Date :
Day :

Unit : 1
Complex Analysis

Complex numbers & Functions

$z = x + iy$ is a complex number

Real part of $z = Re(z) = x$
Imaginary part of $z = Im(z) = y$

where i is imaginary unit $i^2 = -1$.

Sum, product, quotient of complex numbers

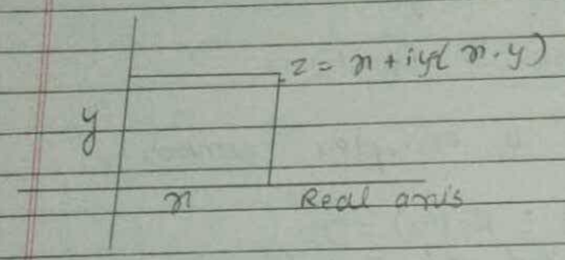
let,
$z_1 = a + ib$, $z_2 = c + id$ be two complex numbers,
then,

$$z_1 + z_2 = (a+ib) + (c+id) = (a+c) + i(b+d)$$

$$z_1 . z_2 = (a+ib).(c+id) = ac + aid + ibc + i^2 bd$$
$$= (ac - bd) + i(ad + bc)$$

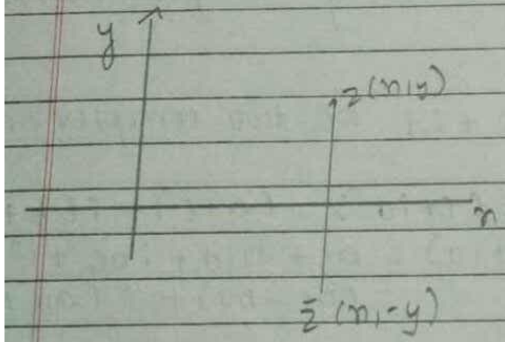
$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+id}{c+id} \times \frac{c-id}{c-id}$$
$$= \frac{ac - iad + ibc + i^2 bd}{c^2 - i^2 d^2}$$
$$= \frac{(ac+bd)}{c^2+d^2} + i \frac{(bc-ad)}{c^2+d^2}$$

Geometric representation



Conjugate of complex numbers

If $z = x+iy$, it conjugate $\bar{z} = x-iy$



Properties of conjugate

$\frac{z+\bar{z}}{2} = Re(z)$

$\frac{z-\bar{z}}{2} = Im(z)$

$\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

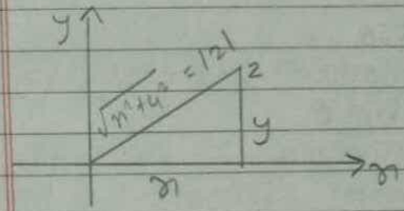
$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Modulus of a complex number

If $z = x+iy$ is a complex number it's modulus is,

$|z| = \sqrt{x^2+y^2}$. $(|x + iy|)$

$= \sqrt{(Re.z)^2 + (Im.z)^2}$



Properties

1. $|z_1 z_2| = |z_1||z_2|$

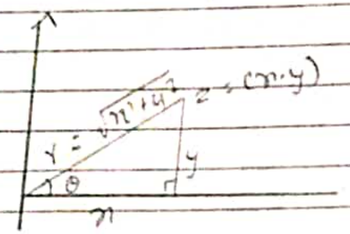
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $|z_2| \neq 0$

3. $|z| = |\bar{z}|$

4. $|z|^2 = z\bar{z}$

5. $|z_1 + z_1| \leq |z_1| + |z_2|$ (Triangle inequalities)

Polar form of a complex number



$\frac{x}{r} = \cos\theta \Rightarrow x = r\cos\theta$

$\frac{y}{r} = \sin\theta \Rightarrow y = r\sin\theta$

$z = x+iy = r\cos\theta + ir\sin\theta$
$= r(\cos\theta + \sin\theta)$

where,

$\frac{\sin\theta}{\cos\theta} = \frac{y/r}{x/r} = \frac{y}{x}$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\frac{y}{x}$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$
$= r^2(\cos^2\theta + \sin^2\theta)$

$x^2+y^2 = r^2$
$r = \sqrt{x^2+y^2}$

Multiplication & division in polar form

If, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then,

$z_1 z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

De-Moivre's theorem

If, n is positive integer or negative integer or a fraction then,

$[r\cos\theta + \sin\theta]^n = r^n(\cos n\theta + \sin n\theta)$

nth root of a complex number

$[r(\cos\theta + i\sin\theta)]^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta + k.360}{n}\right) + i\sin\left(\frac{\theta + k.360}{n}\right)\right]$

where, $k = 0, 1, 2, 3, ----, n-1$
$= 2\pi$

Example

Change $2-2\sqrt{3}i$ into polar form.

Here,

$2-2\sqrt{3}i = x+iy$

$x = 2$, $y = 2\sqrt{3}$

Then polar form is

$r(\cos\theta + i\sin\theta)$

$180-\theta$	θ
$180+\theta$	$(+,-)$ $360-\theta$

$r = \sqrt{x^2+y^2}$

$= \sqrt{2^2+(-2\sqrt{3})^2}$

$= \sqrt{4+12}$

$= 4$

$\theta = \tan^{-1}\left(-\frac{2\sqrt{3}}{2}\right)$

$= \tan^{-1}(-\sqrt{3})$

$= 360-60$

$= 300$

$= 2\pi - \pi/3$

$= 5\pi/3$

Required polar form

$r(\cos\theta + i\sin\theta) = 4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$= 4(\cos 300 + i\sin 300)$

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Complex function:

Let $z = x+iy$ and $w = u+iv$ be two complex numbers in region R of complex plane then $f: R \rightarrow R$ defined by $f(z) = w$ is called complex function. The set R is called domain of function f.

Here, $w = f(z) = u+iv = u(x,y) + iv(x,y)$

Limit and Continuity

A function $f(z)$ is said to have limit l if for given $\varepsilon > 0$ there exist a positive number δ such that $|f(z) - l| < \varepsilon$ for all $|z - z_0| < \delta$

In this case, $\lim_{z \to z_0} f(z) = l$.

A function $f(z)$ is said to be continous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

A function is said to be continous in a domain D if $f(z)$ is continous at all the point of D

Differentiability of complex function

A function $f(z)$ is said to be differentiable at a point z_0 if $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$ exists.

i.e.

$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{\Delta z} \quad \text{where, } z = z_0 + \Delta z$$
$$\text{or, } \Delta z = z - z_0$$

Analytic function

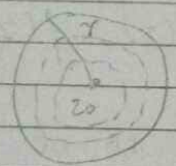
Let $f(z)$ be a complex valued function defined on a domain D. The function $f(z)$ is said to be analytic in D if $f(z)$ is differentiable at all points of D.

In particular,

$f(z)$ is analytic at point z if there is open disk about z on which $f(z)$ is differentiable.

Open circular disk

The set of all z such that $|z - z_0| < r$ is called open circular disk of z_0 of radius r.



Note:

A function is analytic at a point if its derivative exist in some neighborhood of that point.

Cauchy-Riemann equations (C-R equations)

If $f(z) = u(x, y) + iv(x, y)$ is analytic in domain D then the partial derivatives u_x, v_x, u_y, v_y exists and satisfy

$$u_x = v_y, \quad u_y = -v_x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Note:

1. The cauchy Riemann equations are only the necessary conditions for the function to be analytic.
2. Sufficient condition:
   If the four partial derivatives u_x, u_y, v_x, v_y are continous and satisfy cauchy Riemann equations.

Cauchy Riemann equations in polar form

Let $w = f(z)$ be analytic in region D and let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

so, $w = f(re^{i\theta}) = u + iv$

then, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

are known as cauchy Riemann equations in polar form.

Examples

1. Show that $f(z) = z^3$ is analytic for all z.

Soln.
we have,

$f(z) = z^3$

$u + iv = (x+iy)^3$ [$\because z = x+iy$, $f(z) = u+iv$]

$u + iv = x^3 + 3x^2 \cdot iy + 3x(iy)^2 + (iy)^3$

$u + iv = x^3 + 3x^2 y i - 3xy^2 - iy^3$

$u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3)$

$u = x^3 - 3xy^2$ & $v = 3x^2y - y^3$

Diff. partially w.r. to x & y.

$u_x = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$ $\quad v_x = 6xy$

$u_y = \frac{\partial u}{\partial y} = -6xy$ $\quad v_y = 3x^2 - 3y^2$

Here, the above 4 partial derivatives are continuous and satisfy $u_x = v_y$ & $u_y = -v_x$

$\therefore f(z)$ is analytic for all z.

2. Is $f(z) = z\bar{z}$ is analytic for all z?

soln.
we have,

$f(z) = z\bar{z}$ ——— (1)

we know,

$f(z) = u + iv$

$z = x + iy$

$\bar{z} = x - iy$

from (1)

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$u + iv = (x+iy)(x-iy)$

$u + iv = x^2 - i^2y^2$

$u + iv = x^2 + y^2$ [$\because i^2 = -1$]

$\Rightarrow u = x^2 + y^2$ & $v = 0$

Diff. partially w.r. to x & y,

$u_x = 2x$ $\quad v_x = 0$

$u_y = 2y$ $\quad v_y = 0$

Here,

$u_x \neq v_y$

& $u_y \neq -v_x$

$\therefore f(z)$ doesnot satisfy C-R equations.

So, $f(z)$ is not analytic.

Example

1. Test the analyticity of the function $f(z) = \frac{Re(z)}{Im(z)}$

soln;
we have,

$f(z) = \frac{Re(z)}{Im(z)}$

$u + iv = \frac{x}{y}$

$u = \frac{x}{y}$ and $v = 0$

Differentiating partially,

$u_x = \frac{1}{y}$ and $v_y = 0$

$u_y = -\frac{1}{y^2}$ and $v_x = 0$

Here, $u_x \neq v_y$ and $u_y \neq -v_x$
So, $f(z)$ doesnot satisfy C-R equations.
$\therefore$ $f(z)$ is not analytic function.

2. Check the analyticity of $f(z) = \ln z$

Soln;

$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

now,

$f(z) = u + iv$

$\ln z = u + iv$

$\ln(re^{i\theta}) = u + iv$

$\ln r + \ln e^{i\theta} = u + iv$

$\ln r + i\theta = u + iv$ $[\because \ln e^x = x]$

$\Rightarrow u = \ln r$

$\frac{\partial u}{\partial r} = \frac{1}{r}$ & $\frac{\partial v}{\partial r} = 0$

$\frac{\partial u}{\partial \theta} = 0$ $\frac{\partial v}{\partial \theta} = 1$

Here,

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ & $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ (for $r \neq 0$)

$\therefore$ $f(z)$ satisfy C-R eqns (in polar form)

Hence $f(z)$ is analytic except at origin.

Laplace's equation

An equation in the form $\nabla^2\phi = 0$ or $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ or $\phi_{xx} + \phi_{yy} = 0$ is called Laplace's equation.

Theorem:

If $f(z) = u + iv$ is analytic in domain D then u and v satisfies laplace's equation.

i.e. $u_{xx} + u_{yy} = 0$ & $v_{xx} + v_{yy} = 0$

Proof:

let $f(z) = u + iv$ be analytic function then it satisfy C-R eqns.

$u_x = v_y$ and $u_y = -v_x$

Diff. partially,

$u_{xx} = v_{yx}$ & $u_{yy} = -v_{xy}$

$\Rightarrow u_{xx} + u_{yy} = v_{yx} - v_{xy}$ $[\because v_{xy} = v_{yx}]$
$= 0$

Similarly,

$v_{xx} + v_{yy} = 0$

Proved!-

Harmonic Function

Any function $f(x, y)$ which has continuous partial derivatives of 1st and 2nd order and satisfies laplace's equation is called a Harmonic function.

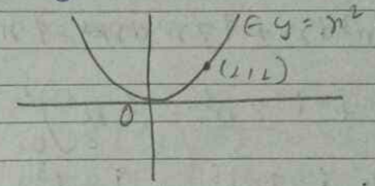
If $f(z) = u + iv$ is analytic then both u and v are harmonic. In this case we say that v is harmonic conjugate of u in the domain.

$$= \left(\frac{3^2}{2} - \frac{1^2}{2}\right) + 3i(2-1)$$

$$= \frac{9}{2} - \frac{1}{2} + 3i$$

$$\int_C Re(z)dz = \frac{9-1}{2} + 3i = 4+3i$$

3. $\int_C \bar{z}dz$ where C is a curve from 0 (origin) along the parabola $y = x^2$ to $1+i$.



Sol^n;

$$\int_C \bar{z}\,dz = \int_C (x-iy)\,d(x+iy)$$

$$= \int_C (x+iy)(dx+idy)$$

$$= \int_C (xdx + ixdy - iydx - i^2 ydy)$$

$$= \int_C (xdx + ydy) + i(xdy - ydx)$$

Since, curve is $y = x^2$

$$\frac{dy}{dx} = 2x$$

$$dy = 2xdx$$

and x run from 0 to 1 (according to the question)

Now,

$$\int_c \bar{z}\,dz = \int_0^1 \left[(x\,dx + x^2 . 2x\,dx) + i(x^2 x\,dx - x^2 dx) \right]$$

$$= \int_0^1 x\,dx + 2x^3 dx + i 2x^2 dx - i x^2 dx$$

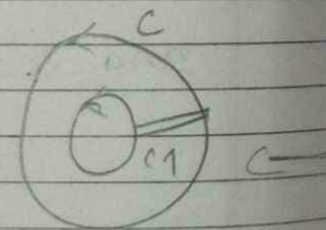
$$= \left[\frac{x^2}{2} + \frac{2x^4}{4} + i\frac{2x^3}{3} - i\frac{x^3}{3} \right]_0^1$$

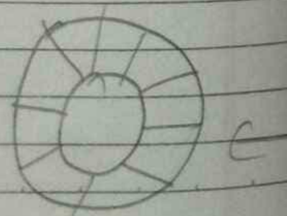
$$= \frac{1^2}{2} + \frac{1^4}{2} + i\frac{2 \cdot 1^3}{3} - i\frac{1^3}{3}$$

$$= \frac{1}{2} + \frac{1}{2} + i \cdot \frac{2}{3} - i \cdot \frac{1}{3}$$

$$= 1 + i\frac{1}{3}$$

$$= 1 + \frac{i}{3}$$





Simply connected domain

A domain D is called simply connected if any curve c which lies in D is closed without having any passout of this domain.

Cauchy's integral theorem:

If $f(z)$ is analytic in a simply connected domain D and $f'(z)$ is continuous in D then $\int_c f(z)dz = 0$.

Cauchy's integral formula:

let $f(z)$ be analytic in a simply connected domain D, then for any point z_0 in D,

$$f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z - z_0} dz \quad \Big/ \text{ as } \int \frac{f(z)}{z - z_0} = 2\pi i f(z_0)$$

where c is any simply closed path that encloses z_0.

Extension to the multiply connected domain.

let D be a doubly connected domain bounded by two simple closed curve C and C_1 and let $f(z)$ be

$$f(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \int_{c_1} \frac{f(z)}{z - z_0} dz$$

Note: A domain which is not simply connected is called multiply connected.

Example: annulus (domain betn two concentric circles.

2k8

Theorem:

If the harmonic functions u and v satisfy C-R equations then $u+iv$ is analytic function.

Examples:

1. Check whether the function $u = \sin x \cosh y$ is harmonic or not. If yes, find the correspon-ding analytic function $f(z) = u+iv$.

Soln;

We have,

$u = \sin x \cosh y$

Diff. partially,

$u_x = \cos x \cosh y$, $u_y = \sin x \sinh y$

$u_{xx} = -\sin x \cosh y$, $u_{yy} = \sin x \cosh y$

Here,

$u_{xx} + u_{yy} = 0$ (Laplace's eqn)

So, u is harmonic function.

Suppose $f(z) = u+iv$ is analytic function.

So that v is harmonic conjugate of u.

now,

$f(z)$ must satisfy C-R eqns.

$u_x = v_y$ and $u_y = -v_x$.

$\Rightarrow \cos x \cosh y = v_y$

$\Rightarrow v_y = \cos x \cosh y$

Integrating w.r. to y,

$v = \cos x \sinh y + h(x)$ [$\because$ constant may contain]

Diff. partially w.r. to x,

$v_x = -\sin x \sinh y + h'(x)$

$-u_y = -\sin x \sinh y + h'(x)$ $\left[\because \begin{array}{l} u_y = -v_x \\ v_x = -u_y \end{array}\right]$

$-\sin x \sinh y = -\sin x \sinh y + h'(x)$

$\Rightarrow h'(x) = 0$

Integrating,

$h(x) = c$

now ① becomes

$v = \cos x \sinh y + c$ is required harmonic conjugate of u.

Reqd. analytic function is

$f(z) = u + iv$

$= \sin x \cosh y + i(\cos x \sinh y + c)$

2. Check whether the function $u = y^3 - 3x^2 y$ is harmonic or not. If yes, find the corresponding analytic function $f(z) = u+iv$.

Soln;

We have,

$u = y^3 - 3x^2 y$

Diff. partially

$u_x = -6xy$ & $u_y = 3y^2 - 3x^2$

$u_{xx} = -6y$ $u_{yy} = 6y$

Here,

$u_{xx} + u_{yy} = 0$ (Laplace's eqn)

So u is harmonic function.

Suppose $f(z) = u+iv$ is analytic function so that v is harmonic conjugate of u.

now, $f(z)$ must satisfy C-R eqns,

$u_x = v_y$ and $u_y = -v_x$

$u_x = v_y$

$-6xy = v_y$

$v_y = -6xy$

$i^2 = -1$ $\quad i^3 = -i$

$i^4 = 1$

Integrating w.r. to y

$V = -\frac{6xy^2}{2} + h(x) = -3xy^2 + h(x)$ —(1)

[constant may contain x]

Diff. partially w.r. to x,

$V_x = -3y^2 + h'(x)$

$-U_y = -3y^2 + h'(x)$

$-(3y^2 - 3x^2) = -3y^2 + h'(x)$ $\quad [\because U_y = -V_x,\ V_x = -U_y]$

$-3y^2 + 3x^2 = -3y^2 + h'(x)$

$h'(x) = 3x^2$

Integrating,

$h(x) = \frac{3x^3}{3} + C = x^3 + C$

now (1) becomes,

$V = -3xy^2 + x^3 + C$ is required harmonic conjugate of u.

Req. analytic function is

$f(z) = u + iV$

$= y^3 - 3x^2y + i(-3xy^2 + x^3 + C)$

To express $f(z)$ in terms of z.

$f(z) = y^3 - 3x^2y - 3i\,xy^2 + ix^3 + ic$

$= i^4y^3 + 3x^2y\,i^2 + 3i^3xy^2 + ix^3 + ic$

$= i(i^3y^3 + 3x^2iy + 3xy^2i^2 + x^3) + ic$

$= i[(iy)^3 + 3(iy)x^2 + 3(iy)^2x + (x)^3] + ic$

$= i[(iy + x)^3] + ic$

$= i(x + iy)^3 + ic$

$= iz^3 + ic$ (when the corresponding func in terms of z).

For harmonic $\quad$ C-R eqns

$u_{xx} + u_{yy} = 0 \quad u_x = v_y \quad u_y = -v_x \quad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

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Examples:

P4. Given that $u = x^2 - y^2$ and $V = -\frac{y}{x^2+y^2}$. Prove that both u and v are harmonic function but $u + iv$ is not analytic function of z.

Soln,

$U = x^2 - y^2$

Diff. partially,

$U_x = 2x, \quad U_y = -2y$

$U_{xx} = 2, \quad U_{yy} = -2$

So,

$U_{xx} + U_{yy} = 2 + (-2) = 0$

$\therefore$ u is harmonic function.

Also we have,

$V = -\frac{y}{x^2+y^2}$

Diff. partially,

$V_x = -y(-1)(x^2+y^2)^{-2} \cdot 2x$

$V_x = \frac{2xy}{(x^2+y^2)^2}$

$V_{xx} = \frac{(x^2+y^2)^2 \cdot 2y - 2xy \cdot 2(x^2+y^2)}{(x^2+y^2)^4}$

$= \frac{2y(x^2+y^2)[x^2+y^2 - 4x^2]}{(x^2+y^2)^4}$

Also,

$V_y = -y(-1)(x^2+y^2)^{-2} \cdot 2y$

$V_y = \frac{2y^2}{(x^2+y^2)^2}$

$$V_{yy} = \frac{(x^2+y^2)(-1)-(-y).2y}{(x^2+y^2)^2}$$

$$V_{yy} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$V_{yy} = \frac{(x^2+y^2)^2.2y-(y^2-x^2).2(x^2+y^2).2y}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)[x^2+y^2-2(y^2-x^2)]}{(x^2+y^2)^4}$$

$$= \frac{2y[x^2+y^2-2y^2+2x^2]}{(x^2+y^2)^3}$$

$$V_{yy} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

now,

$$U_{xx}+V_{yy} = \frac{2y}{(x^2+y^2)^3}[y^2-3x^2+3x^2-y^2] = 0$$

$\therefore$ V is harmonic function.

Here,

$$U_x = 2x \ \& \ V_y = \frac{y^2-x^2}{(x^2+y^2)^2} \Rightarrow U_x \neq V_y$$

so C-R eqn is not satisfied. so $u+iv$ is not an analytic function.

5. Is the function $u = \frac{x}{x^2+y^2}$ harmonic?

If yes, find a corresponding analytic function $f(z) = u+iv$.

Soln,

$$u = \frac{x}{x^2+y^2}$$

$$u_x = \frac{(x^2+y^2)1.-x.2x}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$u_x = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_y = x(-1)(x^2+y^2)^{-2}.2y$$

$$= -\frac{2xy}{(x^2+y^2)^2}$$

Again,

$$u_{xx} = \frac{(x^2+y^2)^2.(-2x)-(y^2-x^2)2(x^2+y^2).2x}{(x^2+y^2)^4}$$

$$= \frac{2x(x^2+y^2)[-(x^2+y^2)-2(y^2-x^2)]}{(x^2+y^2)^4}$$

$$u_{xx} = \frac{2x}{(x^2+y^2)^3}(-x^2-y^2-2y^2+2x^2)$$

$$u_{xx} = \frac{2x(x^2-3y^2)}{(x^2+y^2)^3}$$

$$u_{yy} = \frac{2x}{(x^2+y^2)^3}(-x^2-y^2+4y^2)$$

$$u_{yy} = \frac{2x}{(x^2+y^2)^3}(3y^2-x^2) \quad \text{--- (2)}$$

from (1) & (2), we get,

$$u_{xx}+u_{yy} = 0$$

$\therefore$ u is harmonic function.

$= \left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$

let $f(z) = u+iv$ be analytic function.
So that it satisfies C-R eqns.

$u_x = v_y$ & $u_y = -v_x$

$v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\left[\frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right]\right] =$

$v_y = -\frac{\partial}{\partial y}\left[-\frac{y}{x^2+y^2}\right]$

Integrating,

$v = \frac{-y}{x^2+y^2} + h(x)$

Diff. partially w.r.to x,

$v_x = -y(-1)(x^2+y^2)^{-2} \cdot 2x + h'(x)$

$v_x = \frac{2xy}{(x^2+y^2)^2} + h'(x)$

$-u_y = \frac{2xy}{(x^2+y^2)^2} + h'(x)$

$\frac{2xy}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2} + h'(x)$

$h'(x) = 0$

Int;

$h(x) = c$

from (3),

$v = \frac{-y}{x^2+y^2} + c$ is

reqd. harmonic conjugate of u.

corresponding analytic function is

$f(z) = u+iv$

$= \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2} + c\right)$

$|z| = \sqrt{x^2+y^2}$

$z = x+iy$ $\quad |z|^2 = z\bar{z}$

$\bar{z} = x-iy$

$= \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} + ic$

$= \frac{x-iy}{x^2+y^2} + ic$

$= \frac{\bar{z}}{|z|^2} + ic$

$= \frac{\bar{z}}{z\bar{z}} + ic$

$f(z) = \frac{1}{z} + ic$ is reqd. analytic function,

6. Determine a and b such that the given function $u = ax^3 + by^3$ is harmonic and find a conjugate harmonic.

Soln;

$u = ax^3 + by^3$

$u_x = 3ax^2$, $u_y = 3by^2$

$u_{xx} = 6ax$, $u_{yy} = 6by$

As u is harmonic, we must have,

$u_{xx} + u_{yy} = 0$

$6ax + 6by = 0$

$ax + by = 0$

$a = b = 0$ [$\because x$ & y are variables]

let $u+iv$ be analytic function.

By C-R eqns.

$u_x = v_y$ & $u_y = -v_x$

$\Rightarrow v_y = u_x$

$\Rightarrow v_y = 3ax^2$

$v_y = 0$ [$\because a = 0$]

Derivative of an analytic function

let $f(z)$ be analytic in a simply connected domain D. Then, for any point z_0 in D and any simple closed path ~~closed~~ enclosing z_0.

$$f'(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^2}\,dz$$

In general,

$$f^n(z_0) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^3}\,dz$$

$$f'''(z_0) = \frac{3!}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^4}\,dz$$

and so on.

Examples:

1. Integrate $f(z)$ counterclockwise (ccw) around the unit circle indicating whether cauchy integral theorem applies.

a. $f(z) = e^{-z^2}$

Here, $f(z)$ is differentiable everywhere being exponential function.

So, $f(z)$ is analytic.

$\therefore$ By cauchy integral theorem,

$$\int_c f(z)\,dz = \int_c e^{-z^2}\,dz = 0$$

b. $f(z) = \frac{1}{(z)^2}$

Here, f does not exist for $z=0$ which lies inside unit circle.

So, $f(z)$ is not analytic in the unit circle.

So, cauchy integral theorem is not applicable.

Now, by cauchy integral formula,

$$\int_c f(z)\,dz = 2\pi i f'(z_0) = 2\pi i f'(0) \quad \text{--- (1)}$$

comparing $\int_c \frac{f(z)}{(z-z_0)}$ with $\int_c \frac{dz}{z^2}$.

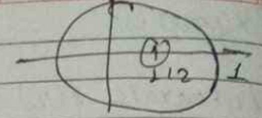
we get, $z_0 = 0$ and $f(z) = 1$

Differentiating $f'(z) = 0 \Rightarrow f'(0) = 0$

from (1)

$$\int_c f(z)\,dz = 2\pi i \times 0 = 0$$

c. $f(z) = \frac{1}{2z-1} = \frac{1}{2(z-1/2)}$



Here,
$f(z)$ doesnot exist at $z = 1/2$ which lies inside unit circle.
So, $f(z)$ is not analytic in unit circle.
So, cauchy's integral theorem is not applicable.

By cauchy's integral formula,

$$\int_c f(z)\,dz = \frac{1}{2}\int \frac{1}{(z-1/2)}\,dz$$
$$= \frac{1}{2} \times 2\pi i$$

Note:
$$\int_c (z-z_0)^n dz = \begin{cases} 2\pi i & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}$$

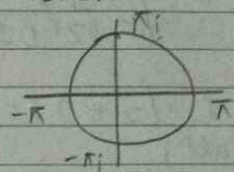
d. $f(z) = \frac{1}{z^4+1.1}$

Here,
$f(z)$ doesnot exist for $\pm\sqrt[4]{1.1} > 1$.
So, $f(z)$ is analytic in unit circle.
$\therefore f(z)$ is analytic in unit circle.
By cauchy integral theorem,

$$\int_c f(z)dz = \int_c \frac{1}{z^4+1}\,dz = 0$$

2. Evaluate
$\int_c \frac{dz}{z-3i}$ where C is the circle $|z| = \pi$.

Hence, $f(z) = \frac{1}{z-3i}$ doesn't exist at $z = 3i$.
which lies inside given circle $|z| = \pi$



So, $f(z)$ is not analytic in C.

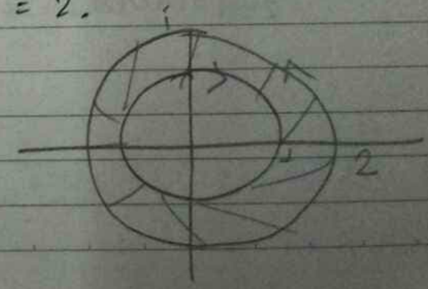
By cauchy integral formula,

$$\int_c \frac{dz}{z-3i} = 2\pi i \quad [\because n = -1]$$

Evaluate

1. $\int_c \frac{e^2}{z}\,dz$ where C consists $|z| = 2$ counter-clockwise and $|z| = 1$ clockwise.

Soln,
$f(z) = \frac{e^2}{z}$ doesn't exist at $z = 0$ which doesn't lie in the region bounded by $|z| = 1$ & $|z| = 2$.



By cauchy integral theorem,

$$\int_c f(z)dz = \int_c \frac{e^z}{z} dz = 0$$

2. $\int_c \frac{dz}{z^2+1}$ where c is the circle (ccw)

i. $|z+i|=1$ ii). $|z-i| = 1$

Soln;

$$f(z) = \frac{1}{z+1}$$

$$\int_c f(z)dz = \int_c \frac{1}{z^2+1} dz$$

$$= \int_c \frac{1}{z^2 - i^2} dz$$

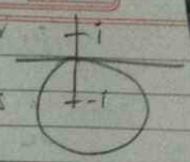
$$= \int_c \frac{1}{(z+i)(z-i)} dz$$

$$\frac{1}{z+i} - \frac{1}{z-i} = \frac{z-i-z-i}{(z+i)(z-i)} = \frac{-2i}{(z+i)(z-i)}$$

$$= \frac{1}{2i} \int_c \frac{1}{z-i} - \frac{1}{z+i} dz \quad \text{--- (1)}$$

The given function doesn't exist at $z=i$ and $z=-i$.

i. Given circle is $|z+i|=1$. $|z-z_0|=r$, z_0 is centre, r = radius



The point $z=i$ doesn't lie in the circle $|z+i| = 1$.
So, by cauchy integral theorem,

$$\int_c \frac{dz}{z-i} = 0.$$

and the point $z=-i$ lies in the circle $|z+i|=1$.

So by cauchy integral formula,

$$\int_c \frac{dz}{z+i} = 2\pi i$$

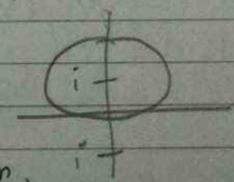
now, eqn (1) becomes,

$$\int_c f(z)dz = \frac{1}{2i}\left[\int_c \frac{1}{z-i} dz - \int_c \frac{1}{z+i} dz\right]$$

$$= \frac{1}{2i}(0 - 2\pi i)$$

$$= -\pi$$

ii. Given circle is $|z-i| = 1$.



The point $z=-i$ doesn't lie in the circle $|z-i|= 1$.
So, by cauchy integral theorem,

$$\int_c \frac{dz}{z+i} = 0$$

and the point $z=i$ lies in the circle $|z-i|= 1$.

So, by cauchy integral formula,

$$\int_c \frac{dz}{z+i} = 2\pi i$$

now,

eqn (1) becomes,

$$\int_c f(z)\,dz = \frac{1}{2i}\left[\int_c \frac{1}{z-i}\,dz - \int_c \frac{1}{z+i}\,dz\right]$$

$$= \frac{1}{2i}\left[2\pi i - 0\right]$$

$$= \pi$$

Cauchy integral formula for derivative of analytic function.

$$f'(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^2}\,dz$$

In general,

$$f^n(z_0) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

Examples.

1. Integrate $\frac{z^2}{z^4-1}$ ccw around the circle.

a. $|z+1| = 1$ b) $|z| = 0.9$

Soln,

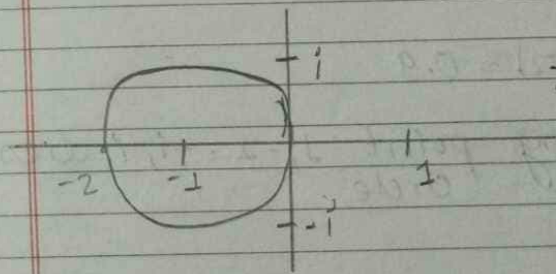
$$\int_c \frac{z^2}{z^4-1}\,dz = \int_c \frac{z^2\,dz}{(z^2-1)(z^2+1)}$$

$$= \int_c \frac{z^2\,dz}{(z+1)(z-1)(z^2-i^2)}$$

$$= \int_c \frac{z^2\,dz}{(z+1)(z-1)(z+i)(z-i)}$$

The function doesn't exist at $z = -1, 1, -i, i$.

i. Given circle is $|z+1| = 1$



when $n = 0$

$$f(z_0) = \frac{0!}{2\pi i}\int_c \frac{f(z)}{z-z_0^{0+1}}$$

$$f(z) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-z_0)}\,dz$$

$$\int_c \frac{f(z)}{(z-z_0)}\,dz = 2\pi i f(z_0)$$

only $z = -1$ lies in circle,

so,

$$\int_c \frac{z^2}{(z-1)(z+1)(z-i)(z+i)}\,dz$$

$$= \int_C \frac{\frac{z^2}{(z-1)(z-i)(z+i)}}{(z+1)} \, dz$$

$= 2\pi i f(z_0)$

$= 2\pi i f(-1)$

$= 2\pi i \dfrac{(-1)^2}{(-1-1)(-1+i)(-1-i)}$

$= 2\pi i \dfrac{1}{(-2)[-(1+i)](i-1)}$

$= \pi i \times \dfrac{1}{i^2 - 1^2}$

$= \pi i \times \dfrac{1}{-1-1}$

$= \dfrac{\pi i}{-2}$

$= -\dfrac{\pi i}{2}$

ii. We have $|z| = 0.9$

Since, no any point $1, -1, -i, i$ lies inside the circle.

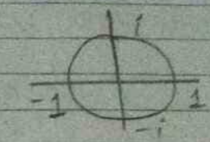
$|z| = 0.9$

So the func is analytic in the given circle.

By Cauchy integral theorem,

$$\int_C \frac{z^2}{z - z_0} \, dz = 0$$

2. Integrate the given function ccw around the unit circle.



a. $\displaystyle\int_C \frac{z^3}{2z-i} \, dz = \int_C \frac{z^3}{2\left(z - \frac{i}{2}\right)} \, dz$

The function doesn't exist at $z = \frac{i}{2}$ which lies inside the unit circle.

By cauchy integral formula,

$$\int_C \frac{z^3}{2\left(z - \frac{i}{2}\right)} \, dz = \frac{1}{2} \int \frac{z^3}{z - i/2} \, dz$$

$= \dfrac{1}{2} \cdot 2\pi i f(z_0)$

$= \pi i f\left(\dfrac{i}{2}\right)$

$= \pi i \left(\dfrac{i}{2}\right)^3$

$= \dfrac{\pi i^4}{2^3}$

$= \dfrac{\pi i^4}{8}$

$= \dfrac{\pi}{8}$

Integrating,
$v = h(x)$ ——— ①
$v_x = h'(x)$
$-u_y = h'(x)$ $[\because u_y = -v_x]$
$-3by^2 = h'(x)$
$0 = h'(x)$ $[\because b = 0]$
Integrating,
$h(x) = c$
from ①
$v = c$ which is harmonic conjugate of u.

Analytic function (If asked)

$u + iv$
$= ax^3 + by^3 + ic$

Complex ~~function~~ Integration

Line integral:
An integral of a complex variable over a curve C in the complex plane is called the complex line integral. It is denoted by
$\int_C f(z)\,dz$.

Reduction to line integrals of Real function.

Let $f(z) = u + iv$ where $z = x + iy$
$dz = dx + i\,dy$

then,
$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$
$$= \int_C [u\,dx + iu\,dy + iv\,dx + i^2 v\,dy]$$
$$= \int_C (u\,dx - v\,dy) + i(u\,dy + v\,dx)$$

Note: If $z = z(t) = x(t) + iy(t)$ $(a \le t \le b)$
then,
$$\int_C f(z)\,dz = \int_a^b f[z(t)]\,z'(t)\,dt$$

Integrate the given function ccw around the unit circle.

1. $\oint \frac{\cosh 3z}{2z}\, dz$

$= \frac{1}{2}\oint \frac{\cosh 3z}{(z-0)}\, dz$

Here, the function doesn't exist at $z=0$. which lies inside the unit circle.

So the function is not analytic in given region.

By cauchy integral formula, $\cosh x = \frac{e^x + e^{-x}}{2}$

$\frac{1}{2}\int_c \frac{\cosh 3z}{(z-0)}\, dz = \frac{1}{2}\, 2\pi i f(0)$ $\qquad \cosh 0 = \frac{e^0 + e^{-0}}{2}$

Here,

$f(z) = \cosh 3z$

so, $f(0) = \cosh 3 0 = 1$

$\int_c \frac{\cosh 3z}{2z}\, dz = \frac{1}{2} \times 2\pi i f(0)$

$= \frac{1}{2} \times 2\pi i \times 1$

$= \pi i$

P4 Integrate the given function over the given contour c clockwise or as indicate.

1. $\frac{1}{z^2+4}$ where C is the ellipse

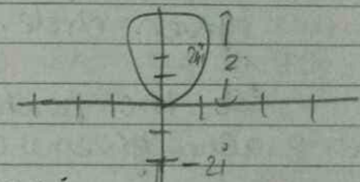
$4x^2 + (y-2)^2 = 4$ $\qquad \frac{x^2}{1} + \frac{(y-2)^2}{4} = 1$

soln,

$\int_c \frac{1}{z^2+4}\, dz$ $\qquad \frac{(x-0)^2}{1} + \frac{(y-2)^2}{2^2} = 1$

$= \int_c \frac{1}{z^2 - 4i^2}\, dz \; [\because i^2 = -1]$ $\qquad$ minor axis $2a = 1$

$= \int_c \frac{1}{(z+2i)(z-2i)}\, dz$ $\qquad$ major axis $2b = 2$



Here, the function doesn't exist at $z = 2i, -2i$.

In given ellipse, point $z = 2i$ lies inside but $z = -2i$ doesn't lies inside it.

By cauchy integral formula,

$\int_c \frac{1/(z+2i)}{z-2i}\, dz = 2\pi i f(2i)$

here,

$f(z) = \frac{1}{z+2i} \Rightarrow f(2i) = \frac{1}{2i+2i} = \frac{1}{4i}$

$\therefore \int_c \frac{1}{z^2+4}\, dz = 2\pi i \times \frac{1}{4i}$

$= \frac{\pi}{2}$

Taylor series

If $f(z)$ is analytic in a with centre at a and radius r then at each point inside c, the series

$$f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3 + \cdots = \frac{f^n(a)}{n!}(z-a)^n$$ converges to $f(z)$.

so,

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n \quad \text{--- (1) where,}$$

$$a_n=\frac{f^n(a)}{n!}$$

This eqn (1) is called Taylor series of $f(z)$ about $z=a$.

In eqn (1), if we put $a=0$ then, Taylor series of $f(z)$ reduces to Maclaurin's series of $f(z)$ and is denoted by

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \text{ where } a_n=\frac{f^n(0)}{n!}$$

i.e, $$f(z)=f(0)+f'(0).z+f''(0)\frac{z^2}{2!}+\frac{f'''}{3!}(0)z^3+ \cdots + \frac{f^n}{n!}(0)z^n$$

note:

1. $$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n = 1+z+z^2+\cdots \text{ if } |z|<1.$$

2. $$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots$$

3. $$\cos z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}$$

4. $$\sin z=\sum_{n=0}^{\infty}\frac{(-1)^n(z)^{2n+1}}{(2n+1)!}$$

5. $$\cosh z=\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$$

6. $$\sinh z=\sum_{n=0}^{\infty}\frac{(z)^{2n+1}}{(2n+1)!}$$

7. $$\log_e(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots$$

8. $$\log_e\left(\frac{1}{1-z}\right)=z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots$$

Examples:

Find the Maclaurin series of.

1. $f(z)=\cos 2z^2$

we know,

$$\cos z=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}$$

so, $$f(z)=\cos(2z^2)=\sum_{n=0}^{\infty}\frac{(-1)^n(2z^2)^{2n}}{(2n)!}$$

$$=1-\frac{(2z^2)^2}{2!}+\frac{(2z^2)^4}{4!}-\cdots$$

$$=1-2z^4+\frac{2}{3}z^8-\cdots$$

is required series.

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} (z)^n \text{ for } |z|<1$$

2. $f(z) = \frac{1}{z+3i}$

$= \frac{1}{3i+z}$

$= \frac{1}{3i\left(1+\frac{z}{3i}\right)}$ $\quad |z|<1$

$= \frac{1}{3i\left[1-\left(-\frac{z}{3i}\right)\right]}$ $\quad \left|\frac{z}{3i}\right|<1$

$= \frac{1}{3i} \sum_{n=0}^{\infty} \left(-\frac{z}{3i}\right)^n$ when $\left|\frac{-z}{3i}\right| < 1$.

$= \frac{1}{3i}\left[1+\left(-\frac{z}{3i}\right)+\left(-\frac{z}{3i}\right)^2+\left(\frac{z}{3i}\right)^3+\cdots\right]$

$= \frac{1}{3i}\left[1-\frac{z}{3i}-\frac{z^2}{9}-\frac{z^3}{27}i+\cdots\right]$

is required series.

Find the mechanism series of

1. $f(z) = \frac{z+2}{1-z^2}$

$= \frac{z+2}{(1-z)(1+z)}$

let $\frac{z+2}{(1-z)(1+z)} = \frac{A}{1-z} + \frac{B}{1+z}$ — (1)

$\Rightarrow z+2 = A(1+z) + B(1-z)$

put $z=1$ then $1+2 = A(1+1)$

$A = 3/2$

put $z=-1$ then $-1+2 = B(1-(-1))$

$1 = 2B$

$B = 1/2$

now from (1)

$\frac{z+2}{(1-z)(1+z)} = \frac{3}{2(1-z)} + \frac{1}{2(1+z)}$

$= \frac{3}{2}\left(\frac{1}{1-z}\right) + \frac{1}{2}\left(\frac{1}{1-(-z)}\right)$

$= \frac{3}{2}\sum_{n=0}^{\infty} (z)^n + \frac{1}{2}\sum_{n=0}^{\infty} (-z)^n$

$= \frac{3}{2}\left(1+z+z^2+z^3+\cdots\right) +$

$\frac{1}{2}\left(1-z+z^2-z^3+\cdots\right)$

$= \frac{1}{2}\left(3+3z+3z^2+3z^3+\cdots 1-z+z^2-z^3+\cdots\right)$

$= \frac{1}{2}\left(4+2z+4z^2+2z^3+\cdots\right)$

$= 2+z+2z^2+z^3+\cdots$ is required expansion.

2. $f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$

let $\frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$ — (1)

$\Rightarrow 1 = A(z-i) + B(z+i)$

put $z = -i$ then $1 = A(-i-i) \Rightarrow A = \frac{-1}{2i}$

put $z = i$ then $1 = B(i+i) \Rightarrow B = \frac{1}{2i}$

now from (1)

$$\frac{1}{z^2+1} = \frac{1}{2i(z+i)} + \frac{1}{2i(z-i)}$$

$$= -\frac{1}{2i}\left[\frac{1}{z+i} - \frac{1}{z-i}\right]$$

$$= -\frac{1}{2i}\left[\frac{1}{i+z} + \frac{1}{i-z}\right]$$

$$= \frac{-1}{2i}\left[\frac{1}{i\left(1+\frac{z}{i}\right)} + \frac{1}{i\left(1-\frac{z}{i}\right)}\right]$$

$$= \frac{-1}{2i \times i}\left[\frac{1}{1-\left(-\frac{z}{i}\right)} + \frac{1}{\left[1-\left(\frac{z}{i}\right)\right]}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(-\frac{z}{i}\right)^n + \sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n\right]$$

(where $|z| < i$ then $\left|\frac{z}{i}\right| < 1$)

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{i}\right)^n + \sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n\right]$$

Taylor series expansion.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$$

$$\frac{1}{2}\sum_{n=\text{even}} 2 \cdot \left(\frac{z}{i}\right)^n$$

$$= \left(\frac{z}{i}\right)^0 + \left(\frac{z}{i}\right)^2 + \left(\frac{z}{i}\right)^4 + \cdots$$

$= 1 - z^2 + z^4 + \cdots$

is required expansion.

Q. Find the Taylor series expansion of the given function at a specified points

1. $f(z) = \frac{1}{z}$ at $z = 2$.

Here,

$$f(z) = \frac{1}{z}$$

$$f'(z) = \frac{-1}{z^2} = \frac{(-1)^1 (1)!}{z^{1+1}}$$

$$f''(z) = \frac{2}{z^3} = \frac{(-1)^2 2!}{z^{2+1}}$$

$$f'''(z) = \frac{-6}{z^4} = \frac{(-1)^3 3!}{(z)^{3+1}}$$

so, $f^n(z) = \frac{(-1)^n n!}{z^{n+1}}$

now, The Taylor series expansion is,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$$

at $a = 2$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(2)}{n!}(z-2)^n$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(2)}{n!}(z-2)^n$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(2)^{n+1} \cdot n!}(z-2)^n$$

Properties :

1. $\int_c [f_1(z) + f_2(z)]\,dz = \int_c f_1(z)\,dz + \int_{c'} f_2(z)\,dz$

2. $\int_c k\,f(z)\,dz = k\int_c f(z)\,dz$

3. $\int_c [k_1 f_1(z) + k_2 f_2(z)]\,dz = k_1\int_c f_1(z)\,dz + k_2\int_c f_2(z)\,dz$

4. $\int_{z_1}^{z_2} f(z)\,dz = -\int_{z_1}^{z_2} f(z)\,dz$

5. If c is composed of two parts c_1 and c_2 then,

$$\int_c f(z)\,dz = \int_{c_1} f_1(z)\,dz + \int_{c_2} f_2(z)\,dz$$

Integrate the given integral in the given path.

1. $\int_c z^2\,dz$ where C is the shortest path joining $1+i$ and $2(1+2i)$.

Sol^n;

$$\int_c z^2\,dz = \int_c (x+iy)^2\,d(x+iy)$$

$$= \int_c (x^2 + 2xyi + i^2y^2)(dx + i\,dy)$$

$$= \int_c (x^2 - y^2 + 2xyi)(dx - i\,dy) \quad [\because i^2 = -1]$$

$$= \int_c (x^2-y^2)\,dx + (x^2-y^2)\,i\,dy + 2xy\,i\,dx + 2xyi^2\,dy$$

$$= \int_c [\{(x^2-y^2)\,dx - 2xy\,dy\}] - i[(x^2-y^2)\,dy + 2ixy\,dx]. \quad \text{—— (1)}$$

Given path c is shortest path joining $1+i = (1,1)$ and $2+4i = (2,4)$

i.e.

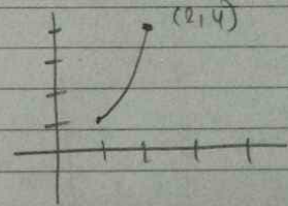
$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1) \quad [\text{straight line}]$$

$$y - y_1 = \frac{4-1}{2-1}(x-1)$$

$$y - 1 = 3(x-1)$$

$$y - 1 = 3x - 3$$

$$y = 3x - 2 \quad \text{—— (2)}$$



diff. $\frac{dy}{dx} = 3 \Rightarrow dy = 3dx$

from (1) (using (2) & (3))

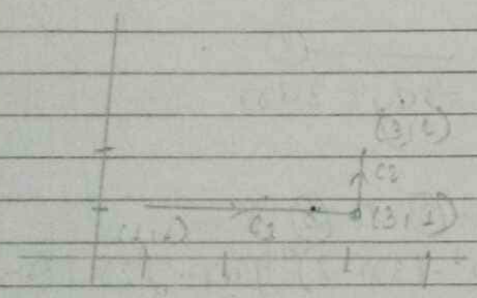
$$\int_c z^2\,dz = \int [\{x^2 - (3x-2)^2\}\,dx - \{2x(3x-2).3dx\}] + i[\{x^2-(3x-2)^2\}3dx + 2x(3x-2)\,dx]$$

$$= \int_{x=1}^{2} (x^2 - 9x^2 + 12 - 4)\,dx - (18x^2 - 12x)\,dx] + i[(x^2 - 9x^2 + 12x - 4).3dx + (6x^2 - 4x)\,dx]$$

$$= \int_1^2 (x^3 - 9x^2 + 12x - 4 - 18x^2 - 12x)\,dx +$$
$$i \int_1^2 3(x^3 - 9x^2 + 12x - 4 + 6x^2 - 4x)\,dx$$

$$= \int_1^2 (-26x^2 + 24x - 4)\,dx + i\int_1^2 (-6x^2 + 24x - 12)\,dx$$
$$= \left(-\frac{26x^3}{3} + \frac{24x^2}{2} - 4x\right) + i\left(-\frac{6x^3}{3} + \frac{24x^2}{2} - 12x\right)_1^2$$
$$= \frac{-86}{3} + 6i$$

2. $\int_c \mathrm{Re}\,z\,dz$ where c is horizontally from $1+i$ and $3+i$ then vertically to $3+2i$,



Sol^n;

Here, the curve is from $(1,1)$ to $(3,2)$ through $(3,2)$.

Let c_1 and c_2 be two horizontal & vertical paths.

now,

$$\int_c \mathrm{Re}\,z\,dz = \int_c x\,d(x+iy).$$
$$= \int_c x\,[dx + i\,dy]$$

For c_1, y is constant so $dy = 0$ & x runs from 1 to 3.

For c_2, x is constant so $dx = 0$ & y runs from 1 to 2.

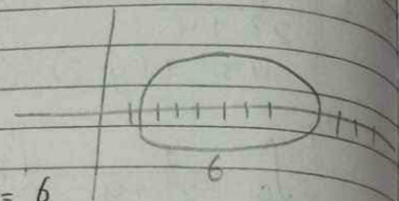
now,

$$\int_c \mathrm{Re}\,z\,dz = \int_c x\,[dx + i\,dy]$$
$$= \int_{c_1} x\,[dx + i\,dy] + \int_{c_2} x\,[dx + i\,dy]$$
$$= \int_{x=1}^3 x(dx + i.0) + \int_{y=1}^2 x(0 + i\,dy)$$
$$= \int_1^3 x\,dx + i\int_1^2 3\,dy$$
$$= \left[\frac{x^2}{2}\right]_1^3 + 3i\,[y]_1^2$$

2. $\frac{\log(z-1)}{z-6}$ where C is the circle $|z-6|=4$

Soln.)

$\int_c \log \frac{(z-1)}{z-6}$



function doesn't exist $z=6$ which lies inside the given circle.

So, the function is not analytic in the given circle.

So the function is not analytic in the given circle.

By cauchy integral formula,

$$\int_c \frac{\log(z-1)}{z-6} dz = 2\pi i f(6) \quad \text{—— (1)}$$

$$f(z) = \log(z-1)$$
$$f(6) = \log(6-1)$$
$$= \log(5)$$

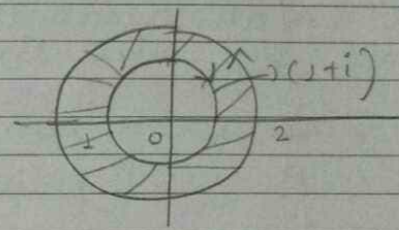
from (1)

$$\int_c \frac{\log(z-1)}{z-6} dz = 2\pi i \log 5 \,//$$

3. $\frac{e^{z^2}}{z^2(z-1-i)}$ where c consist of $|z|=2$ counterclockwise and $|z|=1$ clockwise.

Soln.)

$$\int_c \frac{e^{z^2}}{z^2(z-1-i)}$$



$$= \int_c \frac{e^{z^2}}{z^2\{z-(1+i)\}} dz$$

The function doesn't exist at $z=0$ and $z=1+i$ in which $z=0$ lies outside the region and $z=1+i$ lies inside the region.
So the function is not analytic in given region.
By cauchy integral formula,

$$\int_c \frac{\frac{e^{z^2}}{z^2}}{z-(1+i)} dz = 2\pi i f(1+i) \quad \text{—— (1)}$$

Here,

$$f(z) = \frac{e^{z^2}}{z^2}$$

$$f(1+i) = \frac{e^{(1+i)^2}}{(1+i)^2} = \frac{e^{2i}}{2i}$$

from (i).

$$\int \frac{e^{2z}/z^2}{(z-(1+i))} = 2\pi i f(1+i)$$

$$= 2\pi i \cdot \frac{e^{2i}}{2i}$$

$$= \pi e^{2i}$$

IV. Integrate ccw around the unit circle.

a. $\frac{\sinh 2z}{z^4}$

The given function is not analytic in $z=0$ which lies in the unit circle.

By cauchy integral formula for derivative.

$$\int_C \frac{\sinh 2z}{(z-0)^4} dz = \frac{2\pi i}{3!} f^3(0)$$

Here,

$f(z) = \sinh 2z$

$f'(z) = 2\cosh 2z$

$f''(z) = 4\sinh 2z$

$f'''(z) = 8\cosh 2z$

$\Rightarrow f'''(0) = 8\cosh 2.0$

$= 8\cosh(0)$

$= 8.1$

$= 8$

now,

$$\int_C \frac{\sinh 2z}{(z-0)^4} dz = \frac{2\pi i}{3!} \times 8$$

$$= \frac{4}{3} \cdot 2\pi i$$

$$= \frac{8\pi i}{3}$$

b. $\frac{\cos \pi z}{z^{2n}} = \frac{\cos \pi z}{(z-0)^{2n}}$

This function is not analytic in $z=0$ which lies in the unit circle.

By cauchy integral formula for derivative

$$\int_c \frac{\cos \pi z}{(z-0)^{2n}} = \frac{2\pi i}{B(2n-1)!} f^{2n-1}(0)$$

Here, $f(z) = \cos \pi z$

For $f^{2n-1}(z)$, $(2n-1)$ is odd so the derivative of $\cos \pi z$ include sine function and $\sin 0 = 0$.

So,

$f^{2n-1}(0) = 0 = 0$

$$\int_c \frac{\cos \pi z}{(z-0)^{2n}} = \frac{2\pi i}{(2n-1)!} \cdot 0$$

$= 0$.

Date: 10 May 2024
Day: Friday

Examples

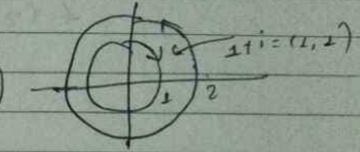
Integrate $f(z) = \frac{2z^2-3}{z(z-1-i)^2}$ around c where c consist of $|z| = 2$ ccw and $|z| = 1$ cw.

Here,

$f(z)$ is analytic in c except at $z=0$ and $z = 1+i$ where $z = 1+i$ lies in the region but $z=0$ doesnot lie inside the region.

So,

$$\int_c \frac{\left(\frac{2z^3-3}{z}\right)}{(z-1-i)^2} dz = \frac{2\pi i}{1!} f'(1+i) - (1)$$



Here,

$f(z) = \frac{2z^3-3}{z}$

$f(z) = 2z^2 - \frac{3}{z}$

$f'(z) = 4z + \frac{3}{z^2}$

$f'(1+i) = 4(1+i) + \frac{3}{(1+i)^2}$

$= 4 + 4i + \frac{3}{2i} \times \frac{i}{i}$

$= 4 + 4i - \frac{3i}{2}$

$= \frac{8+8i-3i}{2}$

$= \frac{8+5i}{2}$

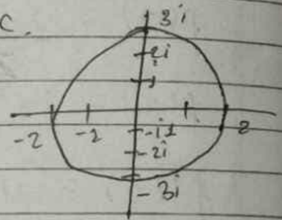
from (i),

$$\int_c \frac{2z^3-3}{z(z-1-i)^2}\,dz = \frac{2\pi i}{1!} \times \frac{8+5i}{2}$$
$$= \pi i\,(8+5i)$$
$$= 8\pi i - 5\pi$$

2. Integrate $\int_c \frac{z^4}{(z+1)(z-i)^2}\,dz$ where $c = 9x^2 + 4y^2 = 36$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1$$
$$\Rightarrow \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1$$

Here, given func is analytic except at $z = -1$ & $z = i$, both lies in c.



Let,

$$\frac{z^4}{(z+1)(z-i)^2} = \frac{A}{z+1} + \frac{B}{(z-i)} + \frac{C}{(z-i)^2}$$

$$\Rightarrow z^4 = A(z-i)^2 + B(z+i)(z-i) + C(z+1)$$

put $z=-1$ then, $(-1)^4 = A(-1-i)^2$
$$= A[-(1+i)]^2$$
$$= A(1+i)^2$$
$$= A.\,2i$$
$$A = \frac{1}{2i}$$

put $z = i$ then $(i)^4 = C\,(i+1)$
$$1 = C\,(1+i)$$
$$C = \frac{1}{1+i}$$

$\left[\int \frac{dz}{z-z_0} = 2\pi i\right]$

put $z=0$ then,
$$0^4 = A(0-i)^2 + B(0+1)(0-i) + C(0+1)$$
$$0 = A.\,i^2 - Bi + C$$
$$0 = \frac{1}{2i} \times i^2 - Bi + \frac{1}{1+i}$$
$$Bi = \frac{i}{2} + \frac{1}{1+i}$$
$$B = \frac{i}{2i} + \frac{1}{i(1+i)} \times \frac{i}{i}$$
$$B = \frac{1}{2} - \frac{i}{1+i}$$

now,

$$\int_c f(z)\,dz = \int_c \frac{z^4}{(z+1)(z-1)^2}\,dz$$
$$= \int_c \left(\frac{A}{z+1} + \frac{B}{z-i} + \frac{C}{(z-i)^2}\right)dz$$
$$= A\int_c \frac{dz}{z+1} + B\int_c \frac{dz}{z-i} + C\int \frac{dz}{(z-i)^2}$$

By cauchy integral formula,

$$A.\,2\pi i + B.\,2\pi i + C\,\frac{2\pi i}{2!}\,f'(i)$$

For third integral,
for 3rd integral, $f(z) = 1$
$$f'(z) = 0$$
$$f'(i) = 0$$

$$= A.\,2\pi i + B.\,2\pi i + C \times 2\pi i \times 0$$
$$= 2\pi i\,(A+B)$$
$$= 2\pi i\left(\frac{1}{2i} + \frac{1}{2} - \frac{i}{1+i}\right)$$

$$f(z) = \frac{(-1)^0(z-2)^0}{2^{0+1}} + \frac{(-1)^1(z-2)^1}{2^{1+1}} + \frac{(-1)^2(z-2)^2}{2^{2+1}} + \frac{(-1)^3(z-2)^3}{2^{3+1}} + \cdots$$

$$= \frac{1}{2} - \frac{(z-2)}{4} - \frac{(z-2)^2}{8} - \frac{(z-3)^3}{16} + \cdots$$

is req. expansion.

2. $f(z) = e^z$ at $z=a$.

Here,

$f(z) = e^z \Rightarrow f^n(z) = e^z$

The taylor series expansion of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$$

at $z=a$

$$f(z) = \sum_{n=0}^{\infty} \frac{e^a(z-a)^n}{n!}$$

$$= e^a \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!}$$

$$= e^a\left[\frac{(z-a)^0}{0!} + \frac{(z-a)^1}{1!} + \frac{(z-a)^2}{2!} + \frac{(z-a)^3}{3} + \cdots\right]$$

$$= e^a\left[1 + (z-a) + \frac{(z-a)^2}{2} + \frac{(z-a)^3}{6} + \cdots\right]$$

is reqd. expansion.

3. $f(z) = \sin z$ at $z = \pi/2$

we have,

$f(z) = \sin z$

$f'(z) = \cos z = \sin\left(z + \frac{\pi}{2}\right)$

$f''(z) = \cos\left(z + \frac{\pi}{2}\right) = \sin\left(z + \frac{\pi}{2} + \frac{\pi}{2}\right) = \sin\left(z + \frac{2\pi}{2}\right)$

$f'''(z) = \cos\left(z + \frac{2\pi}{2}\right) = \sin\left(z + \frac{2\pi}{2} + \frac{\pi}{2}\right) = \sin\left(z + \frac{3\pi}{2}\right)$

so, $f^n(z) = \sin\left(z + \frac{n\pi}{2}\right)$

$= \sin z \cos\frac{n\pi}{2} + \cos z \sin\frac{n\pi}{2}$

now,

$f^n\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2}\cos\frac{n\pi}{2} + \cos\frac{\pi}{2}\sin\frac{n\pi}{2}$

$f^n\left(\frac{\pi}{2}\right) = 1.\cos\frac{n\pi}{2} + 0.\sin\frac{n\pi}{2}$

$= \cos\frac{n\pi}{2}$

now,

Taylor series expansion of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(\pi/2).(z-\pi/2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\cos\frac{n\pi}{2}.(z-\pi/2)^n}{n!}$$

when $n =$ odd, $\cos\frac{n\pi}{2} = 0$

so,

$$f(z) = \sum_{n=\text{even}} \frac{\cos\frac{n\pi}{2}}{n!}\left(z - \frac{\pi}{2}\right)^n$$

$$= \frac{\cos 0}{0!}\left(z - \frac{\pi}{2}\right)^0 + \frac{\cos 2\pi/2}{2!}\left(z - \frac{\pi}{2}\right)^2 +$$

$$\cos\frac{4\pi}{2}\left(z-\frac{\pi}{2}\right)^4 + \dots$$

$$= 1 - \frac{(z-\pi/2)^2}{2!} + \frac{(z-\pi/2)^4}{4!} - \dots$$

4. $f(z) = e^{z^2-2z}$ at $z=1$

$$= e^{z^2-2z+1-1}$$
$$= e^{(z-1)^2-1}$$
$$= \frac{e^{(z-1)^2}}{e}$$

Taylor series of $f(z)$ is

$$f(z) = \frac{e^{(z-1)^2}}{e}$$
$$= \frac{1}{e} \cdot e^{(z-1)^2} \quad \text{--- (1)}$$

we know,

$$e^n = \sum_{n=0}^{\infty} \frac{n^n}{n!}$$

so from (1)

$$f(z) = \frac{1}{e} \sum_{n=0}^{\infty} \left[\frac{(z-1)^2)^n}{n!}\right]$$
$$= \frac{1}{e} \sum_{n=0}^{\infty} \frac{(z-1)^{2n}}{n!}$$
$$= \frac{1}{e}\left[\frac{(z-1)^{1\times 0}}{0!} + \frac{(z-1)^{2\times 1}}{1!} + \frac{(z-1)^{2\times 2}}{2!} + \frac{(z-1)^{1\times 3}}{3!} + \dots\right]$$
$$= \frac{1}{e}\left(1 + \frac{(z-1)^2}{1!} + \frac{(z-1)^4}{2!} + \frac{(z-1)^6}{3} + \dots\right)$$

is req. expansion.

Write the Taylor series expansion for

1. $f(z) = \frac{1}{1-z}$ in $|z| < \frac{1}{2}$.

Here,

$$|z| = \frac{1}{2} < 1.$$

so, $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

2. $f(z) = \frac{1}{z^2+2z+3}$ in $|z|<1$.

soln;

$$f(z) = \frac{1}{z^2+2z+3}$$
$$= \frac{1}{z^2+2 \cdot z \cdot 1 + (1)^2 + (-1)^2 + 3}$$
$$= \frac{1}{(z+1)^2+2}$$
$$= \frac{1}{(z+1)^2 - 2i^2}$$
$$= \frac{1}{(z+1)^2 - (\sqrt{2}i)^2}$$

$$= \frac{1}{(z+1+\sqrt{2}i)(z+1-\sqrt{2}i)} = \frac{A}{(z+1+\sqrt{2}i)} + \frac{B}{(z+1-\sqrt{2}i)} \quad \text{(1)}$$

$\Rightarrow 1 = A(z+1+\sqrt{2}i) + B(z+1+\sqrt{2}i)$

put $z = -1-\sqrt{2}i$ then, $1 = A(-1-\sqrt{2}i+1-\sqrt{2}i)$

$$1 = A(-2\sqrt{2}i)$$
$$A = \frac{-1}{2\sqrt{2}i}$$

put $z = -1+\sqrt{2}i$ then,

$$1 = B(-1+\sqrt{2}i+1+\sqrt{2}i)$$
$$1 = B(2.\sqrt{2}i)$$
$$B = \frac{1}{2\sqrt{2}i}$$

now from (1),

$$f(z) = \frac{-1}{2\sqrt{2}i\,(z+1+\sqrt{2}i)} + \frac{1}{2\sqrt{2}i(z+1-\sqrt{2}i)}$$

$$= \frac{-1}{2\sqrt{2}i}\left(\frac{1}{z+(1+\sqrt{2}i)} - \frac{1}{z+(1-\sqrt{2}i)}\right)$$

$$= \frac{-1}{2\sqrt{2}i}\left[\frac{1}{(1+\sqrt{2}i)\left(1+\frac{z}{1+\sqrt{2}i}\right)} - \frac{1}{(1-\sqrt{2}i)\left(1+\frac{z}{1-\sqrt{2}i}\right)}\right]$$

$$= \frac{-1}{2\sqrt{2}i}\left[\frac{1}{1+\sqrt{2}i}\sum_{n=0}^{\infty}\left(\frac{-z}{1+\sqrt{2}i}\right)^n - \frac{1}{1-\sqrt{2}i}\sum_{n=0}^{\infty}\left(\frac{-z}{1-\sqrt{2}i}\right)^n\right]$$

This expansion is valid only when $\left|\frac{z}{1+\sqrt{2}i}\right| < 1$

& $\left|\frac{z}{1-\sqrt{2}i}\right| < 1$.

or, $|z| < |1+\sqrt{2}i|$ & $|z| < |1-\sqrt{2}i|$

$|z| < \sqrt{1^2+(\sqrt{2})^2}$ & $|z| < \sqrt{1^2+(-\sqrt{2})^2}$

$|z| < \sqrt{3}$ & $|z| < \sqrt{3}$

So the expansion is valid for $|z| < 3$.

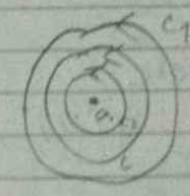
Laurent Series

If $f(z)$ is analytic in two concentric circles C_1 & C_2 with centre at 'a' and also in annular region R bounded by C_1 & C_2 then at any point z in R.

$f(z)$ can be expressed as a convergent series of positive and negative power of $(z-a)$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

where,



$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$\& \; b_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{-n+1}}\,dz$$

where c be any simple closed curve lying within the annular & encircling the inner boundary of region R.

Examples

1. Find the Laurent series that converges for $0<|z|<R$ of the functions.

a. $\frac{\cos z}{z^4}$

we know,

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

now,

$$\frac{\cos z}{z^4} = \frac{1}{z^4}(\cos z)$$

$$= \frac{1}{z^4}\sum_{n=0}^{\infty}\frac{(-1)^n (z)^{2n}}{(2n)!}$$

For pole, $z^3(z-2)=0$

$\Rightarrow z = 0, 2$

Here,

$z=0$ is pole of order 3.

& $z=2$ is pole of order 1.

now,

$$\underset{z=0}{\text{Res}}\, f(z) = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2}\left[z^3 f(z)\right]$$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2}\left[z^3 \frac{(z+1)}{z^3(z-2)}\right]$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left[\frac{(z-2)1-(z+1)1}{(z-2)^2}\right]$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left[\frac{z-2-z-1}{(z-2)^2}\right]$$

$$= \frac{-3}{2} \lim_{z\to 0} \frac{d}{dz}\left[(z-2)^{-2}\right]$$

$$= \frac{-3}{2} \lim_{z\to 0} (-2)\left[z-2\right]^{-3}.1$$

$$= 3. \lim_{z\to 0} \frac{1}{(z-2)^3}$$

$$= 3. \frac{1}{(0-2)^3} = \frac{3}{-8} = -3/8$$

and,

$$\underset{z=2}{\text{Res}}\, f(z) = \lim_{z\to 2} (z-2) f(z)$$

$$= \lim_{z\to 2} (z-2) \frac{(z+1)}{z^3(z-2)} = \frac{2+1}{2^3} = 3/8$$

Here, only $z=0$ is included in the region $|z| < 1/2$

By cauchy Residue theorem,

$$\int_c f(z)\,dz = 2\pi i \times \text{sum of residues}$$

$$= 2\pi i \times \left(-\frac{3}{8}\right)$$

$$= -\frac{3\pi i}{4}$$

3. $$\int_c \frac{1-\cos 2(z-3)}{(z-3)^3}\,dz$$

here, $f(z)$ has poles at $z=3$ of order 3.

Given circle (region) includes the point $z=3$,

now,

$$\underset{z=3}{\text{Res}}\, f(z) = \lim_{z\to 3} \frac{1}{2!} \frac{d^2}{dz^2}\left[(z-3)^3 f(z)\right]$$

$$= \lim_{z\to 3} \frac{1}{2} \frac{d^2}{dz^2}\left[(z-3)^3\left\{\frac{1-\cos^2(z-3)}{(z-3)^3}\right\}\right]$$

$$= \lim_{z\to 3} \frac{1}{2} \frac{d}{dz}\left[2\sin 2(z-3)\right]$$

$$= \lim_{z\to 3} \frac{1}{2}\left[4^2 \cos 2(z-3)\right]$$

$$= 2\cos 2(3-3)$$

$$= 2\cos 0$$

$$= 2\times 1 = 2$$

By cauchy Residue theorem,

$$\int_c f(z)\,dz = 2\pi i \times \text{sum of residue}$$

$$= 2\pi i \times 2$$

$$= 4\pi i$$

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4. Evaluate $\int \frac{4-3z}{z^2-z} dz$ when c is the curve which is counter clockwise simple closed path such that,

a. If 0 and 1 (poles) lies in c.
b. If 0 is inside and 1 outside c.
c. If 1 inside and 0 outside c.
d. If both outside c.

4 Soln; Q1 =

For poles; $z^2-z=0$

$z(z-1)=0$

$z=0, 1$ (simple order or order 1)

a. both inside c

$$\underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0} (z-0) f(z) = \lim_{z\to 0} \cdot \frac{4-3z}{z(z-1)}$$

$$= \frac{4-3.0}{0-1}$$

$$= -4$$

$$\underset{z=1}{\text{Res}}\, f(z) = \lim_{z\to 1} (z-1) f(z)$$

$$= \lim_{z\to 1} (z-1)\frac{4-3z}{z(z-1)}$$

$$= \frac{4-3\times 1}{1}$$

$$= 1$$

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By cauchy residue theorem,

$$\int_c f(z)dz = 2\pi i \times \text{sum of residue}$$
$$= 2\pi i \times (-4+1)$$
$$= 2\pi i \times (-3)$$
$$= -6\pi i$$

b. 0 is inside & 1 is outside c.

Here,

region doesnot include 1.

So, $\underset{z=0}{\text{Res}}\, f(z) = -4$ [from a]

By cauchy residue theorem,

$$\int_c f(z)dz = 2\pi i \times \text{sum of residue}$$
$$= 2\pi i \times (-4)$$
$$= -8\pi i$$

c. 1 is inside & 0 is outside c.

Soln;

c doesnot include 0

So,

$\underset{z=1}{\text{Res}}\, f(z) = 1$ [from (a)]

By C-R theorem,

$$\int_c f(z)dz = 2\pi i \times \text{sum of residue}$$
$$= 2\pi i \times 1$$
$$= 2\pi i$$

d. Both 0 & 1 outside C

Soln,

as 0 & 1 outside c, given function is analytic in c.

By cauchy integral theorem,

$$\int_c f(z)\,dz = \int_c \frac{4-3z}{z^2-z}\,dz = 0$$

Date: 31 May 2024
Day: Friday

Even $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)\,dx$

odd $\int_{-a}^{a} f(x)dx = 0$

Fourier integral and transform:

Fourier integral of $f(x)$

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

where, $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underset{even}{f(t)}\,\underset{even}{\cos\omega t}\,dt$

and, $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underset{even}{f(t)}\,\underset{odd}{\sin\omega t}\,dt$

Fourier cosine and sine integral of $f(x)$.

1. When $f(x)$ is even then $B(\omega)=0$ and $A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,dt$

then, $f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega$ is called Fourier cosine integral of $f(x)$.

2. When $f(x)$ is odd then $A(\omega)=0$ and $B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t\,dt$

then,

$f(x) = \int_0^\infty B(\omega)\sin\omega x\,dx$ is Fourier sine integral.

we observe the ...

$$= \frac{1}{z^4}\left[\frac{(-1)^0(z)^{2\times 0}}{(2\times 0)!} + \frac{(-1)^1(z)^{2\times 1}}{(2\times 1)!} + \frac{(-1)^2(z)^{2\times 2}}{(2\times 2)!} + \frac{(-1)^3(z)^{2\times 3}}{(2\times 3)!} + \cdots\right]$$

$$= \frac{1}{z^4}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right]$$

$$= \frac{1}{z^4} - \frac{1}{z^2 . 2!} + \frac{1}{4!} - \frac{z^2}{6!} + \cdots$$

6. $\frac{e^{z^2}}{z^3}$

Here,

$$f(z) = \frac{e^{z^2}}{z^3} = \frac{1}{z^3}(e^{z^2})$$

$$= \frac{1}{z^3}\sum_{n=0}^{\infty}\frac{(z^2)^n}{n!}$$

$$= \frac{1}{z^3}\sum_{n=0}^{\infty}\frac{z^{2n}}{n!}$$

$$= \frac{1}{z^3}\left(\frac{z^0}{0!} + \frac{z^{2\times 1}}{1!} + \frac{z^{2\times 2}}{2!} + \frac{z^{2\times 3}}{3!} + \cdots\right)$$

$$= \frac{1}{z^3}\left(1 + \frac{z^2}{1!} + \frac{z^4}{2!} + \frac{z^6}{3!} + \cdots\right)$$

$$= \frac{1}{z^3} + \frac{1}{z . 1!} + \frac{z}{2!} + \frac{1}{z^3 . 3!} + \cdots$$

2. Find the laurent series that converges for $0 < |z - z_0| < R$ of the function.

$$f(z) = \frac{1}{z^2+1} \quad \text{at } z_0 = i$$

soln.

$$f(z) = \frac{1}{z^2+1}$$

$$= \frac{1}{z^2 - i^2}$$

$$= \frac{1}{(z+i)(z-i)}$$

$$f(z) = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

$$f(z) = \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{(z-i)+2i}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{2i+(z-i)}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{2i\left(1 + \frac{z-i}{2i}\right)}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{2i}\sum_{n=0}^{\infty}\left\{-\frac{(z-i)}{2i}\right\}^n\right]$$

$$= \frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{2i}\left\{1 - \frac{(z-i)}{2i} + \frac{(z-i)^2}{(2i)^2} - \frac{(z-i)^3}{(2i)^3} + \cdots\right\}\right]$$

Another method,

$$f(z) = \frac{1}{z^2+1}$$

$$= \frac{1}{(z-i)(z+i)}$$

$$= \frac{1}{(z-i)}\frac{1}{(z+i)}$$

$$= \frac{1}{z-i}\left[\frac{1}{z-i+2i}\right]$$

$$= \frac{1}{z-i}\left[\frac{1}{2i\left(1+\frac{z-i}{2i}\right)}\right]$$

$$= \frac{1}{2i(z-i)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n$$

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Find Taylor and laurent series of

$f(z) = \frac{2z-3i}{z^2-3iz-2}$ in

a. $0<|z|<1$ b. $|z|>2$

Sol^n,

let $f(z) = \frac{2z-3i}{z^2-3iz+2i^2}$ $[\because -2 = +2i^2]$

$= \frac{2z-3i}{z^2-2iz-iz+2i^2}$

$= \frac{2z-3i}{z(z-2i)-i(z-2i)}$

$= \frac{2z-3i}{(z-2i)(z-i)}$

$= \frac{A}{z-2i} + \frac{B}{z-i}$ ——(1)

$\Rightarrow 2z-3i = A(z-i) + B(z-2i)$

put $z=2i$ then $2\times 2i - 3i = A(2i-i) \Rightarrow i \Rightarrow Ai \Rightarrow A=1$

put $z=i$ then $2i\times i - 3i = B(i-2i) \Rightarrow -i \Rightarrow -Bi \Rightarrow B=1$.

So from (1).

$f(z) = \frac{1}{z-2i} + \frac{1}{z-i}$

a. we have $0<|z|<1$.

$f(z) = \frac{1}{z-2i} + \frac{1}{z-i}$

$= \frac{1}{-(2i-z)} + \frac{1}{i-z}$

$= \frac{-1}{2i\left(1-\frac{z}{2i}\right)} - \frac{1}{i\left(1-\frac{z}{i}\right)}$

$\frac{1}{1-z} = \sum (z)^n$

Since $|z|<1$ we must have $\left|\frac{z}{2i}\right|<1$ & $\left|\frac{z}{i}\right|<1$.

$= -\frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{z}{2i}\right)^n - \frac{1}{i}\sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n$

which is reqd. Taylor series.

b. $|z|>2$

we have,

$f(z) = \frac{1}{z-2i} + \frac{1}{z-i}$

$= \frac{1}{z\left(1-\frac{2i}{z}\right)} + \frac{1}{z\left(1-\frac{i}{z}\right)}$

$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2i}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n$

which is reqd. laurent series. $\left[\begin{array}{l}\because |z|>2 \\ \frac{2}{z}<1 \\ \Rightarrow \frac{|2i|}{|z|}<1 \text{ \& } \frac{|i|}{|z|}<1\end{array}\right]$

Find the laurent series of

$f(z) = \frac{z^2-1}{z^2+5z+6}$ in the region $2<|z|<3$.

Sol^n,

$f(z) = \frac{z^2-1}{z^2+5z+6}$ By actual division.

$$\begin{array}{r} z^2+5z+6 \,\big)\, z^2-1 \,\big(\, 1 \\ \underline{z^2+5z+6} \\ -5z-7 \end{array}$$

$= 1 + \frac{-5z-7}{z^2+5z+6}$

$= 1 - \frac{(5z+7)}{(z+3)(z+2)}$

let $\dfrac{5z+7}{(z+3)(z+2)} = \dfrac{A}{z+3} + \dfrac{B}{z+2}$

$\Rightarrow 5z+7 = A(z+2) + B(z+3)$

put $z=-3$ then $5\times(-3)+7 = A(-3+2)$

$-8 = -A \Rightarrow A = 8$

put $z=-2$ then $5\times(-2)+7 = B(-2+3) \Rightarrow -3 = B$

now, from ①,

$$\frac{5z+7}{(z+3)(z+2)} = \frac{8}{z+3} - \frac{3}{z+2}$$

so,

$$f(z) = 1 - \left(\frac{8}{z+3} - \frac{3}{z+2}\right)$$

$$= 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

$$= 1 - \frac{8}{3+z} + \frac{3}{z+2}$$

$$= 1 - \frac{8}{3\left(1+\frac{z}{3}\right)} + \frac{3}{2\left(1+\frac{z}{2}\right)}$$

$$= 1 - \frac{8}{3}\sum\left(-\frac{z}{3}\right)^n + \frac{3}{2}\sum\left(-\frac{z}{2}\right)^n$$

$\left[\because 2 < |z| \Rightarrow \left|\frac{2}{z}\right| < 1\right.$

& $|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$

is required Laurent series of $f(z)$.

Find the Laurent series expansion of $f(z) = \dfrac{1}{z^2(1-z)}$ for.

a. $0 < |z| < 1$ b. $1 < |z| < 4$.

a. Here $f(z) = \dfrac{1}{z^2(1-z)}$

$$= \frac{1}{z^2}\left(\frac{1}{1-z}\right)$$

$$= \frac{1}{z^2}\sum_{n=0}^{\infty}(z)^n \quad [\because |z| < 1]$$

$$= \frac{1}{z^2}\left(1 + z + z^2 + z^3 + z^4 + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots$$

is reqd. Laurent series.

b. $f(z) = \dfrac{1}{z^2(1-z)}$ $(1 < |z| < 4)$

$$= -\frac{1}{z^2(z-1)} \qquad \frac{1}{|z|} < 1$$

$$= \frac{-1}{z^2 \cdot z\left(1 - \frac{1}{z}\right)}$$

$$= -\frac{1}{z^3}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \quad \left[\because \left|\frac{1}{z}\right| < 1\right]$$

$$= -\frac{1}{z^3}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

$$= -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \cdots$$

which is reqd. Laurent series.

Q. Find the laurent series expansion for

a. $f(z) = \dfrac{z}{(z-1)(z-3)}$ in $0 < |z-1| < 2$.

b. $\dfrac{e^{2z}}{(z-1)^3}$ in $|z-1| > 1$.

c. $\dfrac{1}{z-z^3}$ in $1 < |z+1| < 2$.

(a) Here,

$$f(z) = \frac{z}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$$\Rightarrow z = A(z-3) + B(z-1)$$

put $z = 1$ then $1 = A(1-3) \Rightarrow A = -\frac{1}{2}$

put $z = 3$ then $3 = B(3-1) \Rightarrow B = \frac{3}{2}$

So,

$$f(z) = \frac{-1}{2(z-1)} + \frac{3}{2(z-3)}$$

$$f(z) = -\frac{1}{2(z-1)} + \frac{3}{2\{(z-1)-2\}}$$

$$= \frac{-1}{2(z-1)} - \frac{3}{2[2-(z-1)]}$$

$$= \frac{-1}{2(z-1)} - \frac{3}{2 \cdot 2\left(1 - \frac{z-1}{2}\right)}$$

$$= \frac{-1}{2(z-1)} - \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n \quad \left[\because |z-1| < 2,\ \left|\frac{z-1}{2}\right| < 1\right]$$

is required laurent series of $f(z)$.

b. $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ in $|z-1| > 1$.

Soln:

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

$$= \frac{e^{2z-2+2}}{(z-1)^3}$$

$$= \frac{e^{2(z-1)+2}}{(z-1)^3}$$

$$= \frac{e^{2(z-1)} \cdot e^2}{(z-1)^3}$$

$$= \frac{e^2}{(z-1)^3} e^{2(z-1)}$$

$$= \frac{e^2}{(z-1)^3} \sum_{n=0}^{\infty} \frac{[2(z-1)]^n}{n!}$$

is reqd. laurent series.

c. $1 < |z+1| < 2$

$$\Rightarrow \left|\frac{1}{z+1}\right| < 1$$

$$\left|\frac{z+1}{2}\right| < 1$$

Day: Friday

Zeros and singularities of a function

Singularity

A function $f(z)$ is said to be a singular or has a singularity at a point $z=z_0$ if $f(z)$ is not analytic at $z=z_0$ but every neighbourhood of the point $z=z_0$ ~~but every neighbourhood of the p~~ contains points at which $f(z)$ is analytic.

Types of singularity

a. Isolated singularity

If $f(z)$ has only one singular point z_0 in the neighbourhood of $f(z)$.

b. Removable singularity

let $f(z)=\sum_{n=0} a_n(z-a)^n$ is analytic for

$|z-a|<R$ but not analytic at $z=a$, then, we call $f(z)$ has removable singularity at $z=a$.

c. Essential singularity

If $f(z)=\sum_{n=0}^{\infty} \frac{a_n}{(z-a)^n}$ then $f(z)$ is said to have essential singularity at $z=a$.

Pole:

If $f(z)=\sum_{n=0}^{\infty} \frac{a_n}{(z-a)^n}$ then $z=a$ is known as pole of $f(z)$.

If $m=1$ then the pole is called a simple pole.

- Zeros of analytic function

If $f(z)$ be a function of z analytic in domain D, A point $z=a$ is said to be zero of $f(z)$ if $f(a)=0$.

If $f'(a)=f''(a)=f'''(a)=\dots=f^{m-1}(a)=0$ but $f^n(a)\neq 0$ then the zero is said to be of order n.

If $f(a)=0$ and $f'(a)\neq 0$ then $z=a$ is called simple zero or zero of order.

Note:

1. If $f(a)=0$ then $f(z)$ has zero at $z=0$.
2. If $f(a)=\infty$ then $f(z)$ has singularity at $z=a$.
3. To find pole, we set denominator $=0$

Examples:

Determine the location and orders of zeros.

1. $\tan \pi z$

Here $f(z)=\tan \pi z$ | If $\tan\theta=\tan\alpha$

for zero, put $f(z)=0$ | then $\theta=n\pi+\alpha$

$\Rightarrow \tan \pi z=0$ | If $\cos\theta=\cos\alpha$

$\Rightarrow \tan \pi z=\tan 0$ | then $\theta=2n\pi\pm\alpha$

$\pi z=n\pi+0$ | If $\sin\theta=\sin\alpha$ then

Example:

1. Show that $\int_0^\infty \left[\frac{\cos xw + \sin xw}{1+w^2}\right] dw = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$

Soln,

Let $f(x) = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$

we know, the fourier integral of $f(x)$ is

$$f(x) = \int_0^\infty [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, dx$$

where, $A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\cos\omega t\, dt$

and

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\sin\omega t\, dt$$

now,

$$A(\omega) = \frac{1}{\pi}\left[\int_{-\infty}^0 0.\cos\omega t\,dt + \int_0^0 \frac{\pi}{2}\cos\omega t\,dt + \int_0^\infty \pi e^{-t}\cos\omega t\,dt\right]$$

$$= \frac{1}{\pi}\int_0^\infty \pi e^{-t}\cos\omega t\, dt$$

$$= \int_0^\infty e^{-t}\cos\omega t\, dt$$

$$= \left[\frac{e^{-t}}{(-1)^2+\omega^2}\{(-1)\cos\omega t + \omega\sin\omega t\}\right]_0^\infty$$

$$= 0 - \left[\frac{e^0}{(-1)^2+\omega^2}\{(-1).\cos 0 + \omega\sin 0\}\right]$$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$= -\left[\frac{1}{1+\omega^2}(-1+0)\right]$$

$$A(\omega) = \frac{1}{1+\omega^2}$$

and $B(\omega) = \frac{1}{\pi}\left[\int_0^\infty \pi e^{-t}\sin\omega t\, dt\right]$

$$= \int_0^\infty e^{-t}\sin\omega t\, dt$$

$$= \left[\frac{e^{-t}}{(-1)^2+\omega^2}\{(-1)\sin\omega t - \omega\cos\omega t\}\right]_0^\infty$$

$$= 0 - \left[\frac{e^0}{(-1)^2+\omega^2}\{(-1)\sin 0 - \omega\cos 0\}\right]$$

$$= \frac{\omega}{1+\omega^2}$$

now (1) becomes,

$$f(x) = \int_0^\infty \left[\frac{1}{1+\omega^2}\cos\omega x + \frac{\omega}{1+\omega^2}\sin\omega x\right] d\omega$$

$$= \int_0^\infty [\cos\omega x + \omega\sin\omega$$

2. $\int_0^\infty \left(\frac{1-\cos \pi\omega}{\omega}\right)\sin x\omega\, d\omega = \frac{\pi}{2}$ if $0<x<\pi$; if $x>\pi$

Soln,

given integral is fourier sine integral.

we know,

Fourier sine integral of $f(x)$ is

$$f(x) = \int_0^\infty B(\omega)\sin\omega x\, d\omega \quad \text{——— (1)}$$

where,

$\pi z = n\pi$ $\qquad \theta = n\pi + (-1)^n \alpha$

$z = n$ $\qquad = n\pi \pm \alpha$

$z = 0, \pm 1, \pm 2, \pm 3, ---$ where $n \in Z$.

Here, the zeros are of simple order.

2. $\cos^2\left(\frac{z}{2}\right)$ $\qquad 2\cos^2\theta - 1$

$= \frac{1+\cos z}{2} = f(z)$

for zeros, put $f(z) = 0$

$\Rightarrow \frac{1+\cos z}{2} = 0$

$\Rightarrow 1 + \cos z = 0$

$\cos z = -1$

$\cos z = \cos \pi$

$z = 2n\pi \pm \pi$

$z = (2n \pm 1)\pi, n \in z$

$z = \pm \pi, \pm 3\pi, \pm 5\pi, ----$

for each z, given function is of 2^{nd} degree so zeros are of second order.

3. $(z^2+1)(e^z-1)$

Here,

$f(z) = (z^2+1)(z^2-1)$

$= (z^2 - i^2)(e^z - 1)$

$= (z-i)(z+i)(e^z-1)$

For zeros, put $f(z) = 0$

$(z-i)(z+i)(e^z-1) = 0$

$\Rightarrow z = i, -i, 0.$

For each z, the function is of degree 1.

So, zeros are of simple order.

Determine the location and type of singularities.

1. $f(z) = 2z^{-3} - z^{-1}$

$= \frac{2}{z^3} - \frac{1}{z}$

$= \frac{2 - z^2}{z^3} = \frac{2-z^2}{(z-0)^3}$

Here, $z = 0$ is a pole of 3^{rd} order.

2. $z^{-2}\sin^2 z$

soln,

$f(z) = z^{-2}\sin^2 z$

for singularity $f(z) = \infty$

$\Rightarrow z^{-2}\sin^2 z = \infty$

$\Rightarrow \frac{\sin^2 z}{z^2} = \infty$

Questions

1.a. Prove Harmonic function & find harmonic conjugate & analytic function.

b. Integrate given function over the given circle.

2.a. Integrate given function over the given curve.

b. Find Taylor series expansion.

3.a. Find the Taylor & Laurent series of $f(z)$ in the given region.

b. Find Laurent series of $f(z)$ in the given region.

4.a. Show $f(z)$ - - - is analytic.

b. Find the location & order of zeros of $f(z)$ - - -

c. Integrate $f(z)$ ccw around a circle by indicating whether Cauchy integral theorem applies

Residue

If $f(z)$ has an isolated singular point at $z=z_0$ then the coefficient of $\frac{1}{z-z_0}$ is called residue of $f(z)$ when $f(z)$ involves the form $\frac{1}{z-z_0}$.

Note:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

valid in $r_1 < |z-z_0| < r_2$

Evaluation of residues

1. If $f(z)$ has a simple pole at $z=z_0$ then residue of $f(z)$ is

$$b_1 = \lim_{z\to z_0} (z-z_0) f(z)$$

2. If $f(z)$ has a pole of order 2 then,

$$b_1 = \lim_{z\to z_0} \left[\frac{\partial}{\partial z} (z-z_0)^2 f(z)\right]$$

3. If $f(z)$ has a pole of order 3 then,

$$b_1 = \frac{1}{2!} \lim_{z\to z_0} \left[\frac{\partial}{\partial z} (z-z_0)^3 f(z)\right]$$

In general,

If $f(z)$ has a pole of order m then,

$$b_1 = \frac{1}{(m-1)!} \lim_{z\to z_0} \left[\frac{\partial^{m-1}}{\partial z^{m-1}} (z-z_0)^m f(z)\right] \text{ at } z=z_0$$

Examples:

Find the residue at singular points from the laurent series as by definition.

1. $f(z) = \frac{4z}{z^2+1}$

$= \frac{4z}{z^2 - i^2}$

$= \frac{4z}{(z+i)(z-i)}$

Here, $f(z)$ has simple poles at $z = i$ & $z = -i$.

now,

$\underset{z=i}{\text{Res}}\, f(z) = \lim_{z \to i} (z-i) f(z)$

$= \lim_{z \to i} (z-i) \frac{4z}{(z+i)(z-i)} = \frac{4i}{i+i} = \frac{4i}{2i} = 2$

and,

$\underset{z=-i}{\text{Res}}\, f(z) = \underset{z \to -i}{\text{Res}} (z+i) f(z)$

$= \underset{z \to -i}{\text{Res}} (z+i) \frac{4z}{(z+i)(z-i)}$

$= \frac{4(-i)}{(-i-i)} = \frac{-4i}{-2i} = 2$

2. $f(z) = \frac{\sin 2z}{z^6}$

$= \frac{\sin 2z}{(z-0)^6}$

Here, $f(z)$ has pole at $z=0$ of order 6.

So, $\underset{z=0}{\text{Res}}\, f(z) = \frac{1}{(6-1)!} \lim_{z \to 0} \frac{\partial^m}{\partial z^{m-1}} \left[(z-z_0)^m f(z) \right]$

$= \frac{1}{5!} \lim_{z \to 0} \frac{\partial^5}{\partial z^5} \left[(z-z_0) f(z) \right]$

$= \frac{1}{5!} \lim_{z \to 0} \frac{\partial^5}{\partial z^5} \left[\frac{z^6 \sin 2z}{z^6} \right]$

$= \frac{1}{5!} \lim_{z \to 0} (32 \cos 2z)$

$= \frac{1}{5!} \times 32 \cos 0 = \frac{32}{120} = \frac{4}{15}$

3. $f(z) = \frac{1}{(z^2-1)^2}$

$= \frac{1}{[(z+1)(z-1)]^2}$

$= \frac{1}{(z+1)^2 (z-1)^2}$

Here, $f(z)$ has poles at $z = 1$ & $z = -1$ both of order 2.

now,

$\underset{z=1}{\text{Res}}\, f(z) = \lim_{z \to 1} \frac{d}{dz} \left[(z-1)^2 f(z) \right]$

$= \lim_{z \to 1} \frac{\partial}{\partial z} \left[(z-1)^2 \frac{1}{(z+1)^2 (z-1)^2} \right]$

$= \lim_{z \to 1} \frac{\partial}{\partial z} \left[\frac{1}{(z+1)^2} \right]$

$= \lim_{z \to 1} \frac{d}{dz} (z+1)^{-2}$

$= \lim_{z \to 1} (-2)(z+1)^{-2-1}$

$= (-2)(1+1)^{-3}$

$= \frac{-2}{2^3}$

$= -1/4$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin\omega t\,dt$$
$$= \frac{2}{\pi}\left[\int_0^\pi \frac{\pi}{2}\sin\omega t\,dt + \int_0^\infty 0\,\sin\omega t\,dt\right]$$
$$= \frac{2}{\pi}\int_0^\pi \frac{\pi}{2}\sin\omega t\,dt$$
$$= \left[-\frac{\cos\omega t}{\omega}\right]_0^\pi$$
$$= -\frac{\cos\omega\pi}{\omega} + \frac{\cos 0}{\omega} = \frac{-\cos\omega\pi + 1}{\omega} = \frac{1-\cos\omega\pi}{\omega}$$

from ①

$$f(x) = \int_0^\infty \left[\frac{1-\cos\omega\pi}{\omega}\right]\sin\omega x\,d\omega$$

proved |-

example:

1. show that

$$\int_0^\infty \left(\frac{\cos x\omega}{1+\omega^2}\right)d\omega = \frac{\pi e^{-x}}{2} \text{ for } x>0$$

The given integral is Fourier cosine integral.
The Fourier cosine integral of $f(x)$ is

$$f(x) = \int_0^\infty A(\omega)\cos\omega x\,d\omega \quad \text{— ①}$$

where

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,dt$$
$$= \frac{2}{\pi}\int_0^\infty \frac{\pi e^{-t}}{2}\cos\omega t\,dt$$
$$= \int_0^\infty e^{-t}\cos\omega t\,dt$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) \qquad e^{-\infty} = \frac{1}{e^\infty} = 0$$

$$= \left[\frac{e^{-t}}{(-1)^2+\omega^2}\{(-1)\cos\omega t + \omega\sin\omega t\}\right]_0^\infty$$

$$A(\omega) = 0 - \frac{1}{1+\omega^2}\left((-1)\times 1 + \omega\times 0\right)$$

$$A(\omega) = \frac{1}{1+\omega^2}$$

now ① becomes,

$$f(x) = \int_0^\infty \frac{1}{1+\omega^2}\cos\omega x\,d\omega$$

proved |-

Fourier cosine Transform

The Fourier cosine integral of $f(x)$ is

$$f(x) = \int_0^\infty A(\omega)\cos x\omega\,d\omega \quad \text{— ①}$$

where, $A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos\omega t\,dt$

now, we set,

$$A(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}f(t).\cos\omega t\,dt$$
$$= \sqrt{\frac{2}{\pi}}\cdot \hat{f}_c(\omega) \text{ where,}$$

$$\hat{f}_c(\omega) = \int_0^\infty \sqrt{\frac{2}{\pi}}f(t)\cos\omega t\,dt \quad \text{— ②}$$

from ①

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cos x\omega\,d\omega \quad \text{— ③}$$

Here,
eqn ② is called Fourier cosine transform of $f(x)$.
and eqn ③ is called Inverse Fourier cosine transform of $\hat{f}_c(\omega)$.

Fourier cosine transform is also denoted by $\mathcal{F}_c(f)$.

Fourier sine transform

The fourier sine integral of $f(x)$ is

$$f(x) = \int_0^\infty B(\omega) \cos x\omega \, d\omega \quad \text{—— (1)}$$

where, $B(\omega) = \frac{2}{\pi} \int_0^\infty f(t) \sin \omega t \, dt$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} f(t) \sin \omega t \, dt$$

$$= \sqrt{\frac{2}{\pi}} \hat{f}_s(\omega)$$

where,

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t \, dt \quad \text{— (2)}$$

is called Fourier sine transform of $f(x)$.

and from (1)

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(\omega) \sin x\omega \, d\omega \quad \text{—— (3)}$$ is

called Inverse fourier sine transform of $\hat{f}_s(\omega)$.

Theorem (statement)

Let $f(x)$ is continuous and absolutely integrable on the x-axis. let $f'(x)$ be piecewise continuous on each finite integral and let $f(x) \longrightarrow 0$ as $x \rightarrow \infty$ then,

$$\mathcal{F}_c[f'(x)] = \omega \mathcal{F}_s(f) - \sqrt{\frac{2}{\pi}} f(0)$$

and, $\mathcal{F}_s[f'(x)] = -\omega \mathcal{F}_c[f(x)]$

$$\hat{f}_c(\omega) = \mathcal{F}_c[f(x)]$$

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example

1. Find the fourier cosine transform of

$$f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

soln;

we know,

Fourier cosine transform of $f(x)$ is

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty x \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{x \sin \omega x}{\omega} - 1 \left(-\frac{\cos \omega x}{\omega^2} \right) \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\left\{ a \frac{\sin \omega a}{\omega} + \frac{\cos \omega a}{\omega^2} \right\} - \left\{ 0 + \frac{\cos 0}{\omega^2} \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a \sin \omega a}{\omega} + \frac{\cos \omega a}{\omega^2} - \frac{1}{\omega^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{a\omega \sin \omega a + \cos \omega a - 1}{\omega^2} \right]$$

is required fourier sine transform of $f(x)$.

2. Find fourier cosine transform of

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 < x < 2 \\ 0 & \text{for } x > 2 \end{cases}$$

soln;

The fourier cosine transform of $f(x)$ is

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \omega x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos \omega x \, dx + \int_1^2 (2-x) \cos \omega x \, dx + \int_0^\infty 0 \cdot \cos \omega x \, dx \right]$$

$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{x \cdot \frac{\sin \omega x}{\omega} - 1\left(\frac{-\cos \omega x}{\omega^2}\right)\right\}_0^1 + \left\{(2-x)\frac{\sin \omega x}{\omega} - (-1)\left(\frac{-\cos \omega x}{\omega^2}\right)\right\}_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left\{\left(1 \cdot \frac{\sin \omega \cdot 1}{\omega} + \frac{\cos \omega \cdot 1}{\omega^2}\right) - \left(0 + \frac{1}{\omega^2}\right)\right\} + \left\{\left(0 - \frac{\cos 2\omega}{\omega^2}\right) - \left(1 \cdot \frac{\sin \omega \cdot 1}{\omega} - \frac{\cos \omega}{\omega^2}\right)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin \omega}{\omega} + \frac{\cos \omega}{\omega^2} - \frac{1}{\omega^2} - \frac{\cos 2\omega}{\omega^2} - \frac{\sin \omega}{\omega} + \frac{\cos \omega}{\omega^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[2\frac{\cos \omega}{\omega^2} - \frac{\cos 2\omega}{\omega^2} - \frac{1}{\omega^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2\cos \omega - \cos 2\omega - 1}{\omega^2}\right] \text{ //}$$

Find the Fourier sine and cosine transform of:

a. $f(x) = 2e^{-5x} + 5e^{-2x}$

we know,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin \omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[2e^{-5x} + 5e^{-2x}\right]\sin \omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left(2\int_0^{\infty} e^{-5x}\sin \omega x\, dx + 5\int_0^{\infty} e^{-2x}\sin \omega x\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(2\left\{\frac{e^{-5x}}{(-5)^2+\omega^2}\left(e^{-5x}\sin \omega x - \omega\cos \omega x\right)\right\}_0^{\infty} + 5\left\{\frac{e^{-2x}}{(-2)^2+\omega^2}\left(e^{-2x}\sin \omega x - \omega\cos \omega x\right)\right\}_0^{\infty}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(2\left\{0 - \frac{1}{25+\omega^2}(0 - \omega \cdot \cos 0)\right\} + 5\left\{0 - \left(0 - \frac{1}{4+\omega^2}(0 - \omega\cos 0)\right\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{2\omega}{25+\omega^2} + \frac{5\omega}{4+\omega^2}\right] \text{ //}$$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

Note:

$$\int_0^{\infty} e^{ax}\cos bx\, dx = \frac{a}{a^2+b^2} \quad \text{when } a<0$$

$$\int_0^{\infty} e^{ax}\sin bx\, dx = \frac{b}{a^2+b^2} \quad \text{when } a<0$$

Fourier sine integral of $f(x)$ is

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos \omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[2\int_0^{\infty} e^{-5x}\cos \omega x\, dx + 5\int_0^{\infty} e^{-2x}\cos \omega x\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[2\left\{\frac{e^{-5x}}{(-5)^2+\omega^2}\left(-5\cos \omega x + \omega\sin \omega x\right)\right\}_0^{\infty} + 5\left\{\frac{e^{-2x}}{(-2)^2+\omega^2}\left(-2\cos \omega x - \omega\sin \omega x\right)\right\}_0^{\infty}\right]$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 f(z)\right]$$

$$= \text{- - - - -}$$

$$= 1/4$$

4. $f(z) = \dfrac{1}{1-e^z} = \dfrac{P(z)}{Q(z)}$ (say)

[Note: For a function $f(z) = \dfrac{P(z)}{Q(z)}$

$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \dfrac{P(z)}{Q'(z)}$

where, $z = z_0$ is simple pole.

Soln:

For pole,

$1 - e^z = 0$

$e^z = 1$

$e^z = e^0$

$z = 0$ is simple pole.

now,

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{P(z)}{Q'(z)}$$

$$= \lim_{z\to 0} \frac{1}{-e^z} = \frac{1}{-e} = \frac{1}{-1} = -1$$

$\left[\because Q(z) = 1-e^z,\ Q'(z) = -e^z\right]$

5. $f(z) = \cot \pi z$

$$= \frac{\cos \pi z}{\sin \pi z} = \frac{f(z)}{Q(z)}$$

Here,

$\sin \pi z = 0 = \sin n\pi$

$\Rightarrow \pi z = n\pi$

$z = n$ (n is integer)

is a pole of order 1 (or simple pole)

now particularly, taking $z = 0$

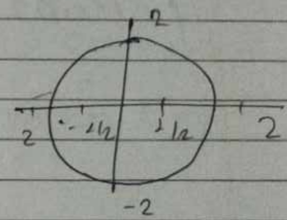
$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{P(z)}{Q'(z)}$$

$$= \lim_{z\to 0} \frac{\cos \pi z}{\pi \cos \pi z}$$

$$= 1/\pi$$

Evaluate the following integral (counter clockwise)

1. $\displaystyle\int_C \tan \pi z$. $C : |z| = 1$



$$= \int_C \frac{\sin \pi z}{\cos \pi z} \left(= \int \frac{P(z)}{Q(z)} \right)$$

for poles,

$\cos \pi z = 0 = \cos \dfrac{n\pi}{2}$

$\Rightarrow \pi z = \dfrac{n\pi}{2}$

$z = n/2 = \pm 1/2,\ \pm 3/2, \text{- - - - -}$

Given region $C : |z| = 1$ includes only $z = \pm 1/2$ which are simple poles.

now at $z = 1/2$

$$\operatorname*{Res}_{z=1/2} f(z) = \lim_{z\to 1/2} \frac{P(z)}{Q'(z)} = \lim_{z\to 1/2} \frac{\sin \pi z}{-\pi \sin \pi z}$$

$$= -1/\pi$$

$$= \sqrt{\frac{2}{\pi}} \left[2 \left\{ 0 - \frac{1}{25+\omega^2} (-5\cos\omega x) \right\} + 5 \left\{ 0 - \frac{1}{4+\omega^2} (-2\cos 0) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{5}{25+\omega^2} + 5 \cdot \frac{2}{4+\omega^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{10}{25+\omega^2} + \frac{10}{4+\omega^2} \right]$$

is required fourier cosine transform.

Note:

let $\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$ then — (1)

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega x}\, d\omega \quad \text{— (2)}$$

Here eqn (1) is called fourier transform of $f(t)$ in complex form and eqn (2) is called inverse fourier transform of $f(t)$.

Fourier transform of derivative

let $f(x)$ be continous on x-axis & $f(x) \to 0$ and $|x| \to \infty$ and $f'(x)$ is absolutely integrable on x-axis then,

1. $\mathcal{F}[f'(x)] = i\omega\, \mathcal{F}[F]$

ii. $\mathcal{F}(f''(x)) = i^2\omega^2\, \mathcal{F}[f]$
$= -\omega^2\, \mathcal{F}[F]$

Convolution

The convolution of two functions $f(x)$ & $g(x)$ is given by,

$$f * g = \int_{-\infty}^{\infty} f(p)\, g(x-p)\, dp$$
$$= \int_{-\infty}^{\infty} f(x-p)\, g(p)\, dp$$

Convolution Theorem:

let $f(x)$ and $g(x)$ are piece wise continous, bounded and absolutely integrable on the x-axis, then

$$\mathcal{F}(f * g) = \sqrt{2\pi}\, \mathcal{F}(f)\, \mathcal{F}(g)$$

Find the fourier transform of

1. $f(x) = \begin{cases} 1 & \text{for } a < x < b \\ 0 & \text{for otherwise} \end{cases}$

soln,

The fourier transform of $f(x)$ is,

$$\hat{f}(\omega) = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b 1\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega b}}{-i\omega} - \frac{e^{-i\omega a}}{-i\omega} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega b} - e^{-i\omega a}}{-i\omega} \right]$$

$$= \frac{1}{\sqrt{2\pi} \times (i\omega)} (e^{-i\omega b} - e^{-i\omega a}) \times \frac{i}{i}$$

$$= \frac{i}{\omega\sqrt{2\pi}} (e^{-i\omega b} - e^{-i\omega a})$$

At $z = -1/2$

$$\underset{z=-1/2}{\text{Res}} f(z) = \lim_{z \to -1/2} \frac{P(z)}{Q(z)}$$

$$= \lim_{z \to -1/2} \frac{\sin \pi z}{-\pi \sin \pi z}$$

$$= -\frac{1}{\pi}$$

From cauchy Residue theorem,

$$\int_c \tan \pi z \, dz = 2\pi i \times (\text{sum of residue})$$

$$= 2\pi i \times \left[\left(-\frac{1}{\pi} \right) + \left(-\frac{1}{\pi} \right) \right]$$

$$= 2\pi i \times -\frac{2}{\pi}$$

$$= -4i$$

Cauchy Residue Theorem

let, $f(z)$ be analytic inside a simple closed path c and on c except at finite no. of singularities $z_1, z_2, z_3, \ldots z_k$ inside c. let the residues at these points be $R_1, R_2, \ldots, R_k$ then

$$\int_c f(z) \, dz = 2\pi i \sum_{n=1} \underset{z=z_a}{\text{Res}} f(z)$$

$$= 2\pi i (R_1 + R_2 + \ldots R_k)$$

$$= 2\pi i \times (\text{sum of Residues})$$

Evaluate the following integrals (ccw)

1. $\int_c \frac{e^z}{\cos z} dz$ $\quad c: |z| = 3$

Soln;

$$f(z) = \frac{e^z}{\cos z} \left(= \frac{P(z)}{Q(z)} \right)$$

For poles,

$$\cos z = 0 = \cos \frac{n\pi}{2} \quad (n \text{ odd})$$

$$z = +\frac{n\pi}{2} = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots$$

Given circle includes the points $\pm \pi/2$

now,

$$\underset{z=\pi/2}{\text{Res}} f(z) = \lim_{z \to \pi/2} \frac{P(z)}{Q(z)} = \lim_{z \to \pi/2} \frac{e^z}{-\sin z} =$$

$$\frac{e^{\pi/2}}{-\sin \pi/2} = -e^{\pi/2}.$$

and,

$$\underset{z=-\pi/2}{\text{Res}} f(z) = \lim_{z \to -\pi/2} \frac{P(z)}{Q(z)} = \lim_{z \to -\pi/2} \frac{e^z}{-\sin z} = \frac{e^{-\pi/2}}{-\sin(-\frac{\pi}{2})}$$

$$= e^{-\pi/2}$$

By cauchy residue theorem,

$$\int_c f(z) \, dz = \int_c \frac{e^z}{\cos z} dz = 2\pi i \times \text{sum of residue}$$

$$= 2\pi i \left(-e^{\pi/2} + e^{-\pi/2} \right)$$

$$= -2\pi i \left(\frac{e^{\pi/2} - e^{-\pi/2}}{2} \right) \times 2$$

$$= -4\pi i \cdot \sinh \pi/2$$

2. $f(x) = \begin{cases} x & \text{for } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

Soln;

The Fourier transform of $f(x)$ is

$$\hat{f}(\omega) = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[x \frac{e^{-i\omega x}}{-i\omega} - 1 \cdot \frac{e^{-i\omega x}}{(-i\omega)(i\omega)} + 0 \cdot \frac{e^{-i\omega x}}{(i\omega)(-i\omega)(-i\omega)} \right] \cdots$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{x e^{-i\omega x}}{-i\omega} - \frac{e^{-i\omega x}}{(-i\omega)^2} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left(\frac{a e^{-i\omega a}}{-i\omega} - \frac{e^{-i\omega a}}{(-i\omega)^2} \right) - \left(-\frac{a e^{-i\omega a}}{-i\omega} - \frac{e^{i\omega a}}{(-i\omega)^2} \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{a e^{-i\omega a}}{i\omega} + \frac{e^{-i\omega a}}{\omega^2} - \frac{a e^{i\omega a}}{i\omega} - \frac{e^{i\omega a}}{\omega^2} \right] //$$

3. $f(x) = \begin{cases} x e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$

The Fourier transform of $f(x)$ is,

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-x} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-x - i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x e^{-(1+i\omega)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{x e^{-(1+i\omega)x}}{-(1+i\omega)} - 1 \frac{e^{-(1+i\omega)x}}{[-(1+i\omega)]^2} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - \left\{ \frac{0 \cdot e^{-(1+i\omega) \cdot 0}}{-(1+i\omega)} - 1 \frac{e^{-(1+i\omega) \cdot 0}}{[-(1+i\omega)]^2} \right\} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{(1+i\omega)^2}$$

$$= \frac{1}{\sqrt{2\pi}\,(1+i\omega)^2} //$$

$\int u v\,dx = uv_1 - u'v_2 + u''v_3 - \cdots$

PYQ

4. Find the fourier transform of $f(x) = e^{-x^2/2}$.

soln;

The fourier transform of $f(x)$ is

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2/2 + i\omega x)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(x^2 + 2i\omega x)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2[(x)^2 + 2x.i\omega + (i\omega)^2 - (i\omega)^2]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2[(x+i\omega)^2 - (i\omega)^2]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(x+i\omega)^2}{2}} . e^{\frac{(i\omega)^2}{2}}\, dx$$

$$= \frac{e^{\omega^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x+i\omega}{\sqrt{2}}\right)^2}\, dx$$

put $\frac{x+i\omega}{\sqrt{2}} = z$ then, $\frac{1}{\sqrt{2}}\, dx = dz$

$\Rightarrow dx = \sqrt{2}\, dz$

when $x \to -\infty$ then $z \to -\infty$

when $x \to \infty$ then $z \to \infty$

now, from (i)

$$F(\omega) = \frac{e^{\omega^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2} . \sqrt{2}\, dz$$

$$= \frac{e^{-\omega^2/2}}{\sqrt{2}\sqrt{\pi}} \times \sqrt{2} \int_{-\infty}^{\infty} e^{-z^2}\, dz$$

$$= \frac{e^{-\omega^2/2}}{\sqrt{\pi}} \times \sqrt{\pi}$$

$$= e^{-\omega^2/2}$$ //

PYQ 5. Find the fourier cosine transform of $f(x) = e^{-mx}$, $m > 0$ and then show that $\int_0^{\infty} \frac{\cos kx}{1+x^2}\, dx = \frac{\pi}{2} e^{-k}$.

soln;

The fourier cosine transform of $f(x)$ is,

$$\mathcal{F}_c(f) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos\omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-mx} \cos\omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-mx}}{(-m)^2+\omega^2} \left((-m)\cos\omega x + \omega \sin\omega x \right) \right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \left\{ \frac{e^0}{m^2+\omega^2} \left(-m\cos 0 + \omega \sin 0 \right) \right\} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{m^2+\omega^2} (-m) \right]$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{m}{m^2+\omega^2}$$

is reqd. fourier cosine transform of $f(x)$.

now,

inverse fourier cosine transform of $f(x)$ is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \mathcal{F}_c(f) \cos\omega x\, d\omega$$

$$e^{-mx} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}}\, \frac{m}{m^2+\omega^2} \cos\omega x\, d\omega$$

$$e^{-mx} = \frac{2}{\pi}\int_0^\infty \frac{m\cos\omega x}{m^2+\omega^2}\,d\omega$$

$$\Rightarrow \int_0^\infty \frac{m\cos\omega x}{m^2+\omega^2}\,d\omega = \frac{\pi}{2}e^{-mx}$$

put $m=1$, $\omega = x$ & $x = k$ then,

$$\int_0^\infty \frac{1.\cos x.k}{1^2+x^2}\,dx = \frac{\pi}{2}e^{-1.k}$$

$$\therefore \int_0^\infty \frac{\cos kx}{1+x^2}\,dx = \frac{\pi}{2}e^{-k}$$

proved/-

6 Find the fourier sine transform of $f(x) = e^{-x}$, $x>0$ then show that $\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$ for $m>0$

soln,

The fourier sine transform of $f(x)$ is,

$$F_s(f) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{(-1)^2+\omega^2}\{(-1)\sin\omega x - \omega\cos\omega x\}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{e^0}{1+\omega^2}(0-\omega\cos 0)\right]$$

$$= \sqrt{\frac{2}{\pi}}\cdot\frac{\omega}{1+\omega^2}$$

is reqd. fourier sine transform of $f(x)$.

now,

inverse fourier sine transform of $f(x)$ is,

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(f)\sin\omega x\,d\omega$$

$$e^{-x} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{\omega}{1+\omega^2}\sin\omega x\,d\omega \quad [\text{using } ①]$$

$$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\omega}{1+\omega^2}\sin\omega x\,d\omega$$

$$\Rightarrow \int_0^\infty \frac{\omega\sin\omega x}{1+\omega^2}\,d\omega = \frac{\pi}{2}e^{-x}$$

set $x = m$ and $\omega = x$ then

$$\int_0^\infty \frac{x\sin x.m}{1+\omega^2} = \frac{\pi}{2}e^{-m}$$

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$$

proved/-

Parseval's identity for fourier transform

let $f(x)$ and $g(x)$ are picewise continous, bounded and absolutely integrable on x-axis with fourier transforms $F(\omega)$ & $G(\omega)$ respectively then,

$$\int_{-\infty}^\infty F(\omega)\bar{G}(\omega)\,d\omega = \int_{-\infty}^\infty f(x)\bar{g}(x)\,dx$$

and,

$$\int_{-\infty}^\infty |F(\omega)|^2\,d\omega = \int_{-\infty}^\infty |f(x)|^2\,dx$$

where, $\bar{G}(\omega)$ is conjugate of $G(\omega)$ & $\bar{g}(x)$ is conjugate of $g(x)$.

Parseval's identity for fourier sine & cosine transform

1.(i). $\int_0^\infty F_c(\omega) G_c(\omega)\, d\omega = \int_0^\infty f(x) g(x)\, dx$

ii. $\int_0^\infty |F_c(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx$

2(i). $\int_0^\infty F_s(\omega) G_s(\omega)\, d\omega = \int_0^\infty f(x) g(x)\, dx$

ii. $\int_0^\infty |F_s(\omega)|^2 d\omega = \int_0^\infty |f(x)|^2 dx$

Example

1. Find the fourier sine transform of e^{-x} for $x > 0$ and then by using parseval's identity. Show that $\int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \frac{\pi}{4}$.

Soln;

let $f(x) = e^{-x}$ for $x > 0$ —— (1)

The fourier transform of $f(x)$ is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{(-1)^2 + \omega^2} \left((-1) \sin \omega x - \omega \cos \omega x \right) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{\omega}{1+\omega^2}$$

By parseval's identity for fourier sine transform

$$\int_0^\infty |F_s\{f(x)\}|^2 d\omega = \int_0^\infty |f(x)|^2 dx$$

$$\int_0^\infty \left| \sqrt{\frac{2}{\pi}} \frac{\omega}{(1+\omega^2)} \right|^2 d\omega = \int_0^\infty |e^{-x}|^2 dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{\omega^2}{(1+\omega^2)^2} d\omega = \int_0^\infty (e^{-x})^2 dx$$

$$\frac{2}{\pi} \int_0^\infty \frac{\omega^2}{(1+\omega^2)^2} d\omega = \left[\frac{e^{-2x}}{-2} \right]_0^\infty$$

$$= \left[0 - \frac{1}{(-2)} \right]$$

$$\frac{2}{\pi} \int_0^\infty \frac{\omega^2}{(1+\omega^2)^2} d\omega = \frac{1}{2}$$

$$\int_0^\infty \frac{\omega^2}{(1+\omega^2)^2} d\omega = \frac{\pi}{4}$$

set $\omega = x$

we get,

$$\int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \frac{\pi}{4} \quad //-$$

(12 hour) u long que

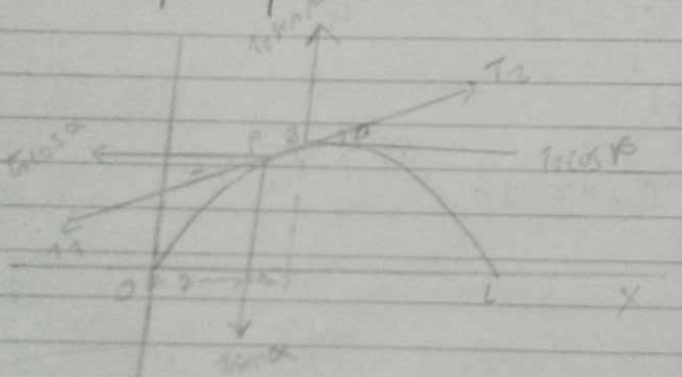
derivation (heat; sation of wave, heat, ...)

Day: Wednesday

One dimensional wave equation

Let a string be stretched to length l and fixed at two points. Let one end of string be the origin and another end be A along the x-axis. Now, we allow it to vibrates, each point of the string moves parallel to the y-axis. We assume that

i. The motion entirely lies in XY plane.
ii. The horizontal displacement of particles of the string is negligible.
iii. The tension of the string is uniform.
iv. The gravitational force acting on string is negligible.
v. The effect of friction is negligible.
vi. The string is perfectly elastic.
vii. The slope of the deflection curve is small.



Let T_1 and T_2 be tensions at points P and Q making angles α, β with the horizontal from figure,

Let $T_1 \cos\alpha = T_2 \cos\beta = T$ ——— ① (constant)

In vertical motion, the components are $-T_1 \sin\alpha$ and $T_2 \sin\beta$

The resultant force is $T_2 \sin\beta - T_1 \sin\alpha$.

By Newton's second law

$$T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2} \quad \text{——— ②}$$

($\because$ force = mass × acceleration)

where ρ is the mass of string per unit length, Δx is length of portion of the string (PQ).

Dividing eqn ② by ①

$$\frac{T_2 \sin\beta - T_1 \sin\alpha}{T} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\frac{T_2 \sin\beta}{T} - \frac{T_1 \sin\alpha}{T} = \frac{\rho}{T} \Delta x \frac{\partial^2 u}{\partial t^2}$$

$$\lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x} = f''(x)$$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} \quad \text{——— ③}$$

We know, $\tan\alpha$ and $\tan\beta$ are slopes of string at x and $x+\Delta x$ respectively.

So, $\tan\alpha = \left.\frac{\partial u}{\partial x}\right|_{x=x}$, $\tan\beta = \left.\frac{\partial u}{\partial x}\right|_{x=x+\Delta x}$

From ③,

$$\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x = \frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{1}{\Delta x}\left[\left\{\frac{\partial u}{\partial x}\right\}_{x+\Delta x} - \left\{\frac{\partial u}{\partial x}\right\}_x\right] = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{\rho}$$

is reqd. one dimensional wave eqn.

Solution of one dimensional wave equation:

The one dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—— (1)}$$

where, $u(x,t)$ is the deflection of the string at x in time t with boundary conditions $u(0,t) = 0$ and $u(l,t) = 0$ for all t —— (2)

$u(x,0) = f(x)$ —— (3) (initial deflection)

and, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$ —— (4) (initial velocity)

Let $u(x,t) = F(x)G(t)$ —— (5) be soln of (1)

Then by differentiation we get,

$$\frac{\partial^2 u}{\partial t^2} = F(x)\ddot{G} \;\&\; \frac{\partial^2 u}{\partial x^2} = F''(x)G(t)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = F\ddot{G} \;\&\; \frac{\partial^2 u}{\partial x^2} = F''G$$

where dot denotes partial derivatives w.r.to t and prime denotes derivative w.r.to x.

from (1),

$$F\ddot{G} = c^2 F''G$$

$$\Rightarrow \frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k \text{ (suppose)}$$

$$\frac{d^2F}{dx^2} - kF = 0$$

which implies

$F'' = kF \Rightarrow F'' - kF = 0$ —— (6)

$\ddot{G} = c^2 kG \Rightarrow \ddot{G} - c^2 kG = 0$ —— (7)

which gives ordinary differential equation. Here, we may have $k>0$, $k<0$ or $k=0$.

(continue----)

Case I: $(k>0)$

- - - - - - - - - - - -

$S_\infty = \frac{a}{1-r}$

Chapter - 2
Z - transform

The Z-transform of a discrete time signal $f(n)$ is defined as

$$f(z) = Z[f(nT)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

The inverse Z-transform is defined as:

$$f(n) = Z^{-1} f(z)$$

If the summation (Σ) runs from $-\infty$ to ∞, it is two sided Z-transform and if it runs from 0 to ∞, it is called one sided Z-transform.

Example

1. Find the Z-transform of $(1)^n$ or 1.

we know,

$$Z(f(n)) = \sum_{n=0}^{\infty} f(n) z^{-n}$$

Here, $f(n) = (1)^n$

So, from (1)

$$Z((1)^n) = \sum_{n=0}^{\infty} (1)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 . z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= z^{-0} + z^{-1} + z^{-2} + z^{-3} + ----$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + -----,$$

$$= \frac{1}{1 - \frac{1}{z}}$$ [for the series to be convergent, using $S_\infty = \frac{a}{1-r}$]

$$= \frac{1}{\left(\frac{z-1}{z}\right)}$$

$$Z[(1)^n] = \frac{z}{z-1}$$

2. $Z[(-1)]^n = \frac{z}{z+1}$

3. $Z(a^n) = \frac{z}{z-a}$ $\quad \therefore Z(na^n) = \frac{az}{(z-a)^2}$

4. $Z(n) = \frac{z}{(z-1)^2}$

5. $Z[na^n] = \frac{a}{z} \cdot \frac{z^2}{(z-a)^2}$

6. $Z\left[\frac{1}{n!}\right] = e^{1/z}$

7. $Z\left[\frac{1}{n+1}\right] = -z \log\left[1 - \frac{1}{z}\right]$

Example

1. Find the Z-transform of $e^{in\pi/2}$ and then find $Z\left[\cos\frac{n\pi}{2}\right]$ and $Z\left[\sin\frac{n\pi}{2}\right]$

we know,

$$Z[a^n] = \frac{z}{z-a}$$ $\quad [e^{i\theta} = \cos\theta + i\sin\theta]$

$$Z\left[e^{\frac{in\pi}{2}}\right] = Z\left[\left(e^{i\pi/2}\right)^n\right]$$

$$= \frac{z}{z - e^{i\pi/2}}$$

$= \dfrac{z}{z - [\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}]}$

$= \dfrac{z}{z - (0 + i.1)}$

$= \dfrac{z}{z - i}$ is required z. transform of $e^{in\pi/2}$.

$= \dfrac{z}{z-i} \times \dfrac{z+i}{z+i}$

$= \dfrac{z^2 + iz}{z^2 - i^2}$

$= \dfrac{z^2 + iz}{z^2 + 1}$, $z\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)$

$= z[\cos\frac{n\pi}{2}] + iz[\sin\frac{n\pi}{2}]$

$= \dfrac{z^2}{z^2+1} + i\dfrac{z}{z^2+1}$

comparing real & imaginary parts, we get,

$z[\cos\frac{n\pi}{2}]$

$= \dfrac{z^2}{z+1}$

and $z[\sin\frac{n\pi}{2}]$

$= \dfrac{z}{z^2+1}$

Defination

let $f(t)$ be function of time period t where t is non-negative defined at $0, T, 2T, 3T, \ldots$ which are discrete values of t then,

$z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$,

The fixed number T is reffered as sampling period.

Theorem:

Show that $z[t^k] = -Tz \dfrac{d}{dz}[z[t^{k-1}]]$

Proof:

By defination,

$z[t^k] = \sum_{n=0}^{\infty} t^k z^{-n}$

$= \sum_{n=0}^{\infty} (nT)^k z^{-n} \left[\because nT = t \ \& \ n = t/T\right]$

$z[t^{k-1}] = \sum_{n=0}^{\infty} (nT)^{k-1} z^{-n}$ [replacing k by k-1]

$z[t^{k-1}] = \sum_{n=0}^{\infty} n^{k-1} T^{k-1} z^{-n}$

Differentiating w.r.to z both side,

$\dfrac{d}{dz}[z[t^{k-1}]] = \sum_{n=0}^{\infty} n^{k-1} T^{k-1} (-n) z^{-n-1}$.

multiplying both side by $-Tz$,

$-Tz \dfrac{d}{dz}[z[t^{k-1}]] = \sum_{n=0}^{\infty} \times (-Tz) n^{k-1} T^{k-1} (-n) z^{-n-1}$

$-Tz \dfrac{d}{dz}[z[t^{k-1}]] = \sum_{n=0}^{\infty} n^k T^k z^{-n}$

$$= \sum_{n=0}^{\infty} (nT)^k z^{-n}$$

$= Z[t^k]$ (from i)

$$\therefore Z[t^k] = -Tz \frac{d}{dz}\left[Z[t^{k-1}]\right] \quad \text{--- (2)}$$

proved/-

Note:

1. when $k=1$ (from (2))

$$Z(t) = -Tz \frac{d}{dz}\left[Z[t^{1-1}]\right]$$

$$= -Tz \frac{d}{dz}[Z(1)]$$

$$= -Tz \frac{d}{dz}\left[\frac{z}{z-1}\right] \quad \left[\because Z(1) = Z(1^n) = \frac{z}{z-1}\right]$$

$$\Rightarrow Z(t) = -Tz\left[\frac{(z-1).1 - z.1}{(z-1)^2}\right] \text{ by quotient rule}$$

$$= -Tz\left[\frac{z-1-z}{(z-1)^2}\right]$$

$$Z(t) = \frac{Tz}{(z-1)^2} \quad \text{--- (3)}$$

(2). when $k=2$ (from 2)

$$Z(t^2) = -Tz \frac{d}{dz}\left[Z(t^{2-1})\right]$$

$$= -Tz \frac{d}{dz}[Z(t)]$$

$$Z(t^2) = -Tz \frac{d}{dz}\left(\frac{Tz}{(z-1)^2}\right) \text{ using (3)}$$

$$= -T^2 z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$\frac{d}{dz}(z-1)^2 = 2(z-1).1$$

$$= -T^2 z\left[\frac{(z-1)^2 . 1 - z . 2(z-1)}{[(z-1)^2]^2}\right]$$

$$= -T^2 z\left[\frac{(z-1)^2 - 2z^2 + 2z}{(z-1)^4}\right]$$

$$= -T^2 z\left[\frac{-z-1}{(z-1)^3}\right]$$

$$\therefore Z(t^2) = T^2 z \frac{(z+1)}{(z-1)^3}$$

First Shifting Theorem:

Statement

If $Z[f(t)] = F(z)$ then $Z[e^{-at} f(t)] = F(e^{aT} z)$

Proof:

we know that,

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} = F(z)$$

By defn,

$$Z[e^{-at} f(t)] = \sum e^{-anT} f(nT) z^{-n}$$

$$= \sum f(nT)[ze^{aT}]^{-n}$$

$$= F[ze^{aT}]$$

$$= F(z)\big|_{z \to ze^{aT}}$$

Theorem:

Prove that $Z[a^n f(t)] = F[z/a]$ where $Z[f(t)] = F(z)$.

Proof:

By definition,

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad [\because Z[f(t)] = F(z)]$$

now,

$$z(a^n f(t)) = \sum_{n=0}^{\infty} a^n f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) \frac{z^{-n}}{a^{-n}}$$

$$= \sum_{n=0}^{\infty} f(nT) \left(\frac{z}{a}\right)^{-n}$$

$$= F\left(\frac{z}{a}\right)$$

$$= F(z) \Big|_{z \to z/a}$$

Example:

Find $z(e^{-at})$ by using First shifting theorem.

By defination,

$$z[e^{-at}]$$

$$= z[e^{-at} \cdot 1]$$

$$= [z(1)]_{z \to ze^{aT}}$$

$$= \left[\frac{z}{z-1}\right]_{z \to ze^{aT}} \quad \left[\because z(1) = \frac{z}{z-1}\right]$$

$$= \frac{ze^{aT}}{ze^{aT}-1}$$

$$[ze^{-at}] = \frac{z}{z-e^{-aT}}$$

2. Find $z(n-1)$

By linearity of z- transform

$$z(n-1) = z(n) - z(1)$$

$$= \frac{z}{(z-1)^2} - \frac{z}{z-1}$$

$$= \frac{z - z(z-1)}{(z-1)^2}$$

$$z(-1)^n = \frac{z}{z-(-1)}$$

$$= \frac{z - z^2 + z}{(z-1)^2}$$

$$z(n-1) = \frac{2z - z^2}{(z-1)^2}$$

3. Find $z(3 \cdot 2^n + 4(-1)^n]$

By defination,

By linearity z- transform

$$z[3 \cdot 2^n] + z[4(-1)^n]$$

$$= 3z(2^n) + 4z(-1)^n$$

$$= 3\left(\frac{z}{z-2}\right) + 4 \cdot \frac{z}{z-(-1)}$$

$$= \frac{3z}{z-2} + \frac{4z}{z+1}$$

4. Find $z(n a^n)$

Since $z[a^n f(t)] = F\left(\frac{z}{a}\right) = F(z)\Big|_{z \to z/a}$

we get,

$$z[n a^n] = z(n)\Big|_{z \to z/a}$$

$$= \frac{z}{(z-1)^2}\Big|_{z \to z/a}$$

$$= \frac{z/a}{(z/a - 1)^2}$$

$$= \frac{z/a}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{z/a \times a^2}{(z-a)^2}$$

$$= \frac{za}{(z-a)^2}$$

Second shifting theorem

If $Z[f(t)] = F(z)$ then $Z[f(t+T)] = z[F(z) - F(0)]$

Proof:

By defination of z transform,

$$Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$Z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T) z^{-n}$$

$$= \sum_{n=0}^{\infty} f\{(n+1)T\} z^{-n}$$

put $n+1 = N$ then $n = N-1$.

when $n=0$ then $N=1$

when $n=\infty$ then $N=\infty$.

now,

$$Z[f(t+T)] = \sum_{N=1}^{\infty} f(NT) z^{-(N-1)}$$

$$= \sum_{N=1}^{\infty} f(NT) z^{-N} \cdot z^{1}$$

$$= z \sum_{N=1}^{\infty} f(NT) z^{-N}$$

$$= z\left[\sum_{n=0}^{\infty} f(NT) z^{-N} - f(0)\right]$$

$$= z[F(z) - f(0)]$$

Inverse z-transforms

The z-transforms of $f(t)$ is $Z[f(t)]$

$$= \sum_{n=0}^{\infty} f(t) z^{-n} = F(z)$$

The inverse z-transform is,

$$Z^{-1}[F(z)] = f(t)$$

$z-2 \overline{)\, 1 - 2/z \,}\ (1/z + 2/z^2$

$\underline{+}$

$2/z$

Method of finding inverse z-transform.

A. Direct division method

Find $Z^{-1}\left(\frac{1}{z-2}\right)$

$z-2 \overline{)\, 1 \,}\ (1/z + 2/z^2 + 4/z^3$

$\underline{1 - 2/z}$

$2/z$

$\underline{2/z - 4/z^2}$

$4/z^2$

we know,

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

now,

$$\frac{1}{z-2} = \frac{1}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \cdots$$

$$= z^{-1} + 2z^{-2} + 4z^{-3} + \cdots$$

now,

$$F(z) = z^{-1} + 2z^{-2} + 4z^{-3} + \cdots$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n}$$

comparing it with $\sum f(n) z^{-n}$.

we get, $f(n) = 2^{n-1}$.

$$\therefore Z^{-1}\left(\frac{1}{z-2}\right) = 2^{n-1}$$

Method of finding inverse z-transform

B. By partial fraction:

Note:

a. $\dfrac{F(z)}{(az+b)(cz+d)} = \dfrac{Az}{az+b} + \dfrac{Bz}{cz+d}$

b. $\dfrac{F(z)}{(az+b)(cz+d)^2} = \dfrac{Az}{az+b} + \dfrac{Bz}{cz+d} + \dfrac{Cz}{(cz+d)^2}$

c. $\dfrac{F(z)}{(az+b)(cz^2+dz+e)} = \dfrac{Az}{az+b} + \dfrac{Bz^2 + Cz + D}{cz^2 + dz + e}$

$$z(a^n) = \frac{z}{(z-a)}$$
$$z^{-1}\left(\frac{z}{z-a}\right) = a^n$$

Find the inverse z-transform of:

$$f(z) = \frac{z^2 - 3z}{(z-5)(z+2)}$$

Soln,

$$f(z) = \frac{z^2 - 3z}{(z-5)(z+2)} \quad \frac{Az}{(z-5)} + \frac{Bz}{z+2} \quad \text{--- (1)}$$

or, $z^2 - 3z = Az(z+2) + Bz(z-5)$

or put $z = 5$

$25 - 15 = 5A \times 7$

$\frac{10}{35} = A$

$\frac{2}{7} = A$

put $z = -2$

$(-2)^2 - 3\times(-2) = 0 + B\times(-2)(-2-5)$

$4+6 = 14B$

$5/7 = B$

now from (1)

$$f(z) = \frac{2z}{7(z-5)} + \frac{5z}{7(z+2)}$$

$$= \frac{2}{7}\times\left(\frac{z}{z-5}\right) + \frac{5}{7}\left(\frac{z}{z+2}\right)$$

$$= \frac{2}{7}(5)^n + \frac{5}{7}(-2)^n$$

is required inverse-transform of $f(z)$.

c. Residue method

Inversion integral method

let $F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}$

$= f(0) + f(1)z^{-1} + F(2)z^{-2} +$
$F(3)z^{-3} + \dots$

Multiplying b.s by z^{n-1}.

$z^{n-1}f(z) = z^{n-1}f(0) + f(1)z^{n-2} + F(2)z^{n-3}$

It is laurent series ~~transform~~ expansion of $z^{n-1}f(z)$ around the point $z=0$.

So, the coefficient.

$$F(n) = \frac{1}{2\pi i}\int_c z^{n-1}F(z)\,dz$$

$$= \frac{1}{2\pi i} \times 2\pi i \times \text{sum of residue of } z^{n-1}F(z).$$

= sum of residue (By Cauchy residue theorem)

$\therefore z^{-1}F(z) = F(n) =$ sum of residue of $z^{n-1}F(z)$.

Example:

Find the inverse z-transform of.

$$\frac{z^2 - 3z}{(z-5)(z+2)}$$

$$z^{n-1}f(z) = z^{n-1}\left(\frac{z^2-3z}{(z-5)(z+2)}\right)$$

$$= \frac{z^{n+1} - 3z^n}{(z-5)(z+2)}$$

Here, $z^{n-1}F(z)$ has poles at $z=5$ & $z=-2$ of 1st order.

now, we know,

$z^{-1}[F(z)] =$ sum of residues of $z^{n-1}f(z)$.

Residue of $z^{n-1} f(z)$ at $z=5$ is

$$\underset{z=5}{\text{Res}}\ z^{n-1}[f(z)] = \lim_{z\to 5} (z-5) z^{n-1} F(z)$$

$$= \lim_{z\to 5} \frac{(z-5)\, z^{n+1} - 3z^n}{(z-5)(z+2)}$$

$$= 5^{n+1} - 3.5^n$$

$$= \frac{5^n (5-3\times 1)}{5+2}$$

$$= 5^n \times \frac{2}{7}$$

Again,

$$\text{Res}\ z^{n-1}[f(z)] = \lim_{z\to -2} (z+2)(z^{n-1} f(z))$$

$$= \lim_{z\to -2} \frac{z^{n+1} - 3z^n}{(z-5)}$$

$$= \frac{(-2)^{n+1} - 3(-2)^n}{-2-5}$$

$$= \frac{(-2)^n(-2-3)}{-7}$$

$$= \frac{5}{7}(-2)^{n-7}$$

now,

$z^{-1}[f(z)]$ = sum of all residues of $z^{n-1}[f(z)]$

$$= \frac{2}{7} 5^n + \frac{5}{7}(-2)^n.$$

Find the inverse z-transform of

$$f(z) = \frac{3z^2 + 2z + 1}{z^2 + 3z + 2}$$

$$F(z) = \frac{3z^2 + 2z + 1}{(z+2)(z+1)} \text{ (by fractorization)}$$

now,

$$z^{n-1} F(z) = z^{n-1} \frac{(3z^2 + 2z + 1)}{(z+2)(z+1)}$$

$$= \frac{3z^{n+1} + 2z^n + z^{n-1}}{(z+2)(z+1)}$$

clearly, $z^{n-1} F(z)$ has poles at $z=-2$ & $z=1$ of simple order.

Now, we know,

$z^{n-1}[f(z)]$ = sum of all residues of $z^{n-1} f(z)$.

now,

$$\underset{z=-2}{\text{Res}}\ z^{n-1}[f(z)] = \lim_{z\to -2} (z+2) z^{n-1} F(z).$$

$$= \lim_{z\to -2} \frac{(z+2)(3z^{n+1} + 2z^n + z^{n-1})}{(z+2)(z+1)}$$

$$= \frac{3.(-2)^{n+1} + 2(-2)^n + (-2)^{n-1}}{(-2+1)}$$

$$= \frac{(-2)^n [3.(-2) + (-2)^{-1}]}{-1}$$

$$= -(-2)^n \left[-6 + 2 + \frac{1}{(-2)}\right]$$

$$= -(-2)^n \left[-4 - \frac{1}{2}\right]$$

$$= -(-2)^n \left(\frac{-8-1}{2}\right)$$

$$= \frac{9}{2}(-2)^n$$

$$= \frac{9}{2}(-2)^n$$

Again,

$$\operatorname{Res}_{z=-1} z^{n-1} F(z) = \lim_{z \to -1} (z+1) z^{n-1} F(z)$$

$$= \lim_{z \to -1} \frac{(z+1)\, 3z^{n+1} + 2z^n + z^{n-1}}{(z+2)(z+1)}$$

$$= \frac{3.(-1)^{n+1} + 2(-1)^n + (-1)^{n-1}}{-1+2}$$

$$= (-1)^n(-3+2-1)$$

$$= (-2)(-1)^n$$

$$z^{-1}[F(z)] = \text{sum of residues of } z^{n-1}[F(z)]$$

$$= \frac{9}{2}(-2)^n + (-2)(-1)^n$$

$$= \frac{9}{2}(-2)^n - 2(-1)^n$$

Difference equation.

The difference equation is the equation between the difference of an unknown function.

Theorem:

If $z(y_n) = \bar{y}$ then $z(y_{n+k}) = z^k\left[\bar{y} - y_0 - \frac{y_1}{z} - \cdots\cdots \frac{y_{k-1}}{z^{k-1}}\right]$

Special cases:

1. If $k=1$ then $z(y_{n+1}) = z(\bar{y} - y_0)$
2. If $k=2$ then $z(y_{n+2}) = z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right)$
3. If $k=3$ then $z(y_{n+3}) = z^3\left(\bar{y} - y_0 - \frac{y_1}{z} - \frac{y_2}{z^2}\right)$

$z(a^n) = \frac{z}{z-a}$.

Example:

Use z-transform to solve the difference equation.

$y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ given $y_0 = y_1 = 0$.

we have,

$$y_{n+2} + 6y_{n+1} + 9y_n = 2^n$$

Taking z-transform on bs:

$$z(y_{n+2}) + 6z(y_{n+1}) + 9z(y_n) = z(2^n)$$

or, $z^2\left(\bar{y} - y_0 - \frac{y_1}{z}\right) + 6z(\bar{y} - y_0) + 9\bar{y} = \frac{z}{z-2}$

$$\left[\because z(a^n) = \frac{z}{z-a}\right]$$

or, $z^2\bar{y} + 6z\bar{y} + 9\bar{y} = \frac{z}{z-2}$ $\quad [\because y_0 = y_1 = 0]$

or, $\bar{y}(z^2 + 6z + 9) = \frac{z}{z-2}$

or, $\bar{y} = \frac{z}{(z-2)(z^2+6z+9)}$

or, $\bar{y} = \frac{z}{(z-2)(z+3)^2}$ [$\because$ It can be done with residue as partial]

let $F(z) = \frac{z}{(z-2)(z+3)^2}$

now,

$$z^{n-1} F(z) = \frac{z^n}{(z-2)(z+3)^2} = \bar{y} = z(y_n)$$

Clearly $z^{n-1}F(z)$ has poles $z=2$ of 1st and $z=-3$ of 2nd order.

now,

$$\operatorname*{Res}_{z=2} z^{n-1}F(z) = \lim_{z\to 2} (z-2)\left[z^{n-1}F(z)\right]$$

$$= \lim_{z\to 2} (z-2)\frac{z^n}{(z-2)(z+3)^2}$$

$$= \frac{(2)^n}{(2+3)^2}$$

$$= \frac{2^n}{25}$$

Again,

$$\operatorname*{Res}_{z=-3} z^{n-1}F(z) = \lim_{z\to -3}\frac{d}{dz}\left[(z+3)^2 z^{n-1} F(z)\right]$$

$$= \lim_{z\to -3}\frac{d}{dz}\left[\frac{(z+3)^2 z^n}{(z-2)(z+3)^2}\right]$$

$$= \lim_{z\to -3}\frac{d}{dz}\left[\frac{z^n}{(z-2)}\right]$$

$$= \lim_{z\to -3}\left[\frac{(z-2)nz^{n-1} - z^n \cdot 1}{(z-2)}\right]$$

$$= \lim_{z\to -3}\left[\frac{(z-2)n z^{n-1} - z^n}{(z-2)^2}\right]$$

$$= \frac{(-3-2)\cdot n(-3)^{n-1} - (-3)^n}{(-3-2)^2}$$

$$= \frac{-5n(-3)^{n-1} - (-3)^n}{25}$$

$$= \frac{-5n(-3)^{n-1}}{25} - \frac{(-3)^n}{25}$$

$$= \frac{-5n(-3)^n}{25\times(-3)^1} - \frac{(-3)^n}{25}$$

$$= \frac{n(-3)^n}{15} - \frac{(-3)^n}{25}$$

$$\frac{z}{(z-2)(z+3)^2} = \frac{Az}{z-2} + \frac{Bz}{z+3} + \frac{Cz}{(z+3)^2}$$

$$= \frac{n(-3)^n}{15} - \frac{(-3)^n}{25}$$

$$y_n = z^{-1}[y]$$
$$= \text{sum of residues}$$
$$= \frac{2^n}{25} + \frac{n(-3)^n}{15} - \frac{(-3)^n}{25}$$

is required solution.

2. Solve using z-transform.
$y_{n+2} - 3y_{n+1} + 2y_n = 4^n$, $y_0 = 0$, $y_1 = 1$.

Given, difference eqn is:

$$y_{n+2} - 3y_{n+1} + 2y_n = 4^n \quad \text{--- (1)}$$
$$y_0 = 0,\quad y_1 = 1 \quad \text{--- (2)}$$

Applying z-transform on (1)

$$z(y_{n+2}) - 3(y_{n+1}) + 2z[y_n] = z(4^n)$$

or, $z^2\left[\bar{y} - y_0 - \frac{y_1}{z}\right] - 3z\left[\bar{y} - y_0\right] + 2\bar{y} = \frac{z}{z-4}$

$$z^2\left[\bar{y} - 0 - \frac{y_1}{z}\right] - 3z\left[\bar{y} - 0\right] + 2\bar{y} = \frac{z}{z-4}$$

or, $z^2\left(\bar{y} - \frac{y_1}{z}\right) - 3z\bar{y} + 2\bar{y} = \frac{z}{z-4}$

or, $z^2\bar{y} - 3z\bar{y} + 2\bar{y} = \frac{z}{z-4} + z$

$$\bar{y}\,(z^2 - 3z + 2) = \frac{z + z^2 - 4z}{z-4}$$

or, $\bar{y}\,[z^2 - 2z - z + 2] = \frac{z^2 - 3z}{z-4}$

$$\frac{2^n \times (-1)}{(-2)} = -2^n$$

or, $\bar{y}\,[(z-1)(z-2)] = \dfrac{z^2-3z}{z-4}$

$$\bar{y} = \frac{z^2-3z}{(z-1)(z-2)(z-4)}$$

Here, $F(z)$ has poles

let, $F(z) = \dfrac{z^2-3z}{(z-1)(z-2)(z-4)}$

Clearly, $F(z)$ has poles at $z = 1, 2, 4$.

now,

$$z^{n-1}F(z) = z^{n-1}\left(\frac{z^2-3z}{(z-1)(z-2)(z-4)}\right)$$

$$= \frac{z^n(z-3)}{(z-1)(z-2)(z-4)}$$

$$z^{n-1}F(z) = \frac{z^{n+1}-3z^n}{(z-1)(z-2)(z-4)}$$

Here, $z^{n-1}F(z)$ has poles at

$z=1$, $z=2$ & $z=4$, of simple order.

now,

from cauchy residue theorem,

$$\operatorname{Res}_{z=1} = \lim_{z\to 1}\left[(z-1)\,z^{n-1}F(z)\right]$$

$$= \lim_{z\to 1}\left[(z-1)\frac{(z^{n+1}-3z^n)}{(z-1)(z-2)(z-4)}\right]$$

$$= \frac{(1)^{n+1}-3\cdot 1^n}{(1-2)(1-4)}$$

$$= \frac{1-3}{(-1)(-3)}$$

$$= \frac{-2}{3}$$

Also,

$$\operatorname{Res}_{z=2} = \lim_{z\to 2}\left[(z-2)\,z^{n-1}F(z)\right]$$

$$= \lim_{z\to 2}\left[(z-2)\cdot\frac{(z^{n+1}-3z^n)}{(z-1)(z-2)(z-4)}\right]$$

$$= \frac{2^{n+1}-6^n}{1\times -2}$$

$$= \frac{2^n(2^1-3)}{1\times(-2)}$$

$$= \frac{2^n\times(-1)}{-2} = \frac{\cancel{+}2^n}{\cancel{+}2}$$

$$= 2^{n-1}$$

$$\operatorname*{Res}_{z=4}\left(z^{n-1}F(z)\right) = \lim_{z\to 4}(z-4)\, z^{n-1}f(z)$$

$$= \lim_{z\to 4} \frac{(z^{n+1} - 3z^{n})}{(z-1)(z-2)}$$

$$= \frac{4^{n+1} - 3\times 4^{n}}{3\times 2}$$

$$= \frac{4^{n}(4-3)}{3\times 2}$$

$$= \frac{4^{n}}{6}$$

$$= \frac{(2^{2})^{n}}{2\times 3}$$

$$= \frac{2^{2n}}{2\times 3}$$

$$= \frac{2^{2n-1}}{3}$$

now,

$$y_n = z^{-1}(\bar{y}) = \text{sum of all residues}$$

$$= \frac{-2}{3} + 2^{n-1} + \frac{2^{2n-1}}{3}$$

3 Solve using z- transform.

$y_{n+2} + 2y_{n+1} + y_n = n$ with $y_0 = y_1 = 0$.

we have,

$y_{n+2} + 2y_{n+1} + y_n = n$ —— (i)

$y_n = y_1 = 0$ —— (ii)

Applying z- transform on both side,

$z[y_{n+2}] + 2z[y_{n+1}] + z[y_n] = z[n]$

$z^2\left[\bar{y} - y_0 - \frac{y_1}{z}\right] + 2z[\bar{y} - y_0] + \bar{y}0 = \frac{z}{(z-1)}$

or, $z^2\left[\bar{y}-y_0-\frac{1}{2}\right]+2z\left[\bar{y}-y_0\right]+\bar{y}=\frac{z}{(z-1)^2}$

or, $z^2\bar{y}-z^2y_0-z+2z\bar{y}-2zy_0+\bar{y}=\frac{z}{(z-1)^2}$

or, $\bar{y}(z^2-2z+1)=\frac{z}{(z-1)^2}+z^2y_0+2zy_0$

$\bar{y}(z-1)^2=\frac{z}{(z-1)^2}+\cancel{z}^{2\times 0}+2z\times 0$

$\bar{y}(z-1)^2=\frac{z}{(z-1)^2}$

or, $\bar{y}=\frac{z}{(z-1)^4}$

now,

Bilinear map

The mapping of the form $w=\frac{az+b}{cz+d}$ is called bilinear map where a,b,c,d are constants and $cz+d\neq 0$.

Conformal mapping

Let w and z be two complex planes then the mapping of z on w plane is called transformation or mapping. If the mapping preserves angle between the curves both in magnitude and sense is called conformal mapping.

Cross ratio

The cross ratio of the four points: z_1,z_2,z_3,z_4 is defined by

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Determination of bilinear transformation

If z_1,z_2,z_3 map into w_1,w_2,w_3 in w-plane then the bilinear map is given by the transformation.

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Example:

1. Find the bilinear transformation which maps the points $z_1 = 2$, $z_2 = i$ and $z_3 = -2$ into the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$

Soln;

To determine bilinear map, we know that,

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$$\Rightarrow (w-1)(i+1)(2-i)(-2-z) = (z-2)(i+2)(1-i)(-1-w)$$

$$\Rightarrow (w-1)[(-2-z)(2i-i^2+2-i)] = (z-2)[(-1-w)(i-i^2+2-2i)]$$

$$\Rightarrow (w-1)[(-2-z)(i+3)] = (z-2)[(-1-w)(-i+3)]$$

$$\Rightarrow (w-1)[-2i-6-zi-3z] = (-1-w)[-zi+3z+2i-6]$$

$$\Rightarrow w(-2i-6-zi-3z) - 1(-2i-6-zi-3z) = -1(-zi+3z+2i-6) - w(-zi+3z+2i-6)$$

$$\Rightarrow w(-2i-6-zi-3z) + w(-zi+3z+2i-6) = -1(-zi+3z+2i-6) + 1(-2i-6-zi-3z)$$

$$\Rightarrow w(-2i-6-zi-\cancel{3z}-zi+\cancel{3z}+2i-6) = \cancel{1} -2i-6-zi-3z+zi-3z-2i+\cancel{6}$$

$$\Rightarrow w(-12-2zi) = (-4i-6z)$$

$$w = \frac{-6z-4i}{-12-2zi} = \frac{-2(3z-2i)}{-2(6+zi)} = \frac{3z+2i}{6+zi}$$

is required bilinear transformation.

$i(1-2-2i+z)$

i^2-1-2i^2+2i

$-1-i+2+2i$

2. Find the bilinear transformation which maps the points $z_1 = 0$, $z_2 = 1$, & $z_3 = \infty$ into the points $w_1 = i$, $w_2 = -1$ & $w_3 = -i$.

To determine bilinear ~~eqn~~ map, we know that,

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\text{or, } \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{\cancel{z_3}(z-z_1)(z_2/z_3-1)}{\cancel{z_3}(z_1-z_2)(1-z/z_3)}$$

$$\text{or, } \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-0)(1/\infty-1)}{(0-1)(1-z/\infty)}$$

$$\Rightarrow \frac{(w-i)(-1+i)}{(i+1)(-i-w)} = \frac{-z}{-1} \quad \left[\because \frac{\text{finite}}{\infty} = 0\right]$$

$$\frac{(w-i)(-1+i)}{(i+1)(-i-w)} = z$$

or, $(w-i)(-1+i) = z(i+1)(-i-w)$

or, $w(-1+i) - i(-1+i) = (zi+z)(-i-w)$

or, $w(-1+i) - i(-1+i) = -i(zi+z) - w(zi+z)$

or, $w(i-1) - i(i-1) = -i(zi+z) - w(zi+z)$

or, $w(i-1) + w(zi+z) = i(i-1) + (zi+z)i$

or, $w(i-1+zi-z) = i(i-1+zi+z)$

or, $w(i(1+z)-1+z) = -1-i+z-zi$

$$w = \frac{-1-i+z-zi}{i(1+z)-1+z}$$

is required bilinear transformation.

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Partial differential equations

An equation involving one or more partial derivatives of unknown function of two or more dependent variables is called partial differential equations.

Linear partial differential equations.

A partial differential eqn is said to be linear if it is of first degree in the dependent variable and its partial derivative.

Some important PDE

1. One dimensional wave eqn $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
2. One dimensional heat eqn $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
3. Two dimensional laplace eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
4. Two dimensional heat eqn $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$
5. Two dimensional wave eqn $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$
6. Three dimensional laplace eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

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Examples

Verify the given function to satisfy one dimensional wave eqn.

$u = \sin 9t . \sin \frac{x}{4}$ ——— (1)

Diff. partially,

$\frac{\partial u}{\partial t} = 9 \cos 9t . \sin \frac{x}{4}$

$\frac{\partial^2 u}{\partial t^2} = -81 \sin 9t . \sin \frac{x}{4}$

Also,

$\frac{\partial u}{\partial x} = \frac{1}{4} \sin 9t . \cos \frac{x}{4}$

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{16} \sin 9t . \sin \frac{x}{4}$ ——— (2)

One dimensional wave eqn is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

using (1) & (2)

$-81 \sin 9t \sin \frac{x}{4} = c^2 \left[-\frac{1}{16} \sin 9t \sin \frac{x}{4} \right]$

$\Rightarrow c^2 = 81 \times 16$

$c^2 = 9^2 \times 4^2$

$c^2 = \pm 36$

The given eqn satisfy one dimensional wave eqn taken $c = \pm 36$.

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2. Verify the given function to satisfy one dimensional heat eqn.

$u = e^{-t} \sin x$ $\qquad \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

Diff. partially,

$\frac{\partial u}{\partial t} = -e^{-t} \sin x$

Again,

$\frac{\partial u}{\partial x} = e^{-t} \cos x$

$\frac{\partial^2 u}{\partial x^2} = -e^{-t} \sin x$

One dimensional heat eqn is:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$-e^{-t} \sin x = c^2 (-e^{-t} \sin x)$

$c^2 = 1$

$c = \pm 1$

$\therefore$ u satisfy one dimensional heat equation if $c = \pm 1$.

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Solve the following PDE.

1. $u_y = u$

$\Rightarrow \frac{du}{dy} = u$

$\Rightarrow \frac{du}{u} = dy$

Integrating,

$\ln u = y + \ln c$

$\ln u - \ln c = y$

$\ln\left(\frac{u}{c}\right) = y$

$\frac{u}{c} = e^y$

$u = ce^y$ is required soln (where c is function of x or constant).

2. $u_{yy} = 0$

$\frac{d^2u}{dy^2} = 0$

Integrating,

$\frac{du}{dy} = c$

$u = x^2 + 2y$

$\frac{du}{dy} = 2$

$du = 2dy$

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$u = 2y + c(x) \rightarrow$ may contain x to term.

Integrating again,

$u = c(x) . y + D(x)$

where $c(x)$ and $D(x)$ are function of x or constant.

3. $u_{xy} = u_x$ —— (1)

let $u_x = v$ then $(u_x)_y = v_y$.

now (1) becomes,

$v_y = v$

$\frac{dv}{dy} = v$

$\Rightarrow \frac{dv}{v} = dy$

Integrating,

$\ln v = y + \ln c$

$\ln v - \ln c = y$

$\ln\left(\frac{v}{c}\right) = y$

$\frac{v}{c} = e^y$

$v = ce^y$

$u_x = ce^y$

$\frac{du}{dx} = ce^y$

$du = ce^y dx$

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Integrating

$u = Ce^{f} \cdot x + D$

where,

C & D are function of x or constant.

4. $u_y = 2xyu$

$\frac{\partial u}{\partial y} = 2xyu$

$\frac{\partial u}{u} = 2xy\,dy$

Integrating,

$\ln u = 2x \cdot \frac{y^2}{2} + \ln c(x)$

$\ln\left[\frac{u}{c(x)}\right] = xy^2$

$\frac{u}{c(x)} = xy^2$

$u = xy^2 C(x)$ is required soln.

5. $u_{xx} - u = 0$

$\frac{\partial^2 u}{\partial x^2} - u = 0$

Its auxilliory eqn is:

$m^2 - 1 = 0$

$m = \pm 1$

The required soln is

$u = A(y)e^{x} + B(y)e^{-x}$

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6. Solve:

$3u_x + 2u_y = 0$

$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ —— (1)

let $u = F(x)G(y)$ —— (2) be soln of (1).

where $F(x)$ is function of x only & $G(y)$ is function of y only.

now,

$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial x} \cdot G$ —— (3)

$\frac{\partial u}{\partial y} = \frac{\partial G}{\partial y} \cdot F$ —— (4)

using (3) & (4) in (1),

$3 \cdot \frac{\partial F}{\partial x} \cdot G + 2 \cdot \frac{\partial G}{\partial y} \cdot F = 0$

$3\frac{\partial F}{\partial x} \cdot G = -2\frac{\partial G}{\partial y} \cdot F$

$\frac{3}{F}\frac{\partial F}{\partial x} = -\frac{2}{G}\frac{\partial G}{\partial y} = k$

now,

$\frac{\partial F}{F} = \frac{k}{3}\partial x$ (from 1st ratio = k)

$\frac{\partial G}{G} = \frac{-k}{2}dy$ (from 2nd ratio = k)

Integrating above eqns:

$\ln G = \frac{K}{2} y + c_2 \Rightarrow G = e^{-K/2\, y + c_2}$

$\ln F = \frac{K}{3} x + c \Rightarrow F = e^{K/3\, x + c_1}$

Using above values of F & G in eqn ②, we get,

$u = e^{(K/3\, x + c_1)} \cdot e^{(-K/2\, y + c_2)}$

$= e^{\left(\frac{Kx}{3} - \frac{Ky}{2}\right)} \cdot e^{(c_1 + c_2)}$

$u = c e^{K(x/3 - y/2)}$

where,

$c = e^{c_1 + c_2}$ is required solution.

(continue ----)

Case I : $(K > 0)$

let $K = \lambda^2$ then

from ⑥,

$F'' - \lambda^2 F = 0$

It's auxiliary eqn is $m^2 - \lambda^2 = 0$

$\Rightarrow m^2 = \pm \lambda$

The soln is $F = Ae^{\lambda x} + Be^{-\lambda x}$ —— ⑧

Also, from ⑦, $G'' - c^2 \lambda^2 G = 0$

It's auxiliary equation is $m^2 - c^2\lambda^2 = 0$

$\Rightarrow m = \pm c\lambda$.

The soln is $G = Ce^{\lambda t} + De^{-c\lambda t}$ —— ⑨

now, from eqn ⑤, ⑧ & ⑨.

$u(x, t) = F(x)\, G(t)$

$= (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{c\lambda t} + De^{-c\lambda t})$ ⑩

This soln is not compatible with wave because wave motion is periodic in nature.

$Ae^{m_1 x} + Be^{m_1 x}$.

Case II : $(K < 0)$

let $K = -\lambda^2$ then,

from ⑥,

$F'' + q^2 F = 0$ —— ⑪

It's auxiliary eqn is $m^2 + q^2 = 0$

$\therefore m = \pm iq$

The soln is $F = A\cos qx + B\sin qx$ — (12)

Again,

from (7)

$G'' + c^2 q^2 G = 0$ —— (13)

It's auxiliary eqn is $m^2 + c^2 q^2 = 0$.

$\Rightarrow m = \pm icq$

now, soln is $G = C\cos qct + D\sin qct$ — (14) $m = \pm iq$

from (5)

$u(x,t) = F(x)\, G(t)$

$= (A\cos qx + B\sin qx)(C\cos qct + D\sin qct)$

is required soln.

case III $(k=0)$

From (6)

$F'' = 0$ $F'' = 0$

$\Rightarrow F = Ax + B$ $d^2F/dx^2 = 0$

from (7) $G'' = 0 \Rightarrow G = Ct + D$ $dF/dx = A$

from (5) $u(x,t) = (Ax + B)(Ct + D)$ — (16) $F = Ax + B$

This eqn doesn't fulfill the condn of wave motion because it is not periodic in nature.

Since wave motion is periodic in nature, the feasible solution is eqn (15)

$u(x,t) = (A\cos qx + B\sin qx)(C\cos qct + D\sin qct)$

It is general soln of one dimensional wave equation.

now, the boundary conditions are:

$u(0,t) = 0$ $u(l,t) = 0$

If the soln $u(x,t) = F(x)\, G(x,t)$ satisfy boundary conditions

$u(0,t) = F(0)\, G(t)$

$= 0$

Since, $G(t) \neq 0$ we have $F(0) = 0$

Also, $u(l,t) = F(l)\, G(t) = 0$

Since, $G(t) \neq 0$, $F(l) = 0$

now, for $k \geq 0$, $u = 0$

for $k < 0$ or $k = -q^2$ the general soln of (6) is

$F = A\cos qx + B\sin qx$ —— (17)

now,

$F(0) = 0 \Rightarrow A = 0$

$F(l) = 0 \Rightarrow B\sin ql = 0$

$\Rightarrow \sin ql = 0$

$\Rightarrow ql = n\pi$

$\Rightarrow q = n\pi / l$ $(n \in \mathbb{Z})$

So, $F(x) = B\sin \frac{n\pi x}{l}$ [from (17)]

Set $B = 1$ then $F(x) = F_n(x) = \sin \frac{n\pi x}{l}$ — (18)

Also, general soln of (7) is

$$G = C\cos qct + D\sin qct$$

since $q = \frac{n\pi}{L}$,

$$G(t) = G_n(t) = C\cos\frac{cn\pi}{L}t + D\sin\frac{cn\pi}{L}t$$

$$G_n(t) = B_n\cos\lambda_n(t) + B_n^*\sin\lambda_n t \quad \text{—} \quad (9)$$

where $\lambda_n = \frac{cn\pi}{L}$ & $B_n = C$ & $B_n^* = D$

now, the required soln is,

$$u_n(x,t) = F_n(x)G_n(t)$$
$$= (B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin\frac{n\pi x}{L}$$

These functions are called the eigen functions or characteristics functions and the values $\lambda_n = \frac{cn\pi}{L}$ are called eigen values of vibrating string.

The set $(\lambda_1, \lambda_2, ----)$ is called spectrum.

As a single soln $u_n(x,t)$ will not satisfy the initial condition, so we consider series of these eigen functions.

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$$u(x,t) = \sum_{n=1}^{\infty}(B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin\frac{n\pi x}{L}$$

This result is obtained through series of deriv-atives with boundary conditions where $\lambda_n = \frac{cn\pi}{L}$.

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

is the soln of the wave eqn satisfying both conditions.

-

The soln of one dimensional ~~value~~ wave eqn

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$ is,

$$u = \sum_{n=1}^{\infty}(B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin\frac{n\pi x}{L}$$

where $\lambda_n = \frac{cn\pi}{L}$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$u(x,0) = f(x) = $ initial deflection

$\frac{\partial u}{\partial t}\Big|_{t=0} = g(x) = $ initial velocity

Examples

1. A tight string of length 20cm fastened at both ends is displaced from its position imparting to each of its points an initial velocity

$$g(x) = \begin{cases} x & \text{for } 0 \le x < 10 \\ 20-x & \text{for } 10 \le x \le 20 \end{cases}$$

where x being distance from one end. Find the displacement at any time t.

soln, we have soln of one dimensional wave eqn.

$$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t)\sin\frac{n\pi x}{L}$$

where,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= 0 \quad \left[\because \text{ If } g(x) \text{ is given then } f(x) = 0\right]$$

and,

$$B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{cn\pi}\int_0^{20} g(x)\sin\frac{n\pi}{20}x\,dx$$

$$\left[\because L = 20 \;\&\; \lambda_n = x \cdot \frac{cn\pi}{L} = cn\pi\right]$$

$$= \frac{2}{cn\pi}\left[\int_0^{10} x\sin\frac{n\pi}{20}x\,dx + \int_{10}^{20}(20-x)\sin\frac{n\pi}{20}x\,dx\right]$$

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$

$$= \frac{2}{cn\pi}\left[\left\{x\left(\frac{-\cos\frac{n\pi}{20}x}{\frac{n\pi}{20}}\right) - 1\cdot\left(\frac{-\sin\frac{n\pi}{20}x}{\left(\frac{n\pi}{20}\right)^2}\right)\right\}_0^{10} + \left\{(20-x)\left(\frac{-\cos\frac{n\pi}{20}x}{\frac{n\pi}{20}}\right) - (-1)\frac{-\sin\frac{n\pi}{20}x}{\left(\frac{n\pi}{20}\right)^2}\right\}_{10}^{20}\right]$$

$$= \frac{2}{cn\pi}\left[\left\{\frac{-20x}{n\pi}\cos\frac{n\pi}{20}x + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{20}x\right\}_0^{10} + \left\{\frac{-20(20-x)}{n\pi}\cos\frac{n\pi}{20}x - \frac{400}{n^2\pi^2}\sin\frac{n\pi}{20}x\right\}_0^{20}\right]$$

$$= \frac{2}{cn\pi}\left[\left\{-\frac{200}{n\pi}\cos\left(\frac{n\pi}{20}\times 10\right) + \frac{400}{n^2\pi^2}\sin\left(\frac{n\pi}{20}\times 10\right)\right\} + \left\{0 - \left(-\frac{20\times 10}{n\pi}\cos\left(\frac{n\pi}{20}\times 10\right) - \frac{400}{n^2\pi^2}\sin\left(\frac{n\pi}{20}\times 10\right)\right)\right\}\right]$$

$$= \frac{2}{cn\pi}\left[\cancel{-\frac{200}{n\pi}\cos\frac{n\pi}{2}} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{200}{n\pi}\cos\frac{n\pi}{2}} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{cn\pi}\times 2\times\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$B_n^* = \frac{1600}{cn^3\pi^3}\sin\frac{n\pi}{2}$$

Using these values of B_n & B_n^* in (1) we get, Note: If initial velocity = 0 then $g(x) = 0$

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$$u(x,t) = \sum_{n=1}^{\infty} \frac{1600}{Cn^3\pi^3} \sin\frac{n\pi}{2} \cdot \sin\lambda_n t \sin\frac{n\pi}{20} x.$$

is reqd. soln of given problem.

2. Find $u(x,t)$ of the string of length $L = \pi$ when $c^2 = 1$, the initial velocity is zero and the initial deflection $(0.1)\, x(\pi - x)$.

Soln.

We know, soln of one dimentional wave eqn is,

$$u(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right) \sin\frac{n\pi x}{L} \quad \text{—(1)}$$

where,

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x\, dx$$

$$B_n^* = \frac{2}{L\lambda_n}\int_0^L g(x) \sin\frac{n\pi}{L} x\, dx$$

and $g(x) = 0$

Since initial velocity $g(x) = 0$, we have,

$B_n^* = 0$

now,

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi}{L} x\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (0.1)\, x(\pi - x) \sin\frac{n\pi}{\pi} x\, dx$$

$\int u\, v\, dx = uv_1 - u'v_2 + u''v_3 - \ldots$

$$= \frac{0.2}{\pi}\int_0^{\pi} \underset{(u)}{(\pi x - x^2)}\, \underset{(v)}{\sin nx}\, dx$$

$$= \frac{0.2}{\pi}\left[(n\pi - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) - (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{0.2}{\pi}\left[-2\frac{\cos n\pi}{n^2} - (-2)\frac{\cos 0}{n^3}\right]$$

$$= \frac{0.2}{\pi}\left[-\frac{2\cos n\pi}{n^3} + \frac{2}{n^3}\right]$$

$$= \frac{0.2}{\pi n^3}\left(2 - 2\cos n\pi\right)$$

$$= \frac{2 \times 0.2}{\pi n^3}\left(1 - \cos n\pi\right)$$

$$= \frac{0.4}{\pi n^3}\left(1 - \cos n\pi\right)$$

now from (1)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{0.4}{\pi n^3}\left(1 - \cos n\pi\right) \cos\lambda_n t \sin\frac{n\pi}{\pi} x$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{0.4}{\pi}\left(\frac{1 - \cos n\pi}{n^3}\right) \cos nt \sin nx \quad \text{—(2)}$$

$$\left(\because \lambda_n = \frac{cn\pi}{L} = \frac{1 \cdot n\pi}{\pi} = n\right)$$

Eqn (2) is reqd. soln,

3. Find $u(x,t)$ of the string of length $l=\pi$ when $c^2=1$, the initial velocity is zero and the initial deflection is $k(\sin x - 1/2 \sin 2x)$

Soln;

$f(x) = k(\sin x - 1/2 \sin x)$ (initial deflection)

$l=\pi$, $c^2=1$ and initial velocity $g(x)=0$

Also,

$\lambda_n = \frac{cn\pi}{l} = \frac{cn\pi}{\pi} = cn = \pm n$ $\left(\because \begin{array}{l} c^2=1 \\ c=\pm 1 \end{array}\right)$

The soln of one dimensional wave eqn is,

$u(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{l} x$ —(1)

where,

$B_n^* = 0$ ($\because g(x)=0$) & $B_n = \frac{2}{\lambda n l} \int_0^l f(x) \sin \frac{n\pi}{l} x\, dx$

now (1) becomes,

$u(x,t) = \sum B_n \cos \lambda_n t \sin n \frac{\pi x}{l}$ $(\because B_n^* = 0)$

$= \sum B_n \cos (\pm nt) \sin n \frac{\pi x}{\pi}$ $\left[\because \begin{array}{l} \lambda_n = \pm n \\ l=\pi \end{array}\right]$

$u(x,t) = \sum B_n \cos nt \sin x$ —(2)

$u(x,0) = \sum B_n \cos 0 \sin nx$

$k\left(\sin x - \frac{1}{2}\sin 2x\right) = \sum_{n=1}^{\infty} B_n \sin n\pi$ $\left[\because u(x,0) = \text{initial deflection}\right]$

$k \sin x - \frac{k}{2} \sin 2x = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3nt$ ---

comparing coefficients of $\sin x, \sin 2x, \sin 3x$ ———

we get,

$B_1 = k$, $B_2 = -k/2$, $B_3 = B_4 = B_5 = \cdots = 0$

from (2)

$u(x,t) = \sum_{n=1}^{\infty} B_n \cos nt \sin nx$

$u(x,t) = B_1 \cos t \sin x + B_2 \cos 2t \sin 2x + 0 + 0$ ---

$= k \cos t \sin x - \frac{k}{2} \cos 2t \sin 2x$

is the required soln.

Example:

Solve by seperating variables.

1. $u_{xy} - u = 0$

Soln;

let the soln of (1) be $u(x,y) = F(x)G(y)$

where $F(x)$ is function of x only &

$G(y)$ is function of y only.

Now,

$u_x = \frac{dF}{dx} G(y)$

$u_x = \dot{F}G$

Again diff. partially w.r. to y

$u_{xy} = \dot{F}G'$ where dot & prime represent partial derivative w.r. to x & y respectively.

Using this value in (1),

$\dot{F}G' - FG = 0$

$\dot{F}G' = FG$

$\frac{\dot{F}}{F} = \frac{G}{G'} = k$ (say)

$\Rightarrow \frac{\dot{F}}{F} = k$ & $\frac{G}{G'} = k$

Integrating.

$\log F = kx + \log c$

$\log F - \log c = kx$

$\log\left(\frac{F}{c}\right) = kx$

$\frac{F}{c} = e^{kx}$

$F = ce^{kx}$

$\Rightarrow \frac{G'}{G} = \frac{1}{k}$

Integrating

$\log G = \frac{1}{k} y + \log D$

$\log G - \log D = \frac{y}{k}$

$\log\left(\frac{G}{D}\right) = \frac{y}{k}$

$\frac{G}{D} = e^{y/k}$

$G = De^{(y/k)}$

now (1) becomes,

$u(x,y) = ce^{kx} \cdot De^{y/k}$

$= cDe^{kx + y/k}$

$= Ee^{(kx + y/k)}$

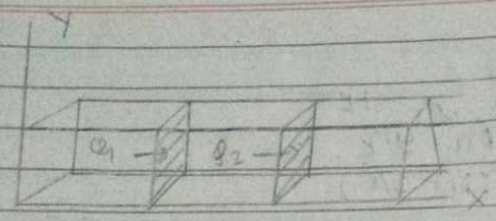
(where $E = cD$ is constant) is required soln.

Derivation of one dimensional Heat equation.

$$\left(\frac{du}{dt} = c^2 \frac{\partial^2 u}{\partial x^2}\right)$$

consider the temperature in a long thin bar oriented along x-axis and is perfectly insulated laterally so that heat,

- the heat flows from higher to lower temperature
- the amount of heat is proportional to the mass and change in temperature.
- the constant of proportionality is the specific heat capacity of the material.
- The rate at which heat flows is proportional to the area & temp gradient normal to the area.
- The constant of proportionally is known as thermal conductivity of the material.
- The heat flows in linear direction along x-axis & loss of heat through other surface is negligi-ble.



Let us suppose the flow of heat by conduction in uniform bar. Let the sides of bar are insulated so that the loss of heat by conduction or radiation is negligible.

Let one end of bar as origin & direction of flow is positive x-axis. The temperature u at any point depends on distance x and the time t. Also the temperature at all points of any cross section is same.

The quantity of heat flowing into the section at the distance is,

$$Q_1 = -KA\left(\frac{\partial u}{\partial x}\right)_x \text{ per second}$$ [As x increase u decrease so $-$ve sign is taken]

Again, quantity of heat flowing out of the section at distance $x+\Delta x$ is,

$$Q_2 = -KA\left[\frac{\partial u}{\partial x}\right]_{x+\Delta x} \text{ per second.}$$

So, total amount of heat retained by the slab with thickness Δx is:

$$Q_1 - Q_2 = -KA\left[\frac{\partial u}{\partial x}\right]_x + KA\left[\frac{\partial u}{\partial x}\right]_{x+\Delta x}$$

$$\left.\frac{f(x+\Delta x)-f(x)}{\Delta x}\right|$$

ρ rho
$m = v\rho$

$$= KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right] \quad \text{— (1)}$$

The rate of change of heat in the slab is,

$$s\rho A\,\Delta x\,\frac{\partial u}{\partial t} \quad \text{— (2)}$$

Where s is specific heat & ρ is density of material from (1) & (2)

$$s\rho A\,\Delta x\,\frac{\partial u}{\partial t} = KA\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

Taking lim Δx we get,

$$s\rho\,\frac{\partial u}{\partial t} = K\,\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = \frac{K}{s\rho}\,\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2}$$ (where $c^2 = \frac{K}{s\rho}$ is called diffusivity of material)

is reqd. one dimensional heat eqn.

Solution of one dimensional heat equation under boundary conditions $u(0,t)=0$, $u(l,t)=0$ is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x\, e^{-\lambda_n^2 t}$$

where $\lambda_n = \frac{cn\pi}{l}$

& $B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x\,dx$

Proof:

The one dimensional heat eqn is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{——} \ (1)$$

where $u(x,t)$ represents the temperature at x in time t. We assume that the ends of bar are kept at temperature 0 (zero) so that,

$$u(0,t) = 0 \ \ \& \ \ u(L,t) = 0 \quad \forall t \quad \text{——} \ (2)$$

and the initial condition is $u(x,0) = f(x)$ — (3)

Let $u(x,t) = F(x)\,G(t)$ be the soln of (1).

now,

$\frac{\partial u}{\partial t} = F\dot{G}$ and $\frac{\partial^2 u}{\partial x^2} = F''G$ where dot(.) & prime(') represents differential w.r.to t & x. from (1),

$$F\dot{G} = c^2 F'' G$$

$$\frac{F''}{F} = \frac{\dot{G}}{c^2 G} = k \text{ (say)} \quad \text{——} \ (5)$$

then,

$$F'' = kF \quad \text{——} \ (6)$$

and,

$$\dot{G} = kc^2 G$$

Case 1 $(k > 0)$

let $k = p^2$ then,

$$F'' = p^2 F \quad \text{——} \ (6)$$

$$F'' - p^2 F = 0 \quad \text{——} \ (7)$$

It's auxiliary eqn is;

$$m^2 - p^2 = 0$$

$$m = \pm p$$

$$\therefore F = Ae^{px} + Be^{-px}$$

from (7).

$$\dot{G} = p^2 c^2 G$$

$$\frac{\dot{G}}{G} = p^2 c^2$$

Integrating $\ln G = p^2 c^2 t + \ln E$

$$\ln \frac{G}{E} = p^2 c^2 t$$

$$\Rightarrow \frac{G}{E} = e^{p^2 c^2 t}$$

$$G = E e^{p^2 c^2 t}$$

$$\therefore u(x,t) = F(x)\,G(t)$$

$$= (Ae^{px} + Be^{-px})\,E e^{p^2 c^2 t} \quad \text{——} \ (8)$$

case II $(k < 0)$

let $k = -p^2$

from (6)

$$F'' = -p^2 F$$

$$F'' + p^2 F = 0$$

from (7)

$$\dot{G} = -p^2 c^2 G$$

$$\frac{\dot{G}}{G} = -p^2 c^2$$

It's auxiliary eqn is

$m^2 + p^2 = 0$

$m = \pm pi$

$\therefore F = A\cos px + B\sin px$

$\ln G = -p^2c^2t + \ln B_n$

$\left(\frac{G}{B_n}\right) = e^{-p^2c^2t}$

$G = B_n e^{-p^2c^2t}$

$\therefore u(x,t) = (A\cos px + B\sin px) B_n e^{-p^2c^2t}$ —— (9)

case III $(K=0)$

from (6) and (7),

$F'' = 0$ and $\dot{G} = 0$

$F' = A$ $\quad G = C$

$F = Ax + B$

$\therefore u(x,t) = C(Ax+B)$ —— (10)

Since u doesn't become infinite as $t \to \infty$, the feasible solution is $u(x,t) = (A\cos px + B\sin px) B_n e^{-p^2c^2t}$.

now we solve eqn (6) & (7) under boundary & initial conditions.

The boundary conditions are:

$u(0,t) = F(0)G(t) = 0$. Since $G(t) \neq 0$. So,

$F(0) = 0$.

Secondary boundary condition gives $F(L) = 0$.

For $K = -p^2$ the general soln of (6) is,

$F(x) = A\cos px + B\sin px$

now,

$F(0) = 0 \Rightarrow A = 0$.

and $F(L) = 0 \Rightarrow B\sin pL = 0$

$\Rightarrow \sin pL = 0$

$\Rightarrow p = \frac{n\pi}{L}$

then,

$F'' = KF$ —— (6) and $\dot{G} = Kc^2G$ —— (7)

$\therefore F(x) = F_n(x) = B\sin\frac{n\pi x}{L}$

$= \sin\frac{n\pi x}{L}$ (setting $B = 1$)

The getting general soln of (7) is,

$G(t) = B_n e^{-p^2c^2t}$

Since $p = \frac{n\pi}{L}$ so, $G(t) = G_n(t) = B_n e^{\left(-\frac{n\pi c}{L}\right)^2 t}$

or, $G(t) = G_n(t) = B_n e^{-\lambda_n^2 t}$ where $\lambda_n = \frac{n\pi c}{L}$

So, $u(x,t) = F(x)G(t)$.

$u(x,t) = F_n(x)G_n(t)$

$= B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}$

This is general soln of (1) satisfying initial condition consider infinite series.

To obtain the soln satisfying initial condition consider finite series.

$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} e^{-\lambda_n^2 t}$ —— (11)

here,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

where $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$ — (12)

The eqn (1) represents the soln of the heat eqn satisfying boundary as well as initial condition with the value of B_n given by eqn (2).

Example

1. Find the temperature $u(x,t)$ in a bar of silver [length 10cm, constant cross sectional area 1cm², density 10.6 gm/cm², thermal conductivity 1.04 cal/cm sec°c, specific heat 0.056 whose ends are kept at temperature 0°c and whose initial temperature (in °c) is $f(x)$ where $f(x) = K \sin(0.2)\pi x$.

Soln,

$K = 1.04$, $s = 0.056$, $\rho = 10.6$, $L = 10cm$

and $f(x) = K \sin(0.2)\pi x$.

now,

$$c^2 = \frac{K}{s\rho}$$

$$= \frac{1.04}{(0.056)(10.6)}$$

$$= 0.175 \text{ cm}^2/\text{sec}$$

We know, the soln of one dimensional heat eqn is,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t}$$ — (1)

where, $B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$ — (2)

and,

$$\lambda_n = \frac{cn\pi}{L}$$

$$= \frac{\sqrt{0.175}}{10} n\pi$$

$$= 0.041 n\pi$$

⟶

1-D heat eqn

$\frac{du}{dt} = c^2\frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{K}{S\rho}$

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now from ②

$B_n = \frac{2}{10}\int_0^{10} K\sin(0.2)\pi x \sin\frac{n\pi}{10}x\,dx$

$= \frac{K}{10}\int_0^{10} 2\sin(0.2\pi)x\sin\left(\frac{n\pi}{10}\right)x\,dx$

using $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$B_n = \frac{K}{10}\int_0^{10}\left[\cos\left(0.2\pi - \frac{n\pi}{10}\right)x - \cos\left(0.2\pi + \frac{n\pi}{10}\right)x\right]dx$

$= \frac{K}{10}\left[\frac{\sin\left(0.2\pi - \frac{n\pi}{10}\right)x}{0.2\pi - \frac{n\pi}{10}} - \frac{\sin\left(0.2\pi + \frac{n\pi}{10}\right)x}{0.2\pi + \frac{n\pi}{10}}\right]_0^{10}$

$= \frac{K}{10}\left[\frac{\sin\left(0.2\pi - \frac{n\pi}{10}\right)10}{0.2\pi - \frac{n\pi}{10}} - \frac{\sin\left(0.2\pi + \frac{n\pi}{10}\right)10}{0.2\pi + \frac{n\pi}{10}}\right]$

$= \frac{K}{10}\times 10\left[\frac{\sin(2\pi - n\pi)}{2\pi - n\pi} - \frac{\sin(2\pi + n\pi)}{2\pi + n\pi}\right]$

now ① becomes,

$u(x,t) = \sum_{n=1}^{\infty} K\left[\frac{\sin(2\pi - n\pi)}{2\pi - n\pi} - \frac{\sin(2\pi + n\pi)}{2\pi + n\pi}\right]. \sin\frac{n\pi}{10}x\, e^{-(0.0421n\pi)^2 t}$

is required solution.

2. Find the temperature distribution in a laterally insulated thin copper bar ($c^2 = 1.158\ cm^2/sec$), of 100 cm long and of constant cross section whose end-points at $x=0$ and $x=100$ are kept at 0°c and whose initial temperature is

i) $f(x) = \sin(0.01)\pi x$

ii) $f(x) = \sin^3(0.01)n\pi$

soln;

we have,

$c^2 = 1.158$, $L = 100$,

$u(x,t) = u(100,t)$

i) Solution of one dimensional heat equation is

$u(x,t) = \sum_{n=1}^{\infty} B_n \frac{n\pi}{L}x\, e^{-\lambda_n^2 t}$ —— ①

where, $B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

from ① when $t=0$,

$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{100}x \cdot e^{-\lambda_n^2 \times 0}$.

$f(x) = \sum_{n=1}^{\infty} B_n \sin(0.01)n\pi x$

$\sin(0.01)\pi x = B_1\sin(0.01)n\pi + B_2\sin(0.02)n\pi + \dots$ (given)

comparing coefficients,

$B_1 = 1$, $B_2 = B_3 = \dots = 0$.

Also,

$\lambda_n = \frac{cn\pi}{L} \Rightarrow \lambda_n^2 = \frac{c^2 n^2\pi^2}{L^2} = \frac{1.158}{(100)^2}n^2\pi^2$

now (1) becomes,

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

$$= B_1 \sin \frac{1.\pi x}{100} e^{-\left[\frac{1.158}{(100)^2}\right] 1^2 \pi^2 t}$$

$$= \sin(0.01)n\pi\, e^{-\left[\frac{1.158}{(100)^2}\right]\pi^2 t} \quad (\because B_1 = 1)$$

is read. soln of (i).

$\sin 3x = 3\sin x - 4\sin^3 x$

$\Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$

$\Rightarrow \sin^3 x = \frac{3\sin x - \sin 3x}{4}$

ii). Given that

$$f(x) = u(x,0) = \sin^3(0.02)\pi x$$

$$\sum_{n=1}^{\infty} B_n \sin(0.02)n\pi x = \frac{3\sin(0.02)\pi x - \sin 3(0.02)\pi x}{4}$$

(using (2))

$\Rightarrow B_1 \sin(0.01)n\pi + B_2 \sin(0.02)\pi x + B_3 \sin(0.03)\pi x + \cdots$

$= \frac{3}{4}\sin(0.02)\pi x - \frac{1}{4}\sin(0.03)\pi x$

comparing coefficients,

$B_1 = \frac{3}{4}$, $B_2 = 0$, $B_3 = -\frac{1}{4}$, $B_4 = B_5 = \cdots = 0$

So the read. soln is

$$u(x,t) = \frac{3}{4}\sin(0.01)n x\, e^{-\lambda_1^2 t} - \frac{1}{4}\sin(0.03)\pi x\, e^{-\lambda_3^2 t}$$

(from (3))

where, $\lambda_n^2 = \frac{1.158}{(100)^2} n^2\pi^2$

~~Laplacian in polar~~

Laplacian in polar coordinates:

The Laplacian in cartesian coordinates is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{--- (1)}$$

or, $\nabla^2 u = u_{xx} + u_{yy}$

put, $x = r\cos\theta$, $y = r\sin\theta$ then,

$$r = \sqrt{x^2+y^2} \quad \text{and} \quad \theta = \tan^{-1}(y/x).$$

now,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= u_r r_x + u_\theta \theta_x$$

Again,

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = u_{rr} r_x^2 + u_r r_{xx} + u_{\theta\theta}\theta_x^2 + u_\theta \theta_{xx} \quad \text{--- (2)}$$

now,

$r = \sqrt{x^2+y^2}$

$r^2 = x^2+y^2$

Diff. partially w.r to x,

$2 r r_x = 2x$

$r_x = \frac{x}{r}$

Similarly, $r_y = \frac{y}{r}$

now,

$$r_{xx} = (r_x)_x = \left(\frac{x}{r}\right)_x = \frac{r\frac{\partial x}{\partial x} - x\cdot\frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x\cdot\frac{x}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r\cdot r^2} = \frac{x^2+y^2-x^2}{r^3}$$

$$= \frac{y^2}{r^3}$$

Similarly,

$$r_{yy} = \frac{x^2}{r^3}$$

and $\theta_x = \left(\tan^{-1} y/x\right)_x = \frac{1}{1+(y/x)^2}\cdot y\cdot\left(\frac{-1}{x^2}\right)$

$$\Rightarrow \theta_x = \frac{-y}{x^2+y^2} = \frac{-y}{r^2}$$

now,

$$\theta_{xx} = \frac{r^2\cdot 0 - (-y)\,2r r_x}{(r^2)^2} = \frac{2\cdot r\cdot y\cdot\frac{x}{r}}{r^4} = \frac{2xy}{r^4}$$

Also,

$$\theta_y = \left[\tan^{-1}(y/x)\right]_y = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x}$$

$$= \frac{x}{x^2+y^2}$$

$$\theta_y = \frac{x}{r^2}$$

and, $\theta_{yy} = \left(\frac{x}{r^2}\right)_y = \frac{r^2\cdot 0 - x\cdot 2r r_y}{(r^2)^2} = \frac{-x\cdot 2r\cdot\frac{y}{r}}{r^4} = \frac{-2xy}{r^4}$

Using the above values in eqn ②,

$$u_{xx} = u_{rr} r_x^2 + u_r r_{xx} + u_{\theta\theta}\theta_x^2 + u_\theta \theta_{xx}$$

$$u_{xx} = u_{rr}\left(\frac{x}{r}\right)^2 + u_r\left(\frac{y^2}{r^3}\right) + u_{\theta\theta}\left(-\frac{y}{r^2}\right)^2 + u_\theta\left(\frac{2xy}{r^4}\right) \quad \text{(A)}$$

Similarly,

$$u_{yy} = u_{rr}\left(\frac{y}{r}\right)^2 + u_r\left(\frac{x^2}{r^3}\right) + u_{\theta\theta}\left(\frac{x}{r^2}\right)^2 + u_\theta\left(-\frac{2xy}{r^4}\right) \quad \text{(B)}$$

Adding (A) and (B)

$$u_{xx}+u_{yy} = u_{rr}\left[\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right] + u_r\left(\frac{y^2}{r^3} + \frac{x^2}{r^3}\right) + u_{\theta\theta}\left[\left(-\frac{y}{r^2}\right)^2 + \left(\frac{x}{r^2}\right)^2\right] + u_\theta\left[\frac{2xy}{r^4} - \frac{2xy}{r^4}\right]$$

$$= u_{rr}\left[\frac{x^2+y^2}{r^2}\right] + u_r\left[\frac{y^2+x^2}{r^3}\right] + u_{\theta\theta}\left(\frac{y^2+x^2}{r^4}\right) + u_\theta$$

$$\Rightarrow u_{xx}+u_{yy} = u_{rr}\left(\frac{r^2}{r^2}\right) + u_r\left(\frac{r^2}{r^3}\right) + u_{\theta\theta}\left(\frac{r^2}{r^4}\right) + 0$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

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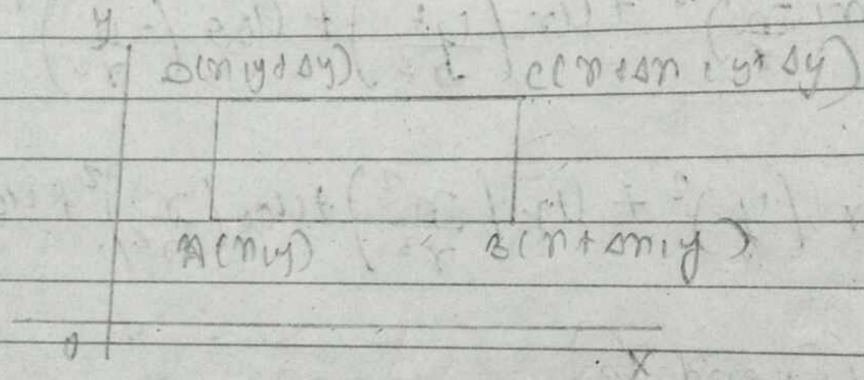
is required laplacian in polar coordinates.

Note:
Laplacian in cylindrical coordinates is
$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}$

Derivation of two dimensional heat eqn.



Let us suppose that the flow of heat in a metal plate in xOy-plane. The tempr at any point be independent of z. Then the flow depends on x, y, and t, only.

Let a rectangle element of plate with sides Δx and Δy and thickness α.

Now the quantity of heat enters the plate from sides AB and AD per second is

$-k\alpha\Delta x\left(\frac{\partial u}{\partial y}\right)_y$ and $-k\alpha\Delta y\left(\frac{\partial u}{\partial x}\right)_x$ respectively.

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The quantity of heatflow out from the sides DC and BC per second is

$-K\alpha\Delta x\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y}$ and $-K\alpha\Delta y\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$ respectively

Total gain of heat by plate per second is,

$$-K\alpha\Delta x\left(\frac{\partial u}{\partial y}\right)_{y} - K\alpha\Delta y\left(\frac{\partial u}{\partial x}\right)_{x} + K\alpha\Delta x\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} + K\alpha\Delta y\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

$$= -K\alpha\Delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] + K\alpha\Delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_{y}\right]$$

$$= K\alpha\Delta y.\Delta x\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] + K\alpha\Delta x.\Delta y\left[\frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_{y}}{\Delta y}\right]$$

$$= K\alpha\Delta x\Delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)}{\Delta y}\right]$$

Also,
the rate of gain of heat by the plate is

$s\rho\alpha\Delta x\Delta y\frac{\partial u}{\partial t}$ —— (2) where s = specific heat
ρ = density
α = thickness of plate

from (1) and (2).

$$s\rho\alpha\Delta x\Delta y\frac{\partial u}{\partial t} = K\alpha\Delta x\Delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\Delta y} - \left(\frac{\partial u}{\partial y}\right)_{y}}{\Delta y}\right]$$

Taking $\lim_{\Delta x \to 0}$ we get,

$$s\rho \frac{\partial u}{\partial t} = k\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{\partial u}{\partial t} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \quad \text{—(3)}$$ is the required two heat eqn.

where $c^2 = \frac{k}{s\rho}$

Special cases

1. If the streamlines are parallel to x-axis then the rate of change $\frac{\partial u}{\partial y}$ of temperature in y direction will be zero. Hence, $\frac{\partial^2 u}{\partial y^2} = 0$. Then the eqn reduces to $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ which is one dimensional heat eqn.

2. For three dimensional heat flow the heat eqn is.

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

When the steady state is reduced. The temperature u will depend only on x and y but not on t then $\frac{\partial u}{\partial t} = 0$

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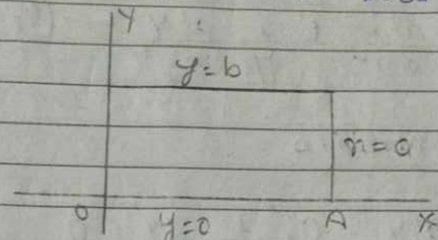
Then the eqn reduces to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace eqn)

Solution of two dimensional heat equation under initial and boundary conditions:



Consider a thin rectangular plate whose surface is permissable for heat flow has $t=0$, an arbitary distribution of tempr $f(x,y)$. It's four edges $x=0, x=a, y=0, y=b$ are kept at zero tempr.

The two dimensional heat eqn is

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{—(1)}$$

The boundary conditions are:

$u(0,y) = u(a,y) = u(x,0) = u(x,b)$ —(2)

and the initial condition is

$u(x,y) = f(x,y)$ for $t=0$ —(3)

let the solution of eqn (1) be

$u = XYT$ —(4)

where X is the function of x and Y is function of y only and T is function of t only.

Let us replace derivatives of (4) in (1)

$$xy\frac{\partial T}{\partial t} = c^2\left(yT\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2}\right)$$

Dividing b.s on XYc^2T we get,

$$\frac{1}{c^2T}\frac{\partial T}{\partial t} = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \quad \text{—— (5)}$$

Here are three possibilities

1. $\frac{1}{X}\frac{\partial^2 X}{\partial t^2} = 0, \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0, \frac{1}{c^2T}\frac{\partial T}{\partial t} = 0$

2. $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = K^2, \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = K_1^2, \frac{1}{c^2T}\frac{\partial T}{\partial t} = K^2$

3. $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -K^2, \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = K_2^2, \frac{1}{c^2T}\frac{\partial T}{\partial t} = -K^2$

where $K^2 = K_1^2 + K_2^2$

Out of the above three solution third is consistent and is accepted.

Then,

$X = (A\cos K_1x + B\sin K_1x)$

$Y = (C\sin K_1 y + D\sin K_2 y)$

and,

$T = Ee^{-c^2K^2t}$

Replacing these values in (1), we get,

$$u = (A\cos K_1x + B\sin K_1x)(C\cos K_1y + D\sin K_2y)Ee^{-c^2K^2t} \quad \text{—— (6)}$$

Applying boundary conditions (2), using $u=0$ and $x=0$ in (6)

$$0 = A(C\cos K_2y + D\sin K_2y)Ee^{-c^2K^2t}$$

$\Rightarrow A = 0$

Using this value in (6), we get,

$$u = B\sin K_1 a(C\cos K_2y + D\sin K_2y)e^{-c^2K^2t}$$

or, $B\sin K_1 a = 0 = \sin n\pi$

$K_1 a = n\pi \Rightarrow K_1 = \frac{n\pi}{a}$ —— (7)

Using value of K_1 in (7), we get,

$$u = B\sin\frac{n\pi x}{a}(C\cos K_2y + D\sin K_2y)e^{-c^2K^2t} \quad \text{—— (8)}$$

On putting $u=0$, $y=0$ in (8) we get,

$$0 = \left(B\sin\frac{n\pi x}{a}\right)Ce^{-c^2K^2t}$$

$\Rightarrow C = 0$

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Substituting values of c in 8, we get,

$$u = \left(B \sin \frac{n\pi x}{a}\right)\left(D \sin k_2 y\right) \cdot e^{-c^2 k^2 t} \quad — (9)$$

on putting $u=0$, $y=b$ in (9).

$$0 = \left(B \sin \frac{n\pi x}{a}\right)\left(D \sin k_2 b\right) e^{-c^2 k^2 t}$$

$$\sin k_2 b = 0 = \sin m\pi$$

$$k_2 b = m\pi$$

$$k_2 = \frac{m\pi}{b} \quad — (8)$$

on putting value of k_2 in (9)

$$u = \left(B \sin \frac{n\pi x}{a}\right)\left(D \sin \frac{m\pi y}{b}\right) e^{-c^2 k^2 t}$$

$$u = A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-c^2 k^2 t} \quad — (10)$$

where,

$$k^2 = k_1^2 + k_2^2$$

$$k^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

$$k^2_{mn} = \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)$$

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Thus we have,

$$u = A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-c^2 k^2_{mn} t}$$

This is solution of two dimensional heat equation satisfying boundary conditions.

Consider the series,

$$u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-c^2 k^2_{mn} t}$$

This is solution of two dimensional heat eqn satisfying boundary conditions.

To obtain a soln that also satisfies initial conditions,

Consider the series,

$$u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} e^{-c^2 k^2_{mn} t}$$

on applying initial condition (3) ($u(x,y) = f(x,y)$ for $t=0$)

$$f(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

$\therefore (t=0)$ (Double Fourier sine series

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where,

$$A_{mn} = \frac{2}{a}\frac{2}{b}\int_0^b\int_0^a \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} f(x,y)\, dx dy$$

Hence the soln of two dimensional heat eqn is,

$$u = \sum_{n=1}^{\infty}\cdot\sum_{m=1}^{\infty} A_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\, e^{-c^2 k^2 mn t}$$

where,

$$A_{mn} = \frac{4}{ab}\int_0^b\int_0^a \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} f(x,y)\, dx dy$$

and $k_{mn}^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)$.