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# Unit-1

## Complex Analysis



(1) **Complex number:** An order pair (a,b) where a and b are real numbers is called complex number. Where a is called the real part and b is called the imaginary part of the complex number; (a,b). A complex number is usually denoted by z or w. Thus if z = (a,b) is a complex number, then a = Real part of z = Re(z)

b = Imaginary part of z = Im(z)

The complex number (a,b) can be plotted as a point in the cartesian plane.

The complex number z = (a,b) is also written as z = a+ib where $i = \sqrt{-1}$

(2) **Some basic operations on complex numbers:**

(i) Equality of two complex numbers.
(ii) Sum of two complex numbers.
(iii) The product of two complex numbers.
(iv) The quotient of two " " .

(3) **Imaginary unit:** The complex number (0,1) with real part 0 and imaginary part 1 is called the imaginary unit. It is denoted by i.

$i^2 = -1$.

$i^2 = i \cdot i$

$= (0,1) \cdot (0,1)$

$= (0 \cdot 0 - 1 \cdot 1,\ 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -(1,0)$

$= -i$

i.e $z \cdot w = (ac - bd,\ ad + bc)$ [i.e. (1,0) = 1]

(4.) The complex number $(a,b)$ can be written as,
$(a,b) = a+ib$

Now, $a+ib = (a,0) + i(b,0)$
$= (a,b)$.

(5.) conjugate of a complex number:

If $z = a+ib$ be a complex number, then its complex conjugate is denoted by $\bar{z}$ and is defined by $\bar{z} = a-ib$.

Properties of conjugate:

If $z = a+ib$ and $w = c+id$ be two complex numbers, then

(i) $\overline{z+w} = \bar{z} + \bar{w}$

(ii) $\overline{zw} = \bar{z} \cdot \bar{w}$

Note If $z$ is real then $z = \bar{z}$

(6.) Absolute value of a complex number:
The absolute value of a complex number $z = (a,b) = a+ib$ is the non-negative real number $|z|$ defined by
$|z| = \sqrt{a^2+b^2} = \sqrt{(\text{Real part})^2 + (\text{Ima. part})^2}$

## (7) properties of absolute value:

If $z = a+ib$ and $w = c+id$ be two complex number, then we have the following properties

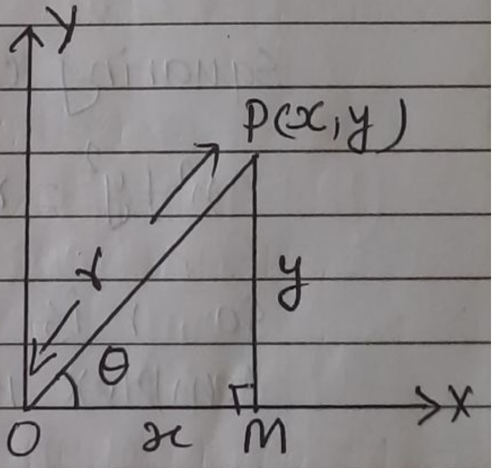
(i) $|zw| = |z||w|$ (ii) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

(iii) If z and w are two complex numbers, then $|z+w| \le |z|+|w|$ (triangle in in equality).

## (8) Expression of a complex number into polar form:

Express $z = x+iy$ into polar form

Ans: let $z = x+iy$ be a complex number and let it be represented by the point in cartesian plane. Join OP and draw perpendicular PM on x-axis.




let $OP = r$ and $\angle POM = \theta$

Then in $\triangle POM$,

$\cos\theta = \frac{OM}{OP} = \frac{x}{r}$

$x = r\cos\theta$ --- (i)

Again $\sin\theta = \frac{PM}{OP} = \frac{y}{r}$

$y = r\sin\theta$ -- (ii)

Now, $z = x+iy = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

Which is the polar representation of complex number. It can be written as,
$z = r[\cos(\theta + K \cdot 360^{\circ}) + i\sin(\theta + K\,360)]$,
Where $\theta = 0, 1, 2, 3, \ldots$

Dividing (ii) by (i) we get

$$\frac{y}{x} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Here $\theta$ is known as amplitude or argument of the complex number $z$ and is denoted by Amp.$(z)$ or Arg.$(z)$.

Squaring (i) and (ii) and adding them.

$$x^2 + y^2 = r^2 \quad \text{i.e.} \quad r = \sqrt{x^2+y^2} = |z| = op$$

So, $r$ is known as the modulus of the complex number.

## 9. Cube root of unity:

The cube root of unity are

$$z = 1, \; \frac{-1+i\sqrt{3}}{2}, \; \frac{-1-i\sqrt{3}}{2}.$$

## Properties of cube root of unity:

(i) Each imaginary cube root of unity is the square of the other.

i.e. $w = \frac{-1+i\sqrt{3}}{2}$ then $w^2 = \frac{-1-i\sqrt{3}}{2}$

So the cube roots of unity are usually denoted by $1, w, w^2$.

(ii) The product of the two imaginary cube roots of unity is 1.

i.e. $w \cdot w^2 = 1$.
$w^3 = 1$.

(iii) The sum of three cube roots of unity is zero.

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$1 + w + w^2 = 0$

10. **limit of a function of a complex variable:** A Singled valued function $f(z)$ of a complex variable $z$ is said to has a limit $l = \alpha + i\beta$

if $\lim_{z \to z_0} f(z) = l$

11. **Continuity of a complex function:** A single valued function $f(z)$ of a complex variable $z$ is called differentiable at a function $z = z_0$ if

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

where $\Delta z = z - z_0$ exists.

i.e. $\lim_{z \to z_0} \frac{f(z - z_0) - f(z_0)}{z - z_0}$ exists.

It is denoted by $f'(z_0)$

or

A complex valued function $f(z)$ is differentiable at $z = z_0$ if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z - z_0) - f(z_0)}{z - z_0}$$

(12) **Analytic function**: A complex valued function $f(z)$ is called analytic at a point $z = z_0$ in the domain D if $f(z)$ is defined and differentiable at each point in a neighbourhood of $z_0$.

A complex valued function $f(z)$ is called analytic in the domain D if $f(z)$ is defined and differentiable at each point in D.

(13.) **Cauchy-Riemann equations (C-R) eq's**:
A function $f(z) = u + iv$, where $u = u(x, y)$ and $v = v(x, y)$ is analytic in a domain D. Then we say

f(z) satisfies the cauchy Riemann eqns (i.e C-R) if the partial derivatives $u_x, u_y, v_x, v_y$ exists and $u_x = v_y$ and $u_y = -v_x$.

Note: A function f(z) is analytic in D iff it satisfies the C-R eqns.

(14) Laplaces Equation: A function $f(z) = u + iv$ where $u = u(x,y)$ and $v = v(x,y)$ is analytic in a domain D. Then we say that the function u satisfies the laplaces equation if

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

As similar, the function v satisfies the laplaces equation if

$$\nabla^2 v = v_{xx} + v_{yy} = 0$$

(15) Harmonic function: A function $f(z) = u + iv$ where $u = u(x,y)$ and $v = v(x,y)$ is analytic in a domain D. Then we say the function u is harmonic function if it satisfies the laplaces equation

$\nabla^2 u = u_{xx} + u_{yy} = 0$. In such case v is called complex conjugate of u.

As similar, the function $v$ is harmonic function if it satisfies the laplace eqⁿ

i.e. $\nabla^2 v = v_{xx} + v_{yy} = 0$

In such case $u$ is called complex conjugate of $v$.

(16) Theorem: (Cauchy - Riemann eqⁿs)

(Necessary condition for analyticity of a function)

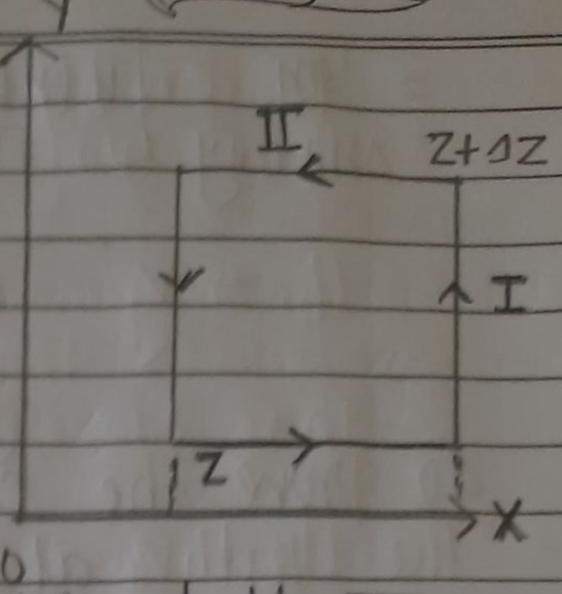
Let $f(z) = u(x,y) + iv(x,y)$ be defined and continuous in some neighbourhood of a point $z = x+iy$ and differentiable at $z$ itself. Then at that point $u_x = v_y$ and $u_y = -v_x$.

Hence, if $f(z) = u+iv$ is analytic in a domain D, those partial derivatives exists and satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at all points of D.

Sol: $z_0$ को neighbourhood point is differentiable $z = z_0$ भयो भने function Analytic हुन्छ।

$f'(z)$ को help ले हामीले C-R. eqⁿ satisfied गराउँछौं।

Proof:
Necessary condition for analyticity of a function.




We have, $f(z) = u+iv$ is differentiable. Then $f'(z)$ exists at $z$ itself. Where

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \ldots \quad (*)$$

For this, let $z = x+iy$

$$f(z) = u+iv = u(x,y) + i\,v(x,y)$$

$$\Delta z = \Delta x + i\Delta y.$$

$$z + \Delta z = x+iy + \Delta x + i\Delta y$$

$$= [x+\Delta x + i(y+\Delta y)]$$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

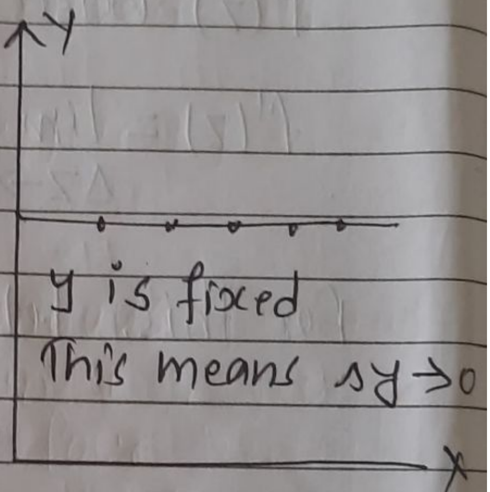
Then from (*) we have

$$f'(z) = \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta z}$$

$$= \lim_{\Delta x + i\Delta y} \frac{\left[u(x+\Delta x, y+\Delta y) - u(x,y)\right] + i\left[v(x+\Delta x, y+\Delta y) - v(x,y)\right]}{\Delta x + i\Delta y} \quad --- \quad (1)$$

To prove the C-R eqⁿ we need to choose the path along x- axis and along y-axis

Case- I : Along x-axis




$\Delta z = \Delta x + i\Delta y.$

$\Delta z = \Delta x + i \cdot 0$

$\Delta z = \Delta x$

if $\Delta z \to 0$

then $\Delta x \to 0$

Then from (1)

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}\right]$$

$$+ \lim_{\Delta x \to 0} i \left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{———} \quad (2)$$

(Note, here diff. w.r.t. x, so it means partial diff. takes place here

i. e. $f'(z) = u_x + i v_x$ .. (A)



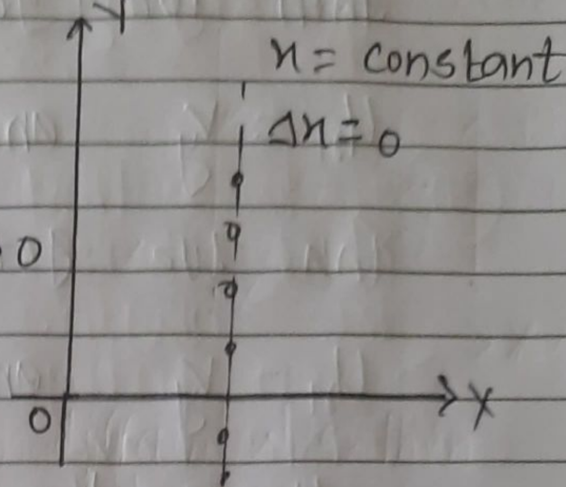
Similarly Along Y- axis

i.e. $\Delta z = \Delta x + i \Delta y$

$\Delta z = 0 + i \Delta y$

$\Delta z = i \Delta y$

if $\Delta z \to 0 \Rightarrow \Delta y \to 0$




Then,

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i \Delta y} + i\left[\frac{v(x, y+\Delta y) - v(x,y)}{i \Delta y}\right]$$

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{i \partial v}{i \partial y}$$

$$= +\frac{1}{i} \times \frac{i}{i} \cdot \frac{\partial u}{\partial y} + \frac{i}{i}\frac{\partial v}{\partial y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\cancel{i} \partial v}{\cancel{i} \partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{--- (3)}$$

$f'(z) = -i u_y + v_y$ . ---- (B)

We have $f'(z)$ exists then equate both A and B then we get

$u_x + i v_x = v_y - i u_y$.

Comparing the real and imaginary part then we get,

$u_x = v_y$ and $u_y = -v_x$

Thus we get $f(z)$ is analytic, then

$u_x = v_y$ and $u_y = -v_x$.

Hence the theorem is proved. ||

(17) Sufficient condition for Analyticity of a function:

Theorem: The single valued continuous function $f(z) = u+iv$ is analytic in a region, R of the z-plane, if the four partial derivatives $u_x, v_x, u_y, v_y$ exists, continuous and satisfy the C-R eqns $u_x = v_y$ and $u_y = -v_x$ at each point of R.

(18.) Polar form of Cauchy-Riemann condition

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \text{and } \frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r} \end{aligned}\right\}$$

These eqns are said to be C-R condition in polar form. ||

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## (19) Some examples:

a. Show that the function $f(z) = z^n$, where $n$ is positive integer, is an analytic function.

Sol: We have $f(z) = z^n$

$$= (re^{i\theta})^n$$

$$= r^n e^{in\theta}$$

$$= r^n (\cos n\theta + i \sin n\theta)$$

Comparing with $f(z) = u + iv$

We get $u = r^n \cos n\theta$ and $v = r^n \sin n\theta$.

$$\frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta$$

$$\frac{\partial u}{\partial \theta} = -nr^n \sin n\theta$$

$$\frac{\partial v}{\partial r} = nr^{n-1} \cdot \sin n\theta$$

$$\frac{\partial v}{\partial \theta} = nr^n \cos n\theta$$

Here, $\frac{\partial u}{\partial r} = nr^{n-1} \cos n\theta = \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right) = nr^{n-1} \cos n\theta$

and $\frac{\partial u}{\partial \theta} = -nr^n \sin n\theta = -r\left(\frac{\partial v}{\partial r}\right)$

$$= -nr^n \sin n\theta$$

Hence the given function is analytic by satisfying C-R equation in polar form.

(20) Show that $f(z) = \log z$ is analytic everywhere in the complex plane except at origin.

Sol: We have $f(z) = \log z$

$= \log(re^{i\theta})$

$= \log r + \log e^{i\theta}$

$= \log r + i\theta \log e$

$= \log r + i\theta$

By comparing with $f(z) = u + iv$, we get,
$u = \log r$ and $v = \theta$

Hence $\frac{\partial v}{\partial r} = 0$, $\frac{\partial v}{\partial \theta} = 1$

$\frac{\partial u}{\partial r} = \frac{1}{r}$, $\frac{\partial u}{\partial \theta} = 0$

Also, $\frac{\partial u}{\partial r} = \frac{1}{r} \cdot = \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right) = \frac{1}{r} \cdot 1$

and $\frac{\partial v}{\partial r} = -r\left(\frac{\partial u}{\partial \theta}\right) = -r \cdot 0 = 0$

Thus the given function is analytic everywhere except $r = 0$.

(21) Check $u = \sin x \cosh y$ is harmonic or not? If yes, find corresponding analytic function $f(z) = u + iv$.

Solⁿ: We have $u = \sin x \cosh y$

Diff. partially

$$\frac{\partial u}{\partial x} = \cos x \cdot \cosh y$$

$$\frac{\partial u}{\partial y} = \sin x \cdot \sinh y$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cdot \cosh y$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x \cdot \cosh y$$

Here, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Thus, we get $u$ is harmonic. We have to find $v$. By C-R eqⁿ $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow \cos x \cosh y = v_y$

Integrating w.r.t. $y$ then we get

$v = \cos x \cdot \sinh y + h(x)$ . . . (1)

Where $h(x)$ is function of $x$ only.

Diff. w.r.t. $x$ we get

$v_x = -\sin x \sinh y + h'(x)$

$-u_y = -\sin x \cdot \sinh y + h'(x)$

$-\sin x \cdot \sinh y = -\sin x \cdot \sinh y + h'(x)$

$\Rightarrow h'(x) = 0$

$\Rightarrow$ on integration we get

$h(x) = c$

Thus From (1), We get,

$v = \cos x \sinh y + c$ is the required harmonic conjugate of $u$.

Therefore $f(z) = u + iv = f(z) = \sin x \cdot \cosh y + i(\cos x \sinh y + c)$ is the required analytic function.

(22.) Show that the function $u(x,y) = 3x^2y + x^2 - y^3 - y^2$ is a harmonic function. Find a function $v(x,y)$ such that $u+iv$ is an analytic function.

Soln: Let $f(z) = u + iv$ be an analytic function with

$u(x,y) = 3x^2y + x^2 - y^3 - y^2$

Here, $u_x = 6xy + 2x$

$u_{xx} = 6y + 2$

$u_y = 3x^2 - 3y^2 - 2y$

$u_{yy} = -6y - 2$

We have Laplaces equation

$u_{xx} + u_{yy} = 6y + 2 - 6y - 2$
$= 0$

Hence we get $u$ is harmonic. For harmonic conjugate,

$u_x = v_y$

$\Rightarrow 6xy + 2x = v_y$

Integrating w.r.t. $y$, we get

$\frac{2 \cdot 3xy^2}{2} + 2xy = v$

Where $h(x)$ is a function of $x$ or constant

Again diff. w.r.t. $x$ we get,

$v_x = 3y^2 + 2y + h'(x)$

$-u_y = 3y^2 + 2y + h'(x)$

$-[3x^2 - 3y^2 - 2y] = 3y^2 + 2y + h'(x)$

$-3x^2 = h'(x)$

on integration w.r.t. $x$, we get

$h(x) = -x^3 + c$

Thus the required harmonic conjugate of $u$ is

$v = 3xy^2 + 2xy - x^3 + c$

And hence the required analytic function is $f(z) = u + iv$

$= (3xy^2 + x^2 - y^3 - y^2) + i(3xy^2 + 2xy - x$

Ans

leave ??

Q.23 Given that $f(z) = u + iv$ is an analytic function and $u + v = e^x(\cos y + \sin y)$. Find $f(z)$.

Soln: We have $u + v = e^x(\cos y + \sin y)$ --- (1)

Diff. (1) w.r.t. $x$ we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = e^x(\cos y + \sin y) \quad \text{---} (2)$$

Again diff (1) w.r.t. $y$, we get

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = e^x(-\sin y + \cos y) \quad \text{---} (3)$$

By replacing $u_y = -v_x$ and $u_x = v_y$,

Eqn (3) reduces to,

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$$-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = e^{x}(-\sin y + \cos y) \quad --- (4)$$

Adding (2) and (4) we get,

$$2\frac{\partial u}{\partial x} = 2e^{x}\cos y$$

or, $\frac{\partial u}{\partial x} = e^{x}\cos y$

Also we get, $\frac{\partial v}{\partial x} = e^{x}\sin y$

But we have,

$$f'(z) = u_x + i v_x$$

$$= u_x(x, y) + i v_x(x, y)$$

$$f'(z) = e^{z}$$

on integration, we get,

$f(z) = e^{z} + c$. where $c$ is complex constant

Thus the required function $f(z) = u + iv$

$$= e^{z} + c$$

$$= e^{x+iy} + c.$$

$$= e^{x}(\cos y + i\sin y) + c$$

$\therefore f(z) = e^{x}\cos y + c + i e^{x}\sin y$. Ans

Q.24 If $v = \arg z$ is harmonic? If yes, find a corresponding harmonic conjugate.

Sol: We have, $v = \arg z$

$$= \theta$$

$$= \tan^{-1}(y/x)$$

Diff. we get,

$$v_x = \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(-\frac{y}{x^2}\right)$$

$$v_x = -\frac{y}{x^2+y^2}$$

$$v_{xx} = \frac{y}{(x^2+y^2)^2}\cdot 2x$$

$$v_y = \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\cdot\frac{1}{x}$$

$$= \frac{x}{x^2+y^2}$$

and $v_{yy} = \frac{x \cdot 2y}{(x^2+y^2)^2}$

Here $v_{xx} + v_{yy} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}$

$= 0.$

Thus $v$ is harmonic function. Then we have to find harmonic conjugate.

Here, $v_x = -\frac{y}{x^2+y^2}$

$\Rightarrow -u_y = \frac{-y}{x^2+y^2}$

$\Rightarrow u_y = \frac{y}{x^2+y^2}$

Integrating w.r.t. $y$ we get

$u = \frac{1}{2} \log(x^2+y^2) + h(x)$

Again diff. w.r.t. $x$ we get

$u_x = \frac{x}{(x^2+y^2)} + h'(x)$

$v_y = \frac{x}{x^2+y^2} + h'(x)$

$\frac{x}{x^2+y^2} = \frac{x}{x^2+y^2} + h'(x) \Rightarrow h'(x) = 0$

$h(x) = c.$

Then required harmonic conjugate of $v$ is

$u = \frac{1}{2} \log(x^2+y^2) + c.$ [1]

Q.25 Is $v = (x^2 - y^2)^2$ is harmonic? If yes, find its harmonic conjugate and analytic function $f(z) = u + iv$.

Sol: We have $v = (x^2 - y^2)^2$

Diff. we get

$$v_x = 2(x^2 - y^2) \cdot 2x$$

$$v_{xx} = 4[2x^2 + (x^2 - y^2)]$$

Also $v_y = -4y(x^2 - y^2)$

$$v_{yy} = -4[x^2 - y^2 - 2y^2]$$

$$= -4(x^2 - 3y^2)$$

Here, $v_{xx} + v_{yy} = 12x^2 - 4y^2 - 4x^2 + 12y^2$

$$= 8x^2 + 8y^2$$

$$\neq 0$$

Thus the given function is not harmonic so we can not evaluate harmonic conjugate of $v$.

Q.26 If $u = ax^3 + bxy$, evaluate a and b such that the given function is harmonic and find its harmonic conjugate

Sol: We have, $u = ax^3 + bxy$, which is harmonic (given)

Then $u_{xx} + u_{yy} = 0$

$\Rightarrow 6ax = 0$ $\qquad u_y = bx$

$\Rightarrow a = 0$

For harmonic conjugate,

$u_x = v_y$

$v_y = 3ax^2 + by$

$v_y = by \quad (\because a = 0)$

on integration w.r.t. $y$, we get,

$v = \frac{by^2}{2} + h(x)$

and diff. w.r.t. $x$

$v_x = h'(x)$

$-u_y = h'(x)$

$\Rightarrow h'(x) = -bx$

on integration w.r.t. $x$ we get

$h(x) = -\frac{bx^2}{2} + c$

Thus,

$v = \frac{by^2}{2} - \frac{bx^2}{2} + c$

$v = \frac{b}{2}(y^2 - x^2) + c$

which is also harmonic. Then

$v_{xx} + v_{yy} = 0$

$\frac{b}{2}(-2) + \frac{b}{2}(2) = 0$

$-b + b = 0$

$0 = 0.$

This gives b is free.

Therefore required value of 'a' and 'b' are 0 and b respectively, ~~then~~ u is harmonic.

Q.27. Determine the value of a, when $u = \cos ax . \cosh 2y$ is harmonic and find its harmonic conjugate.

Sol: We have

$u = \cos ax . \cosh 2y$

Diff. partially we get,

$u_x = -a \sin ax . \cosh 2y$

$u_{xx} = -a^2 \cos ax . \cosh 2y$

$u_y = 2 \cos ax . \sinh 2y$

and $u_{yy} = 4 \cos ax . \cosh 2y$

We have, u is harmonic, then

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$\Rightarrow -a^2 \cos ax . \cosh 2y + 4 \cos ax . \cosh 2y = 0$

$\Rightarrow (4-a^2) \cos ax . \cosh 2y = 0$

Since, $\cos ax . \cosh 2y \neq 0$, then $4-a^2 = 0$.

i.e. $a = \pm 2$.

For harmonic conjugate,

$u_x = v_y$

$v_y = -a \sin ax \cosh 2y$

Integrating w.r.t. $y$, then we get

$v = -\frac{a}{2} \sin ax \sinh 2y + h(x)$

where $h(x)$ is function of $x$ or constant.

Differentiating with respect to $x$, we get,

$v_x = -\frac{a^2}{2} \cos ax . \sinh 2y + h'(x)$

$-u_y = -\frac{a^2}{2} \cos ax . \sinh 2y + h'(x)$

$\Rightarrow -2 \cos ax . \sinh 2y = -\frac{a^2}{2} \cos ax \sinh 2y + h'(x)$

$\Rightarrow -2 \cos ax \sinh 2y = -2 \cos ax . \sinh 2y + h'(x)$

$h'(x) = 0$

on integration,

$h(x) = c$.

Thus the required harmonic conjugate is

$v = -\frac{a}{2} \sin ax \cdot \sinh 2y + c.$

Where $a = \pm 2$ and $c$ is a complex constant.

Q.28. check for analyticity by using C-R eq.

(a) $f(z) = z^6$

$= [r(\cos\theta + i\sin\theta)]^6$ $[\because z = r(\cos\theta + i\sin\theta)]$

$= r^6 [\cos 6\theta + i \sin 6\theta]$ [$\because$ using Demoivre's theorem]

comparing it with $f(z) = u + iv$, then we get

$u = r^6 \cos 6\theta$ and $v = r^6 \sin 6\theta$

$u_r = 6r^5 \cos 6\theta$ $\quad v_r = 6r^5 \sin 6\theta$

$u_\theta = -6r^6 \sin 6\theta$ $\quad v_\theta = 6r^6 \cos 6\theta$

$u_r = 6r^5 \cos 6\theta$

$= \frac{1}{r} 6r^6 \cos 6\theta$

$= \frac{1}{r} v_\theta$

And $u_\theta = -6r^6 \sin 6\theta$

$= -r\, 6r^5 \sin 6\theta$

$= -r v_r$. Thus, the C-R eqn is satisfied. Therefore the function $f(z)$ is analytic.

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(b) $f(z) = e^{x}(\cos y + i\sin y)$ ...(i)

Sol$^n$: Comparing it with $f(z) = u+iv$ then we get,

$u = e^{x}\cos y$ and $v = e^{x}\sin y$

$u_x = e^{x}\cos y$ $v_x = e^{x}\sin y$

$u_y = -e^{x}\sin y$ $v_y = e^{x}\cos y$

$u_x = v_y$ and $u_y = -v_x$

Thus the C-R. equation is satisfied.
Therefore the function $f(z)$ is analytic.

(C) $f(z) = z\bar{z}$

Sol$^n$: Here, $f(z) = z\bar{z}$

$= (x+iy)(x-iy)$

$u+iv = x^2+y^2$

$u = x^2+y^2$ and $v=0$

$u_x = 2x$ $v_x = 0$

$u_y = 2y$ $v_y = 0$

$u_x \neq v_y$. That is $f(z)$ does not satisfy the C-R eq$^n$. Therefore, the function $f(z)$ is not analytic.

d. $f(z) = \log|z| + i\,Arg(z)$

$f(z) = \log r + i\theta$

$u + iv = \log r + i\theta$

Then $u = \log r$ and $v = \theta$

$u_r = \frac{1}{r}$ $\qquad$ $v_\theta = 1$

$u_\theta = 0$ $\qquad$ $v_r = 0$

$u_r = \frac{1}{r} v_\theta$

$\frac{1}{r} = \frac{1}{r} \cdot 1$

$\frac{1}{r} = \frac{1}{r}$

Similarly $u_\theta = r v_r$

$0 = r \cdot 0$

$0 = 0$

Thus $f(z)$ satisfies the C-R eqn.
Therefore, the function $f(z)$ is analytic.

Note: (a) $\frac{d}{dx}(\sinh x) = \cosh x$ $\qquad$ (c) $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

(b) $\frac{d}{dx}(\cosh x) = \sinh x$ $\qquad$ (d) $\frac{d}{dx}(\coth x) = -\text{cosech}^2 x$

(e) $\frac{d}{dx}(\text{cosech}\, x) = -\text{cosech}\, x \cdot \coth x$ $\qquad$ (vi) $\frac{d}{dx}(\text{sech}\, x)$

(e) $f(z) = \frac{Re z}{Im z}$

Sol^n: Here $f(z) = \frac{Re(z)}{Im(z)}$

$= \frac{x}{iy}$

$u+iv = -\frac{ix}{y}$

$u = 0, \quad v = -\frac{x}{y}$.

$u_y = 0$ and $v_x = -\frac{1}{y}$.

This shows that $u_y \neq -v_x$

This means $f(z)$ does not satisfy C-R eq^n
Therefore, the function $f(z)$ is not analytic.

(vii) $\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}$

(viii) $\frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$

(ix) $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$

(x) $\frac{d}{dx}(\operatorname{cosech}^{-1}x) = -\frac{1}{x\sqrt{x^2+1}}$

(xi) $\frac{d}{dx}(\operatorname{sech}^{-1}x) = -\frac{1}{x\sqrt{1-x^2}}$

(xii) $\frac{d}{dx}(\coth^{-1}x) = -\frac{1}{x^2-1}$.

(f) $f(z) = Re(z)^3$.

Sol: $= Re[(x+iy)^3]$

$= Re[x^3 + 3x^2 iy + 3x i^2 y^2 + i^3 y^3]$

$= Re[x^3 + 3x^2 iy + 3x(-y^2) - i y^3]$

By question,

$u+iv = x^3 - 3xy^2$

$u = x^3 - 3xy^2$ and $v = 0$

$u_x = 3x - 3y^2 \qquad v_y = 0$

This shows $u_x \neq v_y$. This means $f(z)$ is not analytic. $\boxed{11}$

$z = x + iy$
$\bar{z} = x - iy$

(9) $f(z) = |z^2|$
$= |z\bar{z}|$

Sol$^n$: Here, $f(z) = x^2 - y^2$
$u + iv = x^2 - y^2$
comparing then we get,

$u = x^2 - y^2$ and $v = 0$
$u_x = 2x$ $\qquad v_y = 0$

$u_x \neq v_y$

so it does not satisfy the C-R eq$^n$.
Therefore, it is not an analytic function.

(29) Show that the function $f(z) = xy + iy$ is everywhere continuous but not analytic.

Sol$^n$: $f(z) = u + iv$
$u + iv = xy + iy$ ——— (1)
$u = xy$ and $v = y$

$u_x = y$ and $v_y = 1$.

This shows the partial derivatives of $f(z)$ are exist, so $f(z)$ is continuous everywhere.
But $u_x \neq v_y$. This means $f(z)$ does not satisfy C-R eq$^n$. So the function $f(z)$ is not analytic.

Q.30 Show that $\sinh z$ is an analytic function.

Sol$^n$: $f(z) = \sinh z$

$$= \frac{e^{z} - e^{-z}}{2}$$

$$= \frac{e^{x+iy} - e^{-(x+iy)}}{2}$$

$$= \frac{e^{x}[\cos y + i\sin y] - e^{-x}[\cos y - i\sin y]}{2}$$

$$= \left(\frac{e^{x} - e^{-x}}{2}\right)\cos y + i\left(\frac{e^{x} + e^{-x}}{2}\right)\sin y$$

$$= \sinh x \cdot \cos y + i\cosh x \cdot \sin y.$$

comparing it with $f(z) = u + iv$ then we get,

$u = \sinh x \cdot \cos y$ $\qquad$ $v = \cosh x \cdot \sin y$

$u_x = \cosh x \cdot \cos y$ $\qquad$ $v_y = \cosh x \cdot \cos y.$

$u_y = -\sinh x \cdot \sin y$ $\qquad$ $v_x = \sinh x \cdot \sin y$

Now $u_x = v_y$ and $u_y = -v_x$.

which satisfies the C-R eq$^n$. Therefore $f(z)$ is analytic function. |||

1.1.14 ✓

Q.31 Show that $f(z) = z^2$ is analytic and show that $f'(z) = 2z$.

Sol: Here $f(z) = z^2$

$= (x+iy)^2$

$u+iv = x^2 - y^2 + 2ixy$.

$u = x^2 - y^2$ and $v = 2xy$.

$u_x = 2x$ $\quad$ $v_y = 2x$

$u_y = -2y$ $\quad$ $v_x = 2y$

$u_x = v_y$ and $u_y = -v_x$.

Hence C-R eqn is satisfied. This means that the function is analytic.

Also, $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{\cancel{z^2} + 2z \cdot \Delta z + \Delta^2 z^2 - \cancel{z^2}}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{\cancel{\Delta z}(2z + \Delta z)}{\cancel{\Delta z}} = 2z + 0$

$= 2z$. $\boxed{//}$

Q.32. Show that $v = 2xy - \frac{y}{x^2+y^2}$ is harmonic function. Find harmonic conjugate $u$ of $v$.

$$v = 2xy - \frac{y}{x^2+y^2}$$

$$v_x = 2y + \frac{2xy}{(x^2+y^2)^2}$$

$$v_y = 2x - \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= 2x - \frac{x^2 - y^2}{(x^2+y^2)^2}$$

and

$$v_{xx} = \frac{(x^2+y^2)^2 \cdot 2y - 2xy\,2(x^2+y^2)\cdot 2x}{(x^2+y^2)^2}$$

$$v_{yy} = -\left[\frac{(x^2+y^2)^2(-2y) - 4x^2y(x^2+y^2)}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{(x^4+2x^2y^2+y^4)(-2y) - 4x^4y - 4x^2y^3}{(x^2+y^2)^2}\right]$$

$$= -\left[\frac{-2x^4y - 2\cdot 2\cdot x^2y^3 - 2y^5 - 4x^4y - 4x^2y^3}{(x^2+y^2)^2}\right]$$

$$= \frac{6x^4y + 8x^2y^3 + 2y^5}{(x^2+y^2)^2}$$

$$V_{xx} = (x^2+y^2)\cdot 2y\left[\frac{x^2+y^2-4x^2}{(x^2+y^2)^4}\right]$$

$$V_{xx} = \frac{2y\,(x^2+y^2-4x^2)}{(x^2+y^2)^3}$$

$$V_{xx} = \frac{2y\,(y^2-3x^2)}{(x^2+y^2)^3}$$

$$V_{yy} = 0 - \left[\frac{(x^2+y^2)^2\cdot(-2y) - (x^2-y^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}\right]$$

$$= -\left[\frac{-2y(x^2+y^2)\left\{x^2+y^2+(x^2-y^2)2\right\}}{(x^2+y^2)^4}\right]$$

$$= -\left[\frac{-2y\,(x^2+y^2+2x^2-2y^2)}{(x^2+y^2)^3}\right]$$

$$= \frac{2y\,[-y^2+3x^2]}{(x^2+y^2)^3} = -\frac{2y\,(y^2-3x^2)}{(x^2+y^2)^3}.$$

i.e. $v_{xx} + v_{yy} = \frac{2y(y^2-3x^2)}{(x^2+y^2)^3} - \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$

$= 0.$

Suppose that $f(z) = u+iv$ be analytic. so $f(z)$ satisfies the C-R eq$^n$. Therefore,

$u_y = -v_x$

$= -2y - \frac{2xy}{(x^2+y^2)^2}$

Integrating w.r.t. y we get,

$u = -2\frac{y^2}{2} + \frac{x}{x^2+y^2} + h(x)$

$u = -y^2 + \frac{x}{(x^2+y^2)} + h(x) \quad \ldots (1)$

Diff. w.r.t. x then we get

$u_x = v_y.$

So, $\frac{(x^2+y^2).1 - x(2x)}{(x^2+y^2)^2} + h'(x)$

$= 2x - \frac{(x^2-y^2)}{(x^2+y^2)^2}$

$h'(x) + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = 2x - \frac{(x^2-y^2)}{x^2+y^2}$

Rough

$x\int \frac{2y}{(x^2+y^2)^2}\,dy$

$x^2+y^2 = p$

$2y\frac{dy}{dp} = 1$

$2y\,dy = dp.$

$= x\int p^{-2}\,dp$

$= x.\frac{p^{-2+1}}{(-2+1)} = -\frac{x}{p}$

$= -\frac{x}{(x^2+y^2)}$

$h'(x) = 2x$

$h(x) = \frac{2x^2}{2} + c$

$h(x) = x^2 + c.$

Then from (1)

$u = -y^2 + \frac{x}{(x^2+y^2)} + x^2 + c.$ Ans

Q.33. Show that $u = e^{2x}(x\cos 2y - y\sin 2y)$ is a harmonic function. Find an analytic function for which $u(x,y)$ is the real part.

Sol$^n$: $u = e^{2x}(x\cos 2y - y\sin 2y)$

$u_x = 2e^{2x}(x\cos 2y - y\sin 2y) + e^{2x}\cos 2y$

$u_{xx} = 4e^{2x}(x\cos 2y - y\sin 2y) + 2e^{2x}\cos 2y + 2e^{2x}\cos 2y$

$u_y = e^{2x}[-2x\sin 2y - \sin 2y - 2y\cos 2y]$

$u_{yy} = e^{2x}[-4x\cos 2y - 2\cos 2y - 2\cos 2y + 4y\sin 2y]$

$u_{xx} + u_{yy} = 4xe^{2x}\cancel{\cos 2y} - 4ye^{2x}\cancel{\sin 2y} + \cancel{2e^{2x}\cos 2y} + 2e^{2x}\cancel{\cos 2y} - 4xe^{2x}\cancel{\cos 2y} - 2e^{2x}\cancel{\cos 2y} - 2e^{2x}\cancel{\cos 2y} + 4e^{2x}y\cancel{\sin 2y}$

i.e. $u_{xx} + u_{yy} = 0$.

This shows that the function $u$ is a harmonic function.

Suppose that $f(z) = u + iv$ be an analytic function, where $u$ is given in (i). so, it Satisfies C-R eq$^n$ i.e.

$$u_x = v_y \quad , \quad u_y = -v_x.$$

$$v_y = u_x = e^{2x}[2x\cos 2y + \cos 2y - y 2\sin 2y]$$

Integrating w.r.t. $y$ we get,

$$v = e^{2x}\left[2x \int \cos 2y\, dy + \int \cos 2y\, dy - 2\int y \sin 2y\, dy\right]$$

$$v = e^{2x}\left[2x.\left(\frac{\sin 2y}{2}\right) + \frac{\sin 2y}{2} - 2\{y \int \sin 2y\, dy - \int \{1 \int \sin 2y\, dy\} dy\}\right]$$

$$v = e^{2x}\left[x\sin 2y + \frac{\sin 2y}{2} + y\frac{\cos 2y}{2} + \int \frac{\cos 2y}{2} dy\right] + h(x)$$

$$v = e^{2x}\left[x \sin 2y + \frac{\sin 2y}{\cancel{2}} + y\frac{\cos 2y . \cancel{2}}{\cancel{2}} - \frac{2\sin 2y}{4\,2}\right] + h(x)$$

$$v = e^{2x}\left[4x\sin 2y + \sin 2y + 2y\cos 2y\right] + h(x)$$

$$v = e^{2x}[x\sin 2y + y\cos 2y] + h(x) \quad \text{——} \; (*)$$

$$v_x = e^{2x}[\sin 2y] + 2e^{2x}(x\sin 2y + y\cos 2y) + h'(x)$$

$V_x = e^{2x} \cdot \sin 2y + 2e^{2x}(x \sin 2y + y \cos 2y) + h'(x)$

$-u_y = e^{2x} \sin 2y + 2e^{2x} \cdot x \sin 2y + 2e^{2x} y \cos 2y + h'$

$\cancel{2xe^{2x} \sin 2y + e^{2x} \sin 2y + 2y e^{2x} \cos 2y}$

$\cancel{= e^{2x} \sin 2y + 2xe^{2x} \sin 2y + 2e^{2x} y \cos 2y}$
$\cancel{+ h'(x)}$

$h'(x) = 0$

on integration

$h(x) = c.$

Then from (*)

$v = e^{2x}[x \sin 2y + y \cos 2y] + c.$ Ans.

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(Q.35) Show that $e^x(x \cos y - y \sin y)$ is harmonic function. Find the analytic function for which the function is the imaginary part.

Sol: $v = e^x(x \cos y - y \sin y)$ - - - (i)

$V_x = e^x(x \cos y - y \sin y) + e^x \cdot \cos y$

$V_x = e^x(x \cos y - y \sin y + \cos y)$

$V_{xx} = e^x(x \cos y - y \sin y + \cos y) + e^x \cos y$

$$v_{xx} = e^x[x\cos y - y\sin y + 2\cos y]$$

$$v_y = e^x(-x\sin y - \sin y - y\cos y)$$

$$v_{yy} = e^x[-x\cos y - \cos y - \cos y + y\sin y]$$

$$v_{yy} = e^x[-x\cos y - 2\cos y + y\sin y]$$

$$v_{xx} + v_{yy} = \cancel{xe^x\cos y} - \cancel{e^x y\sin y} + \cancel{2e^x\cos y}$$
$$\cancel{-xe^x\cos y} - \cancel{2e^x\cos y} + \cancel{e^x \cdot y\sin y}$$
$$= 0$$

$$v_{xx} + v_{yy} = 0$$

Thus the function $v$ is a harmonic.

Suppose that $f(z) = u + iv$ be an analytic function for which $v$ is given in (1). Then it satisfies the C-R eq i.e.
$u_x = v_y$ and $u_y = -v_x$.

So that $u_x = v_y$.

$$u_x = e^x[-x\sin y - \sin y - y\cos y]$$

Then $\cancel{u} = -e^x \cdot x\sin y - \cancel{e^x}\sin y -$

$$u = -\sin y\int xe^x\,dx - \sin y\int e^x\,dx - y\cos y\int dx$$

$= -\sin y \int x e^x\,dx - \sin y \int e^x\,dx - y\cos y \int \ldots$

$= -\sin y[xe^x - e^x] - \sin y\, e^x - y\cos y \cdot x] + h(y)$

$u = -xe^x\sin y + e^x\sin y - \sin y\, e^x - xy\cos y + h(y)$

$u_y = -xe^x\cos y + \cancel{e^x\cos y} - \cancel{e^x\cos y} - x[-y\sin y + \cos y]$
$+ h'(y)$

$u_y = -xe^x\cos y + xy\sin y - x\cos y + h'(y)$

$u_x = v_y$

$u_x = e^x[-x\sin y - \sin y - y\cos y]$

$u = -\sin y \int xe^x\,dx - \sin y \int e^x\,dx - y\cos y \int e^x\,dx.$

$u = -\sin y[xe^x - e^x] - e^x\sin y - y\cos y \cdot e^x + h(y)$

$u = -\sin y\, xe^x + \cancel{e^x\sin y} - \cancel{e^x\sin y} - e^x \cdot y \cdot \cos y + h(y)$

$u_y = -\cos y\, xe^x + e^x\cos y - e^x\cos y - e^x(-y\sin y + \cos y)$
$+ h'(y)$

$u_y = -xe^x\cos y + \cancel{e^x\cos y} - \cancel{e^x\cos y} + ye^x\sin y - e^x\cos y$
$+ h'(y)$

$u_y = -xe^x \cos y - e^x\cos y + ye^x \sin y + h'(y)$

$-v_x = -xe^x \cos y - e^x \cos y + ye^x \sin y + h'(y)$

$\cancel{xe^x \cos y} + \cancel{ye^x \sin y} - \cancel{e^x \cos y} = \cancel{-xe^x \cos y}$
$\cancel{- e^x \cos y} + \cancel{ye^x \sin y} + h'(y)$

$h'(y) = 0.$

$h(y) = c.$

Then from (*) we can write,

$u = -e^x[x\sin y + y\cos y] + c.$

$\therefore f(z) = u + iv$

$= -e^x[x\sin y + y\cos y] + i\, e^x(x\cos y - y\sin y)$

$= -e^x[x\sin y + y\cos y - ix\cos y + iy\sin y] + c$

$= -e^x[ye^{iy} - ixe^{iy}] + c$

$= -e^x.e^{iy}[y - ix] + c$

$= -e^{x+iy}[-i(x+iy)] + c$

$= -e^z(-iz) + c$

$= i.e^z.z + c = i.e^z.z + c.$ Ans

Given that $u = x^2 - y^2$ and $v = \left[\frac{-y}{x^2+y^2}\right]$. prove that both u and v are harmonic function but u+iv is not an analytic function of z.

| | |
|---|---|
| $u = x^2 - y^2$ | $u_y = -2y$ |
| $u_x = 2x$ | $u_{yy} = -2$ |
| $u_{xx} = 2$ | |

$u_{xx} + u_{yy} = 0$ Thus u is a harmonic function.

$$v = -y/(x^2+y^2)$$

and $v_x = -\left[\frac{(x^2+y^2)\cdot 0 - y\cdot 2x}{(x^2+y^2)^2}\right]$

$$v_x = \frac{2xy}{(x^2+y^2)^2}$$

$$v_{xx} = \frac{(x^2+y^2)^2\cdot 2y - 2xy\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$v_{xx} = \frac{(x^2+y^2)^2\cdot 2y - 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$v_{xx} = \frac{2y(x^2+y^2 - 4x^2)}{(x^2+y^2)^3}$$

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$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$v_y = -\frac{[(x^2+y^2)(+1) - y(2y)]}{(x^2+y^2)^2}$$

$$v_y = -\left[\frac{(x^2+y^2-2y^2}{(x^2+y^2)^2}\right]$$

$$v_y = -\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right]$$

$$v_{yy} = -\left[\frac{(x^2+y^2)^2(-2y) - (x^2-y^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}\right]$$

$$v_{yy} = -\left[\frac{(x^2+y^2)(-2y) - (x^2-y^2)(4y)}{(x^2+y^2)^3}\right]$$

$$v_{yy} = +\left[\frac{2y(x^2+y^2+2x^2-2y^2)}{(x^2+y^2)^3}\right]$$

$$v_{yy} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

i.e. $v_{xx} + v_{yy} = 0.$

$$v_{yy} = -\frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

Hence $u$ and $v$ are harmonic functions.

For the next part,

$u_x \neq v_y$

This shows that $f(z)$ does not satisfy the C-R eq$^n$. Therefore, the function $f(z) = u + iv$ is not analytic.

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Q.37 Determine whether the following functions are harmonic. If your answer is yes, find a corresponding analytic function $f(z) = u + iv$

a. $u = \frac{x}{x^2+y^2}$

b. $u = \log|z|$

c. $v = -e^{-x} \sin y$

d. $v = x^3 - 3x^2$.

Sol$^n$: $u = \frac{x}{x^2+y^2}$

$u_x = \frac{(x^2+y^2)\cdot 1 - x \cdot 2x}{(x^2+y^2)^2}$

$u_x = \frac{y^2-x^2}{(x^2+y^2)^2}$

$u_{xx} = \frac{(x^2+y^2)^2(-2x) - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$

$$= \frac{-2x(3y^2-x^2)}{(x^2+y^2)^3}.$$

Also $u_y = \dfrac{(x^2+y^2)\cdot 0 - x(2y)}{(x^2+y^2)^2}$

$u_y = \dfrac{-2xy}{(x^2+y^2)^2}$

$u_{yy} = -2x\dfrac{\left[(x^2+y^2)^2\cdot 1 - y\cdot 2(x^2+y^2)\cdot 2y\right]}{(x^2+y^2)^4}$

$u_{yy} = -2x\left[\dfrac{(x^2+y^2)\{x^2+y^2-4y^2\}}{(x^2+y^2)^4}\right]$

$u_{yy} = -\dfrac{2x\left[x^2-3y^2\right]}{(x^2+y^2)^3}$

$u_{yy} = \dfrac{2x(3y^2-x^2)}{(x^2+y^2)^3}$

$u_{xx}+u_{yy} = -\dfrac{2x(3y^2-x^2)}{(x^2+y^2)^3} + \dfrac{2x(3y^2-x^2)}{(x^2+y^2)^3}$

$= 0$

Thus the function is harmonic function.

for harmonic conjugate, suppose

) = u+iv be an analytic function. so

satisfies the C-R eqn i.e.

$u_x = v_y$ and $u_y = -v_x$.

ble $u_x = v_y$ that means

$v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$

$= -\frac{[(x^2+y^2) - y.2y]}{(x^2+y^2)^2}$

$= + -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}$

$- \int \frac{1}{x^2+y^2}\, dy + \int \frac{2y^2}{(x^2+y^2)^2}\, dy$

$- \frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) +$

$u_y = -v_x$

$v_x = \frac{-2xy}{(x^2+y^2)^2}$

Integrating w.r.t. x then we get,

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$$-v = -y \int \frac{2x}{(x^2+y^2)^2} dx$$

let $x^2+y^2 = t$

$2x\,dx = dt$

$$-v = -y \int \frac{dt}{t^2}$$

$$-v = -y \cdot \left(-\frac{1}{t}\right) + h(y)$$

$$-v = \frac{y}{x^2+y^2} + h(y) \quad \text{---} \ *$$

$$-v_y = \frac{(x^2+y^2)\cdot 1 - y(2y)}{(x^2+y^2)^2} + h'(y) \quad \text{---} \ **$$

$$-v_y = \frac{x^2-y^2}{(x^2+y^2)^2} + h'(y) \quad \text{---} \ **$$

$$-v_y = -\frac{(y^2-x^2)}{(x^2+y^2)^2} + h'(y)$$

$$\cancel{-\frac{(y^2-x^2)}{(x^2+y^2)^2}} = \cancel{-\frac{(y^2-x^2)}{(x^2+y^2)^2}} + h'(y)$$

$h'(y) = 0 \Rightarrow h(y) = c.$

Then from (**)

$-\cancel{xy}$

$$-v = \frac{y}{y^2+x^2} + c.$$

$$v = -\frac{y}{y^2+x^2} - c.$$

$$f(z) = u + iv.$$

$$= \frac{x}{x^2+y^2} + i\left(-\frac{y}{y^2+x^2} - c\right)$$

$$= \frac{x-iy}{x^2+y^2} - ic.$$

b. $u = \log|z|$

Sol: $u = \log\left(\sqrt{x^2+y^2}\right)$

So $u_x = \frac{1}{(x^2+y^2)^{1/2}} \cdot \frac{1}{2}(x^2+y^2)^{1-1/2} \cdot 2x = \frac{x}{x^2+y^2}$

and $u_{xx} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

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Also $u_y = \frac{y}{x^2+y^2}$

$$u_{yy} = \frac{(x^2+y^2).1 - y.(2y)}{(x^2+y^2)^2}$$

$$u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Now, $u_{xx} + u_{yy} = \frac{y^2-x^2}{(x^2+y^2)} + \frac{x^2-y^2}{(x^2+y^2)^2}$

$$u_{xx} + u_{yy} = 0.$$

Therefore, u is harmonic. Therefore u is analytic. so it satisfies the C-R eqn. i.e.

$u_x = v_y$ and $u_y = -v_x$.

Therefore $v_y = u_x = \frac{x}{x^2+y^2}$

integrating w.r.t. y then we get.

$$v = x\int \frac{1}{x^2+y^2} dy + h(x)$$

$$= x.\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) + h(x)$$

$$= \tan^{-1}\left(\frac{y}{x}\right) + h(x)$$

Then, $v_x = \frac{d}{dx}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] + h'(x)$

$$v_x = -\frac{y}{y^2+x^2} + h'(x)$$

or, $$-\cancel{\frac{y}{(y^2+x^2)}} = -\cancel{\frac{y}{(y^2+x^2)}} + h'(x)$$

or, $h'(x) = 0$

$\therefore h(x) = c.$

Hence $v = \tan^{-1}\left(\frac{y}{x}\right) + c.$

Thus $f(z) = u + iv$

$$= \log|z| + i\tan^{-1}(y/x) + c.$$

$$= \log z + i\arg z + c \quad \left[\theta = \tan^{-1}\left(\frac{y}{x}\right)\right.$$

c. $v = -e^{-x}\sin y$

Sol$^n$: $v = -e^{-x}\sin y.$

$v_x = e^{-x}\sin y.$

$v_{xx} = -e^{-x}\sin y.$

$v_y = -e^{-x}\cos y$

$v_{yy} = e^{-x}\sin y$

i.e. $v_{xx} + v_{yy} = 0$

Therefore, $v$ is a harmonic function,

Suppose that $f(z) = u + iv$ be analytic function. so it satisfies the C-R eq$^n$.

i.e. $u_x = v_y$ and $u_y = -v_x$

Therefore, $u_x = -e^{-x}\cos y$

So that $u = +e^{-x}\cos y + h(y)$ — (*)

Then $u_y = -e^{-x}\sin y + h'(y)$

Since $u_y = -v_x$

So $-e^{-x}\cancel{\sin y} = -e^{-x}\cancel{\sin y} + h'(y)$

$h'(y) = 0$

$h(y) = c.$

Then from (*)

$u = e^{-x}\cos y + c.$

$v = -e^{-x}\sin y.$

Hence $f(z) = u + iv$

$= e^{-x}\cos y + c + i(-e^{-x}\sin y)$

$= e^{-x}(\cos y - i\sin y) + c$

$= e^{-x}.e^{-iy} + c$

$= e^{-x-iy} + c$

$= e^{-z} + c.$ Ans.

d. $v = x^3 - 3x^2$.

Soln: $v_x = 3x^2 - 6x$ $\quad u = 0$.

$v_{xx} = 6x - 6$

$v_y = 0$

$v_{yy} = 0$.

$v_{xx} + v_{yy} = 6x - 6 + 0$

$= 6x - 6 \neq 0$

Then $u_{xx} + u_{yy} = 0$.

This function is not harmonic and so the function is not analytic.

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Q 15 Determine a and b such that the given functions are harmonics and find a conjugate harmonic.

(a) $u = ax^3 + by^3$

(b) $u = e^{ax} \cos 5y$.

Soln: Here, $u = ax^3 + by^3$

Then $u_x = 3ax^2 + 3b \cdot 0 = 3ax^2$

$u_{xx} = 6ax$

$u_y = 3by^2$

$u_{yy} = 6by$.

Since $u$ is harmonic, so

$u_{xx} + u_{yy} = 0$

$6ax + 6by = 0$

$ax + by = 0.$

This condition will satisfy if $a = 0$ and $b = 0$

Therefore, $u = 0.$

By given hypothesis $u$ is harmonic. So, it is analytic and therefore satisfies the C-R eqn

i.e. $u_x = v_y$ and $u_y = -v_x$.

So, $v_y = u_x = 0$

Integrating w.r.t. $y$ we get,

$v = h(x)$ —— (A)

Then, $v_x = h'(x)$

Since $u$ satisfies C-R eqn so we can write,

$u_y = -v_x$

so $v_x = h'(x)$

$-u_y = h'(x)$ —— (*)

$\Rightarrow$ from (*)

$h'(x) = 0$

$\Rightarrow h(x) = c.$

$\Rightarrow v = c$

from (A)

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(b) $u = e^{ax} \cos 5y$

Sol$^n$: Here, $u = e^{ax} \cos 5y$

So, $u_x = a e^{ax} \cos 5y$

$u_{xx} = a^2 e^{ax} \cos 5y$

and $u_y = -5 e^{ax} \sin 5y$

$u_{yy} = -25 e^{ax} \cos 5y$

Since, the function $u$ is harmonic. So

$u_{xx} + u_{yy} = 0$

$\Rightarrow a^2 - 25 = 0$

$\Rightarrow (a+5)(a-5) = 0$

$\Rightarrow a = -5, 5.$

Also, for harmonic conjugate of $u$. Since the function $f(z) = u + iv$ be analytic.

So it satisfies the C-R. eq$^n$ i.e.

$u_x = v_y$ and $u_y = -v_x$.

Then, $v_x = 5e^{\pm 5x} \sin 5y.$

$v = \frac{5e^{5x}}{5} \sin 5y + h(y)$

$v = e^{\pm 5x} \sin 5y + h(y)$ —— ⊛

$v_y = u_x = e^{5x} \sin 5y \cancel{+h(y)}\ h'(y)$

so We can write.

$e^{\pm 5x} \cancel{\sin 5y} = e^{5x} \cancel{\sin 5y} + h'(y)$

$h'(y) = 0$

on integration then we get.

$h(y) = c.$

Then from ⊛

$v = e^{\pm 5x} \sin 5y + c.$ Ans.

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Q.12 Show that the function $u = \cos x . \cosh y$ is harmonic and find its harmonic conjugate.

: Here, $u = \cos x . \cosh y$

$u_x = -\sin x \cosh y$

$u_{xx} = -\cos x \cosh y.$

And $u_y = \cos x . \sinh y$

$u_{yy} = \cos x . \cosh y$.

Now, $u_{xx} + u_{yy} = -\cos x \cosh y + \cos x . \cosh y$
$= 0$

Thus $u$ is a harmonic function. And for harmonic conjugate, suppose that

$f(z) = u + iv$ is analytic.
so it satisfies the C-R eqn i.e.

$u_x = v_y$ and $u_y = -v_x$

Since $u_x = v_y$

so $v_y = -\sin x . \cosh y$

Then $v = -\sin x . \sinh y + h(x)$

so $v_x = -\cos x . \sinh y + h'(x)$

since $u_y = -v_x$

So that $h'(x) = 0$

Therefore $h(x) = c$.

Hence $v = -\sin x . \sinh y + c$.

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Q.40. Prove that $u = y^3 - 3x^2y$ is a harmonic function. Determine its harmonic conjugate and find the corresponding analytic function $f(z)$ interms of z.

Sol: Here, $u = y^3 - 3x^2y$

$$u_x = -6xy$$

$$u_{xx} = -6y$$

and $u_y = 3y^2 - 3x^2$

$$u_{yy} = 6y$$

clearly, $u_{xx} + u_{yy} = 0$. So, u is a harmonic function.

For harmonic conjugate of u, suppose that $f(z) = u + iv$ be analytic. So it is satisfies the C-R eq$^n$ i.e.

$u_x = v_y$ and $u_y = -v_x$

Since $u_x = v_y$

$$\Rightarrow v_y = -6xy$$

Integrating w.r.t. y we get,

$$v = -3xy^2 + h(x) \quad \text{--- (i)}$$

$v_x = -3y^2 + h'(x)$

Since $u_y = -v_x$

so $h'(x) = 3x^2$

Then $h(x) = x^3 + c$.

Hence (1) becomes,

$v = -3xy^2 + x^3 + c$.

Thus $f(z) = u + iv$

$= y^3 - 3x^2y - i.3xy^2 + ix^3 + ic$

$= i[x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3] + ic$

$= i(x+iy)^3 + ic$

$= iz^3 + ic$. Ans

Q14. If $u = x^3 - 3xy^2$, show that there exists a function $v(x,y)$ such that

$w = u + iv$ is analytic in a finite region.

Sol: Here $u = x^3 - 3xy^2$ ---- (1)

so $u_x = 3x^2 - 3y^2$

$u_{xx} = 6x$

And $u_y = -6xy$

$u_{yy} = -6x$

clearly, $u_{xx} + u_{yy} = 6x - 6x = 0$.

so $u$ is a harmonic function.

Thus $u$ is harmonic function. Therefore, it is analytic and satisfies the C-R eqn i.e. $v_y = u_x = 3x^2 - 3y^2$

Integrating w.r.t. $y$ then,

$v = 3x^2 y - \frac{3y^3}{3} + h(x)$ - - (ii)

Then $v_x = 6xy + h'(x)$

Since, being $u$ satisfies C-R eqn, we have,

$u_y = -v_x \Rightarrow 6xy + h'(x) = 6xy$

$h'(x) = 0$.

Integrating w.r.t. $x$ then we get

$h(x) = c$.

so $u_x = 3x^2 - 3y^2$

$u_{xx} = 6x$

And $u_y = -6xy$

$u_{yy} = -6x$

clearly, $u_{xx} + u_{yy} = 6x - 6x = 0$.

so $u$ is a harmonic function.

Thus $u$ is harmonic function. Therefore, it is analytic and satisfies the C-R eq^n i.e. $v_y = u_x = 3x^2 - 3y^2$

Integrating w.r.t. $y$ then,

$v = 3x^2y - \frac{3y^3}{3} + h(x)$ - - (ii)

Then $v_x = 6xy + h'(x)$

Since, being $u$ satisfies C-R eq^n, we have,

$u_y = -v_x \Rightarrow 6xy + h'(x) = 6xy$

$h'(x) = 0$.

Integrating w.r.t. $x$ then we get

$h(x) = c$.

Thus (ii) becomes,

$v = 3x^2y - y^3 + c.$

And $w = u + iv$

$= x^3 - 3xy^2 + i(3x^2y - y^3 + c)$ **Ans**

Assignment-1

. Define harmonic function. Is a function $v = 2xy - \frac{y}{x^2+y^2}$ is harmonic? If yes,

Find a corresponding harmonic conjugate and the analytic function. [8]

. Define harmonic function. prove that the function $v = \arg z$ is harmonic.

Also find its conjugate and the corresponding analytic function. [8]

Define harmonic function. show that the function $u = 3x^2y + x^2 - y^3 - y^2$ is a harmonic function. Find the analytic function for which the given function is a real part. [8]

. state and prove the necessary condition for analyticity. Test the analyticity of the function $f(z) = \log z$. [8]

Q.5 Define Laplace equation and harmonic function. Determine a and b such that $u = ax^3 + by^3$ is harmonic and also find the harmonic conjugate. [8]

Q.6 Check $u = \sin x . \cosh y$ is harmonic or not? If yes, find corresponding harmonic conjugate $v$ of $u$. [8]

Q.7. What do you mean by analyticity of function $f(z)$. State Cauchy Riemann $eq^n$ and hence show that it is necessary condition for the function to be analytic. [8]

Q.8 At first, verify the harmonic function. then find harmonic conjugate and corresponding analytic function of the following functions.

a. $u = y^3 - 3x^2y$

b. $u = \cos x . \cosh y$

c. $u = \ln|z| = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$

Q.9 Test the analyticity of the function $f(z) = \frac{\mathrm{Re}\, z}{\mathrm{Im}\, z}$.

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Q.37(a)

$$u = \frac{x}{x^2+y^2}$$

$$u_x = \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$u_{xx} = \frac{(x^2+y^2)^2(-2x) - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2x) - 4x(y^2-x^2)}{(x^2+y^2)^3}$$

$$= \frac{-2x^3 - 2xy^2 - 4xy^2 + 4x^3}{(x^2+y^2)^3}$$

$$= \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$= \frac{2x(x^2-3y^2)}{(x^2+y^2)^3}$$

Similarly, $u = \frac{x}{x^2+y^2}$

$$u_y = \frac{(x^2+y^2)\cdot 0 - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$u_y = \frac{-2xy}{(x^2+y^2)^2}$$

$$y = -\frac{\left[(x^2+y^2)^2 (2x) - 2xy \cdot 2(x^2+y^2)\cdot 2y\right]}{(x^2+y^2)^4}$$

$$y = -\left[\frac{(x^2+y^2)(2x) - 8xy^2}{(x^2+y^2)^3}\right]$$

$$= -\left[\frac{2x^3 + 2xy^2 - 8xy^2}{(x^2+y^2)^3}\right]$$

$$yy = \frac{-(2x^3 - 6xy^2)}{(x^2+y^2)^3}$$

$$yy = -\frac{2x(x^2-3y^2)}{(x^2+y^2)^3}$$

i.e. $u_{xx} + u_{yy} = 0$

Thus $u$ is harmo function.

Now, for analytic function,

$f(z) = u + iv$

(I) $u_x = v_y$ and (II) $u_y = -v_x$

$u_y = -v_x$

$$-v_x = -\frac{2xy}{(x^2+y^2)^2}$$

$$v_x = \frac{2xy}{(x^2+y^2)^2}$$

Integrating w.r.t. $x$ then we get,

$$v = y\int \frac{2x}{(x^2+y^2)^2} \cdot dx + h(y)$$

let $x^2+y^2 = t$

$2x\,dx = dt$

$$v = y\int \frac{1}{t^2} \cdot dt + h(y)$$

$$v = -\frac{y}{t} + h(y)$$

$$v = -\frac{y}{(x^2+y^2)} + h(y) \quad \text{——— (1)}$$

$$v_y = -\left[\frac{(x^2+y^2)\cdot 1 - y\,(2y)}{(x^2+y^2)^2}\right]$$

$$v_y = -\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right] + h'(y)$$

$$v_y = \frac{y^2-x^2}{(x^2+y^2)^2} + h'(y)$$

$$u_x = \frac{y^2-x^2}{(x^2+y^2)^2} + h'(y)$$

$$\cancel{\frac{y^2-x^2}{(x^2+y^2)^2}} = \cancel{\frac{y^2-x^2}{(x^2+y^2)^2}} + h'(y)$$

$$0 = h'(y)$$

$$h(y) = c.$$

Then from (1)

$$v = -\frac{y}{(x^2+y^2)} + c$$

Then the corresponding analytic function is

$$f(z) = u + iv$$

$$= \frac{x}{(x^2+y^2)} + \frac{i(-y)}{(x^2+y^2)} + ic. = \frac{x - iy}{x^2+y^2} + i\ldots$$ Ans