2y+z = y$ $-①$

$x+y+z = 0$

$y+z = -2x$

$2x+y+z = x$

$\cancel{2x - 2x = x}$

$\cancel{x = 0}$

$x+y+z = 0$

$x - 2x = 0$

$-x = 0$

$x = 0$

$x+y+z = 0$

$y = -x-z$

$= 0-z$

$= -z$

$$\begin{bmatrix} 0 \\ z \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ z \\ z \end{bmatrix}$$

At $x = 3$

$$\begin{bmatrix} 2x + y + z \\ x + 2y + z \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore$

$2x + y + z = 3x$	$x + 2y + z = 3y$
$x = y + z$	$y = x + z$
	$y = y + z + z$
	$2z = 0$
	$\therefore z = 0$

$x = y$

$y = x$

$z = 0$

$$\therefore X = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix}$$

$\therefore$ at $\lambda = 1 \Rightarrow X = \begin{bmatrix} 0 \\ z \\ z \end{bmatrix}$ and

At $\lambda = 3 \Rightarrow X = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix}$

(8 marks or 2.5 marks ($7_{/10}$) or 4 marks

Linear transformation or not

To be Linear: $T(au_1 + bu_2) = aTu_1 + bTu_2$

Q:N

$T: R^2 \to R^2$ defined

$T(x, y) = (x, -2y)$ check linear or not.

Solution

$u_1 = (x_1, y_1)$ and

$u_2 = (x_2, y_2)$

$$au_1 + bu_2 = a(x_1, y_1) + b(x_2, y_2)$$
$$= (ax_1, ay_1) + (bx_2, by_2)$$
$$= (ax_1 + bx_2, ay_1 + by_2)$$

$$T_{au_1 + bu_2} = T(ax_1 + bx_2, ay_1 + by_2) \quad \{T(x, y) = (x, -2y)\}$$
$$= (ax_1 + bx_2, -2(ay_1 + by_2))$$

$$aTu_1 + bTu_2 = aT(x_1, y_1) + bT(x_2, y_2)$$
$$= a(x_1, -2y_1) + b(x_2, -2y_2)$$
$$= (ax_1, -2y_1a) + (bx_2, -b2y_2)$$
$$= (ax_1 + bx_2, -a2y_1 - b2y_2)$$
$$= (ax_1 + bx_2, -2(ay_1 + by_2))$$

$\therefore T_{au_1 + bu_2} = aTu_1 + bTu_2$ so it is Linear