Tutorial 2

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1. Find Z-transform using the properties:

a) $na^n u[n]$

Sol^n.

we know,

$$u[n] \xrightarrow{z} \frac{1}{1-z^{-1}} \qquad |z|>1$$

$$a^n u[n] \xrightarrow{z} \frac{1}{1-az^{-1}} \qquad |z|>a$$

and,

$$n a^n u[n] \xrightarrow{z} -z \frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right)$$

$$= -z(-1)(1-az^{-1})^{-2} \times -a(-1)z^{-2}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z|>a$$

b) $2^n u[n-2]$

we know,

$$u[n] \xrightarrow{z} \frac{1}{1-z^{-1}} \qquad |z|>1$$

using time shifting property,

$$u[n-2] \xrightarrow{z} z^{-2} \times \frac{1}{1-z^{-1}} \qquad |z|>1$$

$$= \frac{z^{-2}}{1-z^{-1}} \qquad |z|>1$$

$$2^n u[n-2] \xrightarrow{z} = \frac{(a^{-1}z)^{-2}}{1-(a^{-1}z)^{-1}} \qquad |z|>1$$

$$= \frac{4z^{-2}}{1-2z^{-1}} \times \frac{z}{z}$$

$$= \frac{4z^{-1}}{z-2}$$

c) $n^2 a^n u[n]$

$$\mathcal{Z}(a^n u[n]) = \frac{1}{1-az^{-1}} \quad |z|>|a|$$

$$n^2 a^n u[n] = z^2 \frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right)$$

$$= z^2 . (-1)(1-az^{-1})^{-2} \times (-a)(-1) z^{-2}$$

$$= \frac{-a}{(1-az^{-1})^2}$$

d) $3\left(-\frac{1}{2}\right)^n u[n] - 2(3)^n u[-n-1]$

Soln.

$X(z) = 3X_1(z) - 2X_2(z)$

Then,

$$X_1(z) = \mathcal{Z}\left(\left(\frac{-1}{2}\right)^n u[n]\right)$$

$$= \frac{1}{1+\frac{1}{2}z^{-1}} \quad |z|>|-1/2|$$

$$X_2(z) = \mathcal{Z}(3^n u[-n-1])$$

$$= \frac{1}{1-3z^{-1}} \quad |z|<|3|$$

$$\therefore X(z) = 3 \times \frac{2}{2+z^{-1}} \overset{|z|>1/2}{-} 2 . \frac{1}{1-3z^{-1}} \quad |z|<3$$

2. Find the system function $H(z)$ and unit sample response $h[n]$ of the following system. $y[n] = \frac{1}{2} y[n-1] + 2x[n]$

Soln.

Taking Z transform on both side

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

$$Y(z)\left(1 - \frac{1}{2} z^{-1}\right) = 2X(z)$$

or, $$\frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

$$H(z) = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

Taking inverse z transform.

$$h[n] = 2 \cdot \left(\frac{1}{2}\right)^n u[n]$$

3. Determine the response of the system $y[n] = \frac{5}{6} y[n-1] - \frac{1}{6} y[n-2] + x[n]$ to the input signal.

$$x[n] = \delta[n] - \frac{1}{3}\delta[n-1]$$

Soln.

Taking z transform on both side of system.

$$Y(z) = \frac{5}{6} z^{-1} Y(z) - \frac{1}{6} z^{-2} Y(z) + X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

$$X(z) = 1 - \frac{1}{3} z^{-1}$$

$$H(z) = \frac{1}{1 - \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

$$Y(z) = X(z) \times H(z)$$

$$= 1 - \tfrac{1}{3}z^{-1} \times \frac{1}{1 - \tfrac{5}{6}z^{-1} + \tfrac{1}{6}z^{-2}}$$

$$= \frac{1 - 0.33z^{-1}}{1 - 0.833z^{-1} + 0.666z^{-2}}$$

$Z = 0.33$ $\quad P = 0.416 + 0.701\,i = 0.815\angle(59.31)$

$P^* = 0.416 - 0.701\,i = 0.815\angle(-59.31)$

$$Y(z) = \frac{A}{(1 - pz^{-1})} + \frac{A^*}{(1 - p^*z^{-1})}$$

$$A = \left.\frac{1 - 0.33z^{-1}}{1 - p^*z^{-1}}\right|_{z=p} = \frac{1 - 0.33p^{-1}}{1 - p^*p^{-1}}$$

$$= \frac{1 - \dfrac{0.33}{0.416 + 0.701i}}{1 - \dfrac{(0.416 - 0.701i)}{0.416 + 0.701i}}$$

$$= 0.5 - 0.061i$$

$$= 0.503\, e^{-j6.99}$$

$$A^* = 0.503\, e^{j6.99}$$

$$Y(z) = \frac{0.503e^{-j6.99}}{1 - pz^{-1}} + \frac{0.503e^{j6.99}}{1 - p^*z^{-1}}$$

$$y[n] = 0.503e^{-j6.99}(p^n)u[n] + 0.503e^{j6.99}(p^*)^n u[n]$$

$$= 0.503e^{-j6.99}\, 0.815^n . e^{j59.31n} u[n] + 0.503e^{j6.99}\, 0.815^n$$

$$e^{-j59.31n} * u[n]$$

$$= 0.503 . 0.815^n \left(e^{j(59.31n - 6.99)} + e^{-j(59.31n - 6.99)}\right) u[n]$$

$$= 0.503 . 0.815^n\, 2 \cos(59.31n - 6.99)\, u[n]$$

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$-n\,u[n$

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4. Find inverse Z transform.

a) $X(z) = \dfrac{z^3}{(z+1)(z-1)^2}$, find causal signal.

Soln.

$$X(z) = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$A = \left.\frac{z^3}{(z-1)^2}\right|_{z=-1} = \frac{-1}{4}$$

$$C = \left.\frac{z^3}{z+1}\right|_{z=1} = \frac{1}{2}$$

$$B = \frac{d}{dz}\left(\frac{z^3}{z+1}\right)_{z=1}$$

$$= \left.\frac{(z+1)\,3z^2 - z^3 \cdot 1}{(z+1)^2}\right|_{z=1}$$

$$= \frac{6-1}{4} = \frac{5}{4}$$

$$X(z) = \cancel{\frac{-1}{4} \times \frac{1}{1-(-z)}} + \cancel{\frac{B}{}}$$

$$= \frac{-1}{4}\,\frac{z^{-1}}{1+z^{-1}} + \frac{1}{2}\,\frac{z^{-1}}{1-z^{-1}} + \frac{5}{4}\,\frac{z^{-2}}{(1-z^{-1})^2}$$

$$= \frac{-1}{4}(-1)^n u[n-1] + \frac{1}{2}\,u[n-1] + \frac{5}{4}\,n\,u[n-2]$$

6b. $X(z) = \frac{z}{3z^2-4z+1}$, find anticausal signal,

$$= \frac{z^{-1}}{3-4z^{-1}+z^{-2}}$$

$$= \frac{z^{-1}}{(1-z^{-1})(1-0.33z^{-1})}$$

$$X(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-0.33z^{-1}}$$

$$A = \left.\frac{z^{-1}}{1-0.33z^{-1}}\right|_{z=1} = 1.492$$

$$B = \left.\frac{z^{-1}}{1-z^{-1}}\right|_{z=0.33} = -1.492$$

$$X(z) = \frac{1.492}{1-z^{-1}} - \frac{1.492}{1-0.33z^{-1}}$$

$$= 1.492 - u[-n-1] + 1.492\,(0.33)^n u[-n-1]$$

5. Determine the step response of the causal system $y[n] = y[n-1] + x[n]$. Is this stable system?

Sol^n.

Since input is step response so replace $x[n]$ by $u[n]$

$$y[n] = y[n-1] + u[n]$$

$$Y(z) = z^{-1}Y(z) + \frac{1}{1-z^{-1}}$$

$$Y(z) = \frac{\frac{1}{1-z^{-1}}}{1-z^{-1}}$$

$$Y(z) = \frac{1}{(1-z^{-1})^2}$$

$\cancel{n\,1^n u[n] \xrightarrow{Z} \frac{1}{(1-z^{-1})^2}}$ $\qquad (n+1)1^{n+1}u[n+1] \xrightarrow{Z} \frac{z \cdot z^{-1}}{(1-z^{-1})^2}$

$\cancel{y[n] = n\,u[n]}$ $\qquad y[n] = n+1$

Since input n is independent in time domain so the system is unstable.

6. A LTI system is described by the difference equation
$y[n] = x[n] + 2x[n-1] + x[n-2]$

a. Determine the impulse response of the system.

- The impulse response of the system $h[n]$ is output when the input is discrete time unit impulse function.
The unit impulse function is defined as:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Substituting $x[n] = \delta[n]$

$$h[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$$

The impulse response is

$$h[n] = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ 1, & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

b. Is this stable ~~style~~ system?
Yes this system is stable because sum of all impulse ~~unit~~ response is finite. ie $|1| + |2| + |1|) = 4$ which is finite.

c. Determine frequency response of the system ie $H(e^{j\omega})$
Soln.

$$H(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-2j\omega}$$

$$= 1 + e^{-j\omega} + e^{-j\omega} + e^{-2j\omega}$$

$$= 1 + e^{-j\omega} + e^{-j\omega}(1 + e^{-j\omega})$$

$$= 1 + e^{-j\omega}$$

$$= 1 + e^{-j\omega}(2 + e^{-j\omega})$$

7 When the input to an LTI system is $x[n] = (\frac{1}{2})^n u[n] + 2^n u[-n-1]$;
the output is found to be $y[n] = 6(\frac{1}{2})^n u[n] - 6(\frac{3}{4})^n u[n]$

a. Find transfer function $H(z)$ and plot pole zero.

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad |z| > \frac{1}{2} \quad + \frac{1}{1-2z^{-1}} \quad |z| < 2$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} \qquad \frac{1}{2} < |z| < 2$$

$$Y(z) = 6\frac{1}{1-\frac{1}{2}z^{-1}} - 6\frac{1}{1-\frac{3}{4}z^{-1}} \qquad |z| > 3/4$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\dfrac{6 - 18/4\,z^{-1} - 6 + 3z^{-1}}{(1-1/2\,z^{-1})(1-3/4\,z^{-1})}}{\dfrac{1-2z^{-1} + 1 - 1/2\,z^{-1}}{(1-1/2\,z^{-1})(1-2z^{-1})}}$$

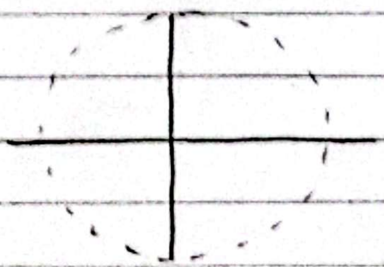
$$= \frac{-6/4\,z^{-1} + 2 - 5/2\,z^{-1}}{(1-2z^{-1})(1-3/4\,z^{-1})}$$

$$= \frac{2 - 4z^{-1}}{(1-2z^{-1})(1-3/4\,z^{-1})}$$

$$= \frac{2(1-2z^{-1})}{(1-2z^{-1})(1-3/4\,z^{-1})}$$

$P_1 = 2 \qquad P_2 = 3/4$

$Z_1 = 2$



b. Find impulse response

$$H(z) = \frac{3}{1-\frac{3}{4}z^{-1}}$$

Using inverse z transform

$$3\left(\frac{3}{4}\right)^n u[n] \xrightarrow{z} \frac{3}{1-\frac{3}{4}z^{-1}}$$

$$\therefore h[n] = 3\left(\frac{3}{4}\right)^n u[n]$$

c. Write LCCD equation that characterize the system.

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$

d. Is the system stable? Is it causal?

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b. Find impulse response.

$$H(z) = \frac{2}{1 - 3/4 z^{-1}}$$

Taking inverse z transform.

$$2 \cdot \left(\frac{3}{4}\right)^n u[n] \xrightarrow{z} \frac{2}{1 - 3/4 z^{-1}}$$

$$\therefore h[n] = 2 \cdot \left(\frac{3}{4}\right)^n u[n]$$

c. Write LCCD equation that characterize the system.

$$y[n] = -\sum_{k=1}^{N} a_k \, y[n-k] + \sum_{k=0}^{M} b_k \, x[n-k]$$

d. Is the system stable? Is it causal?

8. Consider the LTI system with transfer function $H(z) = \dfrac{z^{-2}}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})}$

a. Suppose that the system is known to be stable, determine the output when the input is unit step.

Soln.

Since input is unit step so $x[n] = u[n]$.

$$X(z) = \frac{1}{1-z^{-1}} \quad |z|>1$$

And,

$$Y(z) = H(z) \times X(z)$$

$$= \frac{z^{-2}}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})} \times \frac{1}{1-z^{-1}}$$

$$= \frac{z^{-2}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1-3z^{-1})}$$

Now,

$$Y(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-\frac{1}{2}z^{-1}} + \frac{C}{1-3z^{-1}}$$

And,

$$A = \left.\frac{z^{-2}}{(1-\frac{1}{2}z^{-1})(1-3z^{-1})}\right|_{z=1}$$

$$= -1$$

$$B = \left.\frac{z^{-2}}{(1-z^{-1})(1-3z^{-1})}\right|_{z=\frac{1}{2}}$$

$$= \frac{4}{5}$$

$$C = \left.\frac{z^{-2}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})}\right|_{z=3}$$

$$= \frac{1}{5}$$

$$Y(z) = \frac{-1}{1-z^{-1}} + \frac{4/5}{1-\frac{1}{2}z^{-1}} + \frac{1/5}{1-3z^{-1}}$$

$$y[n] = -u[n] + \frac{4}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{5}(3)^n u[n]$$