Vector Algebra [6 Hrs]

Scalar ~~Triple~~ Product (OR Dot product) of two vector

Let $\vec{a}$ and $\vec{b}$ are two vectors then scalar product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ and is given by:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

i.e.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{a\cdot b} \qquad \left[\because \begin{array}{l} |\vec{a}| = a \\ |\vec{b}| = b \end{array}\right]$$

Let, $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and
$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Then.

$$\vec{a}\cdot\vec{b} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})(b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$
$$= a_1b_1 + a_2b_2 + a_3b_3$$

$$\left[\begin{array}{l} \because \vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1 \quad \& \\ \vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{k}\cdot\vec{i} = 0 \end{array}\right]$$

For eg:

Let $\vec{a} = 2\vec{i} + 3\vec{j} + 5\vec{k}$ &
$\vec{b} = \vec{i} + 2\vec{j} + 3\vec{k}$

Then, $\vec{a}\cdot\vec{b} = 2\times1 + 3\times2 + 5\times3$
$= 2 + 6 + 15$
$= 23.$

CamScanner

Properties of Scalar Product of two vectors

i) Scalar vector of two vector is commutative
i.e. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

ii) Two non-zero vectors are perpendicular (orthogonal) if and only if (iff) their scalar product is zero.
i.e.
$\vec{a} \cdot \vec{b} = 0$. &
iff $\theta = 90^\circ$ or $\theta = \pi/2$

iii) If two vectors $\vec{a}$ and $\vec{b}$ are parallel if
$\theta = 0$
or $\theta = \pi$
i.e.
$\vec{a} = \lambda \vec{b}$

iv) $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = a^2$

v) The scalar product of two vector is distributive
i.e.
$\vec{a} (\vec{b} + \vec{c}) = \vec{a} \vec{b} + \vec{a} \cdot \vec{c}$
&
$(\vec{a} + \vec{b})(\vec{c} + \vec{d}) = (\vec{a} + \vec{b}) \vec{c} + (\vec{a} + \vec{b}) \vec{d}$
$= \vec{a} \vec{c} + \vec{b} \vec{c} + \vec{a} \vec{d} + \vec{b} \vec{d}$

vi) Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

CamScanner

Modulus of Vector

let $\vec{a} = a_1\vec{i} + a_2\vec{j}$

$$|\vec{a}| = a = \sqrt{a_1^2 + a_2^2}$$

Similarly,

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$|\vec{a}| = a = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Unit Vector along $\vec{a}$

Let,

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

&

Unit vector along $\vec{a} = \hat{a}$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{a_1\vec{i} + a_2\vec{j} + a_3\vec{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

$$= \frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\vec{i} + \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\vec{j} + \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}\vec{k}$$

Q. 1. If $\vec{a} = 2\vec{i} + \vec{j} - 3\vec{k}$

$\vec{b} = 3\vec{i} - 2\vec{j} - \vec{k}$

Find $\vec{a}\cdot\vec{b}$ and angle between them.

Sol^n:-

Given,

$$\vec{a} = 2\vec{i} + \vec{j} - 3\vec{k}$$

$$\vec{b} = 3\vec{i} - 2\vec{j} - \vec{k}$$

$$\vec{a}\cdot\vec{b} = (2\vec{i}+\vec{j}-3\vec{k})(3\vec{i}-2\vec{j}-\vec{k})$$

$$= 2\times3 + 1(-2) + (-3)(-1)$$
$$= 6-2+3$$
$$= 7$$

$$|\vec{a}| = \sqrt{2^2+1^2+(-3)^2} = \sqrt{4+1+9} = \sqrt{14}$$
$$|\vec{b}| = \sqrt{3^2+(-2)^2+(-1)^2} = \sqrt{9+4+1} = \sqrt{14}$$

Now,

Angle between them is

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{7}{\sqrt{14}\cdot\sqrt{14}} = \frac{7}{14} = \frac{1}{2}$$

$$\Rightarrow \cos\theta = \frac{1}{2}$$

$$\Rightarrow \cos\theta = \cos 60^\circ$$
$$\therefore \theta = 60^\circ$$

Q.2 Determine the value of 'λ'. So, that

$$\vec{a} = 2\vec{i} + \lambda\vec{j} + \vec{k} \quad \&$$
$$\vec{b} = 4\vec{i} + (-2\vec{j}) - 2\vec{k}$$

Sol^n.

$$\vec{a}\cdot\vec{b} = 0$$
$$\Rightarrow (2\vec{i}+\lambda\vec{j}+\vec{k})(4\vec{i}+(-2\vec{j})-2\vec{k}) = 0$$
$$\Rightarrow 2\times4 + \lambda\times-2 + 1(-2) = 0$$
$$\Rightarrow 8 + (-2\lambda) - 2 = 0$$
$$\Rightarrow 6 - 2\lambda = 0$$
$$\Rightarrow 2\lambda = 6$$
$$\therefore \lambda = 3$$

Q.3. If $\vec{a} = 5\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{j} - 5\vec{k}$ & $\vec{c} = -15\vec{i} + 3\vec{j} - 3\vec{k}$ which pairs of vectors are i) Perpendicular
ii) Parallel

Soln.
Given,

$\vec{a} = 5\vec{i} - \vec{j} + \vec{k}$

$\vec{b} = \vec{j} - 5\vec{k}$

$\vec{c} = -15\vec{i} + 3\vec{j} - 3\vec{k}$

i) perpendicularity

$\cancel{\vec{a} \cdot \vec{b} = 0}$

$\cancel{(5\vec{i} - \vec{j} + \vec{k})(\vec{j} - 5\vec{k}) = 0}$

$(\cancel{0 + (-1)(-0) + 1(-5) = 0}$

$\vec{a} \cdot \vec{b} = (5\vec{i} - \vec{j} + \vec{k})(\vec{j} - 5\vec{k})$
$= 5 \times 0 + (-1)(1) + 1(-5)$
$= 0 - 1 - 5$

$\vec{a} \cdot \vec{b} = -6$

$\vec{a} \cdot \vec{c} = (5\vec{i} - \vec{j} + \vec{k})(-15\vec{i} + 3\vec{j} - 3\vec{k})$
$= 5(-15) + (-1)3 + 1(-3)$
$= -75 - 3 - 3$

$\vec{a} \cdot \vec{c} = -81$

& $\vec{b} \cdot \vec{c} = (\vec{j} - 5\vec{k})(-15\vec{i} + 3\vec{j} - 3\vec{k})$
$= \cancel{1(-15) + (-5}$
$= 0 \times (-15) + 1(+3) + (-5)(-3)$
$= 0 + 3 + 15$
$= 18$

~~No, pair of vectors are perpendicular~~

ii) Parallel

$\vec{c} = -15\vec{i} + 3\vec{j} - 3\vec{k}$

$= -3(5\vec{i} - \vec{j} + \vec{k})$

$\vec{c} = -3\vec{b}$

$\vec{c}$ & $\vec{b}$ are parallel to each other.

Q. 4. Find vector projection of $\vec{a}$ on to $\vec{b}$ where,

$\vec{a} = 3\vec{i} - \vec{j} + \vec{k}$ &

$\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$

Soln:

$\vec{a} \cdot \vec{b} = (3\vec{i} - \vec{j} + \vec{k})(2\vec{i} + \vec{j} - 2\vec{k})$

$= 3 \times 2 + (-1)(1) + 1(-2)$

$= 6 - 1 - 2$

$= 3$

$|\vec{b}| = \sqrt{2^2 + 1^2 + (-2)^2}$

$= \sqrt{4 + 1 + 4}$

$= \sqrt{9}$

$= 3$

$\therefore$ Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$= \dfrac{3}{3}$

$= 1$

Vector Product (OR cross product) of two vector

The vector product or cross product of $\vec{a}$ & $\vec{b}$ is defined by the equation

$$\vec{a} \times \vec{b} = \hat{n}\,|\vec{a}|\,|\vec{b}|\,\sin\theta$$

Where,

$\hat{n}$ be the unit vector perpendicular to the plane containing vector $\vec{a}$ and $\vec{b}$ & direction of $\hat{n}$ being taken according to right handed screw system.

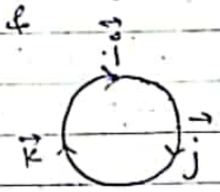
i.e.

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

Note:

$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$

&



let,

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Properties:

i) $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

ii) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

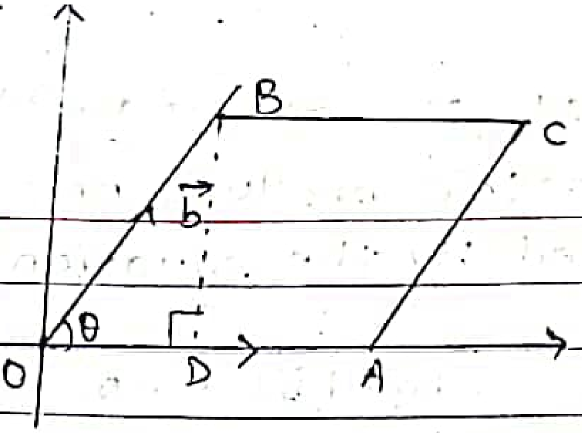
IMP

Geometrical Interpretation of Vector Product

Let, OACB be a parallelogram whose adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represents the vector $\vec{a}$ and $\vec{b}$. $\overrightarrow{BD}$ perpendicular to $\overrightarrow{OA}$

let,

$\angle AOB = \theta$



Here,

Area of parallelogram OACB is:

$$(OA)(BD) = ab\sin\theta$$

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

$$= (OA)(DB)\left(\frac{BD}{DB}\right)\hat{n}$$

$$\Rightarrow |\vec{a}\times\vec{b}| = (OA)(BD) \quad [\because \hat{n} = 1]$$

$$= \text{Area of parallelogram.}$$

$\therefore$ Area of parallelogram $= |\vec{a}\times\vec{b}|$

& Area of Triangle $= \frac{1}{2}|\vec{a}\times\vec{b}|$

Q.1 If $\vec{a} = 4\hat{i}+3\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}+\hat{j}+2\hat{k}$ Find a unit vector $(\hat{n})$ perpendicular to vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\hat{n}$ form a right handed system. Also find the angle between vectors $\vec{a}$ & $\vec{b}$.

Sol^n:-

We know,

$$\hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} \quad \& \quad \sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

Now,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= \vec{i}(6-(-1)) - \vec{j}(8-2) + \vec{k}(-4-6)$$

$$= 7\vec{i} - 6\vec{j} - 10\vec{k}$$

$$|\vec{a}| = \sqrt{4^2+3^2+1^2} = \sqrt{16+9+1} = \sqrt{26}$$

$$|\vec{b}| = \sqrt{2^2+(-1)^2+2^2} = \sqrt{9} = 3$$

&

$$|\vec{a}\times\vec{b}| = \sqrt{7^2+(-6)^2+(-10)^2} = \sqrt{49+36+100} = \sqrt{185}$$

$$\therefore \hat{n} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \frac{7\vec{i} - 6\vec{j} - 10\vec{k}}{\sqrt{185}}$$

and,

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{185}}{\sqrt{26}\times 3}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{185}}{3\sqrt{26}}\right)$$

Scalar Triple Product (STP)

Definition:

Let, $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors then the scalar triple product $\vec{a}$, $\vec{b}$ & $\vec{c}$ is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$ and is a scalar quantity defined by the dot product of $\vec{a}$ and $\vec{b}\times\vec{c}$.

i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$

Let,

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

& $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

Now

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note:

i) If $\vec{a}$, $\vec{b}$ & $\vec{c}$ ~~are~~ is a right handed system then,

$$(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a} = (\vec{c}\times\vec{a})\cdot\vec{b}$$

i.e.

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

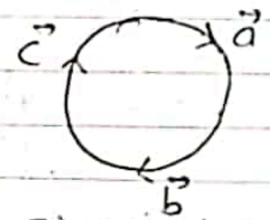


Fig: Right handed system

ii) $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]$

iii) $[\vec{a}\ \vec{b}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{b}] = 0$

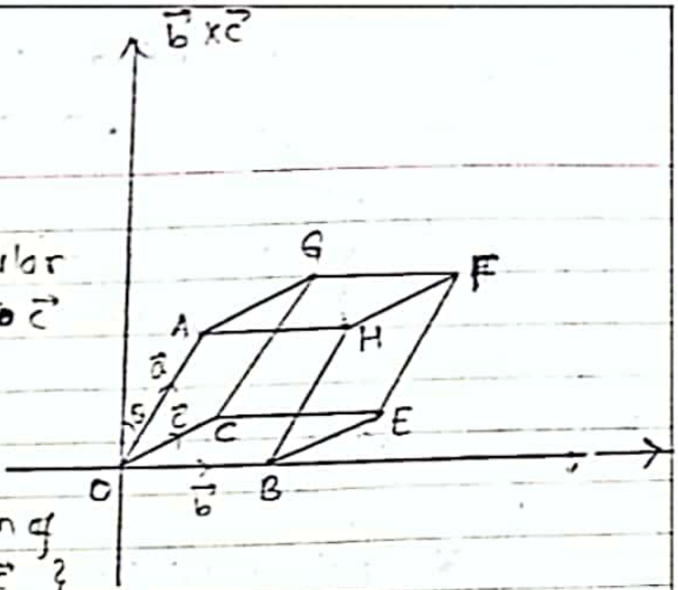
[दुईवटा same भएमा zero हुन्छ]

IMP.

Geometrical Meaning of Scalar Triple Product

Let, OABCEFGH be a parallelopiped where, $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$ are sides of parallelopiped.

Let, 'V' be the volume of the parallelopiped.



We know,

$\vec{b} \times \vec{c}$ = A vector perpendicular to the plane of $\vec{b}$ & $\vec{c}$

Then, By definition.

$\vec{a} \cdot (\vec{b} \times \vec{c}) = |\vec{b} \times \vec{c}| \{$projection of $\vec{a}$ on $\vec{b} \times \vec{c}\}$

= Area of parallelogram OBEC x $|\vec{a}| \cos\theta$

$\therefore \vec{a} \cdot (\vec{b} \times \vec{c})$ = Area of parallelogram OBEC x Height of parallelopiped

Therefore,

Volume of parallopiped with edges $\vec{a} \cdot \vec{b}$ & $\vec{c}$

Thus, Scalar triple product gives the volume of parallelopiped.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = |\vec{b} \times \vec{c}||\vec{a}| \cos\theta$$

$$\cos\theta = \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{a}||\vec{b} \times \vec{c}|}$$

Vector Triple Product

Definition:

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the three vectors then the vector triple product of $\vec{a}$ and $\vec{b} \times \vec{c}$ is called vector triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ and is denoted by $\vec{a} \times (\vec{b} \times \vec{c})$.

Geometrical meaning of $\vec{a} \times (\vec{b} \times \vec{c})$

The vector $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector which is perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$ but we know $\vec{b} \times \vec{c}$ is a vector perpendicular to $\vec{b}$ and $\vec{c}$. Thus, $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector coplanar with $\vec{b}$ and $\vec{c}$.

V.V.IMP

Show that:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

Soln:.

Proof:

Case: I

If one of the vector is zero then it is true.

Case: II

If non- of the vector is zero but vector $\vec{a} = s\vec{b}$ for some scalar 's'. Then both sides are zero so it is true.

Case: III

Suppose that non of vector is zero and $\vec{b}$ and $\vec{c}$ are not parallel. The vector $\vec{a} \times (\vec{b} \times \vec{c})$ is parallel to the plane determined by vector $\vec{b}$ and $\vec{c}$. So, it is possible to find scalars 'm' and 'n' such that:

$$\vec{a} \times (\vec{b} \times \vec{c}) = m\vec{b} + n\vec{c} \quad \text{------} (i)$$

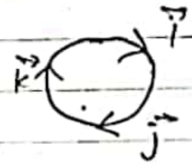
To calculate 'm' and 'n' we introduce orthogonal unit vectors $\vec{j}$ and $\vec{k}$ in the plane of vector $\vec{b}$ and $\vec{c}$. Also, introduce a third vector $\vec{i} = \vec{j} \times \vec{k}$ and write all vectors ~~in terms~~ interms of these unit vectors $\vec{i}, \vec{j}$ & $\vec{k}$.

We choose axes in such a way that y-axis in along the direction of $\vec{b}$ and 'yz' plane coincides with plane containing $\vec{b}$ & $\vec{c}$. Then we assume,

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$
$$\vec{b} = b_2\vec{j}$$
$$\vec{c} = c_2\vec{j} + c_3\vec{k}$$

Now.

$$\vec{b} \times \vec{c} = b_2\vec{j} \times (c_2\vec{j} + c_3\vec{k}) \qquad \left[\because \vec{j}\times\vec{j} = 0 \ \ \& \ \ \vec{j}\times\vec{k} = \vec{i}\right]$$
$$= b_2 c_3 \vec{i}$$



$$\vec{a} \times (\vec{b} \times \vec{c}) = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times b_2 c_3 \vec{i}$$
$$= a_2 b_2 c_3 (-\vec{k}) + a_3 b_2 c_3 \vec{j}$$
$$\therefore \vec{a} \times (\vec{b} \times \vec{c}) = a_3 b_2 c_3 \vec{j} - a_2 b_2 c_3 \vec{k} \quad \text{- - - (ii)}$$

From eq^n (i) & (ii)

$$m\vec{b} + n\vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$$
$$m\, b_2\vec{j} + n(c_2\vec{j} + c_3\vec{k}) = a_3 b_2 c_3 \vec{j} - a_2 b_2 c_3 \vec{k}$$

$$m b_2 \vec{j} + n c_2 \vec{j} + n c_3 \vec{k} = a_3 b_2 c_3 \vec{j} - a_2 b_2 c_3 \vec{k}$$
$$(m b_2 + n c_2)\vec{j} + n c_3 \vec{k} = a_3 b_2 c_3 \vec{j} - a_2 b_2 c_3 \vec{k}$$

$$\Rightarrow \quad n c_3 = - a_2 b_2 c_3$$
$$n = - a_2 b_2$$

and

$$\Rightarrow m b_2 + n c_2 = a_3 b_2 c_3$$

$$\Rightarrow m b_2 + (-a_2 b_2) c_2 = a_3 b_2 c_3$$

$$\Rightarrow m b_2 - a_2 b_2 c_2 = a_3 b_2 c_3$$

$$\Rightarrow m b_2 = a_3 b_2 c_3 + a_2 b_2 c_2$$

$$\Rightarrow m b_2 = b_2 (a_3 c_3 + a_2 c_2)$$

$$\Rightarrow m = a_3 c_3 + a_2 c_2$$

$$\therefore m = a_2 c_2 + a_3 c_3$$

Now.

From eq^n (i)

$$\vec{a} \times (\vec{b} \times \vec{c}) = (a_2 c_2 + a_3 c_3) \vec{b} - a_2 b_2 \vec{c}$$

$$= (\vec{a} \cdot \vec{c}) . \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$$

(ii)

$$(\vec{a} \times \vec{b}) \times \vec{c} = -\{ \vec{c} \times (\vec{a} \times \vec{b}) \}$$

$$= -\{ (\vec{c} \cdot \vec{b}) \vec{a} - (\vec{c} \cdot \vec{a}) \vec{b} \}$$

$$= (\vec{c} \cdot \vec{a}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}$$

Product of four vectors

(i) Scalar Product of four vectors

Expression of

$$(\vec{a} \times \vec{b})(\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$$

$$= \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a}\,\vec{d} \\ \vec{b}\,\vec{c} & \vec{b}\,\vec{d} \end{vmatrix}$$

(ii) Vector Product of four vectors

Expression for

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

Let,

$\vec{a}\times\vec{b} = \vec{v}$

Then,

$$(\vec{a}\times\vec{b})(\vec{c}\times\vec{d}) = \vec{v}\times(\vec{c}\times\vec{d})$$
$$= (\vec{v}\cdot\vec{d})\,\vec{c} - (\vec{v}\cdot\vec{c})\cdot\vec{d}$$
$$= [(\vec{a}\times\vec{b})\cdot\vec{d}\,]\,\vec{c} - [(\vec{a}\times\vec{b})\vec{c}\,]\,\vec{d}$$
$$= [\vec{a}\,\vec{b}\,\vec{d}\,]\,\vec{c} - [\,\vec{a}\,\vec{b}\,\vec{c}\,]\,\vec{d}$$

Note:

i) If three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar then scalar triple product

$$[\vec{a}\;\vec{b}\,\vec{c}\,] = 0$$

Reciprocal System of Vectors

Definition:

Let $\vec{a}$, $\vec{b}$ & $\vec{c}$ are three non-coplanar set of vectors then the vector

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}\,]}$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}\,]}$$

$$\vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}\,]}$$

Which are perpendicular to the planes containing $\vec{b}$ and $\vec{c}$, $\vec{c}$ and $\vec{a}$, $\vec{a}$ and $\vec{b}$ respectively, are called reciprocal system of vectors $\vec{a}$, $\vec{b}$ & $\vec{c}$.

Properties of reciprocal system of vectors

(i) If $\vec{a}, \vec{b}$ and $\vec{c}$ are reciprocal system of vectors $\vec{a}, \vec{b}, \vec{c}$ then

$$\vec{a} \cdot \vec{a}' = \vec{b} \cdot \vec{b}' = \vec{c} \cdot \vec{c}' = 1$$

We have,

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Then,

$$\vec{a}\,\vec{a}' = \frac{\vec{a}\cdot(\vec{b} \times \vec{c})}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = 1$$

$$\vec{b}\,\vec{b}' = \frac{\vec{b}\cdot(\vec{c} \times \vec{a})}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{[\vec{b}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = 1$$

$\therefore$ $\vec{c}\,\vec{c}' = 1$.

(ii) If $\vec{a}', \vec{b}', \& \vec{c}'$ are reciprocal system of vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Then,

$$\vec{a}\,\vec{b}' = \vec{b}\,\vec{c}' = \vec{c}\,\vec{a}' = \vec{b}\,\vec{a}' = \vec{a}\,\vec{c}' = \vec{c}\,\vec{b}' = 0$$

Proof:

$$\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Now,

$$\vec{a}\,\vec{b}' = \frac{\vec{a}\,(\vec{c} \times \vec{a})}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{[\vec{a}\,\vec{c}\,\vec{a}]}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{0}{[\vec{a}\,\vec{b}\,\vec{c}]} = 0$$

Similarly,

$$\vec{b}\,\vec{c}' = \vec{c}\,\vec{a}' = \vec{b}\,\vec{a}' = \vec{a}\,\vec{c}' = \vec{c}\cdot\vec{b}' = 0$$

(iii) $\vec{a}', \vec{b}', \& \vec{c}'$ are reciprocal system of $\vec{a}, \vec{b}, \vec{c}$. Then $\vec{a}, \vec{b}, \vec{c}$ in which the reciprocal of system $\vec{a}, \vec{b}, \vec{c}$.

Q.1 Find the set of reciprocal vectors of

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\vec{b} = \hat{i} - \hat{j} - 2\hat{k}$

& $\vec{c} = -\hat{i} + 2\hat{j} + 2\hat{k}$

Soln.

We have,

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\vec{b} = \hat{i} - \hat{j} - 2\hat{k}$

$\vec{c} = -\hat{i} + 2\hat{j} + 2\hat{k}$

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \quad \vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} \quad \& \quad \vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$$

$\cancel{= 2(-2+4) - 3(2+2) - 1(2+1)}$

$\cancel{= 2 \times 2 - 3 \times 4 - 1 \times 3}$

$\cancel{= 4 - 12 - 3}$

$\cancel{=}$

$= 2(-2+4) - 3(2-2) - 1(2-1)$

$= 2 \times 2 - 0 - 1$

$= 4 - 1$

$= 3$

Now.

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = \hat{i}(-2+4) - \hat{j}(2-2) + \hat{k}(2-1)$$

$= 2\hat{i} + 1\hat{k}$

$$\vec{c} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix} = \hat{i}(-2-6) - \hat{j}(1-4) + \hat{k}(-3-4)$$

$= -8\hat{i} + 3\hat{j} - 7\hat{k}$

CamScanner

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -1 & 1 & -2 \end{vmatrix}$$

$\cancel{\vec{i}(6+2)-\vec{j}(4-1)+\vec{k}(4+3)}$

$\cancel{-9\vec{i}}$

$$= \vec{i}(-6-1)-\vec{j}(-4+1)+\vec{k}(-2-3)$$

$$= -7\vec{i}+3\vec{j}-5\vec{k}$$

$$\therefore \vec{a}' = \frac{2\vec{i}+\vec{k}}{3}$$

$$\vec{b}' = \frac{-8\vec{i}+3\vec{j}-7\vec{k}}{3}$$

$$\vec{c}' = \frac{-7\vec{i}+3\vec{j}-5\vec{k}}{3}$$

Exercise 13.2

(1) Find the volume of parallelopiped whose concurrent edges are represented by the vectors $\vec{i}+\vec{j}+\vec{k}$, $\vec{i}-\vec{j}+\vec{k}$ & $\vec{i}+2\vec{j}-\vec{k}$.

Sol:

Let,

$$\vec{a} = \vec{i}+\vec{j}+\vec{k}$$
$$\vec{b} = \vec{i}-\vec{j}+\vec{k}$$
$$\vec{c} = \vec{i}+2\vec{k}-\vec{j}$$

Now,

Volume of parallelopiped (V) = $[\vec{a}\ \vec{b}\ \vec{c}]$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 1(1-2)-1(-1-1)+1(2+1)$$

$$= -1+2+3 = 4 \text{ unit.}$$

(2) Prove that following four points are coplanar

(i) Let 'O' be the origin & A, B, C & D are four points.

Then,

$$\overrightarrow{OA} = 2\vec{i} + 3\vec{j} - \vec{k}$$

$$\overrightarrow{OB} = \vec{i} - 2\vec{j} + 3\vec{k}$$

$$\overrightarrow{OC} = 3\vec{i} + 4\vec{j} - 2\vec{k}$$

& $\overrightarrow{OD} = \vec{i} - 6\vec{j} + 6\vec{k}$

Now,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (\vec{i} - 2\vec{j} + 3\vec{k}) - (2\vec{i} + 3\vec{j} - \vec{k})$$

$$= -\vec{i} - 5\vec{j} + 4\vec{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= (3\vec{i} + 4\vec{j} - 2\vec{k}) - (\vec{i} - 2\vec{j} + 3\vec{k})$$

$$= 2\vec{i} + 6\vec{j} - 5\vec{k}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$$

$$= (\vec{i} - 6\vec{j} + 6\vec{k}) - (3\vec{i} + 4\vec{j} - 2\vec{k})$$

$$= 2\vec{i} - 10\vec{j} + 8\vec{k}$$

Now,

$$[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}] = \begin{vmatrix} -1 & -5 & 4 \\ 2 & 6 & -5 \\ 2 & -10 & 8 \end{vmatrix}$$

$$= -1(48 - 50) + 5(16 + 10) + 4(-20 - 12)$$

$$= -1(-2) + 130 - 128$$

$$= 2 + 130 - 128$$

(i) $2\vec{i}+3\vec{j}-\vec{k}$, $\vec{i}-2\vec{j}+3\vec{k}$, $3\vec{i}+4\vec{j}-2\vec{k}$ & $\vec{i}-6\vec{j}+6\vec{k}$

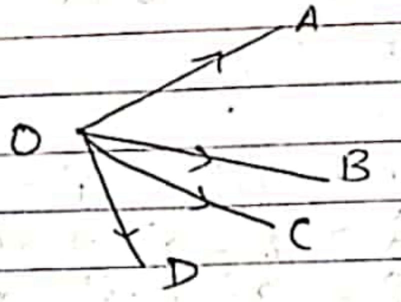
let,

$\vec{OA} = 2\vec{i}+3\vec{j}-\vec{k}$

$\vec{OB} = \vec{i}-2\vec{j}+3\vec{k}$

$\vec{OC} = 3\vec{i}+4\vec{j}-2\vec{k}$

$\vec{OD} = \vec{i}-6\vec{j}+6\vec{k}$



Now,

$\vec{AB} = \vec{OB} - \vec{OA}$

$= (\vec{i}-2\vec{j}+3\vec{k}) - (2\vec{i}+3\vec{j}-\vec{k})$

$= -\vec{i}-5\vec{j}+4\vec{k}$

$\vec{BC} = \vec{OC} - \vec{OB}$

$= (3\vec{i}+4\vec{j}-2\vec{k})-(\vec{i}-2\vec{j}+3\vec{k})$

$= 2\vec{i}+6\vec{j}-5\vec{k}$

$\vec{CD} = \vec{OD} - \vec{OC}$

$= (\vec{i}-6\vec{j}+6\vec{k}) - (3\vec{i}+4\vec{j}-2\vec{k})$

$= -2\vec{i}-10\vec{j}+8\vec{k}$

Now,

$$[\vec{AB}\ \vec{BC}\ \vec{CD}] = \begin{vmatrix} -1 & -5 & 4 \\ 2 & 6 & -5 \\ -2 & -10 & 8 \end{vmatrix}$$

$= -1(48-50)+5(16-10)+4(-20+12)$

$= 2+30-32$

$\therefore [\vec{AB}\ \vec{BC}\ \vec{CD}] = 0$

The given four points are coplanar.

Q.3 Find the constant 'λ' such that the vectors $2\vec{i}-\vec{j}+\vec{k}$, $\vec{i}+2\vec{j}-3\vec{k}$ & $3\vec{i}+\lambda\vec{j}+5\vec{k}$ are coplanar.

Sol^n:-

let,

$$\vec{a} = 2\vec{i}-\vec{j}+\vec{k}$$
$$\vec{b} = \vec{i}+2\vec{j}-3\vec{k}$$
$$\vec{c} = 3\vec{i}+\lambda\vec{j}+5\vec{k}$$

Since, given vectors are coplanar so.

$$[\vec{a}\ \vec{b}\ \vec{c}] = 0$$

$$\Rightarrow \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0$$

$$\Rightarrow 2(10+3\lambda)+1(5+9)+1(\lambda-6)=0$$
$$\Rightarrow 20+6\lambda+14+\lambda-6=0$$
$$\Rightarrow 7\lambda = -20-14+6$$
$$\Rightarrow 7\lambda = -28$$
$$\Rightarrow \lambda = -28/7$$
$$\therefore \lambda = -4$$

Q.6 Show that vector $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})$ ~~+...~~ $+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is parallel to vector $\vec{a}$.

Sol^n:-

Given, $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$

let,

$$I=(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$$

$$=\vec{V_1}\times(\vec{c}\times\vec{d}) \qquad [\because \text{let, } \vec{V_1}=\vec{a}\times\vec{b}]$$

$$=(\vec{V_1}\cdot\vec{d})\vec{c}-(\vec{V_1}\cdot\vec{c})\vec{d}$$

$$=[(\vec{a}\times\vec{b})\cdot\vec{d}]\vec{c}-[(\vec{a}\times\vec{b})\cdot\vec{c}]\vec{d}$$

$$\therefore I=[\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

$$II=(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})$$

$$=-(\vec{d}\times\vec{b})\times(\vec{a}\times\vec{c}) \qquad [\because \text{direction changed}]$$

$$=-\{\vec{V_2}\times(\vec{a}\times\vec{c})\} \qquad [\because \text{let } \vec{V_2}=\vec{d}\times\vec{b}]$$

$$=-\{(\vec{V_2}\cdot\vec{c})\vec{a}-(\vec{V_2}\cdot\vec{a})\vec{c}\}$$

$$=-\{[(\vec{d}\times\vec{b})\cdot\vec{c}]\vec{a}-[(\vec{d}\times\vec{b})\cdot\vec{a}]\vec{c}\}$$

$$=-\{[\vec{d}\ \vec{b}\ \vec{c}]\vec{a}-[\vec{d}\ \vec{b}\ \vec{a}]\vec{c}\}$$

$$\therefore II=[\vec{d}\ \vec{b}\ \vec{a}]\vec{c}-[\vec{d}\ \vec{b}\ \vec{c}]\vec{a}$$

$$III=(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$$

$$=-(\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d})$$

$$=-\{\vec{V_3}\ \cancel{(\vec{b}\times\vec{c})}\times(\vec{a}\times\vec{d})\} \qquad [\because \vec{V_3}=\vec{b}\times\vec{c}]$$

$$=-\{[\vec{V_3}\cdot\vec{d}]\vec{a}-[\vec{V_3}\cdot\vec{a}]\vec{d}\}$$

$$=-\{[(\vec{b}\times\vec{c})\cdot\vec{d}]\vec{a}-[(\vec{b}\times\vec{c})\cdot\vec{a}]\vec{d}\}$$

$$= -\{[(\vec{b}\times\vec{c})\vec{a}]\vec{d} - [(\vec{b}\times\vec{c})\vec{a}]\vec{d}\}$$

$$= -\{[\vec{b}\,\vec{c}\,\vec{a}]\vec{d} - [\vec{b}\,\vec{c}\,\vec{a}]\}$$

$$= -(\vec{b}\times\vec{c})\times(\vec{a}\times\vec{d})$$

$$= -\{\vec{V_3}\times(\vec{a}\times\vec{d})\}$$

$$= -\{(\vec{V_3}\,\vec{d})\vec{a} - (\vec{V_3}\,\vec{a})\vec{d}\} \quad [\because \vec{V_3} = \vec{b}\times\vec{c}]$$

$$= -\{[(\vec{b}\times\vec{c})\vec{d}]\vec{a} - [(\vec{b}\times\vec{c})\vec{a}]\vec{d}\}$$

$$= -\{[\vec{b}\,\vec{c}\,\vec{d}]\vec{a} - [\vec{b}\,\vec{c}\,\vec{a}]\vec{d}\}$$

$$= [\vec{b}\,\vec{c}\,\vec{a}]\vec{d} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$$

Now,

$I + II + III$

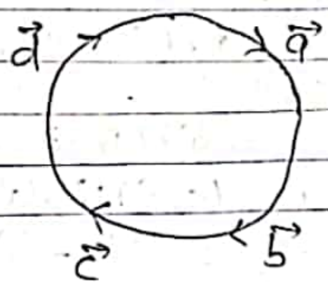
$$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} + [\vec{d}\,\vec{b}\,\vec{a}]\vec{c} - [\vec{d}\,\vec{b}\,\vec{c}]\vec{a} + [\vec{b}\,\vec{c}\,\vec{a}]\vec{d} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$$

$$= [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} + \{-[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}\} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} + [\vec{b}\,\vec{c}\,\vec{a}]\vec{d} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$$

$$= \cancel{[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}} - \cancel{[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}} - \cancel{[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}} + \cancel{[\vec{b}\,\vec{c}\,\vec{a}]\vec{d}} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$$

$$= -2[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$$

$$= \lambda\vec{a} \qquad \therefore \lambda = -2[\vec{b}\,\vec{c}\,\vec{d}]$$



$\therefore$ given vector is parallel with the vector $\vec{a}$.

Q.7 If $\vec{c} = \vec{a} \times \vec{b}$ & $\vec{b} = \vec{a} \times \vec{c}$ with $\vec{a} \neq 0$. Show that $\vec{b} = 0$ & $\vec{c} = 0$

Sol^n:

Here,

$$\vec{c} = \vec{a} \times \vec{b}$$

or $\vec{c} = \vec{a} \times (\vec{a} \times \vec{c})$

or $\vec{c} = (\vec{a} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{a})\vec{c}$

or $\vec{c} = (\vec{a} \cdot \vec{c})\vec{a} - a^2\vec{c}$

$$\vec{c} + a^2\vec{c} = (\vec{a} \cdot \vec{c})\vec{a}$$

$$\vec{c}(1 + a^2) = (\vec{a} \cdot \vec{c})\vec{a}$$

$$\vec{c} = \frac{(\vec{a} \cdot \vec{c})\vec{a}}{(1 + a^2)}$$

$\therefore \vec{c} = \lambda\vec{a}$ $\quad\left[\because \lambda = \frac{(\vec{a} \cdot \vec{c})}{1 + a^2}\right.$

$\therefore$ $\vec{a}$ is parallel to $\vec{c}$.

Now

$$\vec{b} = \vec{a} \times \vec{c}$$

$$= \vec{a} \times \lambda\vec{a}$$

$\therefore \vec{b} = 0$

&

$$\vec{c} = \vec{a} \times \vec{b}.$$

$$= \vec{a} \times 0$$

$\therefore \vec{c} = 0$

Q. 11. Prove that

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0.$$

Sol^n:

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

or, $(\vec{a}\,\vec{c})\vec{b} - (\vec{a}\,\vec{b})\vec{c} + (\vec{b}\,\vec{a})\vec{c} - (\vec{b}\,\vec{c})\vec{a} + (\vec{c}\,\vec{b})\vec{a} - (\vec{c}\,\vec{a})\vec{b} =$

or $\cancel{(\vec{a}\,\vec{c})\vec{b}} - \cancel{(\vec{a}\,\vec{c})\vec{b}} - \cancel{(\vec{a}\,\vec{b})}\vec{c} + \cancel{(\vec{a}\,\vec{b})}\vec{c} - \cancel{(\vec{b}\,\vec{c})}\vec{a} + \cancel{(\vec{c}\,\vec{b})}\vec{a} = 0$

$\therefore 0 = 0$

Q.12 Show that

$(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = [\vec{a}\ \vec{b}\ \vec{c}]\vec{c}$

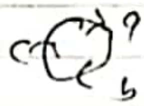
let, ~~th~~

$\vec{v} \times (\vec{c} \times \vec{a})$ $\quad [\because \vec{v} = (\vec{b} \times \vec{c})]$

$= (\vec{v}\,\vec{a})\vec{c} - (\vec{v}\,\vec{c})\vec{a}$

$= [(\vec{b} \times \vec{c})\vec{a}]\vec{c} - [(\vec{b} \times \vec{c})\vec{c}]\vec{a}$

$= [B\,\vec{c}\,\vec{a}]\vec{c} - [B\,\vec{c}\,\vec{c}]\vec{a}$

$= [\vec{b}\,\vec{c}\,\vec{a}]\vec{c}$ $\quad [\because [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]]$ 

$= [\vec{a}\ b\ \vec{c}]\vec{c}$

= RHS

Q.13. Show that

$[\vec{b} \times \vec{c},\ \vec{c} \times \vec{a}.\ \vec{a} \times \vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

Sol:

LHS

$[\vec{b} \times \vec{c},\ \vec{c} \times \vec{a},\ \vec{a} \times \vec{b}]$

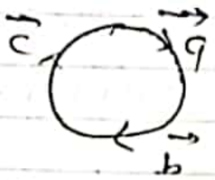
$= (\vec{b} \times \vec{c})\{(\vec{c} \times \vec{a}).(\vec{a} \times \vec{b})\}$

$= (B \times \vec{c})\{[\vec{c}\,\vec{a}\,\vec{b}]\vec{a} - [\vec{c}\,\vec{a}\,\vec{a}]\vec{b}\}$

$= (b \times \vec{c}).\{[\vec{c}\,\vec{a}\,\vec{b}]\vec{a} - 0\}$

$= (\vec{b} \times \vec{c})[\vec{c}\,\vec{a}\,\vec{b}]\vec{a}$

$= (b \times \vec{c})\vec{a}\,[\vec{c}\,\vec{a}\,\vec{b}]$



$= [B\,\vec{c}\,\vec{a}][\vec{c}\,\vec{a}\,\vec{b}]$

$= [\vec{a}\,\vec{b}\,\vec{c}](\vec{a}\,\vec{b}\,\vec{c})$ $\quad [\because$ Same direction $]$

$= [\vec{a}\,\vec{b}\,\vec{c}]^2$

= RHS

Q. 21 Show that:

$$2\vec{a} = \vec{i} \times (\vec{a} \times \vec{i}) + \vec{j} \times (\vec{a} \times \vec{j}) + \vec{k} \times (\vec{a} \times \vec{k})$$

Sol^n:-

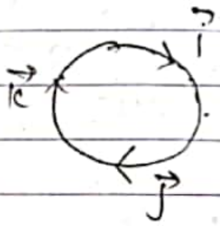
Let,

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

Now,

$$\vec{i} \times (\vec{a} \times \vec{j}) = \vec{i} \times [(a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \times \vec{i}]$$
$$= \vec{i} \times [a_2 (-\vec{k}) + a_3 \vec{j}]$$
$$= \vec{i} \times (a_3 \vec{j} - a_2 \vec{k})$$
$$= a_3 \vec{k} - (-a_2 \vec{j})$$
$$= a_3 \vec{k} + a_2 \vec{j}$$

$$\therefore \vec{i} \times (\vec{a} \times \vec{j}) = a_2 \vec{j} + a_3 \vec{k}$$



Similarly,

$$\vec{j} \times (\vec{a} \times \vec{j}) = \vec{j} \times \{(a_1 \vec{j} + a_2 \vec{j} + a_3 \vec{k}) \times \vec{j}\}$$
$$= \vec{j} \times \{a_1 \vec{k} + a_3 (-\vec{i})\}$$
$$= \vec{j} \times (a_1 \vec{k} - a_3 \vec{i})$$
$$= a_1 \vec{i} - a_3 (-\vec{k})$$
$$= a_1 \vec{i} + a_3 \vec{k}$$

&

$$\vec{k} \times (\vec{a} \times \vec{k}) = \vec{k} \times \{(a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}) \times \vec{k}\}$$
$$= \vec{k} \times \{-a_1 \vec{j} + a_2 \vec{i}\}$$
$$= a_1 \vec{i} + a_2 \vec{j}$$

Now,

From RHS

$$\vec{i} \times (\vec{a} \times \vec{i}) + (\vec{j} \times (\vec{a} \times \vec{j}) + \vec{k} \times (\vec{a} \times \vec{k})$$
$$= a_2 \vec{j} + a_3 \vec{k} + a_1 \vec{i} + a_3 \vec{k} + a_1 \vec{i} + a_2 \vec{j}$$
$$= 2 a_2 \vec{j} + 2a_3 \vec{k} + a_1 \vec{i}$$
$$= 2 (a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k})$$
$$= 2\vec{a}$$
$$= \text{LHS}$$ proved

Q. 23 Show that

$$[\vec{l}\ \vec{m}\ \vec{n}][\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\cdot\vec{c} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\cdot\vec{c} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\cdot\vec{c} \end{vmatrix}$$

Soln:-

let,

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$
$$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$
$$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$

$$\vec{l} = l_1\vec{i} + l_2\vec{j} + l_3\vec{k}$$
$$\vec{m} = m_1\vec{i} + m_2\vec{j} + m_3\vec{k}$$
$$\vec{n} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k}$$

From LHS

$$[\vec{l}\ \vec{m}\ \vec{n}][\vec{a}\ \vec{b}\ \vec{c}]$$

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1a_1 + l_2a_2 + l_3a_3 & l_1b_1 + l_2b_2 + l_3b_3 & l_1c_1 + l_2c_2 + l_3c_3 \\ m_1a_1 + m_2a_2 + m_3a_3 & m_1b_1 + m_2b_2 + m_3b_3 & m_1c_1 + m_2c_2 + m_3c_3 \\ n_1a_1 + n_2a_2 + n_3a_3 & n_1b_1 + n_2b_2 + n_3b_3 & n_1c_1 + n_2c_2 + n_3c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{l}\cdot\vec{a} & \vec{l}\cdot\vec{b} & \vec{l}\cdot\vec{c} \\ \vec{m}\cdot\vec{a} & \vec{m}\cdot\vec{b} & \vec{m}\cdot\vec{c} \\ \vec{n}\cdot\vec{a} & \vec{n}\cdot\vec{b} & \vec{n}\cdot\vec{c} \end{vmatrix}$$ [∵ scalar product of $\vec{l}$ & $\vec{a}$]

CamScanner

Two Dimensional Geometry [6Hrs]

Ellipse

An ellipse is a conic section whose ~~eten~~ eccentricity is less than one.

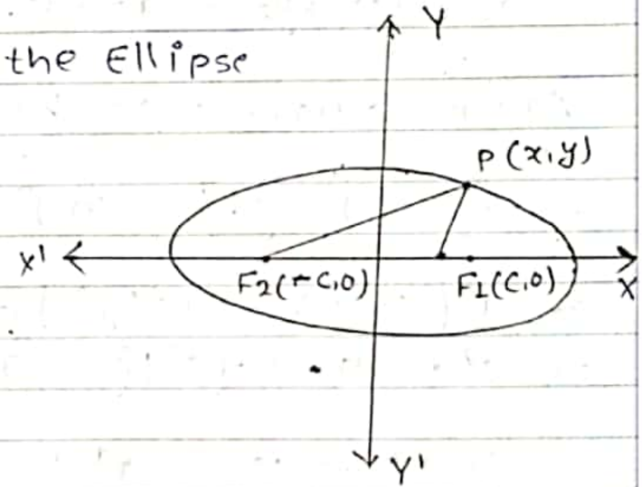
OR

An ellipse is the set of points in a plane whose distance from two fixed points have constant sum. The fixed points are called foci and their mid points ~~ar~~ is called centre of the ellipse.

IMP

Standard Equation of the Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Let, $F_1(c,0)$ and $F_2(-c,0)$ be the foci of an ellipse. So that its centre lies at origin.

Let, $P(x,y)$ be any point on the ellipse. Then by definition of ellipse

$$PF_1 + PF_2 = \text{Constant sum}$$

or. $\sqrt{(x-c)^2+(y-0)^2} + \sqrt{(x-(-c))^2+(y-0)^2} = 2a$ (say)

or $\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$

$$\sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$$

Squaring on both side

$$\left(\sqrt{(x-c)^2+y^2}\right)^2 = \left(2a-\sqrt{(x+c)^2+y^2}\right)^2$$

$$\Rightarrow (x-c)^2+y^2 = (2a)^2 - 2.2a\cdot\sqrt{(x+c)^2+y^2} + \left(\sqrt{(x+c)^2+y^2}\right)^2$$

$$\Rightarrow (x-c)^2+\cancel{y^2} = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + (x+c)^2+y^2$$

$$\Rightarrow \cancel{x^2} - 2xc + c^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + x^2 + 2xc + c^2$$

$$\Rightarrow 4a\sqrt{(x+c)^2+y^2} = 4a^2 + 4xc$$

$$\Rightarrow a\sqrt{(x+c)^2+y^2} = a^2+xc$$

Again squaring on both side

$$a^2[(x+c)^2+y^2] = (a^2)^2 + 2a^2xc + x^2c^2$$

$$\Rightarrow a^2[x^2+2xc+c^2+y^2] = a^4 + 2a^2xc + x^2c^2$$

$$\Rightarrow a^2x^2 + \cancel{2a^2xc} + a^2c^2 + a^2y^2 = a^4 + \cancel{2a^2xc} + x^2c^2$$

$$\Rightarrow a^2x^2 + a^2c^2 + a^2y^2 = a^4 + x^2c^2$$

$$\Rightarrow a^2x^2 - a^2y^2 - x^2c^2 = a^4 - a^2c^2$$

$$\Rightarrow a^2x^2 - x^2c^2 + a^2y^2 = a^2(a^2-c^2)$$

$$x^2(a^2-c^2) + a^2y^2 = a^2(a^2-c^2)$$

Dividing on both side by $a^2(a^2-c^2)$

$$\frac{x^2}{a^2} + \frac{y^2}{(a^2-c^2)} = 1 \quad \text{---(i)}$$

From ΔPF_1F_2

$PF_1 + PF_2 > F_1F_2$ [$\because$ distance]

$2a > 2c$ [$\because$ in coordinate]

$a > c$

$\Rightarrow a^2 > c^2$

$\Rightarrow a^2 - c^2 > 0$ [$\because$ positive]

Let,

$a^2 - c^2 = b^2$

From eq^n (i)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad [\because b^2 = a^2 - c^2]$$

Equation of Tangent and Normal at (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol^n:-

We have,

Equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{---- (i)}$$

The point (x_1, y_1) lies on the ellipse then;

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \quad \text{--- (ii)}$$

Equation of any line passing through (x_1, y_1) is

$$(y - y_1) = m(x - x_1) \quad \text{---- (iii)}$$

slope of tangent can be determined by

Differentiating eq^n(i) w.r.t. 'x'

$$\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$$

$$\text{or, } \frac{1}{a^2}\cdot 2x+\frac{1}{b^2}\,2\cdot y\,\frac{dy}{dx}=0$$

$$\text{or } \frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\text{or } \frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx}=0$$

$$\text{or } \frac{y}{b^2}\frac{dy}{dx}=-\frac{x}{a^2}$$

$$\text{or } \frac{dy}{dx}=-\frac{x}{a^2}\times\frac{b^2}{y}$$

$$\therefore \frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

Slope of tangent at the point (x_1, y_1) is

$$m=\frac{dy}{dx}=\frac{-b^2x_1}{a^2y_1}$$

Put the value of 'm' in eq^n(iii)

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$\text{or } yy_1-y_1^2=-\frac{b^2}{a^2}(xx_1-x_1^2) \qquad [\because \text{Multiplication}]$$

$$\text{or } \left(\frac{yy_1-y_1^2}{b^2}\right)=-\left(\frac{xx_1-x_1^2}{a^2}\right)$$

$$\text{or } \frac{yy_1}{b^2}-\frac{y_1^2}{b^2}=\frac{-xx_1}{a^2}+\frac{x_1^2}{a^2}$$

or, $\frac{yy_1}{b^2} + \frac{xx_1}{a^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$

$\therefore \frac{yy_1}{b^2} + \frac{xx_1}{a^2} = 1 \quad \left[\because \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1\right]$

This is the equation of tangent of ellipse.

For the equation of normal

Slope of normal $(m) = \frac{a^2 y_1}{b^2 x_1}$

From eq^n (iii)

The equation of normal at the point (x_1, y_1) is:

$y - y_1 = \frac{a^2 y}{b^2 x}(x - x_1)$

Find the condition that the line $y = mx + c$ is tangent on the curve (ellipse) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Sol^n:-

Given line is:

$y = mx + c \quad \text{---} \quad (i)$

Eq^n of curve

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or $b^2x^2 + a^2y^2 = a^2b^2$

$\therefore b^2x^2 + a^2y^2 - a^2b^2 = 0 \quad \text{----} \quad (ii)$

Solving eq^n (i) & (ii)

$b^2x^2 + a^2(mx+c)^2 - a^2b^2 = 0$

$\Rightarrow b^2x^2 + a^2(m^2x^2 + 2mxc + c^2) - a^2b^2 = 0$

$\Rightarrow b^2x^2 + a^2m^2x^2 + 2a^2mxc + c^2a^2 - a^2b^2 = 0$

$\Rightarrow x^2(b^2 + a^2m^2) + 2a^2mxc + (a^2c^2 - a^2b^2) = 0$ ...(iii)

This is a quadratic equation (ax^2+bx+c) in 'x'. The line first tangent to the curve second.

If $b^2 - 4ac = 0$

Comparing eq^n (iii) with $ax^2 + bx + c = 0$

$\therefore a = b^2 + a^2m^2$

$b = 2a^2mc$

$c = a^2c^2 - a^2b^2$

Now,

Putting value of a, b, c in ~~eq~~

$b^2 - 4ac = 0$

$(2a^2mc)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 0$

$\Rightarrow 4a^4m^2c^2 - 4[a^2b^2c^2 - a^2b^4 + a^4c^2m^2 - a^4b^2m^2] = 0$

$\Rightarrow 4a^4m^2c^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4m^2c^2 + 4a^4b^2m^2 = 0$

$\Rightarrow \cancel{a^4m^2c^2} - a^2b^2c^2 + a^2b^4 - \cancel{a^4m^2c^2} + a^4b^2m^2 = 0$

$\Rightarrow a^2b^2[b^2 + a^2m^2 - c^2] = 0$

$b^2 + a^2m^2 - c^2 = 0$

$\therefore b^2 + m^2a^2 = c^2$

$\boxed{\text{i.e. } m^2a^2 = c^2 - b^2}$

Now, for the point of contact

$$x = -\frac{B}{2A} = \frac{-2a^2mc}{2(b^2+a^2m^2)} = \frac{-2a^2mc}{2c^2} = \frac{-a^2m}{c}$$

&

$$y = mx+c = m\left(-\frac{a^2m}{c}\right)+c = -\frac{a^2m^2}{c}+c$$

$$= \frac{a^2m^2+c^2}{c} = \frac{c^2-a^2m^2}{c} = \frac{b^2}{c}$$

$\therefore$ The point of contact $(x, y) = \left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$

Find the condition that the line $lx+my+n=0$ may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Also find the point of contact.

Soln:-

Here,

Given line is

$$lx+my+n=0$$

$$my = -lx-n$$

$$y = -\frac{(lx+n)}{m} \quad \text{---- (i)}$$

Eqn of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

or $$\frac{b^2x^2+a^2y^2}{a^2b^2}=1$$

or $$b^2x^2+a^2y^2 = a^2b^2$$

$\therefore$ $$b^2x^2+a^2y^2-a^2b^2=0 \quad \text{--- (ii)}$$

Solving eq^n (i) & (ii)

$$b^2x^2 + a^2\left(-\frac{lx+n}{m}\right)^2 - a^2b^2 = 0$$

or $b^2x^2 + \frac{a^2}{m^2}(l^2x^2 + 2lnx + n^2) - a^2b^2 = 0$

or, $m^2b^2x^2 + a^2l^2x^2 + 2a^2lnx + a^2n^2 - m^2a^2b^2 = 0$

or $x^2(m^2b^2 + a^2l^2) + 2a^2lnx + (a^2n^2 - m^2a^2b^2) = 0$..

Where,

$A = m^2b^2 + a^2l^2$

$B = 2a^2ln$

$C = a^2n^2 - m^2a^2b^2$

Since,

The line (i) touches the ellipse (ii)

If $B^2 - 4AC = 0$

or $(2a^2ln)^2 - 4(m^2b^2 + a^2l^2)(a^2n^2 - m^2a^2b^2) = 0$

or $4a^4l^2n^2 - 4[a^2b^2m^2n^2 - a^2b^4m^4 + a^4l^2n^2 - a^4b^2l^2m^2] = 0$

or $\cancel{a^4l^2n^2} - a^2b^2m^2n^2 + a^2b^4m^4 - \cancel{a^4l^2n^2} + a^4b^2l^2m^2 = 0$

or $a^2b^4m^4 + a^4b^2l^2m^2 - a^2b^2m^2n^2 = 0$

or $a^2b^4m^4 + a^4b^2l^2m^2 = a^2b^2m^2n^2$

or $a^2b^2m^2(b^2m^2 + a^2l^2) = a^2b^2m^2n^2$

$\therefore b^2m^2 + a^2l^2 = n^2$

& $a^2l^2 + b^2m^2 = n^2$

$\Rightarrow b^2m^2 = n^2 - a^2l^2$

This is the required condition.

Also, for the point of contact

$$x = -\frac{B}{2A} = -\left\{\frac{2a^2 ln}{2(m^2b^2+a^2l^2)}\right\} = -\left\{\frac{a^2 ln}{n^2}\right\}$$

$$\therefore x = -\frac{a^2 l}{n}$$

& $$y = -\left\{\frac{lx+n}{m}\right\}$$

$$= -\left\{\frac{l\left(\frac{-a^2l}{n}\right)+n}{m}\right\}$$

$$= -\left\{\frac{-a^2l^2+n^2}{mn}\right\}$$

$$= -\left\{\frac{-a^2l^2+n^2}{mn}\right\}$$

$$= -\left\{\frac{n^2-a^2l^2}{mn}\right\}$$

$$= -\frac{b^2m^2}{mn}$$

$$\therefore y = -\frac{b^2 m}{n}$$

The point of contact $(x, y) = \left(\frac{-a^2 l}{n}, \frac{-b^2 m}{n}\right)$

Hyperbola

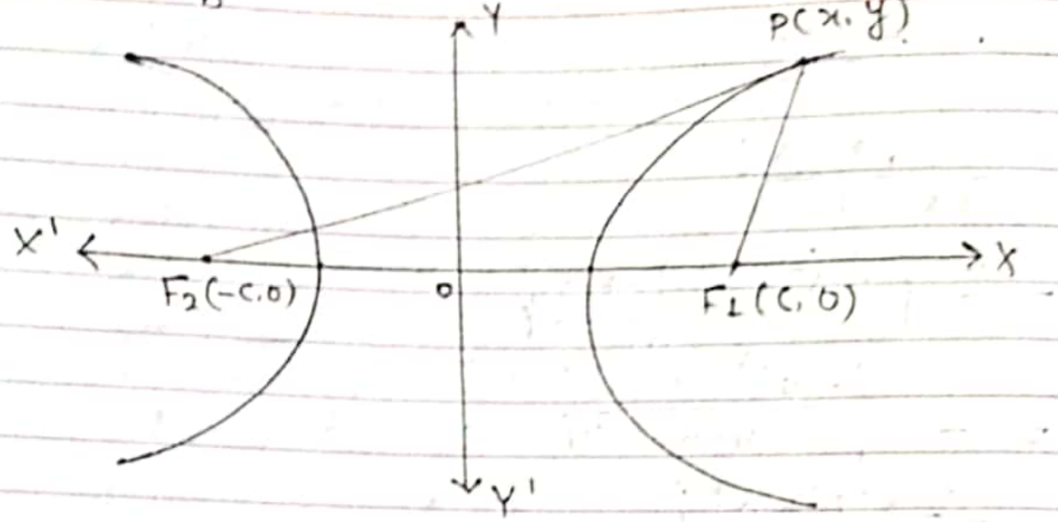
Hyperbola is the conic section whose eccentricity is greater than one.

OR

A hyperbola is the set of points in a plane whose distance from two fixed points have a constant difference.

Derive the equation of hyperbola in the form of

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



let, $F_1(c, 0)$ and $F_2(-c, 0)$ be the foci of hyperbola So, that its centre lies at origin.

Let $P(x, y)$ be any point on the hyperbola. Then by definition of hyperbola.

$PF_2 - PF_1 = \text{constant} = 2a\text{(say)}$

or $\sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} = 2a$

or $\sqrt{(x+c)^2+y^2} = 2a + \sqrt{(x-c)^2+y^2}$

Squaring on both side

or $(x+c)^2+y^2 = 4a^2 + 2 \cdot 2a \cdot \sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$

or $\cancel{x^2}+2cx+\cancel{c^2} = 4a^2+4a\sqrt{(x-c)^2+y^2} + \cancel{x^2}-2cx+\cancel{c^2}$

or $4cx = 4a^2+4a\sqrt{(x-c)^2+y^2}$

$cx = a^2+a\sqrt{(x-c)^2+y^2}$

$$cx - a^2 = a\sqrt{(x-c)^2+y^2}$$

Again, squaring on both side

$$c^2x^2 - 2cxa^2 + a^4 = a^2\left[(x-c)^2+y^2\right]$$

or, $c^2x^2 - 2a^2cx + a^4 = a^2\left[x^2 - 2cx + c^2 + y^2\right]$

or, $c^2x^2 \cancel{- 2a^2cx} + a^4 = a^2x^2 \cancel{- 2a^2cx} + a^2c^2 + a^2y^2$

or, $a^4 + c^2x^2 = a^2x^2 + a^2c^2 + a^2y^2$

or, $a^4 - a^2c^2 = a^2x^2 - c^2x^2 + a^2y^2$

or, $a^2(a^2-c^2) = x^2(a^2-c^2) + a^2y^2$

$$(a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2)$$

Dividing on both side by $a^2(a^2-c^2)$

$$\frac{x^2}{a^2} + \frac{y^2}{(a^2-c^2)} = 1 \quad ---- (i)$$

Now,

From ΔPF_1F_2

$\therefore$ $PF_2 - PF_1 < F_1F_2$

or, $2a < 2c$

or, $a < c$

or, $a^2 < c^2$

$\Rightarrow$ $a^2 - c^2 < 0$ (Negative)

Let, $a^2 - c^2 = -b^2$

From eq^n (i)

$$\frac{x^2}{a^2} + \frac{y^2}{(-b^2)} = 1$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Find the equation of tangent at (x_1, y_1) on hyperbola

Sol^n.

The equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{---- (i)}$$

The (x_1, y_1) lies on the hyperbola. Then,

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1 \quad \text{---- (ii)}$$

Eq^n of line passing through (x_1, y_1) is

$$(y - y_1) = m(x - x_1) \quad \text{---- (iii)}$$

Differentiating eq^n(i) w.r.t. 'x'

$$\frac{d}{dx}\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$$

$$\text{or}\quad \frac{1}{a^2}\,2x - \frac{1}{b^2}\,2y \cdot \frac{dy}{dx} = 0$$

$$\text{or}\quad \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\text{or}\quad 2\left[\frac{x}{a^2} - \frac{y}{b^2}\frac{dy}{dx}\right] = 0$$

$$\text{or}\quad \frac{x}{a^2} = \frac{y}{b^2}\frac{dy}{dx}$$

$$\text{or}\quad \frac{dy}{dx} = \frac{x}{a^2} \times \frac{b^2}{y}$$

$$\therefore \frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

Therefore, slope of tangent at the point (x_1, y_1) is

$$m = \frac{dy}{dx} = \frac{b^2 x_1}{a^2 y_1}$$

putting value of 'm' in equation (iii)

$$(y-y_1)=m(x-x_1)$$

$$(y-y_1) = \frac{b^2 x_1}{a^2 y_1}(x-x_1)$$

$$(y-y_1)y_1 = \frac{b^2}{a^2} x_1 (x-x_1)$$

$$yy_1 - y_1^2 = \frac{b^2}{a^2}(xx_1 - x_1^2)$$

$$\frac{yy_1 - y_1^2}{b^2} = \frac{(xx_1 - x_1^2)}{a^2}$$

$$\frac{yy_1}{b^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{x_1^2}{a^2}$$

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = \frac{xx_1}{a^2} - \frac{yy_1}{b^2}$$

$$\therefore \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

For the eq^n of normal at (x_1, y_1)

$$m = \frac{dy}{dx} = \frac{-a^2 y_1}{b^2 x_1}$$

From eq^n (iii)

Equation of normal

$$(y-y_1) = \frac{-a^2 y_1}{b^2 x_1}(x-x_1)$$

Find the condition that the line $y = mx + c$ may be tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Soln:-

Eqn of line is

$$y = mx + c \text{ ---- (i)}$$

Eqn of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or $b^2x^2 - a^2y^2 = a^2b^2$

or $b^2x^2 - a^2y^2 - a^2b^2 = 0$ --- (ii)

Solving eqn (i) & (ii)

$b^2x^2 - a^2(mx+c)^2 - a^2b^2 = 0$

or $b^2x^2 - a^2[m^2x^2 + 2mxc + c^2] - a^2b^2 = 0$

or $b^2x^2 - [a^2m^2x^2 + 2a^2mcx + a^2c^2] - a^2b^2 = 0$

or $b^2x^2 - a^2m^2x^2 - 2a^2cmx - a^2c^2 - a^2b^2 = 0$

or $(b^2 - a^2m^2)x^2 - 2a^2cmx - (a^2c^2 + a^2b^2) = 0$

$\therefore (b^2 - a^2m^2)x^2 - 2a^2cmx + (-a^2c^2 - a^2b^2) = 0$

The line (i) may touch the eqn (ii)

$A = b^2 - a^2m^2$

$B = 2a^2cm$

$C = -a^2c^2 - a^2b^2$

If $B^2 - 4AC = 0$

$(2a^2cm)^2 - 4[(b^2 - a^2m^2)(-a^2c^2 - a^2b^2)] = 0$

or $4a^4c^2m^2 - 4[a^2b^2c^2 - a^2b^4 + a^4c^2m^2 + a^4b^2m^2] = 0$

$\Rightarrow 4a^4c^2m^2 + 4a^2b^2c^2 + 4a^2b^4 - 4a^4c^2m^2 - 4a^4b^2m^2 = 0$

$\Rightarrow + a^2b^2c^2 + a^2b^4 - a^4b^2m^2 = 0$

$\Rightarrow a^2b^2(c^2 + b^2 - a^2m^2) = 0$

$\Rightarrow c^2 + b^2 - a^2m^2 = 0$

$\therefore c^2 = a^2m^2 - b^2$

Show that the line $lx + my + n = 0$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If $a^2l^2 - b^2m^2 = n^2$

Soln.-

Eqn of line is

$lx + my + n = 0$

$\Rightarrow my = -lx - n$

$\therefore y = -\left(\frac{lx+n}{m}\right)$ --- (i)

Eqn of hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$b^2x^2 - a^2y^2 - a^2b^2 = 0$ --- (ii)

Solving eqn (i) & (ii)

$b^2x^2 - a^2\left[-\left(\frac{lx+n}{m}\right)\right]^2 - a^2b^2 = 0$

$\Rightarrow b^2x^2 - a^2\left[-\frac{(lx+n)}{m}\right]^2 - a^2b^2 = 0$

$\Rightarrow b^2x^2 - a^2\left[\frac{(lx+n)^2}{m^2}\right] - a^2b^2 = 0$

$\Rightarrow b^2x^2 - \frac{a^2}{m^2}[l^2x^2 + 2lxn + n^2] - a^2b^2 = 0$

or $b^2x^2 - \frac{a^2}{m^2}[l^2x^2 + 2lnx + n^2] - a^2b^2 = 0$

or $b^2m^2x^2 - a^2l^2x^2 - 2a^2lnx - a^2n^2 - a^2b^2m^2 = 0$

or, $(b^2m^2 - a^2l^2)x^2 - 2a^2lnx + (-a^2n^2 - a^2b^2m^2) = 0$

The line eq^n (i) ~~& (ii)~~ touches the line (ii) if

$B^2 - 4AC = 0$

Where;

$B = 2a^2ln$

$A = (b^2m^2 - a^2l^2)$

$C = -a^2n^2 - a^2b^2m^2$

So,

$B^2 - 4AC = 0$

or $(2a^2ln)^2 - 4[(b^2m^2 - a^2l^2)(-a^2n^2 - a^2b^2m^2)] = 0$

or, $4a^4l^2n^2 - 4[-a^2b^2m^2n^2 - a^2b^4m^4 + a^4l^2n^2 + a^4b^2l^2m^2] = 0$

or $\cancel{a^4l^2n^2} + a^2b^2m^2n^2 + a^2b^4m^4 - \cancel{a^4l^2n^2} - a^4b^2l^2m^2 = 0$

or $a^2b^2m^2n^2 + a^2b^4m^4 - a^4b^2l^2m^2 = 0$

or $a^2b^2m^2(n^2 + b^2m^2 - a^2l^2) = 0$

or $n^2 + b^2m^2 - a^2l^2 = 0$

or $n^2 + b^2m^2 = a^2l^2$

$\therefore a^2l^2 - b^2m^2 = n^2$

CamScanner

Equation of Ellipse	Center	Vertices	foci	major axis	minor axis	Eccentricity	Length of Latus Rectum	Eq^n of directrix
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b>0$	$(0,0)$	$(\pm a,0)$	$(\pm ae,0)$	$2a$	$2b$	$e=\sqrt{1-\frac{b^2}{a^2}}$	$\frac{2b^2}{a}$	$x=\pm\frac{a}{e}$
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $b>a>0$	$(0,0)$	$(0,\pm b)$	$(0,\pm be)$	$2b$	$2a$	$e=\sqrt{1-\frac{a^2}{b^2}}$	$\frac{2a^2}{b}$	$y=\pm\frac{b}{e}$
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, $a>b>0$	(h,k)	$(h\pm a,k)$	$(h\pm ae,k)$	$2a$	$2b$	$e=\sqrt{1-\frac{b^2}{a^2}}$	$2\frac{b^2}{a}$	$x=h\pm\frac{a}{e}$
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, $b>a>0$	(h,k)	$(h,k\pm b)$	$(h,k\pm be)$	$2b$	$2a$	$e=\sqrt{1-\frac{a^2}{b^2}}$	$2\frac{a^2}{b}$	$y=k\pm\frac{b}{e}$

Eq^n of hyperbola	Vertex	Focus	Transverse axis	Conjugate axis	Eccentricity	eq^n of directrix	length of latus rectum
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$(\pm a,0)$	$(\pm ae,0)$	$2a$	$2b$	$e=\sqrt{1+\frac{b^2}{a^2}}$	$x=\pm\frac{a}{e}$	$2\frac{b^2}{a}$
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$	$(0,\pm b)$	$(0,\pm be)$	$2b$	$2a$	$e=\sqrt{1+\frac{a^2}{b^2}}$	$y=\pm\frac{b}{e}$	$2\frac{a^2}{b}$
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	$(h\pm a,k)$	$(h\pm ae,k)$	$2a$	$2b$	$e=\sqrt{1+\frac{b^2}{a^2}}$	$x=h\pm\frac{a}{e}$	$2\frac{b^2}{a}$
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=-1$	$(h,k\pm b)$	$(h,k\pm be)$	$2b$	$2a$	$e=\sqrt{1+\frac{a^2}{b^2}}$	$y=k\pm\frac{b}{e}$	$2\frac{a^2}{b}$

Find the centre, vertices, foci, of an ellipse.

(i) $25(x-3)^2 + 4(y-1)^2 = 100$

Sol^n:

$$25(x-3)^2 + 4(y-1)^2 = 100$$

$$\text{or, } \frac{25(x-3)^2}{100} + \frac{4(y-1)^2}{100} = 1$$

$$\text{or, } \frac{(x-3)^2}{4} + \frac{(y-1)^2}{25} = 1$$

Comparing this eq^n with

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

We get,

$a^2 = 4 \quad \Rightarrow a = 2$

$b^2 = 25 \quad \Rightarrow b = 5$

$h = 3 \quad \& \quad k = 1$

Here, $b > a > 0$

So,

(i) $e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{(2)^2}{(5)^2}} = \sqrt{\frac{25-4}{25}} = \frac{\sqrt{21}}{5}$

(ii) Centre $(h,k) = (3,1)$

(iii) Vertices $(h, k \pm b) = (3, 1 \pm 5) = (3,-4) \;\&\; (3,6)$

(iv) Foci $= (h, k \pm be) = \left(3, 1 \pm 5 \frac{\sqrt{21}}{5}\right) = \left(3, 1 \pm \sqrt{21}\right)$

(v) major axis $(2a) = 2 \times 2 = 4$

(vi) Minor axis $(2b) = 2 \times 5 = 10$

(vii) Length of latus rectum $\frac{2b^2}{a} = \frac{2(5)^2}{2} = 25$

(viii) Eq^n of Directrix $(x) = k \pm \frac{b}{e}$ $\cancel{= 3 \pm \frac{2 \times 5}{\sqrt{21}} = 3 \pm \frac{10}{\sqrt{21}}}$

$$= 1 \pm \frac{5 \times 5}{\sqrt{21}} = 1 \pm \frac{10}{\sqrt{21}}$$

(ii) $x^2+10x+25y^2=0$

Soln:

$$x^2+10x+25y^2=0$$

$$\text{or}\quad (x)^2+2\cdot x\cdot 5+(5)^2-(5)^2+25y^2=0$$

$$\text{or}\quad (x+5)^2+25y^2=25$$

$$\text{or}\quad (x+5)^2+(5y)^2=25$$

$$\text{or}\quad \frac{(x+5)^2}{25}+\frac{25(y-0)^2}{25}=1$$

$$\therefore \frac{(x+5)^2}{25}+\frac{(y-0)^2}{1}=1$$

Comparing this eqn with $\cancel{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

We get,

$a^2=25 \Rightarrow a=5$

$b^2=1 \Rightarrow b=1$

$h=-5$ & $k=0$

Here, $a>b>0$

So,

(i) Eccentricity $(e)=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{1}{25}}=\frac{\sqrt{24}}{5}$

(ii) Centre $(h,k)=(5,0)$

(iii) Vertices $\cancel{(h, k\pm b}$ $(h\pm a,k)=(-5\pm5,0)=(0,0)$ & $(-10,0)$

(iv) Foci $(h\pm ae,k)=(-5\pm\frac{\sqrt{24}}{5},0)=(-5\pm\sqrt{24},0)$

(v) Major axis $(2b)=2\times1=2$

(vi) Minor axis $(2a)=2\times5=10$

(vii) Length of latus rectum $\cancel{2b}$ $\frac{2a^2}{b}=\frac{2(5)^2}{1}=50$

(viii) Eqn of directrix $(y)=h\pm\frac{a}{e}=0\pm\frac{1}{\sqrt{24}/5}=\frac{5}{\sqrt{24}}$

(iii) $x^2+9y^2-4x+18y+4=0$

Soln.-

$$x^2+9y^2-4x+18y+4=0$$

or $x^2-4x+9y^2+18y+4=0$

or $(x)^2-2.x.2+(2)^2-(2)^2+(3y)^2+2.3y.3+(3)^2-(3)^2+4=0$

or $(x-2)^2-4+(3y+3)^2+4-9=0$

or $(x-2)^2+(3)^2(y+1)^2=9$

or $(x-2)^2+9(y+1)^2=9$

or $\frac{(x-2)^2}{9}+\frac{9(y+1)^2}{9}=1$

$\therefore \frac{(x-2)^2}{9}+\frac{(y+1)^2}{1^2}=1$

Comparing this eq^n with $\cancel{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

We get,

$a^2=9 \Rightarrow a=3$

$b^2=1 \Rightarrow b=1$

$h=2$ & $k=-1$

$a>b>0$

So,

(i) Eccentricity $(e)=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{(1)^2}{(3)^2}}=\frac{\sqrt{9-1}}{\sqrt{3}}=\frac{\sqrt{8}}{3}$

(ii) Centre $(h,k)=(2,-1)$

(iii) Vertices $(h\pm a,k)=(2\pm 3,-1)$

(iv) Foci $(h\pm ae,k)=(2\pm 3\times\frac{\sqrt{8}}{3},-1)=(2\pm\sqrt{8},-1)$

(v) Major axis $(2b)=2\times 1=2$

(vi) Minor axis $(2a)=2\times 3=6$

(vii) Length of latus rectum $\frac{2a^2}{b}=\frac{2\cdot(3)^2}{1}=18$

(viii) Eq^n of directrix $(y)=h\pm\frac{a}{e}=2\pm\frac{3}{\sqrt{8}/3}=2\pm\frac{9}{\sqrt{8}}$

(i) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Solⁿ:

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$\frac{(x)^2}{(4)^2} - \frac{(y)^2}{(3)^2} = 1$$

Comparing this eqⁿ with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

We get,

$a^2 = 4^2 \Rightarrow a = 4$

$b^2 = 9 \Rightarrow b = 3$

$a > b > 0$

(i) Eccentricity $(e) = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$

(ii) ~~Centre~~ Vertex $(\pm a, 0) = (\pm 4, 0)$

(iii) Foci $(\pm ae, 0) = (\pm 4 \times \frac{5}{4}, 0) = (\pm 5, 0)$

(iv) Transverse axis $(2a) = 2 \times 4 = 8$

(v) Conjugate axis $(2b) = 2 \times 3 = 6$

(vi) Eqⁿ of directrix $(x) = \pm \frac{a}{e} = \pm \frac{4}{5/4} = \pm \frac{8}{5}$

(vii) Length of Latus Rectum $(2\frac{b^2}{a}) = \frac{2(3)^2}{\cancel{4}2} = \frac{9}{2}$

(ii) $5x^2 - 4y^2 + 20x + 8y = 4$

Soln:

$$5x^2 - 4y^2 + 20x + 8y = 4$$

or $5x^2 + 20x - 4y^2 + 8y = 4$

or $5(x^2 + 4x) - 4(y^2 + 2y) = 4$

or $5[(x)^2 + 2\cdot x\cdot 2 + (2)^2 - (2)^2] - 4[(y)^2 + 2\cdot y\cdot 1 + (1)^2 - (1)^2] = 4$

or $5[(x+2)^2 - 4] - 4[(y+1)^2 - 1] = 4$

or $5(x+2)^2 - 20 - 4(y+1)^2 + \cancel{4} = \cancel{4}$

or $5(x+2)^2 - 4(y+1)^2 = 20$

or $$\frac{\cancel{5}(x+2)^2}{\cancel{20}\,4} - \frac{\cancel{4}(y+1)^2}{\cancel{20}\,5} = 1$$

or $$\frac{(x+2)^2}{4} - \frac{(y+1)^2}{5} = 1$$

$$\therefore \frac{(x+2)^2}{(2)^2} - \frac{(y+1)^2}{(\sqrt{5})^2} = 1$$

Comparing this eqn with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

We get,

$a^2 = (2)^2 \Rightarrow a = 2$

$b^2 = 5 \Rightarrow b = (\sqrt{5})$

$h = -2$ & $k = -1$

Find the eq^n of tangents to the ellipse $4x^2+3y^2=5$ which have parallel to the line $y=3x+7$

Sol^n:

The given eq^n of ellipse is

$$4x^2+3y^2=5 \quad \text{---(i)}$$

Eq^n of line is

$$3x-y+7=0 \quad \text{---(ii)}$$

Eq^n of any line parallel to the given line is

$$3x-y+k=0 \quad \text{---(iii)}$$

Solving eq^n (i) & (iii)

$$4x^2+3y^2=5$$

$$\text{or}\quad 4x^2+3(3x+k)^2=5$$

$$\text{or}\quad 4x^2+3[9x^2+6kx+k^2]=5$$

$$\text{or}\quad 4x^2+27x^2+18kx+3k^2=5$$

$$\text{or}\quad 31x^2+18kx+3k^2=5$$

$$\therefore\quad 31x^2+18kx+(3k^2-5)=0$$

We get,

$A=31$

$B=18k$

$C=3k^2-5$

The line (iii) tangent to the ellipse (i) is:

$$B^2-4AC=0$$

$$\text{or}\quad (18k)^2-4\cdot31(3k^2-5)=0$$

$$\text{or}\quad 324k^2-124(3k^2-5)=0$$

$$324k^2 - 372k^2 + 620 = 0$$

$$-48k^2 + 620 = 0$$

$$-48k^2 = -620$$

$$k^2 = \frac{620}{48}$$

$$k^2 = \frac{155}{12}$$

$$\therefore k = \sqrt{\frac{155}{12}}$$

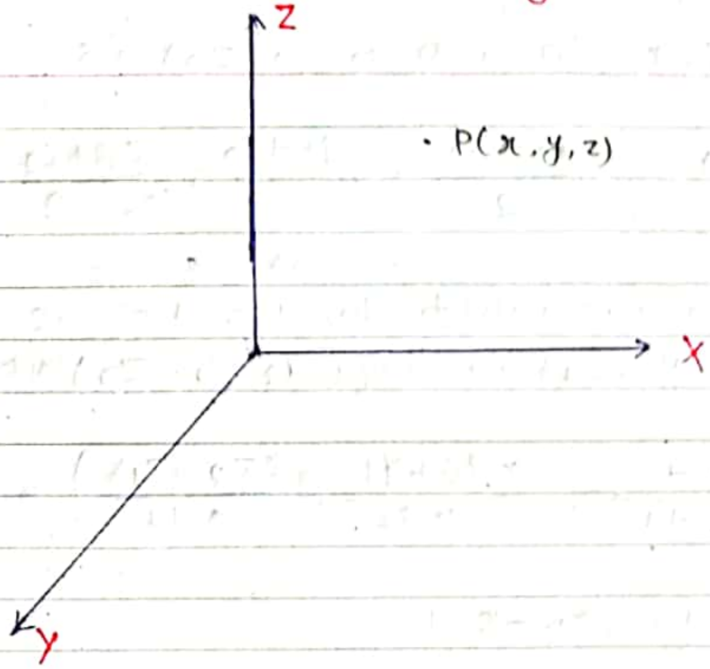
Substituting value of 'k' in eq^n (iii)

$$3x - y \pm \sqrt{\frac{155}{12}} = 0$$

CamScanner

Three Dimensional Geometry [12 Hrs]

3D



1) Distance Formula

The distance between joining the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is:

$$d = PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

2) Section Formula

Let $C(x, y, z)$ divides the line joining the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio: $m_1 : m_2$

(i) Internally

$$(x, y, z) = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2} \right)$$

(ii) Externally

$$(x, y, z) = \left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2} \right)$$

CS CamScanner

3) Midpoint

Midpoint of the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$(x, y, z) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

4) The points which divides the line join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $k:1$ is:

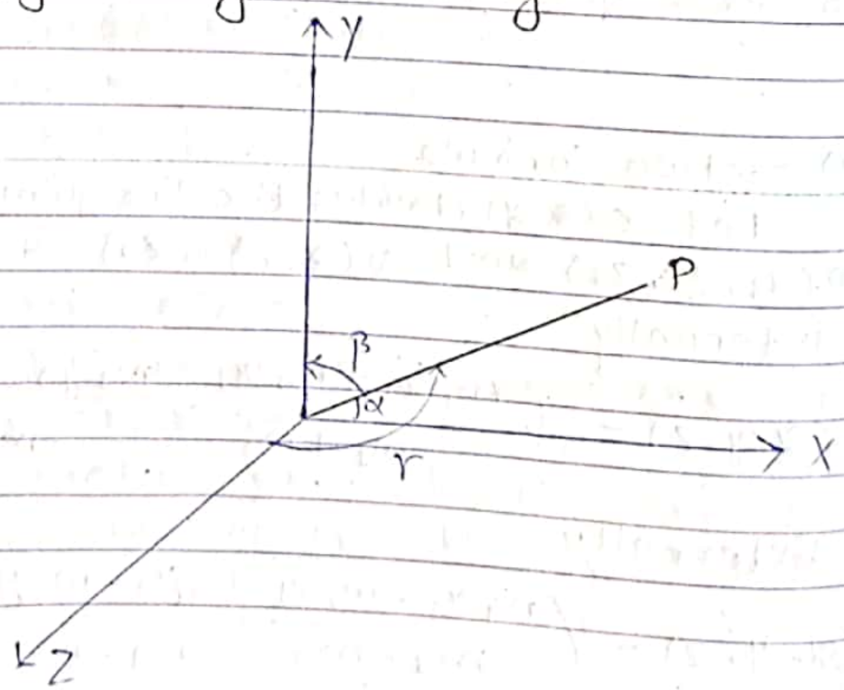
$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$$

As, $m_1 : m_2 = k : 1$

5) Direction cosines of a line (dc's)

Let α, β and γ be the angles made by straight line with positive direction of axes then, $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of the line. They are generally denoted by

$l = \cos\alpha$
$m = \cos\beta$
$n = \cos\gamma$



Note:

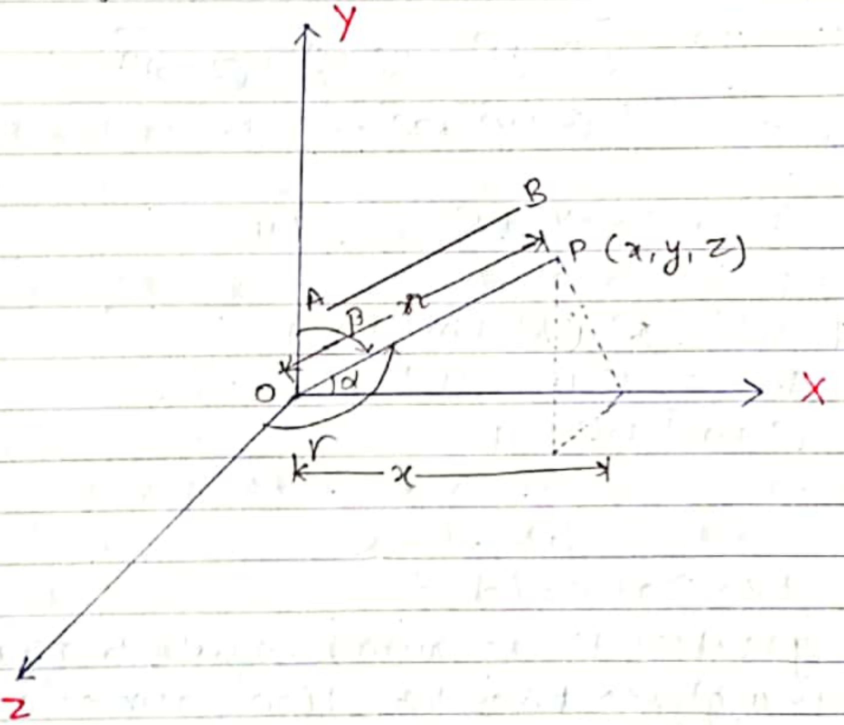
(i) If 'l', 'm', 'n' be the direction cosines of PQ then direction cosines of QP are '$-l$', '$-m$', '$-n$'.

(ii) The direction cosines of x-axis, y-axis and z-axis are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively.

(iii) $\alpha + \beta + r \neq 360^\circ$

6) Show that

$l^2 + m^2 + n^2 = 1$. Where 'l', 'm' and 'n' are direction cosines of a line.



Let AB be a straight line, α, β and r be the angles made by the line AB with axis of coordinate.

Let OP be the parallel to AB. Let, $OP = r$ and let coordinates of 'p' be (x, y, z) then:

$$ON = x = r\cos\alpha$$

or,

$$x = r.l$$

Similarly,

$$y = mr \quad \& \quad z = nr$$

Squaring and adding these relations:

$$x^2+y^2+z^2 = (lr)^2+(mr)^2+(nr)^2$$
$$= r^2(l^2+m^2+n^2) \text{ ---- (i)}$$

But we know,

$$r = OP = \sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$
$$= \sqrt{x^2+y^2+z^2}$$

$$\therefore r^2 = x^2+y^2+z^2 \text{ ---- (ii)}$$

Now, eq^n (i) becomes

$$\therefore x^2+y^2+z^2 = r^2(l^2+m^2+n^2)$$
$$1 = l^2+m^2+n^2$$
$$\therefore l^2+m^2+n^2 = 1$$

7) Direction Ratios (dr's)

Any set of three numbers which are proportional to the direction cosines of a line are called direction ratios of the line if $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$ are dc's of line then 2, 3, 6 are dr's of line

Note: If a, b, c are dr's then

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}} \quad \& \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

l, m & n are dc's.

8) The direction cosines of the line joining the point $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are

$$\left(\frac{x_2-x_1}{r}, \frac{y_2-y_1}{r}, \frac{z_2-z_1}{r}\right)$$

where,

$$r = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Note: The dr's of a line are x_2-x_1, y_2-y_1 and z_2-z_1

9) Angle between two straight lines having direction cosines 'l_1' 'm_1' 'n_1' and 'l_2', 'm_2', 'n_2'.

If 'θ' be the angle between them

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

If two lines are perpendicular to each other. Then,

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \quad [\because \cos 90^\circ]$$

If these two lines are parallel to each other.

$l_1 = l_2$; $m_1 = m_2$ & $n_1 = n_2$

10) Angle between two straight lines having direction ratios $a_1, b_1,$ ~~and~~ c_1 and a_2, b_2, c_2

If 'θ' be the angle between them

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Note:

(i) If two lines are perpendicular to each other then

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

(ii) If two lines are parallel to each other then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = 0$$

Plane

(i) The eq^n of plane in a general form

$$ax + by + cz + d = 0$$

(ii) The eq^n of plane passes through the origin

$$ax + by + cz = 0$$

$[\because d = 0]$

(iii) The eq^n of plane in intercept form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(iv) The eq^n of plane in normal-form

$lx + my + nz = p$

where,

→ l, m, n are dc's of a line, p is the perpendicular form to the plane.

(v) The eq^n of plane passes through the point (x_1, y_1, z_1) is: $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Note:

In the eq^n of plane

$ax + by + cz + d = 0$

Where,

a, b, c are the direction ratios of the normal of the plane.

Exercise

1) The eq^n of plane $2x - y + 2z = 4$. Find

i) The intercept

ii) Distance from the origin to the plane.

Sol^n:.

Given,

(i) Eq^n of plane

$$2x - y + 2z = 4$$

$$\frac{2x}{4} - \frac{y}{4} + \frac{2z}{4} = 1$$

$$\frac{x}{2} + \frac{y}{(-4)} + \frac{z}{2} = 1$$

$\therefore$

x- intercept (a) = 2

y- intercept (b) = -4

z- intercept (c) = 2

(ii) Distance from the origin (0,0,0) to the plane is

$$2x - y + 2z - 4 = 0$$

Now

$$P = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$P = \left| \frac{2\times 0 + (-1)\times 0 + 2\times 0 + (-4)}{\sqrt{2^2 + (-1)^2 + 2^2}} \right|$$

$$\therefore P = \left| \frac{-4}{3} \right| = 4/3$$

2) Find the intercept made on the coordinate axis by the plane: $x+2y-2z=9$. Find also the directions of the normal to this plane.

Soln:

Eqn of plane

$$x+2y-2z=9 \quad \text{---(i)}$$

$$\frac{x}{9}+\frac{2y}{9}-\frac{2z}{9}=1$$

$$\frac{x}{9}+\frac{y}{9/2}+\frac{z}{(-9/2)}=1$$

Comparing this eqn with

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

we get,

$a=9$

$b=9/2$

& $c=-9/2$

~~Also,~~

Also, Direction ratios are $(a,b,c)=1,$

$a=1$

$b=2$ [$\because$ Comparing eqn(i) with $ax+by+cz=0$]

$c=-2$

Now,

Dr's

$$l=\frac{a}{\sqrt{a^2+b^2+c^2}}=\frac{1}{\sqrt{1+4+4}}=\frac{1}{3}$$

$$m=\frac{b}{\sqrt{a^2+b^2+c^2}}=\frac{2}{\sqrt{1+4+4}}=\frac{2}{3}$$

$$n=\frac{c}{\sqrt{a^2+b^2+c^2}}=\frac{-2}{\sqrt{1+4+4}}=-2/3$$

Direction ratios are $(l, m, n) = \left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$

Q. 3 Direction cosines of the line normal to the plane $6x - 3y + 2z = 14$

Sol^n:-

Given,

$$6x - 3y + 2z = 14$$

We get

$$a = 6, \quad b = -3 \quad \& \quad c = 2$$

Now,

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{6}{\sqrt{(6)^2 + (-3)^2 + (2)^2}} = \frac{6}{\sqrt{36 + 9 + 4}} = \frac{6}{7}$$

$$m = \frac{-3}{7} \quad \& \quad n = \frac{2}{7}$$

Q. 4 Direction cosines of the line perpendicular to a plane from the origin are proportional 1, 3, 1 and length of perpendicular is 2. Find eq^n of plane.

Sol^n:- Given,

$$a = 1, \quad b = 3, \quad \& \quad c = 1$$

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{1}{\sqrt{1^2 + 3^2 + 1^2}} = \frac{1}{\sqrt{1 + 9 + 1}} = \frac{1}{\sqrt{11}}$$

$$m = \frac{3}{\sqrt{11}} \quad \& \quad n = \frac{1}{\sqrt{11}}$$

Now,
Eq^n of plane in normal form

$$lx + my + nz = p \text{ ---- (i)}$$

$$\frac{1}{\sqrt{11}}x + \frac{3}{\sqrt{11}}y + \frac{1}{\sqrt{11}}z = 2$$

$$\Rightarrow x + 3y + z = 2\sqrt{11}$$

This is the eq^n of plane in normal form.

Q.5. Find the eq^n of plane containing the lines through the origin with direction cosines proportional to 1, -2, 2 & 2, 3, -1.

Sol^n:

The eq^n of plane through the origin is:

$$ax + by + cz = 0 \text{ ---- (i)}$$

$$(x_1, y_1, z_1) = (1, -2, 2)$$

$$(x_2, y_2, z_2) = (2, 3, -1)$$

The plane (i) contains the line having direction cosines proportional to (1, -2, 2) & (2, 3, -1).

The lines normal to the plane first.
By condition of perpendicularity.

$$1a + (-2)b + 2c = 0$$

$$a - 2b + 2c = 0 \text{ ---- (i)}$$

&

$$2a - 3b - c = 0 \text{ ---- (ii)}$$

Solving (i) & (ii) by cross multiplication

$$\frac{a}{2-6} = \frac{b}{4+1} = \frac{c}{3+4}$$

$$\frac{a}{-4} = \frac{b}{5} = \frac{c}{7} = k \text{ (say)}$$

$\therefore$ $a = -4k$

$b = 5k$

$c = 7k$

Now, substituting these values in eq^n (i)

$-4kx + 5ky + 7kz = 0$

$4x - 5y - 7z = 0$

This is the required eq^n of plane.

Angle between two planes

Let $a_1x + b_1y + c_1z + d_1 = 0$ ---- (i)

$a_2x + b_2y + c_2z + d_2 = 0$ ---- (ii) be two planes

If 'θ' be the angle between them

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Exercise 2.2

Q.1 Find the eq^n of plane through (1,1,1) and parallel to the plane $3x - 4y + 5z = 0$

Sol^n:-

The eq^n of plane is:

$3x - 4y + 5z = 0$ ---- (i)

and point (1,1,1)

The eq^n of plane which is parallel to the parallel to the plane (i).

$$3x-4y+5z+k=0 \quad \text{--- (ii)}$$

This plane passes through the point (1,1,1) then

$$3\times1-4\times1+5\times1+k=0$$
$$-k=3-4+5$$
$$k=-4$$

Putting this value in eq^n (ii)

$$3x-4y+5z+(-4)=0$$
$$3x-4y+5z-4=0 \quad \underline{\underline{Ans}}$$

Q.2. Find the angle between the following pair of planes: $x+3y+5z=0$ and $x-2y+z=20$.

Sol^n:

Eq^n of plane are

$$x+3y+5z=0 \quad \text{--- (i)}$$
$$\& \quad x-2y+z=20 \quad \text{--- (ii)}$$

$\therefore a_1=1, \quad b_1=3 \quad \& \quad c_1=5$

$a_2=1 \quad b_1=-2 \quad \& \quad c_2=1$

If 'θ' be the angle between them, then;

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\cos\theta = \frac{1+(-6)+5}{\sqrt{1+9+25}\sqrt{1+4+1}}$$

$$\Rightarrow \cos\theta = \frac{1-6+5}{\sqrt{35}\sqrt{6}}$$

$$\Rightarrow \cos\theta = \frac{0}{\sqrt{35\times 6}}$$

$$\Rightarrow \cos\theta = 0$$

$$\Rightarrow \cos\theta = \cos \pi/2$$

$$\therefore \theta = \pi/2$$

Q.4. Find the equation of the plane containing the point $(1,-1,2)$ and is perpendicular to the planes $2x+3y-2z=5$ and $x+2y-3z=8$

Soln:- Q. No.3.

The eqn of plane passing through the origin is:

$ax+by+cz=0$ ----(i)

The drs of the line perpendicular to joining the points $(1,1,1)$ & $(3,4,5)$

$1\cdot a+1\cdot b+1\cdot c=0$ ---- (ii)

$3\cdot a+4\cdot b-5\cdot c=0$ ---- (iii)

Solving (ii) & (iii) by cross multiplication method:

$$\frac{a}{-5-4}=\frac{b}{3+5}=\frac{c}{4-3}=k \text{ (say)}$$

$\therefore a=-9k$

$b=8k$

$c=k$

$ax + by + cz = 0$

or $-9kx + 8ky + kz = 0$

$\therefore \quad 9x - 8y - z = 0$

This is the reqrd eq^n of plane.

Q.No.3 Find the equation of the plane passing through the origin & containing the line joining the points (1, 1, 1) and (3, 4, -5)

Sol^n:- Q.No.4

Here,

The eq^n of plane is;

$2x + 3y - 2z = 5 \quad \text{---- (i)}$

and $\quad x + 2y - 3z = 8 \quad \text{---- (ii)}$

Now,

The eq^n of plane passing through the point (1, -1, 2) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\Rightarrow a(x - 1) + b(y + 1) + c(z - 2) = 0 \quad \text{---- (iii)}$

Since, plane (iii) perpendicular to the plane (i) & (ii)

$2a + 3b - 2c = 0$

$1a + 2y - 3c = 0$

Solving these eq^n by cross multiplication method

$$\frac{a}{-9+4} = \frac{b}{-2+6} = \frac{c}{4-3} = k \text{ (say)}$$

$$\text{or}\quad \frac{a}{-5} = \frac{b}{4} = \frac{c}{1} = k$$

$\therefore\ a = -5k\ ,\ b = 4k\ \ \&\ c = k$

So, eq^n (iii) becomes:

$$-5k(x-1) + 4k(y-11) + k(z-2) = 0$$

$$\text{or}\quad -5kx + 5k + 4ky - 44k + kz - 2k = 0$$

$$\text{or}\quad 5x - 4y - z - 7 = 0$$

A.

Q.5 Find eq^n of plane through P(1, 4, −2) at right angle to OP.

Sol^n:

The eq^n of plane through the point (1, 4, −2) is

$$a(x-1) + b(y-4) + c(z+2) = 0 \quad \text{----(i)}$$

The drs is

$a = (1-0) = 1$

$b = (4-0) = 4$

$\&\ c = (-2-0) = -2$

Now, from (i)

$$1(x-1) + 4(y-4) + -2(z+2) = 0$$

$$\text{or}\quad x - 1 + 4y - 16 - 2z - 4 = 0$$

$$\text{or}\quad x - 4y - 2z - 21 = 0$$

A.

1) The equation of plane through the three points (x_1, y_1, z_1), (x_2, y_2, z_2) & (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

2) The condition for four-points (x_1, y_1, z_1) (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) to lie in the plane is;

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

which is condition for coplanarity.

Exercise 2.3

Q.1 Find eq^n of plane through (4,5,1), (3,9,4) & (-4,4,4)

Soln:-

Here,

$(x_1, y_1, z_1) = (4, 5, 1)$

$(x_2, y_2, z_2) = (3, 9, 4)$

$(x_3, y_3, z_3) = (-4, 4, 4)$

The eq^n of plane is:

$$\begin{vmatrix} x & y & z & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \\ -4 & 4 & 4 & 1 \end{vmatrix} = 0$$

Interchanging

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$ & $R_3 \to R_3 - R_4$

$$\begin{vmatrix} x-4 & y-5 & z-1 & 0 \\ 1 & -4 & -3 & 0 \\ 7 & 5 & 0 & 0 \\ -4 & 4 & 4 & 1 \end{vmatrix} = 0$$

$$\text{or} \quad 1 \begin{vmatrix} x-4 & y-5 & z-1 \\ 1 & -4 & -3 \\ 7 & 5 & 0 \end{vmatrix} = 0$$

or $(x-4)15 - (y-5)21 + (z-1) \cdot 33 = 0$

or $15x - 60 - 21y + 105 + 33z - 33 = 0$

or, $15x - 21y + 33z + 12 = 0$

$\therefore \quad 5x - 7y + 11z + 4 = 0$

This is the required eq^n of plane.

Q. 4. Find the eq^n of the plane through the point $(-1, 1, -1)$, $(6, 2, 1)$ and normal to the plane $2x + y + z = 5$.

Soln:

The eq^n of plane through the points $(-1, 1, -1)$ is :

$a(x+1) + b(y-1) + c(z+1) = 0$ ...... (i)

Since, the plane (i) also ~~gives~~ passes through $(6, 2, 1)$

$a(6+1)+b(2-1)+c(1+1)=0$

$7a+b+2c=0 \cdots (ii)$

Since, the plane (i) is normal to the plane $2x+y+z=5$ then,

$2a+b+c=0 \cdots (iii)$

Solving (ii) & (iii) by cross multiplication method:

$$\frac{a}{1-2}=\frac{b}{4-7}=\frac{c}{7-2}=k \text{ (say)}$$

or $\frac{a}{-1}=\frac{b}{-3}=\frac{c}{5}=k$

$a=-k, \quad b=-3k \;\&\; c=5k$

From eq^n (i)

$-k(x+1)+(-3k)(y-1)+5k(z+1)=0$

or $-x-1-3y+3+5z+5=0$

or $-x-3y+5z+7=0$

or $x+3y-5z-7=0$

$\therefore x+3y-5z=7$

The eq^n of plane through the intersection of two plane $a_1x+b_1y+c_1z+d_1=0$ &

$a_2x+b_2y+c_2z+d_2=0$ is

$$(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$$

i.e.

$$(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+(d_1+\lambda d_2)=0$$

Exercise: 2.4

Q. Find the equation of plane through the intersection of $x+2y+3z+4=0$ and $4x+3y+2z+1=0$ passing through $(0,0,0)$.

Sol^n:

The eq^n of plane through the intersection of plane

$x+2y+3z+4=0$ and $4x+3y+2z+1=0$ is

$$(x+2y+3z+4)+\lambda(4x+3y+2z+1)=0 \quad \text{-----(i)}$$

This plane passes through the origin $(0,0,0)$ then,

$$(x+2y+3z+4)+\lambda(4x+3y+2z+1)=0$$

or $4+\lambda=0$

$\therefore \lambda=-4$

Eqⁿ (i) becomes

$x+2y+3z+4+[-4(4x+3y+2z+1)]=0$

or $x+2y+3z+4-16x-12y-8z-4=0$

or $-15x-10y-5z=0$

$\therefore 3x+2y+z=0$

Ans

Q. 3 Find the equation of plane through the intersection of $2x+3y+10z=8$ and $2x-3y+7z=2$ and normal to the plane $3x-2y+4z=5$.

Soln:-

The eqⁿ of plane from the intersection of the plane:

$2x+3y+10z-8=0$ & $2x-3y+7z-2=0$ is;

$(2x+3y+10z-8)+\lambda(2x-3y+7z-2)=0$ ----- (i)

or, $(2+2\lambda)x+(3-3\lambda)y+(10+7\lambda)y-(8+2\lambda)=0$

The plane (i) $\perp^r$ to the plane is

$3x-2y+4z-5=0$

or, $3(2+2\lambda)+(-2)(3-3\lambda)+4(10+7\lambda)=0$

or, $6+6\lambda-6+6\lambda+40+28\lambda=0$

or $40\lambda+40=0$

$\therefore \lambda=-1$

From eqⁿ (i)

$(2x+3y+10z-8)-1(2x-3y+7z-2)=0$

or $2x+3y+10z-8-2x+3y-7z+2=0$

$\therefore 2y+z-2=0$ Ans

④ Equation of planes bisecting the angle between the planes.

Let, the given two planes are:

$a_1x + b_1y + c_1z + d_1 = 0$ ---- (i)

$a_2x + b_2y + c_2z + d_2 = 0$ ---- (ii)

Eq^n of the planes bisector of the angle between the planes are:

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Case-1. Taking '+ve' sign

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \text{--- (iii)}$$

Case-2: Taking '-ve' sign

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = -\frac{a_2x + b_2y + c_2z}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \text{--- (iv)}$$

Exercise 2.5

Q. 1. Show that the origin lies in the ~~house~~ obtuse angle between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$. Find the bisector of the acute angle between the above two planes.

Sol^n.

Eq^n of planes are:

$2x - y + 2z + 3 = 0$ ---- (i)

$3x - 2y + 6z + 8 = 0$ ---- (ii)

Eq^n of bisectors of the angle bet^n the planes are:

$$\frac{2x-y+2z+3}{\sqrt{(2)^2+(-1)^2+(2)^2}} = \pm \frac{3x-2y+6z+8}{\sqrt{(3)^2+(2)^2+(6)^2}}$$

$$\frac{2x-y+2z+3}{3} = \pm \frac{3x-2y+6z+8}{7}$$

Taking '+ve' sign

$$\frac{2x-y+2z+3}{3} = \frac{3x-2y+6z+8}{7}$$

or $14x - 7y + 14z + 21 = 9x - 6y + 18z + 24$

or $5x - y - 4z - 3 = 0$

$\therefore$ $5x - y - 4z = 3$ ---- (iii)

Taking '-ve' sign

$$\frac{2x-y+2z+3}{3} = -\left(\frac{3x-2y+6z+8}{7}\right)$$

or, $$\frac{2x-y+2z+3}{3} = \frac{-3x+2y-6z-8}{7}$$

or $14x - 7y + 14z + 21 = -9x + 6y - 18y - 24$

or, $23x - 13y + 32z + 45 = 0$ ---- (iv)

For the bisector, taking plane (i) & (iii)
[Note: इसको bisector को लागि (i) & (iii)

$a_1 = 2,\ b_1 = -1\ \&\ c_1 = 2$
& $a_2 = 5,\ b_2 = -1\ \&\ c_2 = -4$

Now,
If 'θ' be the angle, then;

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

or $$\cos\theta = \frac{10 + 1 - 8}{\sqrt{4+1+4}\ \sqrt{25+1+16}}$$

or $$\cos\theta = \frac{3}{3\sqrt{42}}$$

$$\therefore \cos\theta = \frac{1}{\sqrt{42}} \left[= \frac{b}{h} \right]$$

So,
$$\tan\theta = \frac{\sqrt{42 - 1}}{1}$$

$$\therefore \tan\theta = \sqrt{41} > 1$$

So, the angle is obtuse angle.

(V) Pair of Plane

The eq^n of second degree $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ will represent a pairs of plane if

$abc+2fgh-af^2+bg^2\ ch^2=0$

If 'θ' be the angle between two plane, then:

$$\tan\theta=\frac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$$

Note: If $\theta=\frac{\pi}{2}$ then

$\therefore\ a+b+c=0$

Exercise 2.6

Q.1. Show $2x^2-y^2+2z^2-yz+5zx+xy=0$ represents the pair of plane and find angle between them.

Sol^n:-

The 2nd degree eq^n is;

$$2x^2-y^2+2z^2-yz+5zx+xy=0 \quad \text{--- (i)}$$

Comparing this eq^n with

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

We get,

$a=2,\quad b=-1,\quad c=2$

$2f=-1$, $2g=5$ & $2h=1$

$f=-1/2$, $g=5/2$, $h=1/2$

Now

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 2\cdot(-1)\cdot 2 + 2\cdot(-1/2)\cdot\frac{5}{2}\cdot\frac{1}{2} - 2\cdot(-1/2)^2 - 1(-5/2)^2 \cdot 2(1$$

$$\Rightarrow -4 - \frac{5}{4} - \frac{1}{2} + \frac{25}{4} - \frac{1}{2} = 0$$

$$\Rightarrow \frac{-4\times 4 - 5 - 2 + 25 - 4}{4} = 0$$

$$\Rightarrow -16 - 7 + 25 - 4 = 0$$

$$\Rightarrow -4 + 2\cdot(-1/2)\times\frac{5}{2}\times\frac{1}{2} - 2(-1/2)^2 - (-1)\left(\frac{5}{2}\right)^2 - 2(1/2)^2$$

$$\Rightarrow -4 - \frac{5}{4} + \frac{1}{2} + \frac{25}{4} - \frac{1}{2} = 0$$

$$\Rightarrow \frac{-4\times 4 - 5 + 25}{4}$$

Now,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 2\times(-1)\times 2 + 2\times(-1/2)\times 5/2 \times 1/2 - 2(-1/2)^2 - (-1)\times\left(\frac{5}{2}\right)^2 - 2(1/2)^2 = 0$$

$$\Rightarrow -4 - \frac{5}{4} - \frac{1}{2} + \frac{25}{4} - \frac{1}{2} = 0$$

$$\Rightarrow \frac{-16 - 5 - 2 + 25 - 2}{4} = 0$$

$$\Rightarrow -25 + 25 = 0$$

$$0 = 0$$

So, eq^n (1) represent the pair of plane.

Now, 'θ' be the angle between them;

$$\tan\theta = \frac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$$

$$= \frac{2\sqrt{(-1/2)^2+(5/2)^2+(1/2)^2+2+2-4}}{2-1+2}$$

$$= \frac{2\sqrt{\frac{1}{4}+\frac{25}{4}+\frac{1}{4}}}{3}$$

$$= \frac{2\sqrt{27}}{3\times 2}$$

$$= \frac{2\sqrt{3\times 3\times 3}}{3\times 2}$$

$$= \frac{\cancel{2}\times\cancel{3}\sqrt{3}}{\cancel{3}\times\cancel{2}}$$

$$= \sqrt{3}$$

i.e. $\tan\theta = \sqrt{3}$

$\tan\theta = \tan \pi/3$

$$\boxed{\theta = \pi/3}$$

Straight Line

The intersection of two planes gives a straight line. Let $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ be two planes then the intersection of these plane gives the equation of straight line.

* Equation of line through the point (x_1, y_1, z_1) having drs a,b,c is:

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \quad [\text{symmetrical form}]$$

* Eq^n of line joining two given points $A(x_1, y_1, z_1)$ & $B(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Exercise 3.1

Q.1 Find the value of 'k' such that the lines: $\frac{x-1}{2} = \frac{y-3}{4k} = \frac{z}{2}$ and $\frac{x-2}{2k} = \frac{y-1}{3} = \frac{z-1}{4}$ are perpendicular.

Sol^n:-

Given,

Eq^n of lines are:

$$\frac{x-1}{2} = \frac{y-3}{4k} = \frac{z}{2} \quad \text{----(i)}$$

& $\frac{x-2}{2k} = \frac{y-1}{3} = \frac{z-1}{4}$ .... (ii)

We get,

$a_1 = 2$ $\quad a_2 = 2k$

$b_1 = 4k$ $\quad b_2 = 3$

& $c_1 = 2$ $\quad c_2 = 4$

Since, the line (i) & (ii) are $\perp^r$ then,

$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$\Rightarrow 4k + 12k + 8 = 0$

$\Rightarrow 16k + 8 = 0$

$\therefore k = -1/2$ Ans

Q.2. Find the distance of the point (1, −3, 5) from the plane $3x - 2y + 6z = 15$ along a line with direction cosines proportional to (2, 1, −2).

Soln:-

The eqn of line passing through (1, −3, 5) having direction ratios (2, 1, −2) is

$$\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-5}{-2} = r \text{ (say)}$$

Then,

Any point on the line lies in the plane

$x = 2r + 1$

$y = r - 3$

$z = -2r + 5$

i.e. $(x, y, z) = (2r+1, r-3, -2r+5)$

then,

$\cancel{a(x-}$

$3(2r+1) - 2(r-3) + 6(-r+15) = 15$

or, $6r+3 - 2r + 6 - 6r + 130 = 15$

$2r = 15 - 30 - 9$

$r = 6/2$

$\therefore r = 3$

$\therefore (x, y, z) = (2r+1, r-3, -r+15)$

$= (2\times3+1, 3-3, -6+15)$

$(x, y, z) = (7, 0, -1)$

This is the required point.

Now,

The distance between joining the point $(1, -3, 5)$ & $(7, 0, -1)$ is;

$d = \sqrt{(7-1)^2 + (-3-0)^2 + (-1-5)^2}$

$= \sqrt{36+9+36}$

$= \sqrt{81}$

$\therefore d = 9$

Q.3. Find the points in which the line $\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2}$ cuts the surface $11x^2 - 5y^2 + z^2 = 0$.

Soln:

The given eqn of line is

$$\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2} = r \text{ (say)} \quad \text{----- (i)}$$

The given eq^n of surface is

$$11x^2 - 5y^2 + z^2 = 0$$

From eq^n (i)

$x+1 = -r$, $y-12 = 5r$ & $z-7 = 2r$

$x = -r-1$ $\quad$ $y = 5r+12$ $\quad$ $z = 2r+7$

As the eq^n(i) cuts the surface eq^n then,

$$11x^2 - 5y^2 + z^2 = 0$$

$$11(-r-1)^2 - 5(5r+12)^2 + (2r+7)^2 = 0$$

or, $11[(-r)^2 + 2r + 1] - 5[25r^2 + 120r + 144] + 4r^2 + 28r + 49 = 0$

or, $11r^2 + 22r + 11 - 125r^2 - 600r - 720 + 4r^2 + 28r + 49 = 0$

or, $-110r^2 - 550r - 660 = 0$

or, $r^2 + 5r + 6 = 0$

or $r^2 + 3r + 2r + 6 = 0$

or $r(r+3) + 2(r+3) = 0$

$(r+3)(r+2) = 0$

$\therefore r = -3$

& $r = -2$

Again, from eq^n(i), general points are:

When $r = -3$

$(x, y, z) = (-(-3)-1, 5(-3)+12, 2(-3)+7)$

$(x, y, z) = (2, -3, 1)$

& when $r = -2$

$(x, y, z) = (-(-2)-1, 5(-2)+12, 2(-2)+7)$

$\therefore (x, y, z) = (1, 2, 3)$

Q. 4 Find the point where the line joining $(1,-3,4)$ $(9,3,-1)$ cuts the plane $x-y+2z=3$.

Sol^n:-

The given two line joining points is;

$(x_1, y_1, z_1) = (1, -3, 4)$

$(x_2, y_2, z_2) = (9, 3, -1)$

Eq^n of line joining points ~~as~~:

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = r \text{ (say)} \quad \text{---} (i)$$

Substituting values of (x_1, y_1, z_1) & (x_2, y_2, z_2) in eq^n (i)

$$\frac{x-1}{9-1} = \frac{y-(-3)}{3-(-3)} = \frac{z-4}{-1-4} = r$$

$$\therefore \frac{x-1}{8} = \frac{y+3}{6} = \frac{z-4}{-5} = r$$

We get,

$x = 8r+1$, $y = 6r-3$ & $z = -5r+4$

The eq^n (i) cuts the plane $x-y+2z=3$

i.e.

$$x-y+2z=3$$

$$(8r+1)-(6r-3)+2(-5r+4)=3$$

$$8r+1-6r+3-10r+8=3$$

$$-8r+12=3$$

$$\therefore r = 9/8$$

So, general points are.

$x = 8\left(\frac{9}{8}\right)+1$ $\quad y = 6\times\frac{9}{8}-3$ $\quad z = -5\times\frac{9}{8}+4$

$= 9+1$ $\quad = \frac{15}{4}$ $\quad = -\frac{13}{8}$

$= 10$

$\therefore (x, y, z) = \left(10, \frac{15}{4}, -\frac{13}{8}\right)$

Q. 5 Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x-y+z=5$.

Soln.

Eqn of line is:

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = r \quad \text{--- (i)}$$

The general point on the line

$(x, y, z) = (3r+2, 4r-1, 12r+2)$

The eqn of line (i) intersects the plane. So, the point lies on the plane $x-y+z=5$ is

$3r+2-(4r-1)+12r+2=5$

or $3r+2-4r+1+12r+2=5$

or $11r+5=5$

or $11r=0$

$\therefore r=0$

So, The required point on the plane is

$(x, y, z) = (3\times0+2, 4\times0-1, 12\times0+2)$

$\therefore (x, y, z) = (2, -1, 2)$

Now, distance between the point $(-1,-5,-10)$ & $(2,-1,2)$ is

$$d = \sqrt{(2-(-1))^2 + (-1+5)^2 + (2+10)^2}$$

$$= \sqrt{3^2 + 4^2 + 12^2}$$

$$\therefore d = \sqrt{169} = 13$$

Q. 6 Find the two points on the line $\frac{x-2}{1} = \frac{y+3}{2} = \frac{z+5}{2}$ either side of $(2,-3,-5)$ and at a distance 3 from it.

Soln:-

Here, eqn of line is;

$$\frac{x-2}{1} = \frac{y+3}{2} = \frac{z+5}{2} = r \text{ (say)}$$

or, $x = r+2$ $\quad y = 2r-3$ $\quad$ & $z = 2r-5$

So, The general point of a line is;

$$(x, y, z) = (r+2, 2r-3, 2r-5)$$

Since,

Distance between the joining the points $(r+2, 2r-3, 2r-5)$ & $(2,-3,-5)$ is 3. So,

$$d = \sqrt{[2-(r+2)]^2 + [-3-(2r-3)]^2 + [-5-(2r-5)]^2}$$

$$3 = \sqrt{r^2 + 4r^2 + 4r^2}$$

$$3 = \pm 3r$$

$$\therefore r = \pm 1$$

The point is at '+1' is
(x, y, z) is $(3, -1, 3)$ &
at '-1'
(x, y, z) is $(1, -5, -7)$

Q. 7. Find the distance of the point $(1, -2, 3)$ from the plane $x - y + z = 5$ measured parallel to $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

Sol^n:-

The eq^n of line through the point $(1, -2, 3)$ & parallel to the line

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{-6} \text{ is}$$

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r \text{ (say)} \quad \text{------ (i)}$$

The general point on the line
$(x, y, z) = (2r+1, 3r-2, -6r+3)$

The line meets the plane. So
$x - y + 5 = 5$

$2r + 1 - (3r - 2) + (-6r + 3) = 5$

or $2r + 1 - 3r + 2 - 6r + 3 = 5$

or $-7r + 6 = 5$

$-7r = -1$

$\therefore r = 1/7$

Now, general point is

$(x, y, z) = \left(2 \times \frac{1}{7} + 1,\ 3 \times \frac{1}{7} - 2,\ -6 \times \frac{1}{7} + 3\right)$

$(x, y, z) = \left(\frac{9}{7},\ \frac{-11}{7},\ \frac{15}{7}\right)$

Distance betn $(1, -2, 3)$ & $\left(9/7, -11/7, 15/7\right)$ is

$$d = \sqrt{\left(\frac{9}{7} - 1\right)^2 + \left(-\frac{11}{7} + 2\right)^2 + \left(\frac{15}{7} - 3\right)^2}$$

$$d = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}}$$

$$d = \sqrt{\frac{4 + 9 + 36}{49}}$$

$d = 1$

Q.8 Find the equation to the line passing through $(-1, -2, -3)$ and the perpendicular to each of the lines

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \text{ and } \frac{x+2}{4} = \frac{y+3}{5} = \frac{z+4}{6}$$

Solⁿ:-

The eqⁿ of line passing through the point $(-1, -2, -3)$ having drs 'a', 'b', 'c'.

$$\frac{x+1}{a} = \frac{y+2}{b} = \frac{z+3}{c} \quad \text{--- (i)}$$

Since,
The line (i) $\perp^r$. to the lines,

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \text{ --- (ii)}$$

and

$$\frac{x+2}{4} = \frac{y+3}{5} = \frac{z+4}{6} \text{ ---- (iii)}$$

Then,
From (i) and (ii)
$3a + 4b + 5c = 0$ --- (iv)

From (i) and (iii)
$4a + 5b + 6c = 0$ --- (v)

On solving (iv) and (v) by cross multiplication method:

$$\frac{a}{24-25} = \frac{b}{20-18} = \frac{c}{15-16} = r \text{ (say)}$$

or $$\frac{a}{-1} = \frac{b}{2} = \frac{c}{-1} = r$$

$\Rightarrow a = -r, \quad b = 2r \quad \& \quad c = -r$

Eq^n (i) becomes

$$\frac{x+1}{-r} = \frac{y+2}{2r} = \frac{z+3}{-r}$$

$$\therefore \frac{x+1}{-1} = \frac{y+2}{2} = \frac{z+3}{-1}$$

This is the required eq^n of line.

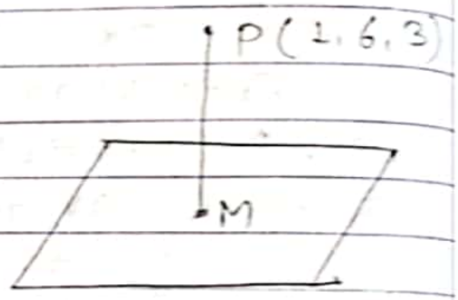
Q.9 Show that the equation of the perpendicular from the point (1,6,3) to the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ is $\frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}$ and the foot of perpendicular is (1,3,5) and the length of the perpendicular is $\sqrt{13}$.

Soln:-

Here,

Eqn of line

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}=r \text{ (say)}$$



The general point on the line

$M(x,y,z)=(r,2r+1,3r+2)$

The direction ratio of the line joining the points (1,6,3) and $(r, 2r+1, 3r+2)$

$$=(x_2-x_1, y_2-y_1, z_2-z_1)$$
$$=(r-1, 2r+1-6, 3r+2-3)$$
$$(a,b,c)=(r-1, 2r-5, 3r-1)$$

Now,

The line PM perpendicular to the line (i). So,

$$a_1a_2+b_1b_2+c_1c_2=0$$
$$\text{or } 1(r-1)+2(2r-5)+3(3r-1)=0$$
$$\text{or } r-1+4r-10+9r-3=0$$
$$\text{or } 14r-14=0$$
$$\therefore r=1$$

The drs of line(a, b, c) = $(1-1, 2-5, 3-1)$

$= (0, -3, 2)$

The eq^n of line through $(1, 6, 3)$ having drs ratio $(0, -3, 2)$ is;

$$\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$$

The foot of the perpendicular is

$= (r, 2r+1, 3r+2)$

$= (1, 2\times1+1, 3\times1+2)$

$= (1, 3, 5)$

Also,

The distance between joining the points $(1, 6, 3)$ and $(1, 3, 5)$ is,

$d = \sqrt{(1-1)^2 + (6-3)^2 + (3-5)^2}$

$= \sqrt{0+9+4}$

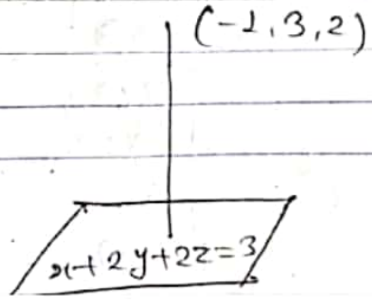
$\therefore d = \sqrt{13}$

Q. 10 Find the equation of the line through $(-1, 3, 2)$ and perpendicular to the plane $x+2y+2z=3$, the length of the perpendicular and the co-ordinate of its foot.

Sol^n:-

The eq^n of line through point $(-1, 3, 2)$ perpen-dicular to the plane $x+2y+2z=3$ is;

$$\frac{x+1}{1} = \frac{y-3}{2} = \frac{z-2}{2} = r \quad \text{---- (i)}$$



The general point on the line is;

$(x,y,z) = (r-1, 2r+3, 2r+2)$

Since,

The line (i) touches the plane $x+2y+2z=3$ Then,

$$x+2y+2z=3$$
$$\text{or } (r-1)+2(2r+3)+2(2r+2)=3$$
$$\text{or } r-1+4r+6+4r+4=3$$
$$\text{or } 9r+9=3$$
$$\text{or } 9r = 3-9$$
$$\text{or } r = \frac{-6}{9}$$
$$\therefore r = -2/3$$

So, the general point is;

$$(x,y,z) = (r-1, 2r+3, 2r+2)$$
$$= \left(-\frac{2}{3}-1, 2\times\left(-\frac{2}{3}\right)+3, 2\left(-\frac{2}{3}\right)+2\right)$$
$$(x,y,z) = \left(-\frac{5}{3}, \frac{5}{3}, \frac{2}{3}\right)$$

The distance of $\perp^r$ to the point $(-5/3, 5/3, 2/3)$ and $(-1,3,2)$ is;

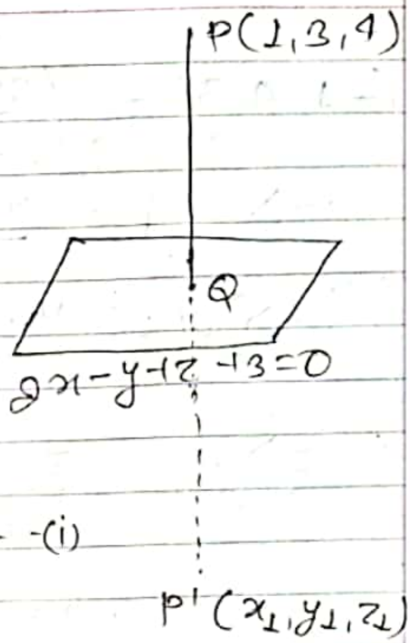
$$d = \sqrt{\left(-\frac{5}{3}+1\right)^2+\left(\frac{5}{3}-3\right)^2+\left(\frac{2}{3}-2\right)^2}$$
$$= \sqrt{\left(-\frac{2}{3}\right)^2+(-4/3)^2+(-4/3)^2}$$
$$= \sqrt{\frac{36}{9}} \quad 4$$
$$\text{or } d = \sqrt{4}$$
$$\therefore d = 2$$

Q.11 Find the image of the point $P(1,3,4)$ in the plane $2x-y+z+3=0$.

Sol^n:-

Let,

The image point of $P(1,3,4)$ is $P'(x_1,y_1,z_1)$



The eq^n of line through the $(1,3,4)$ and $\perp^r$ to the plane $2x-y+z+3=0$ is;

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=r\text{ (say)} \quad \text{---(i)}$$

General point on the line is

$(x,y,z)=(2r+1,\ -r+3,\ r+4)$

Since,

The line (i) touches the plane $2x-y+z+3=0$

So,

$2(2r+1)-(-r+3)+(r+4)+3=0$

or, $4r+2+r-3+r+4+3=0$

or, $6r+6=0$

$\therefore\ r=-1$

So, general point is

$(x,y,z)=(2\times(-1)+1,\ -(-1)+3,\ (-1)+4)$

$(x,y,z)=(-1,4,3)$

This is the mid-point of PP'

Then, by mid-point formula:-

$$(x, y, z) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

$$(-1, 4, 3) = \left(\frac{x_1+1}{2}, \frac{y_1+3}{2}, \frac{z_1+4}{2}\right)$$

$\Rightarrow -1 = \frac{x_1+1}{2} \Rightarrow x_1 = -3$

$\Rightarrow 4 = \frac{y_1+3}{2} \Rightarrow y_1 = 5$

$\Rightarrow 3 = \frac{z_1+4}{2} \Rightarrow z_1 = 2$

$\therefore$ The required image point is:
$(-3, 5, 2)$ A-Q

(*) Transformation of Equation of line from general form to Symmetrical form

Exercise 3.2

Q. 1 Change the equation $x+y+z+1=0=4x+y-2z+2$ in symmetrical form.

Sol:-

Given,
The eq^n of line

$$x+y+z+1=0=4x+y-2z+2 \quad \text{---(i)}$$

Put $z=0$,

$x+y+1=0$ and $4x+y+2=0$

$\Rightarrow x+y=-1$ ---(ii) $\quad \Rightarrow 4x+y=-2$ ---(iii)

Solving (ii) & (iii)

$$\begin{array}{r} x+y=-1 \\ 4x+y=-2 \\ \underline{-\quad - \quad +} \\ -3x=+1 \end{array}$$

$\therefore x = 1/3$

& $x+y=-1$

$\frac{1}{3}+y=-1$

$y=-1-1/3$

$\therefore y=-2/3$

$\therefore$ the point on the line is $(-1/3, -2/3, 0)$

Let, 'a', 'b', 'c' be the drs of the line. Then;

$1a+1b+1c=0$

$4a+1b-2c=0$

$$\frac{a}{-2-1}=\frac{b}{4-(-2)}=\frac{c}{1-4}=k$$

$$\text{or } \frac{a}{-3}=\frac{b}{6}=\frac{c}{-3}=k$$

$\Rightarrow a=k, \quad b=-2k \quad c=k$

Now,

The eq^n of line through the point $\left(-\frac{1}{3}, -\frac{2}{3}, 0\right)$ having drs $k, -2k, k$ is

$$\frac{x-(-1/3)}{k} = \frac{y+2/3}{-2k} = \frac{z}{k}$$

$$\therefore \frac{x+1/3}{1} = \frac{y+2/3}{-2} = \frac{z}{}$$

Q.3 Find the equation of the line through $(2,3,4)$ parallel to line $x-2y+z=4$, $4x+3y-z+4=0$

Sol^n:-

Eq^n of line passing through the point $(2,3,4)$ and having drs a, b, c is

$$\frac{x-2}{a} = \frac{y-3}{b} = \frac{z-4}{c} \quad \text{--- (i)}$$

The line (i) parallel to the given line $x-2y+z=4$, $4x+3y-z+4=0$

Then, drs of line (i) satisfies the condition of perpendicularity with given line

$$1a - 2b + 1c = 0$$
$$4a + 3b - 1c = 0$$

$$\frac{a}{2-3} = \frac{b}{4-(-1)} = \frac{c}{3-(-8)}$$

$$\frac{a}{-1} = \frac{b}{5} = \frac{c}{11} = k$$

$\Rightarrow a = -k, \quad b = 5k, \quad c = 11k$

Now,

$$\frac{x-2}{-k} = \frac{y-3}{5k} = \frac{z-4}{11k}$$

$$\therefore \frac{x-2}{-1} = \frac{y-3}{5} = \frac{z-4}{11}$$

Q.5 Prove that the lines $x = ay+b$, $z = cy+d$ and $x = a'y+b'$, $z = c'y+d'$ are perpendicular if $aa'+cc'+1=0$.

Sol^n:-

Eq^n of lines are

$x = ay+b$

$\Rightarrow y = \frac{x-b}{a}$

$z = cy+d$

$\Rightarrow y = \frac{z-d}{c}$.

Therefore,

$$\frac{x-b}{a} = y = \frac{z-d}{c}$$

$$\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c} \quad \text{---(i)}$$

Again, given,

$x = a'y + b'$

$\Rightarrow y = \frac{x - b'}{a'}$

$z = c' + d'$

$\Rightarrow y = \frac{z - d'}{c'}$

Therefore,

$$\frac{x - b'}{a'} = y = \frac{z - d'}{c'}$$

$$\therefore \frac{x - b'}{a'} = \frac{y - 0}{1} = \frac{z - d'}{c'} \quad \text{------ (ii)}$$

Since,

line (i) & (ii) perpendicular to each other.

If $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$aa' + 1 \cdot 1 + cc' = 0$

$\therefore aa' + cc' + 1 = 0$

Hence proved

Q. 6 Prove that the lines $x = -2y + 7$, $z = 3y + 10$ and $x = 5y - 1$, $z = 3y - 6$ are perpendicular to each other.

Sol^n:-

Eq^n of line are:

$x = -2y+7 \Rightarrow y = \frac{x-7}{-2}$ --- (i)

$z = 3y+10 \Rightarrow y = \frac{z-10}{3}$ --- (ii)

From (i) & (ii)

$$\frac{x-7}{-2} = y = \frac{z-10}{3}$$

or $$\frac{x-7}{-2} = \frac{y-0}{1} = \frac{z-10}{3}$$

&

$x = 5y-1 \quad y = \frac{x+1}{5}$ --- (iii)

$z = 3y-6 \Rightarrow y = \frac{z+6}{3}$ --- (iv)

From (iii) & (iv)

$$\frac{x+1}{5} = y = \frac{z+6}{3}$$

$$\frac{x+1}{5} = \frac{y-0}{1} = \frac{z+6}{3}$$

If the lines are perpendicular then,

$$-2 \times 5 + 1 \times 1 + 3 \times 3 = 0$$

or $$-10 + 1 + 9 = 0$$

$$\therefore 0 = 0$$

This proves that both lines are $\perp^r$.

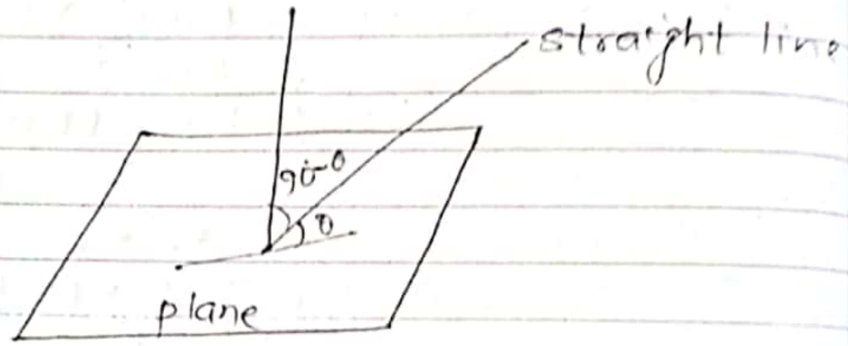
* Angle between line and plane

let $ax+by+cz+d=0$ be a plane and
$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ be a line.

If 'θ' be the angle between them.

Then,

$$\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}=\frac{al+bm+cn}{\sqrt{\Sigma a^2}\sqrt{\Sigma l^2}}$$



Note: 1) The given line is parallel to the given plane. Then
$al+bm+cn=0$.

2) If the line is perpendicular to the plane. Then, it will be parallel to the normal of the plane and

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$$

* Condition for the line to lie on the plane

Let $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ---- (i) be a line

and $ax+by+cz+d=0$ be a plane ---(ii)

If the line (i) lies on the plane (ii). Then,

$$al + bm + cn = 0$$

& $ax_1 + by_1 + cz_1 + d = 0$

Note:

1) The line is parallel to the plane

$$al + bm + cn = 0$$

2) The point (x_1, y_1, z_1) of the line lies on the plane $ax_1 + by_1 + cz_1 + d = 0$.

If the eq^n of the line is;

$$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$$

Then,

The eq^n of plane containing this line is

$$a_1x + b_1y + c_1z + d_1 + k(a_2x + b_2y + c_2z + d_2) = 0$$

$$\Rightarrow (a_1 + a_2k)x + (b_1 + kb_2)y + (c_1 + kc_2)z + (d_1 + kd_2) = 0$$

Exercise 3.3

Q.1 Find the value of 'k' such that the line $\frac{x-2}{2} = \frac{y+3}{5} = \frac{z-5}{k}$ is parallel to the plane $2x - 3y + z = 3$.

Sol^n:-

Here,

Eq^n of line is

$$\frac{x-2}{2} = \frac{y+3}{5} = \frac{z-5}{k} \text{ ---- (i)}$$

& eq^n of plane is

$$2x - 3y + z = 3 \text{ ---- (ii)}$$

The line (i) is parallel to the plane (ii).
Then,

$al + bm + cn = 0$

$\Rightarrow 2\times 2 + 5(-3) + k\cdot 1 = 0$

$\Rightarrow 4 - 15 + k = 0$

$\therefore k = 11$

Q.3. Find the eq^n of a plane containing the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and is perpendicular to the plane $x+2y+z=12$.

Sol^n:

Eq^n of line is

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4} \quad \text{---(i)}$$

The eq^n of plane containing the line 1st is

$a(x-1) + b(y+1) + c(z-3) = 0$ ----(ii)

Since,

Direction ratios of the line normal to the line.
So, it is also normal to the plane (ii).

$a\times 2 + b(-1) + c\times 4 = 0$

$2a - b + 4c = 0$ ---(iii)

Plane (ii) is perpendicular to the plane $x+2y+z=12$. Then

$1\times a + 2\times b + 1\times c = 0$

$a + 2b + c = 0$ ---(iv)

Solving eq^n(iii) & (iv) by cross multiplication method.

$$\frac{a}{-1-8} = \frac{b}{4-2} = \frac{c}{4+1} = k \text{ (say)}$$

$$\frac{a}{-9} = \frac{b}{2} = \frac{c}{5} = k$$

$\therefore a = -9k, \quad b = 2k \quad \& \quad c = 5k$

So,

eq^n(i) becomes :-

$-9k(x-1) + 2k(y+1) + 5k(z-3) = 0$

or $-9x + 9 + 2 + 2y + 5z - 15 = 0$

or $-9x + 2y + 5z - 4 = 0$

$\therefore 9x - 2y - 5z + 4 = 0$

Q.4 Find the eq^n of the plane through $(2, -3, 1)$ normal to the joining $(3, 4, -1)$ and $(2, -1, 5)$.

Sol^n:-

The eq^n of plane through the point $(2, -3, 1)$ is

$a(x-2) + b(y+3) + c(z-1) = 0$ ---- (i)

The eq^n of line through joining the points $(3, 4, -1)$ and $(2, -1, 5)$ is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

or $$\frac{x-3}{2-3} = \frac{y-4}{-1-4} = \frac{z+1}{5+1}$$

or, $\frac{x-3}{-1} = \frac{y-4}{-5} = \frac{z+1}{6}$

$\therefore \frac{x-3}{1} = \frac{y-4}{5} = \frac{z-1}{-6}$ ------ (ii)

If the line (ii) is perpendicular to plane (i) then it is also parallel to the normal of the plane. Then,

$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$

$\frac{1}{a} = \frac{5}{b} = \frac{-6}{c} = k$

$a = k$, $b = 5k$ & $c = -6k$

From (i)

$x - 2 + 5y + 15 - 6z + 6 = 0$

$\therefore x + 5y - 6z + 19 = 0$

Ans

Coplanar Lines

The eq^n of two straight lines are

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad \text{--- (i)}$$

and $$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad \text{--- (ii)}$$

The line (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

The eq^n of plane containing the line (i) & (ii) is

$$\begin{vmatrix} x-x & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

OR

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Note:

(i) If two lines are not in symmetrical then we change them to symmetrical form.

(ii) If the line intercept each other then they will be coplanar.

Exercise 3.4

Chap-1: matrix & System of linear Equn,

Let A

order of matrix:

rows (say m)

columns (say n)

$= m \times n$

A unique real number associated with

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45-48) - 2(36-42) + 3(32-35)$$

$$= -3 + 12 - 9 = 0$$

A squre matrix A is said to be a singular matrix if $|A| = 0$ and A squre matrix is said to be non singular matrix if $|A| \neq 0$

Example.

① Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then $|A| = |$

System of linear equations.

two variables

$$\left.\begin{array}{l} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{array}\right\}$$

Coeff matrix

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

~~Three variable~~

a_1 Augmented Matrix

$$\begin{bmatrix} a_1 & b_1 & : & c_1 \\ a_2 & b_2 & : & c_2 \end{bmatrix}$$

A system of linear eqn is said to be

(1) Consistent and independent.

If the eqn has only one solution

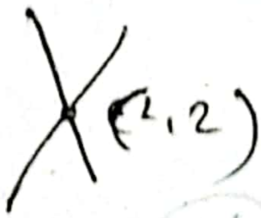
eg:

$$x+y=3$$
$$3x-y=1$$

$$4x=4$$

$$x=1 \qquad (x,y)=(1,2)$$

$$y=2.$$



(11) Consistent & dependendent

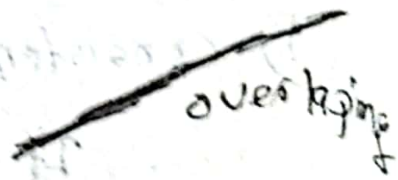
→ A system of linear eqn is said to be..

If the system have infinitely many solution.

eqn $\left.\begin{array}{l} x+y=3 \\ 2x+2y=6 \end{array}\right\} \times 2$

$$2x + 2y = 6$$
$$2x \pm 2y = 6$$
$$0 = 0$$

fig:



$$y = k \text{ (say)}$$

$$x + y = 3$$

$$x = 3 - y$$

$$x = 3 - k$$
$$y = k$$

(no-solution) → no intersection

(III) Inconsistent and Independent

→ A System of linear eqn if The system has no solution

$$\left.\begin{aligned} x + y &= 3 \\ 2x + 2y &= 5 \end{aligned}\right\} \times 2$$

$$2x+2y=6$$
$$2x+2y=5$$

$$6=1$$

Elementary row operation on matrix

(1) replace row by the sum of itself and multiple of another row.

(2) Interchange two rows

(3) multiply all the entry in a row by a non-zero-constant

9) for unique solution

Exercise 1.2

1.

a) $\left[\begin{array}{cc:c} 2 & 3 & h \\ 4 & 6 & 7 \end{array}\right]$

$R_2 \to R_2 - 2R_1$

$\left[\begin{array}{cc:c} 2 & 3 & \\ 0 & 0 & 7-2h \end{array}\right]$

$7 - 2h = 0$

$h = 7/2$

b) $\left[\begin{array}{cc:c} 1 & -2 & 3 \\ 3 & h & -2 \end{array}\right]$

$h + 3\times 2$

$h+6$

Soln:

$R_2 \to R_2 - 3R_1$

$-2 - 3\times(3)$

$-2 - 9$

-11

$\left[\begin{array}{cc:c} 1 & -2 & 3 \\ 0 & h+6 & -2-9 \end{array}\right]$

Exercise 1.2

1.

a)
$$\left[\begin{array}{cc:c} 2 & 3 & h \\ 4 & 6 & 7 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{cc:c} 2 & 3 & h \\ 0 & 0 & 7-2h \end{array}\right]$$

$7-2h=0$

$h = 7/2$

b)
$$\left[\begin{array}{cc:c} 1 & -2 & 3 \\ 3 & h & -2 \end{array}\right]$$

$h+3\times 2$

$h+6$

$-2-3\times(3)$

$-2-9$

-11

Soln:

$R_2 \to R_2 - 3R_1$

$$\left[\begin{array}{cc:c} 1 & -2 & 3 \\ 0 & h+6 & -2-9 \end{array}\right]$$

$$\left[\begin{array}{cc:c} 1 & -2 & 3 \\ 0 & 6+h & -11 \end{array}\right]$$

$$6+h \neq -11$$

$$h \neq -6$$

$$h = \cancel{6+11}$$

$$= \cancel{17}$$

2 a) The system of linear eqn is

02

$$x_1 + hx_2 = 2$$

$$4x_1 + 8x_2 = k$$

The augmented material of the system of linear eqn is:

$$\left[\begin{array}{cc:c} 1 & h & 2 \\ 4 & 8 & k \end{array}\right]$$

$R_2 \to R_2 - 4R_1$

$$\left[\begin{array}{cc:c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right]$$

(i) The system of linear eqn has no solution.

$8-4h = 0$ & $k-8 \neq 0$

$h = 2$ & $k \neq 8$

(ii) The system of linear has unique solution

If $8-4h \neq 0$ But $k-8$ note It may zero or not

$h \neq 2$

iii) If $8-4h=0$ & $k-8=0$

$h=2$ & $k=8$

3a) $\left[\begin{array}{ccc:c} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{array}\right]$

$R_2 \to R_2 - 3R_1$

$\left[\begin{array}{ccc:c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right]$

$-5z = -15$

$z = 3$

y is free

Or, $x+3y+4z=7$

$x+3y+4\times 3=7$

Or, $x+3y=7-12$

$x+3y=-5$

$x=-5-3y$

$x=-5-3y$

y is free

$z=3$

4a)

The system of linear eqn is

$2x+3y+4z=20$

$3x+4y+5z=26$

$3x+5y+6z=31$

The augmented matrix of the system of linear eqn is:

$$2x+3y+4z=20$$

$$3x+4y+5z=26$$

$$3x+5y+6z=31$$

$$\left[\begin{array}{ccc:c} 2 & 3 & 4 & 20 \\ 3 & 4 & 5 & 26 \\ 3 & 5 & 6 & 31 \end{array}\right]$$

$R_1 \to \frac{1}{2} R_1$

$$\left[\begin{array}{ccc:c} 1 & 3/2 & 2 & 10 \\ 3 & 4 & 5 & 26 \\ 3 & 5 & 6 & 31 \end{array}\right]$$

$R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc:c} 1 & 3/2 & 2 & 10 \\ 0 & -1/2 & -1 & 4 \\ 0 & 1/2 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc:c} 1 & 3/2 & 2 & 10 \\ 0 & -1/2 & -1 & -4 \\ 0 & 0 & -1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 3/2 & 2 & 10 \\ 0 & 1/2 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x + \frac{3}{2}y + 2z = 10$ — (I)

$1/2\, y + z = 4$ — (II)

$z = 3$ — (III)

~~put~~ from (III) & (II)

$\frac{1}{2}y = 4 - 3$

$= 1$

$y = 2$ — (IV)

from (iii) (ii) & (i)

$$x + \frac{3}{\cancel{2}} \times \cancel{2} + 2 \times 3 = 10$$

$$x + 3 + 6 = 10$$

$$x = 10 - 9$$
$$= 1$$

5a)

$$x_1 + 3x_3 = 2$$

$$x_2 - 3x_4 = 3$$

$$-2x_2 + 3x_3 + 2x_4 = 1$$

$$3x_1 + 7x_4 = -5$$

$$\left[\begin{array}{cccc:c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{array}\right]$$

$R_4 \to R_4 - 3R_1$ $\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & 11 \end{array}\right]$

$R_4 \to R_4 + 3R_3$

The system of linear eqn of three variables

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$x = \frac{D_1}{D}$$

$$y = \frac{D_2}{D}$$

$$D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$z = D_3/D$$

$$D \neq 0$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} 3 & -13 & 8 \\ 2 & -5 & 9 \\ -4 & 2 & -26 \end{vmatrix} = 0$$

$$D_3 = 101$$

$$x = \cancel{7} \ \frac{D_1}{D} = -7$$

$$y = 0$$

$$z = 1$$

Rank of a matrix

→ The no. of non-zero present in the matrix echecon form is also known as rank of matrix denoted by (ρ)

Properties of Rank of a matrix

(i) only rank of Null matrix is zero.
If A is Null matrix then $\rho(A) = 0$

(2) Rank of matrix $A_{m \times n}$ is

$$\rho(A_{m\times n}) \leq \min(m, n)$$

3. $\rho(A \cdot B) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$

2.A

Let

$$A = \begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$$

$R_1 \to R_1 + 4R_2$

$$\begin{bmatrix} 0 & 0 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$A = \begin{bmatrix} 0 & 0 \\ -2 & 1 \\ 0 & 0 \end{bmatrix}$$

This shoae that A has $\{R_2\}$ as independent rows of a matrix

$\rho(A) = 1$

(9)

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix}$$

$R_3 \to R_3 + R_1$

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ 0 & 5 & 8 \\ 1 & 2 & 4 \end{bmatrix}$$

$R_3 \to R_3 - R_2 \Rightarrow \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$

$R_1 \to R_1 - 3R_4$

$$A = \begin{bmatrix} 0 & -5 & -8 \\ 0 & 5 & 8 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$A = \begin{bmatrix} 0 & -5 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

This show that A has $\{R_1, R_4\}$ as independent row of matrix

$\therefore \rho(A) = 2$

Maths

Consistency of system of linear equations

Let the system of linear equations be

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{ --- } (1)$$

The coefficient matrix is

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_3 \\ a_2 & b_3 & c_3 \end{bmatrix} \text{ --- } (2)$$

and the augmented matrix in $[A:B]$

$$= \begin{bmatrix} a_1 & b_1 & c_1 & : & d_1 \\ a_2 & b_2 & c_2 & : & d_2 \\ a_3 & b_3 & c_3 & : & d_3 \end{bmatrix} \text{ --- } (3)$$

Then the system (1) is called consistent if Rank A = Rank of $[A:B]$

i.e. Rank of coeff. matrix = Rank of Augmented Matrix

otherwise the system is called inconsistent.

Note)

1) The system ① is consistent and has a unique solution if Rank of A = Rank of
(A : B] = no. of variables

2) The system ① is consistent and has infinite solution if rank of A
= rank of of [A : B) ≠ no. of variables

Consistency of system of linear equations.

solution of system of linear eqn by
Gauss elimination method:

Q.N 1

E.x. 1.3

(1) $6x+4y=2$

$3x-5y=-34$

The Augmented matrix of the system of linear equation is

$$[A:B] = \left[\begin{array}{cc:c} 6 & 4 & 2 \\ 3 & -5 & -34 \end{array}\right]$$

$-10-4$

$-34\times2-2$

$-68-2$

Applying $R_2 \to 2R_2 - R_1$

$$\left[\begin{array}{cc:c} 6 & 4 & 2 \\ 0 & -14 & -70 \end{array}\right]$$

$\therefore$ Rank of $A = 2$ & Rank of $[A:B] = 2$

$\therefore$ Rank of $A =$ Rank of $[A:B]$

So, solution is consistent.

The corresponding eqn are:

$$6x+4y=2 \quad \text{— (I)}$$
$$-14y=-70 \quad \text{— (II)}$$

From (II)

$$-14y=-70$$
$$y=5$$

putting value of y in eqn (I)

$$x=-3$$

(3) The system of linear equation is

(i)

$$x-2y+3z=11 \quad \text{— (I)}$$
$$3x+y-z=2 \quad \text{— (II)}$$
$$5x+3y+2z=3 \quad \text{— (III)}$$

The Augmented matrix of system of linear equation is

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & = 11 \\ 3 & 1 & -1 & = 2 \\ 5 & 3 & 2 & = 3 \end{bmatrix}$$

=

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & : & 11 \\ 3 & 1 & -1 & : & 2 \\ 5 & 3 & 2 & : & 3 \end{bmatrix}$$

-9

$R_2 \longrightarrow 3R_2 - 3R_1$ $\quad 2 - 33$

&

$R_3 \longrightarrow R_3 - 5R_1$ $\quad 3 + 10$ $\quad 2 - -13$

we get

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & : & 11 \\ 0 & 7 & -10 & : & -31 \\ 0 & 13 & -13 & : & 52 \end{bmatrix}$$

$R_3 \to 7R_3 - 13R_2$

$$[A:B] = \begin{bmatrix} 1 & -2 & 3 & : & 11 \\ 0 & 7 & -10 & : & -31 \\ 0 & 0 & 39 & : & 39 \end{bmatrix}$$

$\therefore$ Rank of A = Rank of $[A:B] = 3$

Then the corresponding eqn are

$x - 2y + 3z = 11$ — (I)

$7y - 10z = -31$ — (II)

$39z = 39$ — (III)

From (III) $39z = 39 = z = 1$

From (II) $y = -3$

From (I) $x = 2$

$\therefore x = 2$

$y = -3$

$z = 1$

(10)

The system of linear eqn are.

$x + y + z = 3$ — (I)

$x + 2y + 3z = 4$ — (II)

$2x + 3y + 4z = 9$ — (III)

Agumented matrix of given system of linear equation is

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 9 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 2 & 3 & 4 & 9 \end{array}\right]$$

$R_3 \rightarrow R_3 - 2R_1$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

$R_3 \rightarrow R_2 - R_3$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

Rank of $A = 2$ & Rank of $[A:B] = 3$

$\therefore$ Rank $\neq$ equal

In consistent:

7

5

Determine the value p and q for which the system of eqn:

$$x-2y+2z=2 \quad \text{— (I)}$$
$$x+2y+3z=-1 \quad \text{— (II)}$$
$$2x+2y+pz=9 \quad \text{— (III)}$$

a) unique solution (b) no solution (c) infinite solution

Agumented matrix of given matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 1 & 2 & 3 & -1 \\ 2 & 2 & p & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 1 \\ 0 & 4 & 1 & -2 \\ 0 & 0 & p-5 & q-1 \end{array}\right]$$

a) for unique solution

if rank of $A =$ rank of $[A:B] = 3$

$p-5 \neq 0$ & $q-1 \neq 0$

$\rightarrow p \neq 5$ & $q \neq 1$

b) For no solution

$p-5=0$ & $q-1 \neq 0$

$p=5$ & $q \neq 1$

c) For infinitely many solution.

if, $p-5=0$ & $q-1=0$

$p=5$ & $q=1$

Vector space.

Vector-sub space

Let V be the vectorspace in the field F.
a non empty subset W of V is said to be sub space of V for all $u, v \in W$ and a, b belongs $\in F$ then.

$$au + bv \in W$$

2.1

(a)

Here,

$V = R^2$

$W = \{(x, y) : x + 2y = 0\}$

Let $u = (x_1 y_1)$ & $v = (x_2 y_2)$
be element of W
& $a, b \in F$ then

$x_1 + 2y_1 = 0$ || $x_2 + 2y_2 = 0$

Now,

$$au + bv = a(x_1, y_1) + b(x_2 + y_2)$$
$$= (ax_1, ay_1) + (bx_2 + by_2)$$
$$= \underbrace{ax_1 + bx_2}_{x} + \underbrace{ay_1 + by_2}_{y}$$
$$= ax_1 + bx_2 + 2(ay_1 + by_2)$$
$$= ax_1 + bx_2 + 2ay_1 + 2by_2$$
$$= a(x_1 + 2y_1) + b(x_2 + 2y_2)$$
$$= a \times 0 + b \times 0$$

There $au+bv \in W$ is a subspace of v

1

b) Son:

Here

$V = R3$

$W = \{(x, y, z) : x+2y+z = 0\}$

Let $u = (x_1, y_1, z_1)$ & $v = (x_2, y_2, z_2)$

Then

$x_1 + 2y_1 + z_1 = 0$ & $x_2 + 2y_2 + z_2 = 0$

Let

$a, b \in F$

Now, $au + bv = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$

$= ax_1 + ay_1 + az_1 + bx_2 + by_2 + bz_2$

$= \underbrace{ax_1 + bx_2}_{x} + \underbrace{ay_1 + by_2}_{y} + \underbrace{az_1 + bz_2}_{z}$

$= ax_1 + bx_2 + 2ay_1 + 2by_2 + az_1 + bz_2$

$= \underbrace{ax_1 + 2ay_1 + az_1}_{0} + bx_2 + 2by_2 + bz_2$

$= a(x_1 + 2y_1 + z_1) + b(x_2 + 2y_2 + z_2)$

$= 0 + 0$

There for $au + bv \in W$

W is a subspace of $V = R^3$

e) Here $V = R^3$ is a vector space

and

$W = \{(0, y, z) : y, z \in R\}$

Let $u = (0, y_1, z_1)$ and $v = (0, y_2, z_2)$

; $y_1, z_1, y_2, z_2 \in R$

& $a, b \in F$

$au + bv = a(0, y_1, z_1) + b(0, y_2, z_2)$

$= (0, ay_1, az_1) + (0, by_2, bz_2)$

$= 0, ay_1 + by_2, az_1 + bz_2$

$\therefore$ $au + bv \in W$ when

$ay_1 + by_2$ az_1 $bz_2 \in R$

Hence

W is a subspace of $V = R^3$

Vectors

f) we have
$V =$ set of all 2×2 matrices
and
$$W = \left\{ \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} : b, c, d \in R \right\}$$

Let $u = \begin{pmatrix} 0 & b_1 \\ c_1 & d_1 \end{pmatrix}$ & $v = \begin{pmatrix} 0 & b_2 \\ c_2 & d_2 \end{pmatrix} \in W$

where, $b_1, c_1, d_1, b_2, c_2, d_2 \in R$

and $a, b \in F$

Now,
$$au + bv = a\begin{pmatrix} 0 & b_1 \\ c_1 & d_1 \end{pmatrix} + b\begin{pmatrix} 0 & b_2 \\ c_2 & d_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & ab_1 \\ ac_1 & ad_1 \end{pmatrix} + \begin{pmatrix} 0 & bb_2 \\ bc_2 & bd_2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & ab_1 + bb_2 \\ ac_1 + bc_2 & ad_1 + bd_2 \end{pmatrix}$$

where $ab_1 + bb_2$,
$ac_1 + bc_2$,
$ad_1 + bd_2 \in R$

Linear independend and dependend vector

Let $\{u_1, u_2 \cdots u_n\}$ be the set of n-vect, then the expression $C_1 v_1 + C_2 v_2 + \cdots + C_n v_n$ is called linear combination of n-vector where $c_1\, c_2 \cdots c_n$ are scalars

The vector are said to linearly independant if $c_1 v_1 + c_2 v_2 + \cdots c_n v_n = 0$

Impliese $c_1 = c_2 = \cdots = c_n = 0$

and dependent if at least one scalar $c_i \neq 0$

e) Here $V = R^3$ is a vector space and

$W = \{(0, y, z) : y, z \in R\}$

Let $u = (0, y_1, z_1)$ and $v = (0, y_2, z_2)$

$\forall\, y_1, z_1, y_2, z_2 \in R$

& $a, b \in F$

$au + bv = a(0, y_1, z_1) + b(0, y_2, z_2)$

$= (0, ay_1, az_1) + (0, by_2, bz_2)$

$= 0, ay_1 + by_2, az_1 + bz_2$

$\therefore$ $au + bv \in W$ where

$ay_1 + by_2$ $az_1 + bz_2 \in R$

Hence

W is a subspace of $V = R^3$

b) We have $T: R^2 \to R^3$ is defined by $T(x, y)$
$= (x, y, xy)$

let $u_1 = (x_1, y_1)$ & $u_2 = (x_2, y_2)$ & $a, b \in F$

Then $T(u_1) = (x_1, y_1, x_1 \cdot y_1)$, $Tu_2 (x_2, y_2, x_2 \cdot y_2)$

Now,

$$au_1 + bu_2 = a(x_1, y_1) + b(x_2, y_2)$$
$$= (ax_1, ay_1) + (bx_2, by_2)$$
$$= (\underbrace{ax_1 + bx_2}_{x}, \underbrace{ay_1 + by_2}_{y})$$

Also,

$$aT(u_1) + b(T(u_2)) = a(x_1, y_1 \cdot x_1 y_1)$$
$$+ b \cdot (x_2, y_2, x_2 \cdot y_2)$$
$$= (ax_1, ay_1, ax_1 y_1)$$
$$+ (bx_2, by_2, bx_2 y_2)$$
$$= (ax_1 + bx_2, ay_1 + by_2, ax_1 y_1 + bx_2 y_2)$$

Hence,

$$T(au_1 + bu_2) \neq aT(u_1) + b(Tu_2)$$

$T: R^2 \to R^2$ difeniatly

$T(x, y) = (x, -2y)$

Let $u_1 = x_1 y_1$ & $u_2 = x_2, y_2$

$au_1 + bu_2 =$

$a(x_1 y_1) + b(x_2, y_2)$

$(ax_1 \; ay_1) + (bx_2, by_2)$

$\underbrace{ax_1 + bx_2}_{x}, \underbrace{ay_1 + by_2}_{y}$

$T(x, y) = (x, -2y)$

$T(au_1 + bu_2) = (ax_1 + bx_2, -2(ay_1 + by_2))$

$T(u_1) \to (x_1 -2y_1)$

$T(u_2) \to (x_2, -2y_2)$

$aT(u_1) + bT(u_2) \to a(x_1 -2y_1) + b(x_2, -2y_2)$

$= (ax_1 - a2y_1) + \cancel{bx_2x} (bx_2, -b2y_2)$

$= ax_1 + bx_2, + a2y_2 b2y_2)$

$= (ax_1 + bx_2, -2ay_1 - 2by_2)$

$= (ax_1 + bx_2), -2(ay_1 + b$

Eigen Value and Eigen vector

Def: Let A be a $n \times n$ square matrix. Then for any non zero column vector X if $AX = \lambda X$ then λ is called Eigen value of A and x is called Corresponding eigen vector

procedure:

1. Solve the characteristics i.e. $|A - \lambda I| = 0$

2. Obtain the value of λ $\quad AX = \lambda X$

3. Take non-zero column vector $\boxed{Ax = \lambda x}$

4. Solve the eqn For each value of λ

Q.N. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix}.$$

The characteristics eqn is:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) = 0$$

$$\lambda = \pm 1$$

Let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ be non zero column vector.

such that

$$Ax = \lambda x$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{--- (1)}$$

when

$$\lambda = 1, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+0 \\ 0-y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = x$$

$$-y = y$$

$$2y = 0$$

$$\therefore y = 0$$

here for the eign vector corresponding to eigen value $\lambda = 1$ is

$$X = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{ie} \quad (x, 0)$$

when $\lambda = -1$ from (1)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ -y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

$\Rightarrow$ $x = -x \qquad x = 0$

$-y = -y \qquad y = y$

Therefore the eisgen vector corresponding to eigen value $\lambda = -1$ is

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} = \text{i.e } (0, y)$$

4) Find Eigenvalue as well as vector of following matrix (7 marks)

Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$|A - \lambda I| = ?$

$A - \lambda I = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 - \lambda I & 0 & 0 \\ 0 & -8-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{bmatrix}$

Its characteristics eqn is zero

$|A - \lambda I| = \begin{vmatrix} 3-\lambda I & 0 & 0 \\ 0 & -8-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix}$

$0 = 3 - \lambda I \left((4-\lambda)(-8-\lambda) - 0 \right)$

$(3-\lambda I)(4-\lambda)(-8-\lambda)=0$

$\lambda = 3$

$\lambda = 4$

$\lambda = -8$

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is non zero column

$AX = \lambda x$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 3x \\ -8y \\ 4z \end{bmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (1)$$

CamScanner

when $\lambda = 3$

$\cancel{\lambda = 3}$

then,

$$\begin{bmatrix} 3x \\ -8y \\ 4z \end{bmatrix} = 3\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x \\ -8y \\ 4z \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \\ 3z \end{bmatrix}$$

$3x = 3x \quad x = x$

$-8y = 3y \quad \Rightarrow y = 0$

$4z = 3z \Rightarrow z = 0$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$$

ie $(x, 0, 0)$.

CamScanner

...amics

at $\lambda = -2$

and

at $\lambda =$

4.5)

Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

$$|A - \lambda I| = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

or,

$$(2-\lambda)\{(-1-\lambda)(1-\lambda) - 3\} = 0$$

$$(-\lambda+2)\ \cancel{-1-\lambda}\ \{-1(1-\lambda)-\lambda(1-\lambda)\}-3=0$$

$$(-\lambda+2)\ \{-1+\cancel{\lambda}-\cancel{\lambda}+\lambda^2\}-3=0$$

$$(-\lambda+2)\ \{\lambda^2-1-3\}=0$$

$$(-\lambda+2)\ (\lambda^2-4)=0$$

$$(2-\lambda)\ (\lambda^2-4)=0$$

$$\lambda=2 \qquad \lambda=\pm 2$$

Let

$$X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$AX=\lambda X$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+y+2z \\ -y+3z \\ y+z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{— (1)}$$

put

$\lambda = 2$

$$\begin{bmatrix} 2x+y+2z \\ -y+3z \\ y+z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 2x+y+2z \\ -y+3z \\ y+z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

Now

$$2x+y+2z = 2x$$

$$y+2z = 0 \quad \text{— (I)}$$

$$-y+3z = 2y$$

$$3z-3y = 0 \quad \text{— (II)}$$

$$y+z-2z = 0$$

$$\cancel{y+z} \quad y-z = 0 \quad \text{— (III)}$$

$$y = z$$

$y+2y=0$

$3y=0$

$y=0$

$y=0$

$z=y$

$z=0$

From (ii) & (iii)

$3z - 3z = 0$

$y + z = 2z$

$z + z = 2z$

$\cancel{2z} = \cancel{2z}$

$\boxed{z = z}$

$\therefore x = x$ $X = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$

$2x + y + 2z = -3x$ — (v)

$-y + 3z = -2y$ — (vi)

$y + z = -2z$ — (vii)

(vi) $y + 3z = 0$

$y = -3z$

put $y = -3z$ in eq.n (v)

$\cancel{+3z + 3z = +6z}$

Thermodynamics

$$2x + (-3z) + 2z = -2x$$

$$4x + (-z) = 0$$

$$z = 4x$$

$$z = \frac{1}{3}y$$

$$4x = \frac{1}{3}y$$

$$-12x = y$$

5 a)

Let $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

The characteristics matrix is

Let

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}$$

$$(A - \lambda I| = 0$$

$$(2-\lambda)(5-\lambda) - (3 \times 3) = 0$$

$$2(5-\lambda) - \lambda(5-\lambda) - 9 = 0$$

$$\lambda^2 - 7\lambda + 1 = 0$$

to verify caley Hamilation therem. we have to show

2.9 EIGENVALUE AND EIGENVECTOR

The goal of this title is to show how Ax is related to x, where A is n×n matrix and x is column vector in R^n. For example, if A is 2×2 matrix and if x is non-zero vector in ... such that $Ax = \lambda x$ for some scalar λ, then each vector on the line through origin determined by x gets mapped back on to the same line under the multiplication by matrix A.

Definition (Eigenvalue and Eigenvector)

If A is $n \times n$ matrix, then a scalar λ is called an eigenvalue of matrix A if equation $Ax = \lambda x$ has a non-trivial solution. Such an x is called eigenvector corresponding to eigenvalue λ and the corresponding vector $x \in R^n$ is called an eigen-vector of matrix A.

Properties of Eigenvalue and Eigenvector

1. The eigenvalues of a triangular matrix are the entries on its main diagonal.
2. Product of the eigen values is equal to the determinant of the matrix.
3. If λ is an eigenvalue of a matrix A then $\frac{1}{\lambda}$, (for $\lambda \neq 0$) is the eigenvalue of A^{-1}.
4. If λ is an eigenvalue of an orthogonal matrix A, then $\frac{1}{\lambda}$, (for $\lambda \neq 0$) is an eigenvalue A^{-1}.
5. If $v_1, v_2, \dots\dots, v_r$ are eigenvectors that correspond to distinct eigenvalue $\lambda_1, \lambda_2, \dots, \lambda_r$ of an n×n matrix A, then the set $\{v_1, v_2, \dots\dots, v_r\}$ is linearly independent.

Characteristic Polynomial and Characteristic Equation

If λ be an eigenvalue of a square matrix A, then $\det(A - \lambda I)$ is called characteristic polynomial and $\det(A - \lambda I) = 0$ is called characteristic equation of the matrix A.

Statement of Cayley–Hamilton Theorem

Every square matrix satisfies its own characteristic equation. That is, if the characteristic equation of n^{th} order of square matrix A satisfies $|A - \lambda I| = 0$.

i.e. $(-1)^n \lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + \dots + k_n = 0.$

Then $(-1)^n A^n + k_1A^{n-1} + k_2A^{n-2} + \dots + k_nI = 0.$

Example 18: Show that -2 is eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

Solution:

Given,

$$\lambda = -2 \text{ and } A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}.$$

If

$$Ax = \lambda x \Rightarrow Ax = -2x \Rightarrow (A + 2I)x = 0 \qquad \dots\dots(i)$$

has non-trivial solution, then $\lambda = -2$ is eigenvalue of matrix A.

put $\lambda = A$

then eqn be

$$A^2 - 7A + I = 0$$

$$= \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} - 7 \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4+9 & 6+15 \\ 6+15 & 9+25 \end{bmatrix} - \begin{bmatrix} 14 & 21 \\ 21 & 35 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix} - \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix}$$

$$= 0$$

This verify caley - Hamilton theorem to find the inverse, multiplying both side of (i) by A^{-1} we get

$A \times A^{-1} = I$ $\qquad$ $AI = A$

for A^{-1}

$$A^2 - 7A + 1I = 0$$

multiplying both side A^{-1}

$A^{-1} = 7I - A$

$= 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

$= \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

$|A| = 10 - 9 = 1$

$A^{-1} = \frac{1}{|A|} \text{adj } A$

$= \frac{1}{1}\begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$

CamScanner

6) soln:

Given

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{bmatrix}$$

The characteristics eqn,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

or $1-\lambda \{(2-\lambda)(1-\lambda) - 6\} - 3\{4(1-\lambda) - 3\} + 7\{8 - 2(2-\lambda)\} = 0$

or $1-\lambda \{2-\lambda-\lambda+\lambda^2-6\} - 3\{4-4\lambda-3\} +$

$7\{8 - 2 + \lambda\} = 0$

or $\lambda^3 - 4\lambda^2 - 26\lambda - 35 = 0$ —— (1)

To verify caley hamilton theorem
put $x = A$

$$A^3 - 4A^2 - 20A - 35I = 0 \quad (11)$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 23 & 7+9+7 \\ 4+8+3 & 12+4+6 & 28+6+3 \\ 1+8+1 & 3+4+2 & 7+6+1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 153 & 208 \\ 66 & 76 & 111 \end{bmatrix}$$

$$A \times A^{-1} = I$$

L.H.S

$$A^3 - 4A^2 - 20A - 35I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ 0 proved

Multiplying A^{-1} on both side of (11)

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

$$35A^{-1} = A^2 - 4A - 20I$$

Deffni: A squre matrix A is called d iagonalizable if there exisl an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$

8 a) =

Let $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

$$|A - \lambda I| = 0$$

or, $\begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} = 0$ $\lambda = \pm 1$

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$AX = \lambda X$

$$\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{--- (1)}$$

when ~~$\lambda = 1$~~

The model matrix

$$P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$D = P^{-1}AP = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thermod...

10)

Here

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = 0$$

$$7 - 7\lambda - \lambda + \lambda^2 + 8 = 0$$

$$7 - 8\lambda + \lambda^2 + 8 = 0$$

$$15 - 8\lambda + \lambda^2 = 0$$

$$15 - \cancel{5\lambda} - 3\lambda + \lambda^2 = 0$$

$$5(3-\lambda) - \lambda(3-\lambda) = 0$$

$$\lambda = 5 \quad \& \quad 3 = \lambda$$

Let $V = \begin{pmatrix} x \\ y \end{pmatrix}$ be a eigen space vector s.,

$$AX = \lambda X$$

or $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$

or, $\begin{bmatrix} 7x + 2y \\ 4x + y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$ — (1)

when, $\lambda = 3$ then eq^n (1) be

$\begin{bmatrix} 7x + 2y \\ -4x + y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$

$-4x + y = 3y$
$-4x = 2y$
$-2x = y$

or, $\begin{bmatrix} 7x + 2y \\ -4x + y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$

$-4x$
$y = 2x$

$7x + 2y = 3x$ — (11)

$-4x + y = 3y$ — (111)

$4x = -2y$

From (11)

$7x + 2y = 3x$.

$2y = -4x$

Let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be a eigen space vector s,

$$AX = \lambda X$$

or $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$

or, $\begin{bmatrix} 7x + 2y \\ 4x + y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$ — (1)

where, $\lambda = 3$ then eqn (1) be

$\begin{bmatrix} 7x + 2y \\ -4x + y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$

or, $\begin{bmatrix} 7x + 2y \\ -4x + y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$

$7x + 2y = 3x$ — (II)

$-4x + y = 3y$ — (III)

From (II)

$7x + 2y = 3x$.

$2y = -4x$

$-4x$

$y = 2x$

$4x = -2y$

Therm...

$y = -2x$

$x = x$ is

$$X = \begin{bmatrix} x \\ -2x \end{bmatrix} = x\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$7x + 2y = 5x$

$2x + 2y = 0$

$2x + y = 0$

$y = -x$

$x = -y$

when $\lambda = 5$ then eqn (1) become

$$\begin{bmatrix} 7x + 2y \\ -4x + y \end{bmatrix} = 5\begin{bmatrix} x \\ y \end{bmatrix}$$

$-5x = 5y$

$x = -y$

$2x = -2y$

(2) $x = -y$

$$7x + 2y = 5x \quad \text{(I)}$$

$$-4x + y = 5y \quad \text{(II)}$$

$2x + 2y = 0$

$x = -y$

$$-x = y$$

$$x = -y$$

$$v = \begin{bmatrix} -y \\ y \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

modal matrix $p = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$

we know Digonal matrix is

$$D = p^{-1} A p$$

$$= \frac{1}{-1} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^4 = p^{-1} D^4 p$$

$$= \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3^4 & 0 \\ 0 & 5^4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$$

$=$

if $Q = P^{-1}$ then $A = Q^{-1} B Q$

so B is similar to A and we say A & B are similar $(A \sim B)$

The changing A to $P^{-1}AP$ is called similarity transformation.

Linear programming problem.

* Simplex method.

Max. $z = 40x_1 + 88x_2$ subject to constraints

$$2x_1 + 8x_2 \leq 60$$
$$5x_1 + 5x_2 \leq 60$$
$$x_i \geq 0 \quad (i = 1, 2)$$

Soln

Let us introduce x_3 and x_4 ~~slag~~ slack variable

$$z - 40x_1 - 88x_2 = 0$$
$$2x_1 + 8x_2 + x_3 = 60$$
$$5x_1 + 5x_2 + x_4 = 60$$
$$(x_i > 0)$$

The initial simplex table is.

	z	x_1	x_2	x_3	x_4	constant	Ratio
R_1	1	-40	-88	0	0	0.	
R_2	0	2	8	1	0	60.	$60/8 = 7.5$
R_3	0	5	5	0	1	60.	$60/2 = 30$

The greatest negative in R_1 is -88, x_2 is pivot column.

R_2 is pivot row and 8 is the pivot element.

Here, basic variable are, $x_3 = 60$ and $x_4 = 60$.

basic = only one non-zero column.

and non basic variable are,

$x_1 = 0, \quad x_2 = 0$

Applying

$R_1 \to R_1 + 11R_2,$

$R_3 \to 4R_3 - R_2$

	Z	x_1	x_2	x_3	x_4	constant	Ratio.
R_1	1	-18	0	11	0	660	
R_2	0	2	8	1	0	60	60/2 = 30
R_3	0	18	0	-1	4	180	180/18 = 10

Here,

basic variable are,

x_2 and x_4

non-basic variable

are $x_1 = 0, \quad x_3 = 0$

$R_1 \to R_1 + R_3$ and $R_2 \to 9R_2 - R_3$

	Z	x_1	x_2	x_3	x_4	constant	Ratio.
R_1	1	0	0	10	4	840	
R_2	0	0	72	10	-4	360	
R_3	0	18	0	-1	4	180	

Here,

x_1 and x_2 are basic variable

$18x_1 = 180 \qquad 72x_2 = 360$

$x_1 = 10 \qquad x_2 = 5$

x_3 and x_4 are non basic

$x_3 = 0, \quad x_4 = 0$.

$< = +$
$> = -$

max. $z = 840$ at $x_1 = 10$
$x_2 = 5$

max. $z = 15x_1 + 10x_2$
Subject

$2x_1 + 2x_2 \leq 10$
$2x_1 + 3x_2 \leq 10$
$x_1, x_2 \geq 0$

Soln

Let us introduce x_3 and x_4 slack variable,

$z - 15x_1 - 10x_2 = 0$
$2x_1 + 2x_2 + x_3 = 10$
$2x_1 + 3x_2 + x_4 = 10$

The initial simplex table is.

	z	x_1	x_2	x_3	x_4	constant	Ratio.
R_1	1	-15	-10	0	0	0	
R_2	0	2	2	1	0	10	$10/2 = 5$
R_3	0	2	3	0	1	10	$10/2 = 5$

Here,

the basic variable are, x_3 and $x_4 = 10$

and

non basic variable are,
$x_1 = 0$ and $x_2 = 0$

2 10

Applying. $R_1 \rightarrow 2R_1 + 15R_2$ and
$R_3 \rightarrow R_3 - R_2$

T	x_1	x_2	x_3	x_4	constant	ratio.
2	0	10	15	0	150	
0	2	2	1	0	10	
0	0	2	-1	1	0	

Here,

basic variable are, x_1 and x_4.

$2x_1 = 10$ $\quad x_4 = 0$

$x_1 = 5$

and non-basic variable are.

$x_2 = 0$ and $x_3 = 0$

$\max 2Z = 150$

$Z = 75$ at $x_1 = 5$

$x_2 = 0$

d. $\max Z = 6x_1 + 12x_2$ s.to

$0 \le x_1 \le 4, \quad 0 \le x_2 \le 4,$

$6x_1 + 12x_2 \le 72$

Soln :

let us introduce non-negative slack variable, x_3, x_4 and x_5.

$Z - 6x_1 - 12x_2 = 0.$

$x_1 + x_3 = 4$

$x_2 + x_4 = 4$

$6x_1 + 12x_2 + x_5 = 75$

Mathym...

Unit

Linear programming problem: (L.P.P.)

(simplex method)

Example

max z

$$\max z = 40x_1 + 88x_2$$

Constraints. $2x_1 + 8x_2 \leq 60$

$5x_1 + 2x_2 \leq 60$

$x_i \geq 0 \; (i = 1, 2)$

Date
Page

Exercise - 3.1

1. soln

a. max. $z = 15x_1 + 10x_2$

s.t. $2x_1 + 2x_2 \le 10$

$2x_1 + 3x_2 \le 10$

$x_1, x_2 \ge 0$

soln,

Let us introduce non-negative slack variables x_3 & x_4

Then,

$z - 15x_1 - 10x_2 = 0$

$2x_1 + 2x_2 + x_3 = 10$

$2x_1 + 3x_2 + x_4 = 10$

The initial simplex table is:

	z	x_1	x_2	x_3	x_4	constant	ratio
R_1	1	-15	-10	0	0	0	
R_2	0	2	2	1	0	10	$\frac{10}{2} = 5$
R_3	0	2	3	0	1	10	$\frac{10}{2} = 5$

x_1 is pivot column

2 is the pivot element

Here, the basic variable are x_3 & x_4

$x_3 = 10, x_4 = 10$

& non-basic variable are : $x_1 = 0, x_2 = 0$

$2\times -10+15\times 2 = -20+30$ $\quad 2\times 0+15\times 10 = 150$ $\quad 2\times -15+15\times 2 = -30+30 = 0$ $\quad 2\times 1+15\times 0 = 2$

Applying $R_1 \longrightarrow 2R_1 + 15R_2$, $R_3 \longrightarrow R_3 - R_2$

	z	x_1	x_2	x_3	x_4	constant	ratio
R_1	2	0	10	15	0	150	
R_2	0	2	2	1	0	10	
R_3	0	0	1	-1	1	0	

Here, basic variables are x_1 & x_4 $\Rightarrow 2x_1 = 10 \Rightarrow x_1 = 5$

$x_4 = 0$

& non-basic variables are $x_2 = 0, x_3 = 0$

~~max~~ $2z = 150$

$\Rightarrow z = 75$

$\therefore$ max. $z = 75$ at $x_1 = 5$ & $x_2 = 0$

Assignment

d) max. $z = 6x_1 + 12x_2$

$0 \le x_1 \le 4,\ 0 \le x_2 \le 4,\ 6x_1 + 12x_2 \le 72$

Sol^n:

Let us consider non-negative slack variables x_3, x_4 & x_5

Then,

$$z - 6x_1 - 12x_2 = 0$$

$$x_1 + x_3 = 4$$

$$x_2 + x_4 = 4$$

$$6x_1 + 12x_2 + x_5 = 72$$

The initial simplex table is:

	z	x_1	x_2	x_3	x_4	x_5	constant	ratio
R_1	1	-6	-12	0	0	0	0	
R_2	0	1	0	1	0	0	4	Undefined
R_3	0	0	1	0	1	0	4	$\frac{4}{1} = 4$
R_4	0	6	12	0	0	1	72	$\frac{72}{12} = 6$

x_2 is pivot column

1 is the pivot element

Here, basic variables are x_3, x_4 & x_5

$\Rightarrow x_3 = 4, x_4 = 4$ & $x_5 = 72$

& non-basic variables are x_1 & x_2

$\Rightarrow x_1 = 0, x_2 = 0$

Applying: $R_1 \longrightarrow R_1 + 12R_3$, $R_4 \longrightarrow R_4 - 12R_3$

	z	x_1	x_2	x_3	x_4	x_5	constant	ratio
R_1	1	-6	0	0	12	0	48	
R_2	0	1	0	1	0	0	4	4
R_3	0	0	1	0	1	0	4	undefined
R_4	0	6	0	0	-12	1	24	4

-6 is most negative. So, x_1 column is pivot column. Pivot row is R_2 and 1 is pivot element

Basic variables $x_2 = 4$, $x_3 = 4$, $x_5 = 24$

& non- " " $x_1 = 0$, $x_4 = 0$

Applying, $R_1 \rightarrow R_1 + 6R_2$, $R_4 \rightarrow R_4 - 6R_2$

z	x_1	x_2	x_3	x_4	x_5	constant	ratio
1	0	0	6	12	0	72	
0	1	0	1	0	0	4	
0	0	1	0	1	0	4	
0	0	0	-6	-12	1	0	

$\therefore R_1$ is the non-negative entry.
So, the table give optimal solⁿ.

Assume, non-basic variables are ~~zero~~.
~~$x_3 = 0 = x_4 = x_5$~~ $x_1 = 4$, $x_2 = 4$, $x_5 = 0$
& non-basic variables are $x_3 = 0$ & $x_4 = 0$
$\therefore$ ~~The~~ passing variable,

$x_1 = 4$
$x_2 = 4$

Then by R_1, $1z = 72$
$z = 72$

$\therefore$ max. $z = 72$ at $x_1 = 4$, $x_2 = 4$

The simplex Method: Degeneracy

Q) Maximize $z = 4x_1 + x_2 + 2x_3$, s. to $x_1 + x_2 + x_3 \le 1$, $x_1 + x_2 - x_3 \le 0$, $x_1, x_2, x_3 \ge 0$

soln,

Let us introduce non-negative slack variables x_4 & x_5. Then,

$$z - 4x_1 - x_2 - 2x_3 = 0$$
$$x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1 + x_2 - x_3 + x_5 = 0$$

The initial simplex table is:

	z	$x_1 \downarrow$	x_2	x_3	x_4	x_5	constant	ratio
R_1	1	-4	-1	-2	0	0	0	
R_2	0	1	1	1	1	0	1	$\frac{1}{1} = 1$
R_3	0	1	1	-1	0	1	0	$\frac{0}{1} = 0$

x_1 is pivot column
pivot element is 1

$R_1 \to R_1 + 4R_3$, $R_2 \to R_2 - R_3$

	z	x_1	x_2	x_3	x_4	x_5	constant	ratio
R_1	1	0	3	$-6\downarrow$	0	4	0	
R_2	0	0	0	2	1	-1	1	$1/2 = 0.5$
R_3	0	1	1	-1	0	1	0	$0/1 = 0$

0 1 3 5 0

$R_1 \rightarrow R_1 + 3R_2$ & $R_3 \rightarrow 2R_3 + R_2$

	z	x_1	x_2	x_3	x_4	x_5	constant	ratio
R_1	1	0	3	0	3	1	3	
R_2	0	0	0	2	1	-1	1	
R_3	0	2	2	0	1	1	1	

Here, basic variables are $2x_1 = 1 \Rightarrow x_1 = 1/2$

$2x_3 = 1 \Rightarrow x_3 = 1/2$

& non-basic variables are

$x_2 = 0, x_4 = 0, x_5 = 0$

max. $z = 3$ at $x_1 = 1/2, x_2 = 0, x_3 = 1/2$

max $z =$

$x_1 + 2x_2 \leq 6$ slack

$-x_1 + 3x_2 \geq -1$

i.e. $\boxed{x_1 - 3x_2 \leq 1}$

$x_1 + 2x_2 \leq 6$ x_3 surplus

$-x_1 + 3x_2 \geq 2$ x_4

$x_1 + 2x_2 + x_3 = 6$

$x_1 + 3x_2 - x_4 = 2$

Solve LPP, minimize $z = 5x_1 - 20x_2$ s. to
$-2x_1 + 10x_2 \le 5$, $2x_1 + 5x_2 \le 10$

soln.

Let us introduce non-negative slack variables.
Then,

$$z - 5x_1 + 20x_2 = 0$$
$$-2x_1 + 10x_2 + x_3 = 5$$
$$2x_1 + 5x_2 + x_4 = 10$$

The initial simplex table is:

z	x_1	x_2	x_3	x_4	constant	ratio
1	-5	20	0	0	0	
0	-2	10	1	0	5	$5/10 = 1/2$
0	2	5	0	1	10	$10/5 = 1/2$

Applying, $R_1 \to R_1 - 2R_2$ & $R_3 \to 2R_3 - R_2$

z	x_1	x_2	x_3	x_4	constant	ratio
1	-1	0	-2	0	-10	
0	-2	10	1	0	5	
0	6	0	-1	2	15	

Here, basic variables are $10x_2 = 5$
$\Rightarrow x_2 = 1/2$
& $2x_4 = 15$
$\Rightarrow x_4 = 15/2 = 7.5$

& non- basic variables are $x_1 = 0$ & $x_3 = 0$

$\therefore$ min $z = -10$ at $x = 0,\ x_2 = 1/2$

j) max. $f = 6x_1 + 6x_2 + 9x_3$ $\quad x_j \geq 0\ (j = 1, 2, 3, 4, 5)$

subject to,

$$x_1 + x_3 + x_4 = 1$$
$$x_2 + x_3 + x_5 = 1$$