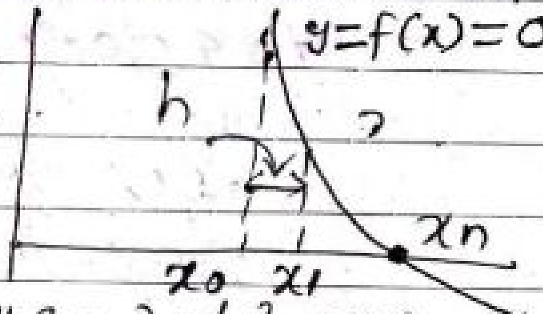
Method - I

Newton - Raphson Method

Let x_0 be an initial approximation to the root of the equation $y = f(x) = 0$. If $x_1 = x_0 + h$ be the exact root then $f(x) = 0$ where h be difference between x_0 & x_1 which is very small.



By using Taylor's series

$$f(x_1) = 0$$

$$f(x_0 + h) = 0$$

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) = 0$$

Since h is very small, neglecting h^2 & higher power of h, we get

$$f(x_0) + hf'(x_0) = 0$$

$$\therefore h = -\frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

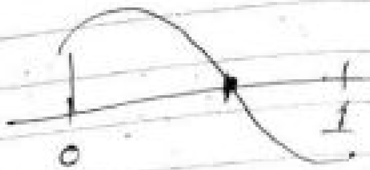
Similarly starting with x_1, better approximation $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

This process is continued to get the root of desired accuracy

In general

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$ This is called N-R formula.

Find root of eq^n $x^3-6x+4=0$ correct to 4 decimal places.



Here

$f(x) = x^3 - 6x + 4$

$f(0) = 4$ (+ve)

$f(1) = -1$ (-ve)

Since $f(0)$ & $f(1)$ have opposite sign. root lies between 0&1. Let initial approximation $x_0 = 0.5$

Now $f(x) = x^3 - 6x + 4$

$f'(x) = 3x^2 - 6$

We know N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The successive approximations are

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{0.5^3 - 6(0.5) + 4}{3(0.5)^2 - 6} = 0.71429$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.71429 - \frac{0.71429^3 - 6\times(0.71429) + 4}{3(0.71429) - 6}$$

$$= 0.73190$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.73190 - \frac{0.73190^3 - 6\times(0.73190) + 4}{3(0.73190) - 6}$$

$$= 0.73205$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.73205 - \frac{0.73205^3 - 6\times(0.73205) + 4}{3(0.73205) - 6}$$

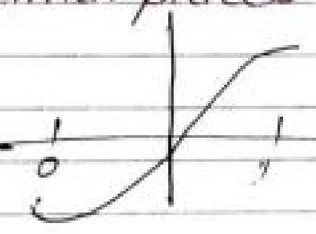
$$= 0.73205$$

Since $x_3 = x_4$

The required root $= 0.73205$ Ans

2) Using N-R method to 3 decimal places $3x = \cos x + 1$

Soln:



Here $f(x) = 3x - \cos x - 1$

$f(0) = -2$ (-ve)

$f(1) = 1.4597$ (+ve)

Then,

Let $x_0 = 0.4$

$f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

By formula

~~$x_0 = x$~~

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The successive approximations are

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.6127$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.6071$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.6071$$

Since $x_2 = x_3$

So Root $= 0.6071$ //

Exercises

14) Compute a root of each

9) $x^2 - 5x + 6 = 0$ $\quad x_0 = 5$

Soln:

Let $f(x) = x^2 - 5x + 6 = 0$

$f(0) = 6$ (+ve)

$f(1) = 2$ (+ve)

$f(2) = 0$

$f(3) = 0$

$f(4) = 2$

$f(5) =$

$\frac{d}{dx} x \log_{10} x$

$x \frac{d \log_{10} x}{dx} + \log_{10} x$

$x \cdot \frac{0.4343}{x} + \log_{10} x$

$0.4343 + \log_{10} x$

shift calc 1

Q.No.3 Compute a real root.

$x \log x = 1.2$ correct to 4 decimal places.

Here

$f(x) = x \log x - 1.2$

$f(1) = -1.20000$ (-ve)

$f(2) = -0.59794$ (-ve)

$f(3) = 0.23136$ (+ve)

Since $f(2)$ & $f(3)$ have opposite sign. A root lies between 2 & 3

Let initial approximation $x_0 = 2.5$

Now, $f(x) = x \log_{10} x - 1.2$

$$f'(x) = \frac{d}{dx}(x \log_{10} x)$$

$$= x \frac{d}{dx}(\log_{10} e \log_e x) + \log_{10} x$$

$= \log_{10} x + 0.43429$

we have, N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

the successive approximations are:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.5 - \frac{2.5\log 2.5 - 1.2}{\log_{10} 2.5 + 0.43429}$$

$$= 2.74651$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.74651 - \frac{2.74651\log 2.74651}{\log_{10} 2.74651 + 0.43429}$$

$$= 2.74065$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.74065 - \frac{2.74065\log 2.74065 - 1.2}{\log_{10} 2.74065 + 0.43429}$$

$$= 2.74065$$

4) Find a real root of square root of 18, correct to 3 decimal places.

Let $x = \sqrt{18}$

$x^2 = 18$

$\therefore f(x) = x^2 - 18$

$f(4) = -2.00000$

$f(5) = 7$ (+ve)

Since $f(4)$ & $f(5)$ have opp. sign. A root lies betn 4 & 5

Initial approximation $x_0 = 4.5$

Here $f(x) = x^2 - 18$

$f'(x) = 2x$

We know,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4.2500$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 4.2426$$

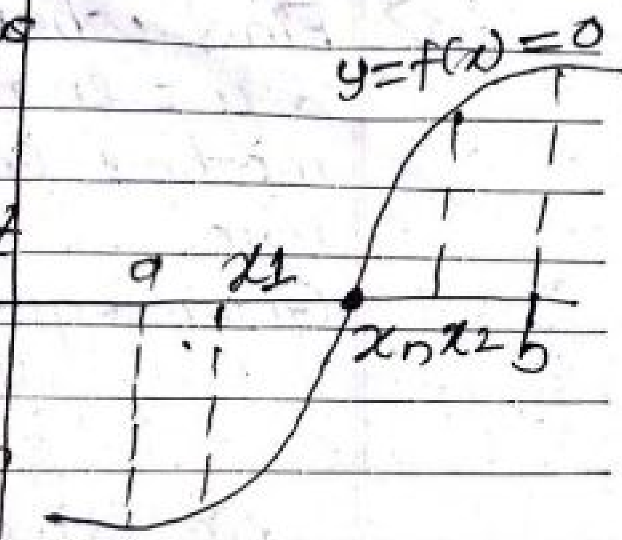
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 4.2426$$

So root $= 4.2426$

Method-II

Bisection method (Bracketing method)

Let $y=f(x)$ be a continuous function between a and b with $f(a)$ & $f(b)$ have opposite sign then a root lies between a & b



Then first approximation

$$x_1=\frac{a+b}{2}$$

If $f(x_1)=0$ then x_1 is the exact root otherwise the root lies between a & x_1 or x_1 & b according to the sign of $f(x_1)$

In fig $f(a)$ is negative and $f(b)$ is +ve then $x_1=\frac{a+b}{2}$. Again $f(x_1)=-ve$ so the root lies between x_1 & b. The second approximation is $x_2=\frac{x_1+b}{2}$. $f(x_2)$ +ve. The root lies betn x_1 & x_2. The third approximation $x_3=\frac{x_1+x_2}{2}$. This procedure is continued to get root of desired accuracy.

In general $\boxed{x_n=\frac{a+b}{2}}$ is Bisection formula.

(1)

setting: $x=\frac{a+b}{2}$: $\ln x - x + 3$

2019 Fall

1) Find where the graph of $y=x-3$ & $y=\ln x$ intersect using bisection method. Get correct to 4 decimal places.

Here

Given equations are

$$y=x-3$$
$$y=\ln x$$

So $\ln x = x-3$

or $\ln x - x + 3 = 0$

$\therefore f(x) = \ln x - x + 3$

$\therefore f(4) = 0.38629(+ve)$

$f(5) = -0.39056(-ve)$

Since $f(4)$ and $f(5)$ have opp. sign.

A root lies between 4 & 5

Let $a=4$, $b=5$

We know, the bisection formula

$$x_n = \frac{a+b}{2}$$

Tabulating for successive approximations, we get,

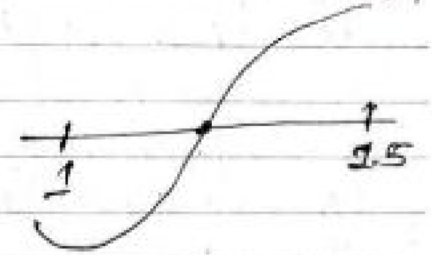
Iteration	a (+ve)	b (-ve)	$x=\frac{a+b}{2}$	$f(x)=\ln x - x + 3$
1.	4	5	4.50000	+ve
2.	4.5	5	4.75000	-ve
3)	4.5	4.75	4.625	-ve
4)	4.5	4.625	4.5625	-ve
5)	4.5	4.5625	4.53125	-ve
6)	4.5	4.53125	4.51563	-ve
7)	4.5	4.51563	4.50782	-ve
8)	4.5	4.50782	4.50391	+ve
9)	4.50391	4.50782	4.50587	-ve
10	4.50391	4.50587	4.50489	+ve
11	4.50489	4.50587	4.50538	-ve
12	4.50489	4.50538	4.50514	+ve
13	4.50514	4.50538	4.50526	-ve
14.	4.50514	4.50526	4.50520	+ve
15	4.50520	4.50526	4.50523	END

Hence $x_{14} = x_{15}$

The required root $= 4.5052$ Ans

2) Find a root of equation $xe^x = 3$ by using bracketing method correct to 3 decimal places ($x_1 = 1, x_2 = 1.5$)

Soln:



$f(x) = xe^x - 3$

$f(1) = -0.2817(-ve)$

$f(1.5) = 3.7225(+ve)$

Since $f(1)$ & $f(1.5)$ have opposite sign.

A root lies between 1 & 1.5

Let $a = x_1 = 1$

$b = x_2 = 1.5$

We know
The bisection formula:

$$x_n = \frac{a+b}{2}$$

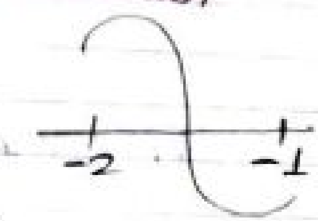
Tabulating for successive approximations, we get.

Iteration	a(+ve)	b(+ve)	$x=\frac{a+b}{2}$	$f(x)=xe^x-3$
1	1	1.5	1.2500	+ve
2	1	1.2500	1.1250	+ve
3	1	1.125	1.0625	+ve
4.	1	1.0625	1.0313	-ve
5)	1.0313	1.0625	1.0469	-ve
6)	1.0469	1.0625	1.0547	+ve
7.	1.0469	1.0547	1.0508	+ve
8)	1.0469	1.0508	1.0489	-ve
9)	1.0489	1.0508	1.0499	-ve
10)	1.0499	1.0508	1.0504	+ve
11	1.0499	1.0504	1.0502	+ve
12.	1.0499	1.0502	1.0501	+ve
13.	1.0499	1.0501	1.0500	+ve
14.	1.0499	1.0500	1.0499	-ve
15)	1.0499	1.0500	1.0499	END

root = 1.0499

3) Find negative root
$x^2-4x-10$ correct to 2 decimal

$f(x)=x^2-4x-10$
$f(0)=-10$ (-ve)
$f(-1)=-5$ (-ve)
$f(-2)=2$ (+ve)



Since $f(-1)$ and $f(-2)$ have opposite sign so root lies between (-2 and -1)
So $a=-2$
$b=-1$

Then By bisection formula

$$x_n = \frac{a+b}{2}$$

Now Tabulating the successive approximation

Iteration	a(+ve)	b(-ve)	$x=\frac{a+b}{2}$	$f(x)=x^2-4x-10$
1)	-2	-1	-1.500	-ve
2)	-2	-1.5	-1.750	+ve
3)	-1.750	-1.5	-1.625	-ve
4)	-1.750	-1.625	-1.688	-ve
5)	-1.750	-1.688	-1.719	-ve
6)	-1.750	-1.719	-1.735	-ve
7)	-1.750	-1.735	-1.743	+ve
8)	-1.743	-1.735	-1.739	-ve
9)	-1.743	-1.739	-1.741	-ve

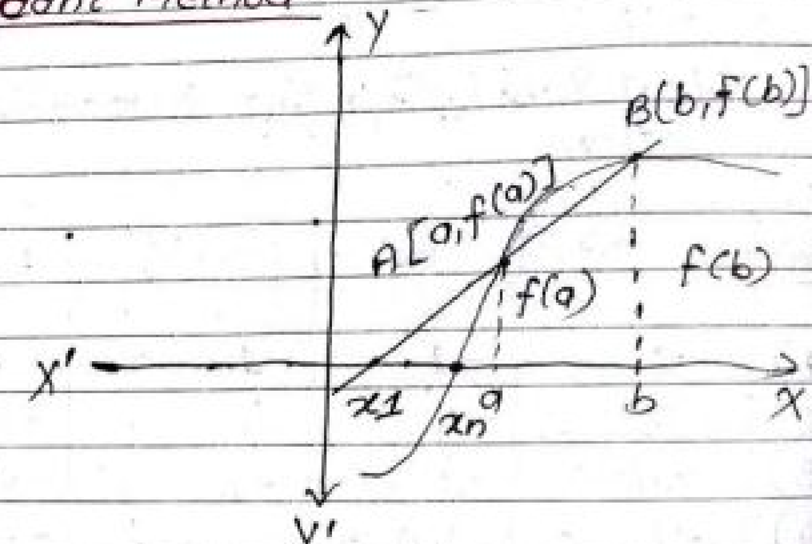
10)	-1.743	-1.741	-1.742	+0.00 +ve
11	-1.742	-1.741	-1.742	- ve

Ans = -1.742

Solution of System of Non Linear Equations

⇓

Method III **Secant Method:**



This method is an important improvement over the method of false position. It does not require the condition f(a) & f(b) must have the opposite sign.

Let a & b are two initial limits of the interval. The equation of secant (chord) joining the points $A[a, f(a)]$, $B[b, f(b)]$ is given by.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\text{or } y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

But this line cuts the x-axis at $(x, 0)$

$$\therefore 0 - f(a) = \frac{f(b) - f(a)}{b - a}(x_1 - a)$$

$$\text{or, } x_1 - a = \frac{-f(a)(b - a)}{f(b) - f(a)}$$

$$\text{or, } x_1 = \frac{-f(a)(b - a)}{f(b) - f(a)} + a$$

$$\text{or, } x_1 = \frac{-bf(a) + af(a) + af(b) - af(a)}{f(b) - f(a)}$$

$$\therefore x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

which is an first approximation. Continuing this procedure we get desire solution

In general $x_n = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

setting $X=$
No sign

Date: | Page:

1) Using secant method, Find a real root of $xe^x = \cos x$ correct to 4 decimal places.

Solⁿ:

Here,

$f(x) = xe^x - \cos x$

$f(0) = -1$ (-ve)

$f(1) = 2.17798$ (+ve)

let $a = 0$, $b = 1$

$f(a) = -1$, $f(b) = 2.17798$ (+ve)

We know, secant formula

$$x_n = \frac{af(b) - f(a)b}{f(b) - f(a)}$$

The Tabulating for successive approximation.

Iteration	a (A)	f(a) (B)	b (C)	f(b) (D)	$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$	$xe^x - \cos x$
1.	0	-1	1	2.17798	0.31467	-0.51987 +ve
2.	0.31467	-0.51987	0	-1	0.65538	0.46938
3.	0.65538	0.46938	0.31467	-0.51987	0.49372	-0.07167
4.	0.49372	-0.07167	0.65538	0.46938	0.51513	-0.00796
5.	0.51513	-0.00796	0.49372	-0.07167	0.51780	0.00014
6.	0.51780	0.00014	0.51513	-0.00796	0.51775	-0.00001
7.	0.51775	-0.00001	0.51780	0.00014	0.51775	-0.00001 (END)
8.						
9.						

Required root $= 0.5177$

setting $x = \frac{AD - CB}{D - B}$: $f(x)$

Date: | Page:

2. Determine root of $e^x = x^3 + \cos 25x$. Using secant method correct to 4 decimal places

Solⁿ:

$f(x) = e^x - x^3 - \cos 25x$

$f(1) = 0.72708$

$f(2) = -1.57591$

Let

$a = 1$, $\Rightarrow f(a) = 0.72708$

$b = 2 \Rightarrow f(b) = -1.57591$

We know secant formula.

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

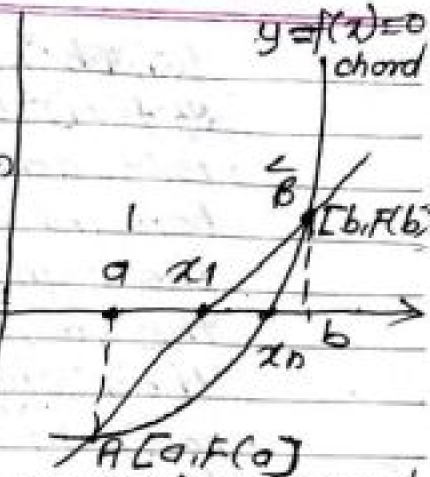
Tabulating for successive approximation

Itⁿ	a (A)	f(a) (B)	b (C)	f(b) (D)	$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$	$e^x - \cos x - x^3$
1)	1	0.72708	2	-1.57591	1.31571	1.35598
2	1.31571	1.35598	1	0.72708	0.63500	2.61704
3)	0.63500	2.61704	1.31571	1.35598	2.04766	-1.43698
4	2.04766	-1.43698	0.63500	2.61704	1.54693	0.43338
5.	1.54693	0.43338	2.04766	-1.43698	1.66295	1.41919
6.	1.66295	1.41919	1.54693	0.43338	1.49593	0.16085
7.	1.49593	0.16085	1.66295	1.41919	1.47458	0.49142
8.	1.47458	0.49142	1.49593	0.16085	1.50632	0.09311
9	1.50632	0.09311	1.47458	0.49142	1.51374	0.08550
10	1.51374	0.08550	1.50632	0.09311	1.59711	1.47622
11	1.59711	1.47622	1.51374	0.08550	1.50861	0.08711

12	1.50861	0.08711	1.59711	1.47622	1.50306	0.10723
13	1.50306	0.10723	1.50861	0.08711	1.53264	0.21454
14	1.53264	0.21454	1.50306	0.10723	1.47350	0.51896
15	1.47350	0.51896	1.53264	0.21454	1.57501	1.02899
16	1.57501	1.02899	1.47350	0.51896	1.37220	2.82855
17	1.37220	2.32855	1.57501	1.02899	1.73559	0.38540
18	1.73559	-0.38540	1.37220	2.32855	1.68399	0.91834
19	1.68399	0.918.34				

Method-IV

False Position Method



This method is oldest method for computing the real roots of an equation $f(x)=0$. For this method we find two numbers a & b such that f(a) & f(b) are of different signs. Hence a root lies between a & b. The curve $y=f(x)=0$ must cross the x-axis betn a & b

The equation of chord joining the points $A[a,f(a)]$ & $B[b,f(b)]$ is given by

$$y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$\text{or } y-f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

But this line cuts x-axis at $(x_1, 0)$

$$\therefore 0-f(a) = \frac{f(b)-f(a)}{b-a}(x_1-a)$$

$$\text{or, } (x_1-a) = \frac{-f(a)(b-a)}{f(b)-f(a)}$$

$$\text{or } x_1 = \frac{-f(a)(b-a)}{f(b)-f(a)}+a$$

$$= \frac{-f(a)b+af(a)+af(b)-af(a)}{f(b)-f(a)}$$

$$= \frac{af(b)-bf(a)}{f(b)-f(a)}$$

which is first approximation
If $f(x_1)=0$ then x_1 is the exact root. otherwise the root lies betn a & x_1 or x_1 & b according the sign of $f(x_1)$ Replacing the value by x_1 & $f(x_1)$ as their sign, we get other approx. This procedure is continued to get root of desired accuracy

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

1) Find a real roots $\log x - \cos x = 0$ correct to 4 decimal places.

$f(x) = \log x - \cos x$.
$f(0) =$ Not defined
$f(1) = -0.54030$ (-ve)
$f(2) = 0.71718$ (+ve)
Since $f(1)$ & $f(2)$ have opp sign. root lies betn 1 & 2
let $a = 1$, $f(a) = -0.54030$
$b = 2$ $f(b) = 0.71718$
We know,
False position formula
$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Tabulating for successive approximation we get

Itn	a (A) (-ve)	f(a) (B)	b (C) (+ve)	f(b) (D)	$x = \frac{af(b)-bf(a)}{f(b)-f(a)}$	f(x)
1)	~~-0.54030~~	~~2~~				
1)	1	-0.54030	2	0.71718	1.42967	0.01458 (+ve)
2)	1	-0.54030	1.42967 ~~0.01458~~	0.01458	1.41838	-0.00003 (-ve)
3)	1.41838	-0.00003	1.42967	0.01458	1.41840	-0.000004 (-ve)
4)	1.41840	-0.000004	1.42967	0.01458	1.41840	END

Reqn root = 1.4184 (Ans)

2017 Spring

1) Find a real root of $e^{\cos x} - \sin x - 1 = 0$ correct to 4 decimal places using False position method.

soln:

$f(x) = e^{\cos x} - \sin x - 1$

$f(0) = 1.71828$ (+ve)

$f(1) = -0.12495$ (-ve)

Now,

$a = 0, \ f(a) = 1.71828$

$b = 1 \ \ f(b) = -0.12495$

Since $f(1)$ and $f(2)$ have opp. sign

So root lies betn 1 and 2

Now,

From False position:

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Tabulating successive approximation

Iteration	a (+ve)	f(a)	b	f(b) (-ve)	$x_n = \frac{af(b)-bf(a)}{f(b)-f(a)}$	f(x)
1	0	1.71828	1	-0.12495	0.93221	0.01201(+ve)
2	0.93221	0.01201	1	-0.12495	0.93815	-0.00017(-ve)
3)	0.93221	0.01201	0.93815	-0.00017	0.93807	0.000005 (+ve)
4)	0.93807	0.000005	0.93815	-0.00017	0.93807	END

Ans = 0.93807

Method V

Fixed Point Iteration Method

→ Let $f(x) = 0$ be the given equation where roots are to be determined. In iteration method we write the given eqn in the form of $x = \phi(x)$.

Let $x = x_0$ be the initial approximation of the required root, then the first approximation is given by $x_1 = \phi(x_0)$

Second, third & other approximations are

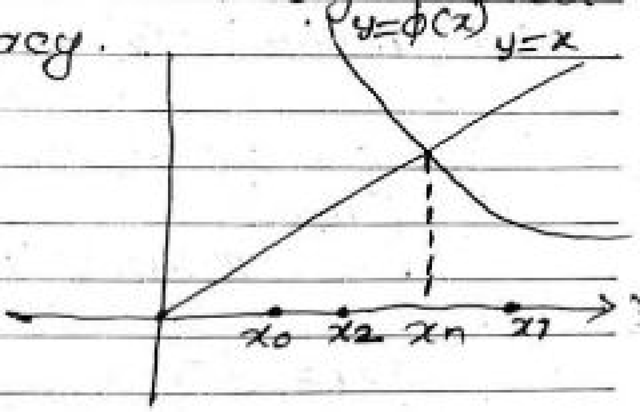
$x_2 = \phi(x_1)$

$x_3 = \phi(x_2)$ & so on

In General

$$\boxed{x_{n+1} = \phi(x_n)}$$

It is called fixed point iteration formula. This process is continued to get the root of desired accuracy.



1) By using fixed point iterat^n Find a positive root of $x^3+x^2-1=0$ correct to 4 decimal places

Soln:

$f(x) = x^3+x^2-1$

$f(0) = -1$ (-ve)

$f(1) = 1$ (+ve)

Since $f(0)$ & $f(1)$ have opp sign A root lies bet^n 0 & 1.

Let initial approximation $x_0 = 0.5$

Now, Rearranging given eq^n we get

$$x^3+x^2-1=0$$

or $x^2(x+1) = 1$

or, $x = \frac{1}{\sqrt{x+1}}$ $\phi(x)$

where $\phi(x) = \frac{1}{\sqrt{x+1}}$

We know fixed point iteration formula

$$x_{n+1} = \phi(x_n)$$

$x_1 = \phi(x_0) = \frac{1}{\sqrt{0.5+1}} = 0.8650$

$x_2 = \phi(x_1) = \frac{1}{\sqrt{0.81650}} = 0.74196$

$x_3 = \phi(x_2) = \frac{1}{\sqrt{0.74196+1}} = 0.75767$

$x_4 = \phi(x_3) = \frac{1}{\sqrt{0.75767+1}} = 0.75428$

$x_5 = \phi(x_4) = \frac{1}{\sqrt{0.75428+1}} = 0.75501$

$x_6 = \phi(x_5) = \frac{1}{\sqrt{0.75501+1}} = 0.75485$

$x_7 = \phi(x_6) = \frac{1}{\sqrt{0.75485+1}} = 0.75485$

Hence $x_6 = x_7$

So req^n root $= 0.7548$

Error $= |0.75488 - 0.75485|$

$= 0.00003$.

2) Using fixed point iteration method. Find square root of 5. correct to 4 decimal places

Here $x = \sqrt{5}$

or $x^2 = 5$

or $x^2-5=0$

$\therefore f(x) = x^2-5$

$f(0) = -5$

$f(2) = -1$ (-ve)

$f(3) = 4$ (+ve)

Since $f(2)$ & $f(3)$ have opp sign
A root lies betn 2&3
Let initial approximation
$x_0 = 2.5$
Now Rearranging the given eqn we get
Now,
$x^2 - 5 = 0$
or $x^2 = 5$
$\therefore x = \frac{5}{x} = \phi(x)$

where $\phi(x) = \frac{5}{x}$
By formula $x_{n+1} = \phi(x_n)$
The successive approximatn are
$x_1 = \phi(x_0) = \frac{5}{2.5} = 2$
$x_2 = \phi(x_1) = \frac{5}{2} = 2.5$
$x_3 = \phi(x_2) = \frac{5}{2.5} = 2$

$x_4 = \phi(x_3) = \frac{5}{2} = 2.5$
This process does not converge to the solution. This is called oscillatory divergence.

Again, rearranging the given eqn we get
$x^2 = 5$
or $x = \frac{5}{x}$

or $x + x = \frac{5}{x} + x$
or $2x = \frac{5 + x^2}{x}$
$\therefore x = \frac{5 + x^2}{2x} = \phi(x)$

Now the successive approximations
$x_1 = \phi(x_0) = \frac{5 + 2.5^2}{2(2.5)} = 2.25000$

$x_2 = \phi(x_1) = \frac{5 + 2.25000^2}{2(2.25000)} = 2.23611$

$x_3 = \phi(x_2) = \frac{5 + 2.23611^2}{2(2.23611)} = 2.23607$

$x_4 = \phi(x_3) = \frac{5 + 2.23607^2}{2(2.23607)} = 2.23607$

c) $2x - \log_{10} x = 7$ 4 decimal

$f(x) = 2x \log_{10} x - 7$

$f(1) = -5$ (-ve)
$f(3) = -1.47712$ (-ve)
$f(4) = 0.39739$ (+ve)

Here $f(3)$ and $f(4)$ have opp sign. So ~~zero~~ root lies bet^n 3 and 4

Now, $x_0 = 3.2$

Rearranging the given eq^n is

$$2x - \log_{10} x = 7$$

$$\therefore x = \frac{7 + \log_{10} x}{2}$$

Then the successive appr:

$$x_{n+1} = \phi(x_n)$$

$$x_1 = \phi(x_0) = \frac{7 + \log(x_0)}{2} = 3.75257$$

$$x_2 = \phi(x_1) = \frac{7 + \log(x_1)}{2} = 3.78716$$

$$x_3 = \phi(x_2) = \frac{7 + \log(x_2)}{2} = 3.78916$$

$$x_4 = \phi(x_3) = \frac{7 + \log(x_3)}{2} = 3.78927$$

$$x_5 = \phi(x_4) = \frac{7 + \log(x_4)}{2} = 3.78928$$

Now,

Required root $= 3.7892$

Chapter-Two

Solution of System of Linear Equations.

Method I> Gauss Elimination Method.

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular matrix form from which the unknowns are found by back substitution.

Let us consider a system of linear equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

The augmented matrix form is

$$\begin{bmatrix} a_1 & b_1 & c_1 & : d_1 \\ a_2 & b_2 & c_2 & : d_2 \\ a_3 & b_3 & c_3 & : d_3 \end{bmatrix}$$

With the help of a_1 make a_2 & a_3 to zero we get

$$\begin{bmatrix} a_1 & b_1 & c_1 & : d_1 \\ 0 & b_2' & c_2' & : d_2' \\ 0 & b_3' & c_3' & : d_3' \end{bmatrix}$$

With the help of b_2' make b_3' to zero,

$$\begin{bmatrix} a_1 & b_1 & c_1 & : d_1 \\ 0 & b_2' & c_2' & : d_2^2 \\ 0 & 0 & c_3''' & : d_3''' \end{bmatrix}$$

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Which is an upper triangular matrix
Using back substitution method
From R_3, $c_3''z = d_3'' \Rightarrow z$
From R_2 $b_2'y + c_2'z = d_2' \Rightarrow y$
From R_1 $a_1x + b_1y + c_2z = d_1 \Rightarrow x$
Hence we get value of $x, y \& z$.

1) Using Gauss Elimination method,

$2x+y+z=4$
$4x+2y+3z=4$
$x-y+z=0$

Here augmented matrix form is

$$\begin{bmatrix} 2 & 1 & 1 & : 4 \\ 4 & 2 & 3 & : 4 \\ 1 & -1 & 1 & : 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to 2R_3 - R_1$

$$\begin{bmatrix} 2 & 1 & 1 & : 4 \\ 0 & 0 & 1 & : -4 \\ 0 & -3 & 1 & : -4 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$, we get

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$$\begin{bmatrix} 2 & 1 & 1 & : 4 \\ 0 & -3 & 1 & : -4 \\ 0 & 0 & 1 & : -4 \end{bmatrix}$$

Then It is an upper triangular matrix
Using back substitution method

From R_3, $1z = -4 \Rightarrow z = -4$
From R_2 $-3y + z = -4$
or $-3y - 4 = -4$
$\therefore y = 0$
From R_1, $2x + y + z = 4$
$2x - 4 = 4$
or $2x = 8$
$\therefore x = 4$

Hence $x = 4$
$y = 0$
$z = -4$

Solve the system of linear equations by using Gauss Elimination method.

Soln:

$6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$
$12x_1 - 8x_2 + 6x_3 + 10x_4 = 26$
$3x_1 - 13x_2 + 9x_3 + 3x_4 = -19$
$-6x_1 + 4x_2 + x_3 - 18x_4 = -34$.

Here augmented matrix form

$$\left[\begin{array}{cccc:c} 6 & -2 & 2 & 4 & 16 \\ 12 & -8 & 6 & 10 & 26 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{array}\right]$$

$R_1 \to R_1 \div 2$

$R_2 \to R_2 \div 2$, we get.

$$\left[\begin{array}{cccc:c} 3 & -1 & 1 & 2 & 8 \\ 6 & -4 & 3 & 5 & 13 \\ 3 & -13 & 9 & 3 & -19 \\ -6 & 4 & 1 & -18 & -34 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 + 2R_1$

$$\left[\begin{array}{cccc:c} 3 & -1 & 1 & 2 & 8 \\ 0 & -2 & 1 & 1 & -3 \\ 0 & -12 & 8 & 1 & -27 \\ 0 & 2 & 3 & -14 & -18 \end{array}\right]$$

$R_2 \to R_3 - 6R_2$

$R_4 \to R_4 + R_2$

$$\left[\begin{array}{cccc:c} 3 & -1 & 1 & 2 & 8 \\ 0 & -2 & 1 & 1 & -3 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 4 & -13 & -21 \end{array}\right]$$

$$\left[\begin{array}{cccc:c} 3 & -1 & 1 & 2 & 8 \\ 0 & -2 & 1 & 1 & -3 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right]$$

which is an upper triangular matrix

Using back substitution we get

From $R_4: -3x_4 = -3 \Rightarrow x_4 = 1$

From $R_3: 2x_3 - 5x_4 = -9 \Rightarrow x_3 = -2$

From $R_2: -2x_2 + x_3 + x_4 = -3 \Rightarrow x_2 = 1$

From $R_1: 3x_1 - x_2 + x_3 + 2x_4 = 8 \Rightarrow x_1 = 3$

Hence $\left.\begin{array}{l} x_1 = 3 \\ x_2 = 1 \\ x_3 = -2 \\ x_4 = 1 \end{array}\right\}$ answer

Gauss Elimination Method with partial pivoting:

In the first step of Gauss Elimination method, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging 1st equation with the equation having largest coefficient of x

In second step, the ~~an~~ numerically largest, coefficient of y is chosen from the remaining equations and brought as the second pivot by interchanging the second equation with the equation having largest coefficient of y. This process is continued till we arrive at equ^n with single variable. This modified procedure is called partial pivoting

1)

Solve the following by using Gauss Elimination with partial pivoting:

$$2x+2y+z=6$$
$$4x+2y+3z=4$$
$$x-y+z=0$$

Sol^n:

Here the augmented matrix form is

$$\begin{bmatrix} 2 & 2 & 1 & : 6 \\ 4 & 2 & 3 & : 4 \\ 1 & -1 & 1 & : 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$, we get

$$\begin{bmatrix} 4 & 2 & 3 & : 4 \\ 2 & 2 & 1 & : 6 \\ 1 & -1 & 1 & : 0 \end{bmatrix}$$

$R_2 \to 2R_2 - R_1$

$R_3 \to 4R_3 - R_1$

$$\begin{bmatrix} 4 & 2 & 3 & : 4 \\ 0 & 2 & -1 & : 8 \\ 0 & -6 & 1 & : -4 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$ we get

$$\begin{bmatrix} 4 & 2 & 3 & : 4 \\ 0 & -6 & 1 & : -4 \\ 0 & 2 & -1 & : 8 \end{bmatrix}$$

$R_3 \to 3R_3 + R_2$

$$\begin{bmatrix} 4 & 2 & 3 & : 4 \\ 0 & -6 & 1 & : -4 \\ 0 & 0 & -2 & : 20 \end{bmatrix}$$

which is an upper triangular matrix.

Using back substitution, we get

From R_3, $-2z = 20 \Rightarrow z = -10$

From R_2 $-6y + z = -4$

$\quad$ or $-6y - 10 = -4$

$\quad$ or, $-6y = 6$

$\quad \therefore y = -1$

From R_1 $4x + 2y + 3z = 4$

$\quad$ or, $4x - 2 - 30 = 4$

$\quad$ or, $4x = 36$

$\quad \therefore x = 9$

Hence

$$\left.\begin{aligned} x &= 9 \\ y &= -1 \\ z &= -10 \end{aligned}\right\}$$

Ans: 1, 0, −1, 2

2) Solve the equations by using Gauss Elimination method with partial pivoting

$x + y + z + w = 2$

$x + y + 3z - 2w = -6$

$2x + 3y - z + 2w = 7$

$x + 2y + z - w = -2$

The augmented matrix form

$$\begin{bmatrix} 1 & 1 & 1 & 1 & : 2 \\ 1 & 1 & 3 & -2 & : -6 \\ 2 & 3 & -1 & 2 & : 7 \\ 1 & 2 & 1 & -1 & : -2 \end{bmatrix}$$

Now, $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & 3 & -1 & 2 & : 7 \\ 1 & 1 & 3 & -2 & : -6 \\ 1 & 1 & 1 & 1 & : 2 \\ 1 & 2 & 1 & -1 & : -2 \end{bmatrix}$$

$R_2 \rightarrow 2R_2 - R_1$

$R_3 \rightarrow 2R_3 - R_1$

$R_4 \rightarrow 2R_4 - R_1$

$$\begin{bmatrix} 2 & 3 & -1 & 2 & : 7 \\ 0 & -1 & 7 & -6 & : -19 \\ 0 & -1 & 3 & 0 & : -3 \\ 0 & 1 & 3 & -4 & : -11 \end{bmatrix}$$

$R_4 \leftrightarrow R_2$

$$\left[\begin{array}{cccc:c} 2 & 3 & -1 & 2 & 7 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & -1 & 3 & 0 & -3 \\ 0 & -1 & 7 & -6 & -19 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$R_4 \to R_3 + R_2$

$$\left[\begin{array}{cccc:c} 2 & 3 & -1 & 2 & 7 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 6 & -4 & -14 \\ 0 & 0 & 10 & -10 & -30 \end{array}\right]$$

$R_3 \leftrightarrow R_4$

$$\left[\begin{array}{cccc:c} 2 & 3 & -1 & 2 & 7 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 10 & -10 & -30 \\ 0 & 0 & 6 & -4 & -14 \end{array}\right]$$

$R_4 \to 5R_4 - \cancel{6}\,3R_3$

$$\left[\begin{array}{cccc:c} 2 & 3 & -1 & 2 & 7 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 10 & -10 & -30 \\ 0 & 0 & 0 & 10 & 20 \end{array}\right]$$

From R_4 ~~so~~ $w = 2$

From R_3 $10z - 10w = -30$

$10z = -10$

$z = -1$

From R_2 $1y + 3z - 4w = -11$

or $y - 3 - 4 \times 2 = -11$

$y = 0$

From R_1 so $2x + 3y - 1z + 2 = 7$

$2x = 7 - 3\times 0 - 1 - 2$

$x = 1$

Hence $\left.\begin{array}{l} x = 1 \\ y = 0 \\ z = -1 \\ w = 2 \end{array}\right\}$

Method II>

Gauss Jordan Method

This method is a modified form of Gauss - Elimination method. In this method the coefficient matrix is reduced to unit matrix and get the unknowns directly without using back substitution.

let us consider a system,

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$.

The augmented matrix form is

$$\left[\begin{array}{ccc:c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

$R_1 \longrightarrow \dfrac{R_1}{a_1}$, we get

$$\left[\begin{array}{ccc:c} 1 & b_1' & c_1' & d_1' \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$

With help of 1 making a_2 & a_3 to zero, we get

$$\left[\begin{array}{ccc:c} 1 & b_1' & c_1' & d_1' \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{array}\right]$$

$R_2 \longrightarrow R_2 \div b_2'$ we get

$$\left[\begin{array}{ccc:c} 1 & b_1' & c_1' & d_1' \\ 0 & 1 & c_2'' & d_2'' \\ 0 & b_3' & c_3' & d_3' \end{array}\right]$$

With help of 1 make b_1' & b_3' to zero, we get

$$\left[\begin{array}{ccc:c} 1 & 0 & c_1'' & d_1'' \\ 0 & 1 & c_2'' & d_2'' \\ 0 & 0 & c_3'' & d_3' \end{array}\right]$$

$R_3 \longrightarrow R_3 \div c_3''$ we get

$$\left[\begin{array}{ccc:c} 1 & 0 & c_1'' & d_1'' \\ 0 & 1 & c_2'' & d_2'' \\ 0 & 0 & 1 & d_3'' \end{array}\right]$$

Making c_1'' & c_2'' to zero with help of 1 we get

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & d_1''' \\ 0 & 1 & 0 & d_2''' \\ 0 & 0 & 1 & d_3''' \end{array}\right]$$

Hence $x = d_1'''$

$y = d_2'''$

$z = d_3'''$

Solution of System of linear Equations

Solve:

$$2x_1 + 4x_2 - 6x_3 = -8$$
$$x_1 + 3x_2 + x_3 = 10$$
$$2x_1 - 4x_2 - 2x_3 = -12$$

Here augmented matrix form is

$$\left[\begin{array}{ccc:c} 2 & 4 & -6 & -8 \\ 1 & 3 & 1 & 10 \\ 2 & -4 & -2 & -12 \end{array}\right]$$

$R_1 \to R_1 \div 2$, we get

$$\left[\begin{array}{ccc:c} 1 & 2 & -3 & -4 \\ 1 & 3 & 1 & 10 \\ 2 & -4 & -2 & -12 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & 2 & -3 & -4 \\ 0 & 1 & 4 & 14 \\ 0 & -8 & 4 & -4 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$

$R_3 \to R_3 + 8R_2$

$$\left[\begin{array}{ccc:c} 1 & 0 & -11 & -32 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 36 & 108 \end{array}\right]$$

$R_3 \to R_3 \div 36$

$$\left[\begin{array}{ccc:c} 1 & 0 & -11 & -32 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 \to R_1 + 11R_3$

$R_2 \to R_2 - 4R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Hence $x_1 = 1$ $x_3 = 3$
$x_2 = 2$

$1 \ -1 \ -1$

$-1 + \frac{1}{1} \times 4$

2) $2x_1 + 2x_2 - x_3 + x_4 = 4$

$4x_1 + 3x_2 - x_3 + 2x_4 = 6$

$8x_1 + 5x_2 - 3x_3 + 4x_4 = 12$

$x_1 + 3x_2 - 2x_3 + 2x_4 = 6$

The augmented matrix form

$$\left[\begin{array}{cccc:c} 2 & 2 & -1 & 1 & 4 \\ 4 & 3 & -1 & 2 & 6 \\ 8 & 5 & -3 & 4 & 12 \\ 1 & 3 & -2 & 2 & 6 \end{array}\right]$$

$R_1 \to R_1 \div 2$

$$\left[\begin{array}{cccc:c} 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 2 \\ 4 & 3 & -1 & 2 & 6 \\ 8 & 5 & -3 & 4 & 12 \\ 3 & 3 & -2 & 2 & 6 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 8R_1$, $R_4 \to R_4 - \overset{3}{R_1}$

$$\left[\begin{array}{cccc:c} 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 2 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & -3 & 1 & 0 & -4 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

$R_2 \to R_2 \div -1$

$$\left[\begin{array}{cccc:c} 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 2 \\ 0 & 1 & -1 & 0 & +2 \\ 0 & -3 & 1 & 0 & -4 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$R_3 \to R_3 + 3R_2$

$0+\frac{1}{2}\times 2$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

$R_3 \to R_3 \div -2$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right]$$

$R_1 \to R_1 - \frac{1}{2} R_3$

$R_2 \to R_2 + R_3$

$R_4 \to R_4 + \frac{1}{2} R_3$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{2} & 1+\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$R_4 \to R_4 * 2$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{2} & 1+\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$4-\frac{1}{2}*-1$ $\quad 0-\frac{1}{2}*$ $\quad 0+\frac{1}{2}$ $\quad 0-\frac{1}{2}*$

$R_1 \to R_1 - \frac{1}{2} R_4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$x_1 = 1$

$x_2 = 1$

$x_3 = -1$

$x_4 = -1$

Inverse of a matrix:

a) By using Gauss Elimination Method

We know that X will be the inverse of A if $AX = I$ where I is the unit matrix of the same order as A

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a given matrix

$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ be inverse of A such that

$AX = I$

The augmented matrix form is

$$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Using Gauss Elimination procedure, it reduces to

$$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & a & 0 & 0 \\ 0 & a_{22}' & a_{23}' & b & c & 0 \\ 0 & 0 & a_{33}''' & d & e & f \end{array}\right]$$

This can be divide into 3 systems:

System I $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} a \\ b \\ d \end{bmatrix}$

Using back substitution we get $\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}$

System II>

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ c \\ e \end{bmatrix}$$

Using back substitution we get $\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}$

System III>

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}$$

1> Find inverse of

$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

Let $X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$

be inverse of A such that

$AX = I$

The augmented matrix form

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to 2R_2 - 3R_1$

$R_3 \to 2R_3 - R_1$

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 7 & 17 & -1 & 0 & 2 \end{array}\right]$$

$R_3 \to R_3 - 7R_2$

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 0 & -4 & 20 & -14 & 2 \end{array}\right]$$

which is upper triangular matrix form

It can write in 3 systems

System

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 20 \end{bmatrix}$$

From back substitution

From R_3 $-4x_{31} = 20 \Rightarrow x_{31} = -5$

From R_2 $x_{21} + 3x_{31} = -3 \Rightarrow x_{21} = 12$

From R_1 $2x_{11} + x_{21} + x_{31} = 1 \Rightarrow x_{11} = -3$

$$\therefore \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ -5 \end{bmatrix}$$

System II

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -14 \end{bmatrix}$$

or, From back substitution

From R_3 $-4x_{32} = -14$

$\therefore x_{32} = \frac{7}{2}$

From R_2 $x_{22} + 3x_{32} = 2$

or, $x_{22} = \frac{-17}{2}$

From R_1 $2x_{12} + x_{22} + x_{32} = 0$

$x_{12} = \frac{+5}{2}$

From system III

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

From R_3

$-4x_{33} = 2$

$\therefore x_{33} = -\frac{1}{2}$

From R_2

$x_{23} + 3x_{33} = 0$

or, $x_{23} = -3x_{33}$

$= \frac{3}{2}$

From R_1

$2x_{13} + x_{23} + x_{33} = 0$

$\therefore x_{13} = \frac{1}{2} - \frac{3}{2}$

$\therefore x_{13} = -\frac{1}{2}$

Hence

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} & -\frac{1}{2} \\ 12 & -\frac{17}{2} & \frac{3}{2} \\ -5 & \frac{7}{2} & -\frac{1}{2} \end{bmatrix}$$

Q) Using Gauss Elimination method. Find inverse of

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

Now, $AX = I$

The augmented matrix form

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 + 2R_1$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$R_3 \rightarrow R_3 + R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

which is upper triangular matrix form.

Now,

system I:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

From R_3,

$$-4x_{31} = 1$$
$$\therefore x_{31} = -\frac{1}{4}$$

From R_2

$$2x_{21} - 6x_{31} = -1$$
$$\text{or } 2x_{21} = -1 + 6 \times \left(-\frac{1}{4}\right)$$
$$\therefore x_{21} = -\frac{5}{4}$$

From R_1

$$x_{11} + x_{21} + 3x_{31} = 1$$
$$\therefore x_{11} = 1 + \frac{5}{4} - 3 \times \left(-\frac{1}{4}\right)$$
$$= 3$$

System II

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

From R_3

$$-4x_{32} = 1$$
$$\therefore x_{32} = -\frac{1}{4}$$

From R_2

$$2x_{22} - 6x_{32} = 1$$
$$\therefore x_{22} = -\frac{1}{4}$$

From R_1

$$x_{12} + x_{22} + 3x_{32} = 0$$
$$\therefore x_{12} = 1$$

System III

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} x_{13} \\ x_{32} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From R_3 $\quad -4x_{33} = 1$

$$\therefore x_{33} = -\frac{1}{4}$$

From R_2 $\quad 2x_{32} - 6x_{33} = 0$

$$\therefore x_{32} = \frac{6x_{33}}{2}$$
$$= -\frac{3}{4}$$

From R_1 $\quad x_{13} + x_{32} + 3x_{33} = 0$

$$\therefore x_{13} = \frac{3}{2}$$

Hence

$$A^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{7}{4} \end{bmatrix}$$

Solution of System of linear Equations.

Inverse of a matrix

b) By Gauss - Jordan Method

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let X be the inverse of A such that

$AX = I$

The augmented matrix form is :

$$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

By using Gauss - Jordan procedure it is reduced to

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{array}\right]$$

Then $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

1) Find inverse of :

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

The augmented matrix form is

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3R_1$

$R_3 \rightarrow R_3 - R_1$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & -5 & -3 & 1 & 0 \end{array}\right]$$

$R_1 \to R_1 + R_2$

$R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 0 & 1 & -3 \end{array}\right]$$

$R_3 \to \frac{R_3}{-5}$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{array}\right]$$

$R_1 \to R_1 - 2R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{array}\right]$$

$$A = \begin{bmatrix} 0 & \frac{2}{5} & -\frac{1}{5} \\ -1 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

$-4-3\neq 0$

Method III > Factorisation Method:

This method is based on the fact that every square matrix A can be expressed as the product of lower triangular matrix and an upper triangular matrix.

Let us consider a system of linear eq^n

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

In matrix form It can be written as

$$AX = B$$

where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

let $LU = A$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$\& \; U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Now eqⁿ (i) becomes

$$LUX = B \longrightarrow (iii)$$

Let $UX = Z \longrightarrow (iv)$

Then $LZ = B \longrightarrow (v)$

solving eqⁿ (v) and then (iv) we get the value of X.

On the basis of diagonal elements there are two methods.

The factorisation with L having unit diagonal is called

a) Dolittle LU Factorisation Method

$$LU = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

b) The factorisation with U having unit diagonals is called Cholesky LU factorisation method

$$LU = A$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Solve the system by using Dolittle LU factorisation method.

$$3x_1 + 2x_2 + x_3 = 10$$
$$2x_1 + 3x_2 + 2x_3 = 14$$
$$x_1 + 2x_2 + 3x_3 = 14$$

Soln:

Here the given system is written in

$$AX = B$$

where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Now

$$LU = A$$

or

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$u_{11} = 3$, $u_{12} = 2$, $u_{13} = 1$

Now,

$l_{21} u_{11} = 2$

$l_{21} = \frac{2}{3}$

$l_{21} u_{12} + u_{22} = 3$

or $\frac{2}{3} \times 2 + u_{22} = 3$

or, $u_{22} = \frac{5}{3}$

$l_{21} u_{13} + u_{23} = 2$

$\frac{2}{3} \times 1 + u_{23} = 2$

$\therefore u_{23} = \frac{4}{3}$

$l_{31} u_{11} = 1$

$\therefore l_{31} = \frac{1}{3}$

$l_{31} u_{12} + l_{32} u_{22} = 2$

$\frac{1}{3} \times 2 + \frac{5}{3} l_{32} = 2$

or, $\frac{5}{3} l_{32} = \frac{4}{3}$

$\therefore l_{32} = \frac{4}{5}$

$l_{31} u_{13} + l_{32} u_{23} + u_{33} = 3$

or, $\frac{1}{3} \times 1 + \frac{4}{3} l_{32} + u_{33} = 3$

or $\frac{1}{3} + \frac{4}{3} \times \frac{4}{5} + u_{33} = 3$

$\therefore u_{33} = \frac{8}{5}$ //

Now,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix}$$

From eq^n (i)

$AX = B$

$LUX = B \longrightarrow$ (ii)

let $UX = Z \longrightarrow$ (iii)

From (ii) $LZ = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Using forward substitution,

From R_1, $z_1 = 10$

From R_2

$\frac{2}{3} z_1 + z_2 = 14 \Rightarrow z_2 = \frac{22}{3}$

From R_3

$\frac{1}{3} z_1 + \frac{4}{5} z_2 + z_3 = 14 \Rightarrow z_3 = \frac{24}{5}$

$$\therefore \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{22}{3} \\ 24/5 \end{bmatrix}$$

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From eq^n (iii)

$Ux = Z$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Using back substition

From R_3 $\frac{8}{5} x_3 = 14$

$\therefore x_3 = 3$

From R_2 : $\frac{5}{3} x_2 + \frac{4}{3} x_3 = \frac{22}{3}$

$\Rightarrow x_2 = 2$

From R_1 : $3x_1 + 2x_2 + x_3 = 10$

$\therefore x_1 = 1$

Hence

$$\left.\begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{array}\right\}$$

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2) Using LU factorisation method, solve the system

$2x + 3y + z = 9$

$x + 2y + 3z = 6$

$3x + y + 2z = 8$

Here the system can be written in matrix form as

$AX = B \longrightarrow (i)$

where $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

let $LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21} u_{11} & l_{21} u_{12} + u_{22} & l_{21} u_{13} + u_{23} \\ l_{31} u_{11} & l_{31} u_{12} + l_{32} u_{22} & l_{31} u_{13} + l_{32} u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Equating the corresponding element

$u_{11} = 2, \quad u_{12} = 3, \quad u_{13} = 1$

$l_{21} u_{11} = 1 \Rightarrow l_{21} = \frac{1}{2}$

$l_{21}u_{12}+u_{22}=2$

or, $u_{22}=2-\frac{1}{2}*3$

$=2-\frac{3}{2}$

$=\frac{1}{2}$

$l_{21}*u_{13}+u_{23}=3$

or, $u_{23}=3-\frac{1}{2}*1$

$=\frac{5}{2}$

~~$l_{31}u_{13}+l_{32}u_{23}+u_{33}=2$~~

or, ~~$u_{33}=2-1*$~~

$l_{31}u_{11}=3$

$\therefore l_{31}=\frac{3}{2}$

$l_{31}u_{12}+l_{32}u_{22}=1$

$\frac{3}{2}*3+\frac{1}{2}l_{32}=1$

$\therefore l_{32}=-7$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=2$

$\therefore u_{33}=18$

$$L=\begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ \frac{1}{2} & +1 & 0\\ \frac{3}{2} & -7 & 1\end{bmatrix}$$

$$U=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}=\begin{bmatrix}2 & 3 & 1\\ 0 & \frac{1}{2} & \frac{5}{2}\\ 0 & 0 & 18\end{bmatrix}$$

From eqⁿ ①

$AX=B$

or $LUX=B$

Let $UX=Z$

$LZ=B$

$$\begin{bmatrix}1 & 0 & 0\\ \frac{1}{2} & 1 & 0\\ \frac{3}{2} & -7 & 1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}9\\ 6\\ 8\end{bmatrix}$$

From R_1, $z_1=9$

R_2, $\frac{1}{2}z_1+z_2=6 \Rightarrow z_2=\frac{3}{2}$

R_3: $\frac{3}{2}z_1-7z_2+z_3=8 \Rightarrow z_3=5$

$$\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}=\begin{bmatrix}9\\ \frac{3}{2}\\ 5\end{bmatrix}$$

From eq^n (iii)

$UX = Z$

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

Using back substition

From R_3 $18z = 5 \Rightarrow z = \frac{5}{18}$

From R_2 $\frac{1}{2}y + \frac{5}{2}z = \frac{3}{2}$

$\therefore y = \frac{29}{18}$

From $2x + 3y + z = 5$

$\therefore x = 5 - \frac{5}{18} - 3 \times \frac{29}{18}$

$= \frac{35}{18}$

Using cholesky (Crout's method)

$2x_1 + 2x_2 + x_3 = 6$ (5, 1-6)

$4x_1 + 2x_2 + 3x_3 = 4$

$x_1 + x_2 + x_3 = 0$

$AX = B$

The augmented matrix as:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Now let $LU = A$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

or
$$\begin{bmatrix} l_{11} & l_{11} \times u_{12} & l_{11}u_{13} \\ l_{21} & l_{21} \times u_{12} + l_{22} & l_{21}u_{13} + l_{22} \times u_{23} \\ l_{31} & l_{31} \times u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Equating the value

$l_{11} = 2$ $\qquad$ $l_{11} \cdot u_{12} = 2$ $\qquad$ $l_{11}u_{13} = 1$

$\therefore u_{12} = 1$ $\qquad$ $\therefore u_{13} = \frac{1}{2}$

$l_{21} = 4$ $\qquad$ $l_{21}u_{12} + l_{23} = 2$ $\qquad$ $l_{31} = 1$

$4 \times 1 + l_{23} = 2$

$\therefore l_{22} = -2$

$l_{21}u_{13} + l_{22} \times u_{23} = 3$

$4 \times \frac{1}{2} + (-2) \times u_{23} = 3$

$u_{23} = \frac{-1}{2}$

$l_{31} \cdot u_{12} + l_{32} = 1$

$1 \times 1 + l_{32} = 1$

$\therefore l_{32} = 0$

$l_{31} u_{13} + l_{32} u_{23} + l_{33} = 1$

or, $1 \cdot \frac{1}{2} + 0 \times \left(\frac{-1}{2}\right) + l_{33} = 1$

or, $l_{33} = 1 - \frac{1}{2} = \frac{1}{2}$

Now,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{23} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Here $AX = B$

$LUX = B$

let $UX = Z$

So $LZ = B$

Now,
$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & -2 & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

or
$$\begin{bmatrix} 2z_1 \\ 4z_1 - 2z_2 \\ z_1 + \frac{1}{2} z_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

Using forward substitution

From R_1 $\quad 2z_1 = 6 \Rightarrow z_1 = 3$

From R_2 $\quad 4z_1 - 2z_2 = 4$

or $4 \times 3 - 2z_2 = 4$

or, $z_2 = 4$

From R_3 $\quad z_1 + \frac{1}{2} z_3 = 0$

or, $3 + \frac{1}{2} z_3 = 0$

or, $\frac{1}{2} z_3 = -3$

$\therefore z_3 = -6$

Again $UX = Z$

or
$$\begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

or,
$$\begin{bmatrix} x_1 + x_2 + \frac{1}{2} x_3 \\ x_2 - \frac{1}{2} x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

From back substitution,

From R_3 $\quad x_3 = -6$

From R_2 $\quad x_2 - \frac{1}{2} x_3 = 4$

or, $x_2 = 4 - 3 = 1$

From R_1 $\quad x_1 + 1 - 3 = 3 \Rightarrow x_1 = 5$

Hence
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$$

Method IV

Iterative Method:

a) Gauss- Seidal Method

b) Gauss- Jacobi's Method:

a) Gauss Seidal Method:

⇒ This method is a modified form of Gauss-jacobi's method.

Let us consider a system of linear Equations

$a_1x+b_1y+c_1z=d_1$

$a_2x+b_2y+c_2z=d_2$

$a_3x+b_3y+c_3z=d_3$

If the system is diagonally dominant, it can be written as,

From eq^n(i) $x=\frac{1}{a_1}[d_1-b_1y-c_1z]$

From eq^n ii) $y=\frac{1}{b_2}(d_2-a_2x-c_2z]$

From eq^n(iii) $z=\frac{1}{c_3}(d_3-a_3x-b_3y)$

Let initial approximation be $x=x_0$, $y=y_0$, $z=z_0$

For 1st approximation

put $y=y_0$ & $z=z_0$ in eq^n (i) we get,

$x_1=\frac{1}{a_1}[d_1-b_1y_0-c_1z_0]$

Put $x=x_1$ & $z=z_0$ in eq^n (ii) we get

$y_1=\frac{1}{b_2}(d_2-a_2x_1-c_2z_0)$

Put $x=x_1$ & $y=y_1$ in eq^n (iii) we get

$z_1=\frac{1}{c_3}[d_3-a_3x_1-b_3y_1]$

As soon as a new approximation for an unknown is found, it is immediately used in next step.

This process is repeated till values of x, y, z are obtained to desired accuracy.

1) Apply Gauss- Seidel Method, solve the systems:

$20x+y-2z=17$

$3x+20y-z=-18$

$2x-3y+20z=25$

Here the given system is diagonally dominant. It can be written as

$x=\frac{1}{20}[17-y+2z$

$y=\frac{1}{20}[-18-3x+z]$

setting

$A = \frac{17-B+2C}{20}$: $B = \frac{-18-2A+C}{20}$: $C = \frac{25-2A+3B}{20}$

$z = \frac{1}{-20}[25-2x+3y]$

Let initial approximation $x_0=y_0=z_0=0$

Tabulating for successive approximations, we get,

Itn	$x=\frac{1}{20}(17-y+2z)$	$y=\frac{1}{20}(-18-3x+z)$	$z=\frac{1}{20}(25-2x-3y)$
Initial	0	0	0
1	0.85	-1.03	1.01
2)	1.00	-1.00	1.00
3)	1.00	-1.00	1.00

From table,

$x=1$

$y=-1$

$z=1$

2) Using Gauss seidal iteration method solve the system.

$10x_1-2x_2-x_3-x_4=3$

$-2x_1+10x_2-x_3-x_4=15$

$-x_1-x_2+10x_3-2x_4=27$

$-x_1-x_2-2x_3+10x_4=-9$

Here the given system is diagonally dominant. It can be written as,

$x_1 = \frac{3+2x_2+x_3+x_4}{10}$

$x_2 = \frac{15+2x_1+x_3+x_4}{10}$

$x_3 = \frac{27+x_1+x_2+2x_4}{10}$

$x_4 = \frac{(-9+x_1+x_2+2x_3)}{10}$

Let initial approximation $x_{1_0}=x_{2_0}=x_{3_0}=x_{4_0}=0$

Tabulating successive approximation

Itn	x_1	x_2	x_3	x_4
Initial	0	0	0	0
1)	0.30	1.56	2.89	-0.14
2)	0.89	1.95	2.96	-0.02
3)	0.98	1.99	2.99	0.00
4)	1.00	2.00	3.00	0.00
5)	1.00	2.00	3.00	0.00

From table

$$\left.\begin{aligned} x_1 &= 1 \\ x_2 &= 2 \\ x_3 &= 3 \\ x_4 &= 0 \end{aligned}\right\} \text{Ans}$$

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3) $10x_1 + 6x_2 - 5x_3 = 27$
$3x_1 + 8x_2 + 10x_3 = 27$
$4x_1 + 10x_2 + 3x_3 = 27$

Rearranging the given eqⁿ to make diagonally dominant we get

$10x_1 + 6x_2 - 5x_3 = 27$
$4x_1 + 10x_2 + 3x_3 = 27$
$3x_1 + 8x_2 + 10x_3 = 27$

Then

$$x_1 = \frac{27 - 6x_2 + 5x_3}{10}$$

$$x_2 = \frac{27 - 4x_1 - 3x_3}{10}$$

$$x_3 = \frac{27 - 3x_1 - 8x_2}{10}$$

Tabulating successive approximation

Itⁿ	x_1	x_2	x_3
Initial	0	0	0
1)	2.70	1.62	0.59
2)	2.03	1.71	0.72
3)	2.03		

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setting: $\frac{17-B+2C}{20}$:

$(-1, 1$

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1) Solve the following set of equations by Gauss-Jacobi method.

$$20x+y-2z=17$$
$$3x+20y-z=-18$$
$$2x-3y+20z=25$$

Here the system is diagonally dominant. It can be written as

$$x=\frac{1}{20}(17-y+2z)$$
$$y=\frac{1}{20}(-18-3x+z)$$
$$z=\frac{1}{20}(25-2x+3y)$$

Let initial approximation $x_0=y_0=z_0=0$.

Tabulating for successive approximation.

Itn	x	y	z
Initial	0	0	0
1	0.85	-0.90	1.25
2)	1.02	-0.97	1.03
3)	1.00	-1.00	1.00
4)	1.00	-1.00	1.00

From table

$$\left.\begin{aligned} x&=1\\ y&=-1\\ z&=1 \end{aligned}\right\}\text{Ans}$$

2) $2x_1 + x_2 + x_3 = 5$

$3x_1 + 5x_2 + 2x_3 = 15$

$2x_1 + x_2 + 4x_3 = 8$

Here the system is diagonally dominant

It can be ~~do~~ written as

$$x_1 = \frac{5 - x_2 - x_3}{2}$$

$$x_2 = \frac{15 - 3x_1 - 2x_3}{5}$$

$$x_3 = \frac{8 - x_2 - 2x_1}{4}$$

Let initial approximation

$x_0 = y_0 = z_0 = 0$

Tabulating successive approximat

Itn	x_1	x_2	x_3
Initial	0	0	0
1	2.50	3.00	2.00
2)	0.00	0.70	0.00
3,	2.15	3.00	1.83
4)	0.09	0.98	0.18
5)	1.92	2.87	1.71
6)	0.21	1.16	0.32
7)	1.76	~~1.16~~ 2.75	1.61
8)	0.32	1.30	0.43
9	1.64	2.64	1.52
10	0.42	1.41	0.52
11	1.54	2.54	1.44
12	0.51	1.5	0.60
13	1.45	2.45	1.37
14	0.59	1.58	0.66
15	1.38	2.38	1.31
16	0.66	1.65	0.72
17	1.32	2.32	1.26
18	0.71	1.70	0.76
19	1.27	2.27	1.22
20	0.76	1.75	0.80
21	1.23	2.22	1.18
22	0.80	1.79	0.83
23	1.19	2.19	1.15
24	0.83	1.83	0.86
25	1.16	2.16	1.13
26	0.86	1.85	0.88
27	1.14	2.13	1.11
28	0.88	1.87	0.90
29	1.12	2.11	1.09
30	0.9	1.89	0.91
31	1.10	2.10	1.08
32	0.91	1.91	0.93
33	1.08	2.08	1.07
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Eigen value & Eigen vector

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Problems by power Method

1↳ If A be any square matrix of order $n*n$ with elements a_{ij} we can find a column matrix X and a constant λ such that $AX=\lambda X$ where λ is called the eigen value & X is called the corresponding eigen vector.

Let A be the given matrix & $X^{(0)}$ be the initial approximation of eigen vector. We evaluate AX^{0} which is written as $\lambda^{(1)}X^{(1)}$. This gives the first approximation. Similarly we evaluate $AX^{(1)}=\lambda^{(2)}X^{(2)}$ which gives the second approximation. we repeat this procedure to ge the repeatation of X as well as λ.

Note:

1) For initial approximation, take

$$X^{(0)}=\begin{bmatrix}1\\0\\0\end{bmatrix} \text{ or } \begin{bmatrix}0\\1\\0\end{bmatrix} \text{ or } \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ or } \begin{bmatrix}1\\1\\0\end{bmatrix} \text{ or } \begin{bmatrix}1\\0\\1\end{bmatrix} \text{ or } \begin{bmatrix}0\\1\\1\end{bmatrix} \text{ or } \begin{bmatrix}1\\1\\1\end{bmatrix}$$

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2) Evaluate $AX^{(0)}$ and take numerically largest value as common & divide the whole element of the vector with largest value

1) Find the largest eigen value and corresponding eigen vector of matrix

$$A=\begin{bmatrix}1&2&0\\2&1&0\\0&0&-1\end{bmatrix}$$

soln:

Here Given matrix is $A=\begin{bmatrix}1&2&0\\2&1&0\\0&0&-1\end{bmatrix}$

Let $X^{(0)}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ be the initial approximation for eigen vector. Then successive approximations are:

$$AX^{(0)}=\begin{bmatrix}1&2&0\\2&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\2\\0\end{bmatrix}=2\begin{bmatrix}0.5\\1\\0\end{bmatrix}$$
$$=\lambda^{(1)}X^{(1)}$$

$$AX^{(1)}=\begin{bmatrix}1&2&0\\2&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0.5\\1\\0\end{bmatrix}=\begin{bmatrix}2.5\\2\\0\end{bmatrix}=2.5\begin{bmatrix}1\\0.8\\0\end{bmatrix}$$
$$=\lambda^{(2)}X^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 2.8 \\ 0 \end{bmatrix}$$

$$= 2.8 \begin{bmatrix} 0.93 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.93 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 2.86 \\ 0 \end{bmatrix}$$

$$= 2.93 \begin{bmatrix} 1 \\ 0.98 \\ 0 \end{bmatrix}$$

$$= \lambda^{(4)} x^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.98 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.96 \\ 2.98 \\ 0 \end{bmatrix} =$$

$$= 2.98 \begin{bmatrix} 0.99 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(5)} x^{(5)}$$

$$Ax^{(5)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.99 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 2.98 \\ 0 \end{bmatrix}$$

$$= 2.99 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{6} x^{(6)}$$

$$Ax^{(6)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(7)} x^{(7)}$$

$$Ax^{(7)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(8)} x^{(8)}$$

Hence, largest eigenvalue $\lambda = 3$ corresponding eigen vector $X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Let $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be initial approximation for eigen vector. Then successive approximation

$$Ax^{(0)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$$

$$= 7\begin{bmatrix} 1 \\ 0.43 \\ 0 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.43 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3.58 \\ 1.86 \\ 0 \end{bmatrix} = 3.58\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$= \lambda^{(4)} x^{(4)}$$

$$Ax^4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$$Ax^5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(6)} x^{(6)}$$

Hence largest eigen value = 4

corresponding eigen vector = $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$

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3) Determine the largest eigen value and the corresponding eigenvector of the matrix using power method.

$$A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$

Let $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be initial approximation for eigen vector. Then successive approximation

$$Ax^{(0)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ -20 \end{bmatrix} = 20\begin{bmatrix} 0.75 \\ -0.5 \\ -1 \end{bmatrix}$$

$$= \lambda_1^{(1)} x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0.75 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 16.25 \\ -7.50 \\ -15 \end{bmatrix} = 16.25\begin{bmatrix} 1 \\ -2.17 \\ -0.92 \end{bmatrix}$$

$$= \lambda^{(2)} x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -2.17 \\ -0.92 \end{bmatrix} = \begin{bmatrix} 26.49 \\ -30.52 \\ -23.16 \end{bmatrix} = 30.52\begin{bmatrix} 0.87 \\ -1 \\ -0.76 \end{bmatrix}$$

$$= \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}\begin{bmatrix} 0.87 \\ -1 \\ -0.76 \end{bmatrix} = \begin{bmatrix} 19.33 \\ -16.14 \\ -16.84 \end{bmatrix} = 19.33\begin{bmatrix} 1 \\ -0.83 \\ -0.87 \end{bmatrix}$$

$$AX^{(4)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.83 \\ -0.87 \end{bmatrix} = \begin{bmatrix} 20.93 \\ -11.74 \\ -21.58 \end{bmatrix}$$

$$= 21.58 \begin{bmatrix} 0.97 \\ -0.68 \\ -1 \end{bmatrix}$$

$$= \lambda^5 X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.97 \\ -0.68 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.27 \\ -11.86 \\ -20.12 \end{bmatrix}$$

$$= 20.27 \begin{bmatrix} 1 \\ -0.59 \\ -0.99 \end{bmatrix}$$

$$= \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.59 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 20.33 \\ -11.14 \\ -20.38 \end{bmatrix}$$

$$= 20.38 \begin{bmatrix} 0.99 \\ -0.55 \\ -1 \end{bmatrix}$$

$$= \lambda^{(7)} X^{(7)}$$

$$AX^{(7)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.99 \\ -0.55 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.05 \\ -10.50 \\ -20.00 \end{bmatrix}$$

$$= 20.05 \begin{bmatrix} 1 \\ -0.51 \\ -1 \end{bmatrix}$$

$$AX^{(8)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.51 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.04 \\ -10.12 \\ -20.04 \end{bmatrix} = 20.04 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$$

$$AX^{(9)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.00 \\ -10 \\ -20.00 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$$

$$AX^{10} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.00 \\ -10.00 \\ -20.00 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$$

Hence largest eigen value $= 20$

corresponding eigen vector $= \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$

CHAPTER- Three

Interpolation and Approximation

Computing the value for a tabulated function at a point is called interpolation.

Suppose we are given the following values of $y=f(x)$ for a set of values of x.

x	x_0	x_1	x_2	----	x_n
$y=f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$		$f(x_n)$

Then the process of finding the value of corresponding to any value of x is called interpolation

Interpolation with unequal intervals

Let $f(x_0), f(x_1), f(x_2), f(x_3), ---$ be the values corresponding to $x = x_0, x_1, x_2, x_3 -----$ where $x_0, x_1, x_2 ---$ are not equispaced. Then we can find value of $f(x)$ by following methods:

1) Lagrange's interpolation method.
2) Divided difference interpolation method.

Method-I

Lagrange's Interpolation method

Let us consider a tabulated function:

x	x_0	x_1	x_2	x_3
$y=f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Then

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)$$

This formula is called lagrange's interpolation formula.

Similarly

x	x_0	x_1	x_2	x_3	x_4
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times f(x_0) + \frac{(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_0)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} f(x_1) + \frac{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} f(x_3)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} f(x_4)$$

1) Using Lagrange interpolation Find f(10) from following table

x	5	6	9	11
y	12	13	14	16

Soln:

Here $x_0 = 5$ $f(x_0) = 12$

$x_1 = 6$ $f(x_1) = 13$

$x_2 = 9$ $f(x_2) = 14$

$x_3 = 11$ $f(x_3) = 16$

$f(x) = ?$

Using Lagrang's interpolation formula

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13 + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$= 14.67$

2)

x	5	7	11	13	17
f(x)	150	392	1452	2366	5202

Using Lagrang's formula evaluate f(9)

Soln:

$x = 9$ $f(x_0) = 15$

$x_0 = 5$ $f(x_1) = 392$

$x_1 = 7$ $f(x_2) = 1452$

$x_2 = 11$ $f(x_3) = 2366$

$x_3 = 13$ $f(x_4) = 5202$

$x_4 = 17$

Now,

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392 + \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366 + \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$= 810$

Hence $f(9) = 810$

1st

Find Eq^n $f(x)$

3)

x	0	1	3	4
y	-12	0	6	12

Soln:

$x_0 = 0$ $f(x_0) = -12$

$x_1 = 1$ $f(x_1) = 0$

$x_2 = 3$ $f(x_2) = 6$

$x_3 = 4$ $f(x_3) = 12$

Now,

$$f(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} * (-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-3)} * 0 + \frac{(x-1)(x-0)(x-4)}{(3-1)(3-0)(3-4)} * 6 + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} * 12$$

$$= \frac{-12(x^2-x+3)(x-4)}{-1*-3*-4} + \frac{6(x^2-x)(x-4)}{2*3*-9} + \frac{12(x^2-x)(x-3)}{4*3*1}$$

$$= \frac{-12(x^3-x^2+3x-4x^2+4x-12)}{-12} + \frac{6(x^3-4x^2-x^2+4x)}{6*-9} + \frac{12(x^3-3x^2-x^2+3x)}{12}$$

$$= (x^3-5x^2+7x-12) + x^3-4x^2+3x + \frac{(x^3-5x^2+4x)}{-9}$$

$$= \frac{-9(2x^3-9x^2+10x-12)+x^3-5x^2+4x}{-9}$$

$$= \frac{-18x^3+81x^2-90x+108+x^3-5x^2+4x}{-9}$$

$$= \frac{-17x^3+76x^2-86x+108)}{-9}$$

$$= \frac{17x^3-76x^2+86x-108}{9}$$ //

4) The function $y = f(x)$ is given at the points (7,3), (8,1), (9,1) & (10,9). Find the value of y for $x = 9.5$ using Lagrange interpolation formula.

Here

Given table is

x	7	8	9	10
$f(x)$	3	1	1	9

Now,

$$f(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10)}{(7-8)(7-9)(7-10)} * 3 + \frac{(9.5-7)(9.5-9)(9.5-10)}{(8-7)(8-9)(8-10)} * 1 + \frac{(9.5-7)(9.5-8)(9.5-10)}{(9-7)(9-8)(9-10)} * 1 + \frac{(9.5-7)(9.5-8)(\ldots}{(10-7)(10\ldots}$$

Find eq^n $f(x)$

3)

x	0	1	3	4
y	-12	0	6	12

Soln:

$x_0 = 0 \quad f(x_0) = -12$

$x_1 = 1 \quad f(x_1) = 0$

$x_2 = 3 \quad f(x_2) = 6$

$x_3 = 4 \quad f(x_3) = 12$

Now,

$$f(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} * (-12) + \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-3)} * 0 + \frac{(x-1)(x-0)(x-4)}{(3-1)(3-0)(3-4)} * 6 + \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} * 12$$

$$= \frac{-12(x^2-x+3)(x-4)}{-1*-3*-4} + \frac{6(x^2-x)(x-4)}{2*3*-9} + \frac{12(x^2-x)(x-3)}{4*3*1}$$

$$= \frac{-12(x^3-x^2+3x-4x^2+4x-12)}{-12} + \frac{6(x^3-4x^2-x^2+4x)}{6*-9} + \frac{12(x^3-3x^2-x^2+3x)}{12}$$

$$= (x^3-5x^2+7x-12) + x^3-4x^2+3x + \frac{(x^3-5x^2+4x)}{-9}$$

$$= \frac{-9(2x^3-9x^2+10x-12) + x^3-5x^2+4x}{-9}$$

$$= \frac{-18x^3+81x^2-90x+108+x^3-5x^2+4x}{-9}$$

$$= \frac{-17x^3+76x^2-86x+108}{-9}$$

$$= \frac{17x^3-76x^2+86x-108}{9} \; //$$

4) The function $y = f(x)$ is given at the points $(7,3)$, $(8,1)$, $(9,1)$ & $(10,9)$. Find the value of y for $x = 9.5$ using Lagrange interpolation formula.

~~Here~~

Given table is

x	7	8	9	10
$f(x)$	3	1	1	9

Now,

$$f(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10)}{(7-8)(7-9)(7-10)} * 3 + \frac{(9.5-7)(9.5-9)(9.5-10)}{(8-7)(8-9)(8-10)} * 1 + \frac{(9.5-7)(9.5-8)(9.5-10)}{(9-7)(9-8)(9-10)} * 1 + \frac{(9.5-7)(9.5-8)(9.5-9)}{(10-7)(10-8)(10-9)} * 9$$

$= -\frac{3}{16} - \frac{5}{16} + \frac{15}{16} + \frac{45}{16}$

$= \frac{58}{16}$

$= 3.63$

PU (2019) Fall

4(b)

Determine the largest eigen value and corresponding eigen vector of matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$

Soln:

Let $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$AX^{(0)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$

$= 6 \begin{bmatrix} 0.17 \\ 0.67 \\ 1 \end{bmatrix} = \lambda' x^{(1)}$

$AX^{(1)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.17 \\ 0.67 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 0.16 \\ 2.36 \\ 8.03 \end{bmatrix} = 8.03 \begin{bmatrix} 0.02 \\ 0.29 \\ 1 \end{bmatrix} = \lambda^{(2)} x^{(2)}$

$AX^{(2)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.29 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.15 \\ 0.24 \\ 6.99 \end{bmatrix} = 5.99 \begin{bmatrix} 0.19 \\ 0.04 \\ 1 \end{bmatrix}$

$= \lambda^{(3)} x^{(3)}$

$AX^{(3)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.19 \\ 0.04 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.07 \\ -0.08 \\ 6.26 \end{bmatrix} = 6.26 \begin{bmatrix} 0.33 \\ -0.01 \\ 1 \end{bmatrix}$

$= \lambda^{(4)} x^{(4)}$

$AX^{(4)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.33 \\ -0.01 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.36 \\ 0.28 \\ 6.95 \end{bmatrix} = 6.95 \begin{bmatrix} 0.34 \\ 0.04 \\ 1 \end{bmatrix}$

$= \lambda^{(5)} x^{(5)}$

$AX^{5} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.34 \\ 0.04 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.22 \\ 0.52 \\ 7.16 \end{bmatrix} = 7.16 \begin{bmatrix} 0.31 \\ 0.07 \\ 1 \end{bmatrix}$

$= \lambda^{(6)} x^{(6)}$

$AX^{6} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.31 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.1 \\ 0.52 \\ 7.07 \end{bmatrix} = 7.07 \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix}$

$$AX^{(7)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.09 \\ 0.48 \\ 7.01 \end{bmatrix}$$

$$= 7.01 \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix}$$

$$AX^{(8)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.09 \\ 0.48 \\ 7.01 \end{bmatrix}$$

$$= 7.01 \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix}$$

Largest eigen value $\doteq 7.01$

corresponding eigen vector

$$= \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} //$$

Method II

Divided Difference Method:-

Divided difference table.

x	$y = f(x)$	$\Delta^1 f(x)$
x_0	$f(x_0)$	$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f(x_0, x_1)$
x_1	$f(x_1)$	$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f(x_1, x_2)$
x_2	$f(x_2)$	$\frac{f(x_3) - f(x_2)}{x_3 - x_2} = f(x_2, x_3)$
x_3	$f(x_3)$	

$\Delta^2 f(x)$	$\Delta^3 f(x)$
$\frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} = f(x_0, x_1, x_2)$	$\frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$
$\frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = f(x_1, x_2, x_3)$	$= f(x_0, x_1, x_2, x_3)$

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3).$$

Construct a divided difference table of

x	0	2	3	5	6
$f(x)$	1	19	55	241	415

Divided difference table

x	$f(x)$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1	$\frac{19-1}{2-0}=9.0$	$\frac{36-9.0}{3-0}=9$	$\frac{19-9.0}{5-0}=2$
2	19	$\frac{55-19}{3-2}=36$	$\frac{93-36}{5-2}=19$	$\frac{27-19}{6-2}=2$
3	55	$\frac{241-55}{5-3}=93$	$\frac{174-93}{6-3}=27$	
5	241	$\frac{415-241}{6-5}=174$		
6	415			

$\Delta^4 f(x)$
$\frac{2-2}{6-0}=0$

Interpolation & Approximation

M-

2)

x	5	7	11	13	17
y	150	392	1452	2366	5202

Evaluate $f(9)$

Soln:

The divided difference table

x	$f(x)=y$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
5	150	$\frac{392-150}{7-5}=121$	$\frac{265-121}{11-5}=24$	$\frac{32-24}{13-5}=1$	$\frac{1-1}{17-5}=0$
7	392	$\frac{1452-392}{11-7}=265$	$\frac{457-265}{13-7}=32$	$\frac{42-32}{17-7}=1$	
11	1452	$\frac{2366-1452}{13-11}=457$	$\frac{709-457}{17-11}=42$		
13	2366	$\frac{5202-2366}{17-13}$ 709			
17	5202				

Using Newton's Divided difference.

$$f(9)=150+(9-5)\times 121+(9-5)\times(9-7)\times 24$$
$$+(9-5)(9-7)(9-11)\times 1$$
$$=810$$

3) The following table give percentage of criminals for different age groups. Using suitable interpolation formula. Find percentage of criminals under age 35

Under age	25	30	40	50
No% of criminals	52	67.3	84.1	94.4

The divided difference table

x	$f(x)$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
25	52	$\frac{67.3-52}{30-25} = 3.06$	$\frac{1.68-3.06}{40-25}$	
30	67.3	$\frac{84.1-67.3}{40-30} = 1.68$	$= -0.092$	$\frac{-0.0325+0.092}{50-25}$
40	84.1	$\frac{94.4-84.1}{50-40} = 1.03$	$\frac{1.03-1.68}{50-30}$	$= 2.38\times10^{-3}$ $= 0.00238$
50	94.4		$= -0.0325$	

$f(35) = 52 + (35-25)\times 3.06 + (35-25)(35-30) \times -0.092 + (35-25)(35-30)(35-40) \times 0.0025$

$= 77.375$

Find polynomial eq^n.

4)

x	0	1	2	5
$f(x)$	2	3	12	147

The divided difference table is:

x	$f(x)$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	2	$\frac{3-2}{1-0} = 1$	$\frac{9-1}{2-0} = 4$	$\frac{9-4}{5-0} = 1$
1	3	$\frac{12-3}{2-1} = 9$	$\frac{45-9}{5-1}$	
2	12	$\frac{147-12}{5-2} = 45$	$= 9$	
5	147			

$f(x) = 2 + (x-0)\times 1 + (x-0)(x-1)\times 4 + (x-0)(x-1)(x-2)\times 1$

$= 2 + x + x(x-1)\times 4 + x(x-1)(x-2)$

$= 2 + x + 4x^2 - 4x + x(x^2 - 2x - x + 2)$

$= 2 + x + 4x^2 - 4x + x^3 - 3x^2 + 2x$

$= 2 + 3x + x^2 + x^3 - 4x$

$= 2 - x + x^2 + x^3 \quad = x^3 + x^2 - x + 2$

PU 2016 (Spring)

5)

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

The divided difference table

x	$f(x)$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1	$\frac{14-1}{1-0}=13$	$\frac{1-13}{2-0}=-6$	$\frac{-2+6}{4-0}=1$	$\frac{1-1}{5-0}=0$
1	14	$\frac{15-14}{2-1}=1$	$\frac{-5-1}{4-1}=-2$	$\frac{2+2}{5-1}=1$	
2	15	$\frac{5-15}{4-2}=-5$	$\frac{1+5}{5-2}=2$	$\frac{6-2}{6-2}=1$	$\frac{1-1}{6-1}=0$
4	5	$\frac{6-5}{5-4}=1$	$\frac{13-1}{6-4}=6$		
5	6	$\frac{19-6}{6-5}=13$			
6	19				

$$f(3) = 1 + (3-0)13 + (3-0)(3-1)\times -6 + (3-0)(3-1)(3-2)\times 1 + (3-0)(3-1)(3-2)(3-4)\times 0$$

$$= 10.$$

Method III Interpolation With Equal Interval
Newton Forward Interpolation Method

Formula:

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

where $p = \frac{x-x_0}{h}$

h = size of intervals.

This formula is used if x is near to the beginning value of given tabulated function. Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$ & others are found in difference table.

Method IV Newton's Backward Interpolation method

Formula: $f(x) = y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \dots$

where $p = \frac{x-x_0}{h}$

h = class size

$\nabla y_0, \nabla^2 y_0, \nabla^3 y_0$ & others are found in difference table.

This formula is used if x is near to the ending value of tabulated function

Q No. 1 From following table, estimate the number of students who obtained marks 45.

Marks (x)	40	50	60	70	80
No. of std $f(x)$	31	73	124	159	190

Using suitable interpolation

Soln:

Here $x = 45$ is near to the beginning value of tabulated functn, Newton's Forward interpolation formula is suitable formula.

The difference table is given below

$-16-5$
$-4+16$

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

From table:

$x = 45$ $\quad\therefore p = \dfrac{x - x_0}{h} = \dfrac{45-40}{10} = 0.5$

$x_0 = 40$

$h = 10$

We know Newton's Forward interpolation formula.

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 31 + 0.5 \times 42 + \frac{0.5(0.5-1)}{2!} \times 9 + \frac{0.5(0.5-1)(0.5-2)}{3!} \times (-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!} \times 37$$

$= 47.8672$

Hence, number of students who obtained marks $45 = 47.8672$

$= 48$ //

2) The following table gives the pop^n of ~~then~~ town during last six census. Estimate using Newton's interpolation formula, the increase in pop^n during perid 1946 & 1948?

Year	1911	1921	1931	1941	1951	1961
pop^n(000)	12	13	20	27	39	52

Here, $x = 1946$ & 1948 are near to the ending value of the tabulated $funct^n$. Newton's backward interpolation formula is suitable method

x	$y=f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1911	12					
		1				
			6			
1921	13				1	
		7		-6		
1931	20		0		11	
		7		5		-20
1941	27		5		-9	
		12		-4		
1951	39		1	$\nabla^3 y_0$	$\nabla^4 y_0$	$\nabla^5 y_0$
		13	$\nabla^2 y_0$			
1961	52	∇y_0				
x_0	y_0					

▷ For 1946

$h = 10$

$x = 1946$

$x_0 = 1961$

$p = \frac{x - x_0}{h} = \frac{1946 - 1961}{10} = -1.5$

We know Newton's interpolation formula

$$f(x) = y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_0 + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_0$$

$-1.5+4$
$=2.5$

$= 52 + (-1.5) \times 13 - 1.5 \frac{(-1.5+1)}{2} \times 1$

$- 1.5 \frac{(-1.5+1)(-1.5+2)(-4)}{3!}$

$\frac{1.5(-1.5+1)(-1.5+2)(-1.5+3)}{4!} \times (-9)$

$- \frac{1.5(-1.5+1)(-1.5+2)(-1.5+3)(-1.5+4)}{5!}$

$\times (-20)$

$= 32.180$

11) For 1948.

$x = 1948$

$x_0 = 1961$

$\therefore p = \frac{x - x_0}{h} = -1.3$

Then

$f(x) =$

$f(1948) = 52 + (-1.3 \times 13) - 1.3 \frac{(-1.3+1)}{2} \times 1$

$- \frac{1.3(-1.3+1)(-1.3+2)}{3!} \times (-4)$

$- \frac{1.3(-1.3+1)(-1.3+2)(-1.3+3)}{4!}$

$\times (-9) - 1.3 \frac{(-1.3+1)(-1.3+2)(-1.3+3)(-1.3+4)}{5!} \times (-20)$

$= 34.7301$

Now,

Increase in $pop^n = 34.7301 - 32.180$
$= 2.5501$

3) From Following data, estimate the no. of persons earning hourly wages between 60 & 70 rupees.

wage Rs	20-40	40-60	60-80	80-100	100-120
pop^n (000)	250	120	100	70	50

Here the difference table is as below

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
" 60	370		-20		
		100		-10	
" 80	470		-30		20
		70		10	
" 100	540		-20		
		50			
" 120	590				

Below 70

From table

$x_0 = 40$ $\therefore p = \frac{x - x_0}{h} = \frac{70-40}{20} = 1.5$

$h = 20$

$x = 70$

From Newton's Forward interpolat^n formula

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$\therefore f(70) = 423.594.$

$f(60) = 370$

Then

No. of person's earning below 60 $= 370$

No. of person's bet^n 60 & 70

$= 423.594 - 370$

$= 53.594$ thousand

$= 53594$

4) The following table gives the displacement of an object at various of time. Find the velocity & acceleration of object at $t = 1.6$ using suitable interpolation

x	t(sec)
y	

4. Estimate the value of $\sin\theta$ at $\theta = 25^\circ$ using Newton's forward interpolation

θ	10	20	30	40	50
$y = \sin\theta$	0.1736	0.3420	0.5000	0.6428	0.7660

The forward difference table is

θ	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	0.1736				
		0.1684			
			-0.0104		
20	0.3420			-0.0048	
		0.1580			0.0004
30	0.5000		-0.0152		
		0.1428		-0.0044	
40	0.6428		-0.0196		
		0.1232			
50	0.7660				

~~f(25) =~~

$x = 25^\circ \qquad h = 10$

$x_0 = 10$

$p = \frac{x - x_0}{h} = \frac{25-10}{10} = 1.5$

Then

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} * \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} * \Delta^4 y_0$$

$= 0.4226$

a) Fitting of a str. line
b) Fitting of a parabola
c) Fitting of a exponential curve

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Fitting of a Curve:

a) Fitting of a straight line:

The equation of straight line is

$y = a + bx \longrightarrow (1)$

Fitting of a curve means finding the value of parameters a & b with help of normal equations

The normal equations are

$na + b\Sigma x = \Sigma y \longrightarrow (i)$

$a\Sigma x + b\Sigma x^2 = \Sigma xy \longrightarrow (ii)$

Tabulating for $\Sigma x, \Sigma y, \Sigma x^2, \Sigma xy$ then solving (i) & (ii) we get value of a & b. Substituting value of a & b in eqn (1) we get required eqn of straight line

1) Fit a straight line by using least square method to following data

x	1	2	3	4	5
y	14	27	40	55	68

Estimate value of y when x=6

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Soln:

Let eqn of straight line to be fit is

$y = a + bx \longrightarrow (1)$

Normal eqn are:

$na + b\Sigma x = \Sigma y$

$a\Sigma x + b\Sigma x^2 = \Sigma xy$

Tabulating for $\Sigma x, \Sigma y, \Sigma x^2 y$ & Σxy, we g

x	y	x^2	Σxy
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340
$\Sigma x =$ 15	$\Sigma y =$ 204	$\Sigma x^2 = 55$	$\Sigma xy =$ 748

$n = 5$

Normal eqn are

$na + b\Sigma x = \Sigma y$

$a\Sigma x + b\Sigma x^2 = \Sigma xy$

Now

$5a + 15b = 204$

$15a + 55b = 748$

we get $a = 0$

$b = 13.6$

From eqn (1) $y = 13.6x$

when $x = 6$, $y = 13.6 * 6 = 81.6$

Hence req^n eq^n of str line is
$y = 13.6x$
at $x=6$, $y=81.6$

PU 2014 (Fall)

The following table gives height x cm & weight y kg of five persons.

x	175	165	160	155	145
y	68	58	55	52	48

Assuming linear relationship bet^n obtain regression line. Obtain x when $y = 40$.

Let linear relationship bet^n x & y is
eq^n of straight line $y = a+bx$
Normal eq^n are
$$na + b\Sigma x = \Sigma y \quad \text{(i)}$$
$$a\Sigma x + b\Sigma x^2 = \Sigma xy \quad \text{(ii)}$$

x	y	x^2	xy
175	68	30625	11900
165	58	27225	9570
160	55	25600	8800
155	52	24025	8060
145	48	21025	6960
$\Sigma x = 800$	$\Sigma y = 281$	$\Sigma x^2 = 128500$	$\Sigma xy = 45290$

Then
$$na + b\Sigma x = \Sigma y$$
$$a\Sigma x + b\Sigma x^2 = \Sigma xy$$
Then
$$5a + b \times 800 = 281$$
$$800a + b \times 128500 = 45290$$
We get
$$\therefore a = -49.4$$
$$b = 0.66$$
Then $y = a+bx$
$$40 = a + bx$$
$$\text{or } 40 = -49.4 + 0.66x$$
$$\therefore x = 135.45$$
The required eq^n is $y = -49.4 + 0.66x$
$x = 135.45$.

3) If p is pull required to lift a load w by means of pulley. Find a linear law of form $p = mw + c$ using following data

w	50	70	100	120
p	12	15	21	25

where p & w are taken in kg

Here

Given eq^n to be fit be

$$p = c + mw$$

Normal eq^{n}s are

$$nc + m\Sigma w = \Sigma p$$

$$c\Sigma w + m\Sigma w^2 = \Sigma wp$$

w	p	w^2	pw
50	12	2500	600
70	15	4900	1050
100	21	10000	2100
120	25	14400	3000
$\Sigma w = 340$	$\Sigma p = 73$	$\Sigma w^2 = 31800$	Σpw 6750

Then

$$4c + m340 = 73$$

$$340c + 31800m = 6750$$

We get $c = 2.276$

$m = 0.188$

So $p = 0.188w + 2.276$ //

4) Find missing value from following data $y = a + bx$

x	3	6	7	9	10
y	168	—	120	72	73

Here.

Given data is

x	3	7	9	10
y	168	120	72	73

We have to find value of y when $x=6$

Given eq^n of st. line be

$$y = a + bx$$

Normal eq^{n}s are

$$na + b\Sigma x = \Sigma y$$

$$a\Sigma x + b\Sigma x^2 = \Sigma xy$$

x	y	x^2	xy
3	168	9	504
7	120	49	840
9	72	81	648
10	73	100	730
$\Sigma x = 29$	$\Sigma y = 433$	$\Sigma x^2 = 239$	$\Sigma xy = 2722$

Then
$4a+29b=433$
$29a+239b=2722$

We get $a=213.46$
$b=-14.513$

Now,
$y=213.46-14.513x$
At $x=6$
$y=213.46-14.513\times6$
$=126.382$

b) Fitting of a parabola

The equation of a parabola of degree two is $y=a+bx+cx^2 \longrightarrow ①$
The normal equations are
$na+b\Sigma x+c\Sigma x^2=\Sigma y \rightarrow (i)$
$a\Sigma x+b\Sigma x^2+c\Sigma x^3=\Sigma xy \rightarrow (ii)$
$a\Sigma x^2+b\Sigma x^3+c\Sigma x^4=\Sigma x^2y \rightarrow (iii)$
Tabulating for $\Sigma x, \Sigma y, \Sigma x^2, \Sigma x^3$, $\Sigma x^4, \Sigma xy$ & Σx^2y and solving normal eq^{ns} we get a, b & c

1) Fit a second degree parabola

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

Find difference between actual value & calculated the value of y when $x=2$
Soln:
Let eq^n of parabola to be fit is:
$y=a+bx+cx^2 \longrightarrow ①$
Normal equations are:
$na+b\Sigma x+c\Sigma x^2=\Sigma y \longrightarrow (i)$
$a\Sigma x+b\Sigma x^2+c\Sigma x^3=\Sigma xy \longrightarrow (ii)$
$a\Sigma x^2+b\Sigma x^3+c\Sigma x^4=\Sigma x^2y \longrightarrow (iii)$
Tabulating for $\Sigma x, \Sigma y, \Sigma xy, \Sigma x^2, \Sigma x^3$, Σx^4 & Σx^2y

x	y	xy	x^2	x^3	x^4	x^2y
0	1	0	0	0	0	0
1	1.8	1.8	1	1	1	1.8
2	1.3	2.6	4	8	16	5.2
3	2.5	7.5	9	27	81	22.5
4	6.3	25.2	16	64	256	100.8
$\Sigma x=$ 10	$\Sigma y=$ 12.9	$\Sigma xy=$ 37.1	$\Sigma x^2=$ 30	Σx^3 =100	$\Sigma x^4=$ 354	$\Sigma x^2y=$ 130.3

$5a+10b+30c=12.9$
$10a+30b+100c=37.1$
$30a+100b+354c=130.3$

we get
$a=1.42$
$b=-1.07$
$c=0.55$

Then $y=a+bx+cx^2$
$y=1.42-1.07x+0.55x^2$

Also when $x=2$
$y=1.48$

Then difference $=1.30-1.48$
$=-0.18$

Hence reqd eqn : $y=1.42-1.07x+0.55x^2$

Difference $=-0.18$

2) Fit the eqn of parabola

x	1	2	3	4
y	6	11	18	27

Let eqn of parabola to be fit is :
$y=a+bx+cx^2$

Normal equations are:
$na+b\Sigma x+c\Sigma x^2=\Sigma y$
$a\Sigma x+b\Sigma x^2+c\Sigma x^3=\Sigma xy$
$a\Sigma x^2+b\Sigma x^3+c\Sigma x^4=\Sigma x^2y$

Tabulating for $\Sigma x, \Sigma x^2, \Sigma x^3, \Sigma x^4, \Sigma xy$, Σx^2y.

x	y	x^2	x^3	x^4	xy	x^2y
1	6	1	1	1	6	6
2	11	4	8	16	22	44
3	18	9	27	81	54	162
4	27	16	64	256	108	432
$\Sigma x=10$	$\Sigma y=62$	$\Sigma x^2=30$	$\Sigma x^3=100$	$\Sigma x^4=354$	Σxy Σxy 190	$\Sigma x^2y=644$

Now,
$4a+10b+30c=62$
$10a+30b+100c=190$
$30a+100b+354=644$

we get
$a=3$
$b=2$
$c=1$

Then $y=a+bx+cx^2$
$=3+2x+x^2$
$=x^2+2x+3$

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c) Fitting of a exponential curve:

i) $y = ax^b$

Taking log on both sides, we get

$\log y = \log (ax^b)$

$\log y = \log a + b \log x$

or $Y = A + bX \longrightarrow (1)$

where $\log y = Y$

$\log a = A$

$\log x = X$

$b = b$

Eq^n (1) is eq^n of str line

Normal equations are

$nA + b\Sigma X = \Sigma Y$

$A \Sigma X + b \Sigma X^2 = \Sigma XY$

Tabulating for $\Sigma X, \Sigma Y, \Sigma X^2, \Sigma XY$ then solving normal eq^n's we get a&b.

ii) $y = ae^{bx}$

Taking log on both sides

$\log y = \log (ae^{bx})$

$\log y = \log a + \log e^{bx}$

$\log y = \log a + bx \log e$

$Y = A + Bx \longrightarrow (1)$

Normal eq^n is

$nA + B\Sigma x = \Sigma y$

$A \Sigma x + B \Sigma x^2 = \Sigma xy$

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iii) $xy^a = b$

Taking log on both sides

$\log (xy^a) = \log b$

or $\log x + a \log y = \log b$

or, $a \log y = \log b - \log x$

$\log y = \frac{\log b}{a} - \frac{\log x}{a}$

1) Fit a curve of form $y = ae^{bx}$ to the following data

x	0	2	4
y	5.012	10	31.62

Here the curve to be fit

$y = ae^{bx}$

Taking log on both sides

$\log y = \log ae^{bx}$

or $\log y = \log a + \log e^{bx}$

or $\log y = \log a + bx \log e$

or $\log y = \log a + b \log e . x$

$\therefore Y = A + Bx$

The eq^n (i) is eq^n of straight lines

The normal eq^{ns} are

$nA + B\Sigma X = \Sigma Y \longrightarrow (i)$

$A \Sigma X + B \Sigma X^2 = \Sigma XY \longrightarrow (ii)$

x	y	$Y=\log y$	xY	x^2
0	5.01	0.7	0.7	0
2	10	1	2	4
4	31.62	1.5	6	16
$\Sigma x=6$		$\Sigma Y=3.2$	$\Sigma xY=8$	$\Sigma x^2=20$

From Normal equations:

$3A+6B=3.2$

$6A+20B=8$

We get $A=0.6667$, $B=0.2$

Now,

$\log a = A \Rightarrow a = \text{Antilog } A$
$= 4.6419$

$b\log e = B \Rightarrow b = \frac{B}{\log e} = 0.4605$

Then

$y = ae^{bx}$
$= 4.6419e^{0.4605x}$

2) Fit a power curve $y=ax^b$ to the following data

x	61	26	7	2.6
y	350	400	500	600

Here $y=ax^b$

Taking log on both sides

$\log y = \log a + b\log x$

or, $Y = A + bX \longrightarrow (1)$

Where

$Y=\log y$

$A=\log a$

$b=b$

$X=\log x$

Then eqn (1) is straight line

$nA + b\Sigma X = \Sigma Y$

$A\Sigma X + b\Sigma X^2 = \Sigma XY$

Finding ΣX, ΣX^2, ΣY, ΣXY

x	y	X	Y	XY	X^2
61	350	1.7853	2.5441	4.5420	3.1873
26	400	1.4150	2.6021	3.6820	2.0022
7	500	0.8451	2.6990	2.2809	0.7142
2.6	600	0.4150	2.7782	1.1530	0.1722
		$\Sigma X = 4.4604$	$\Sigma Y = 10.6234$	$\Sigma XY = 11.6579$	$\Sigma X^2 = 6.0759$

Normal eq^n

$4A + 4.4604 b = 10.6234$

$4.4604 A + 6.0759 b = 11.6579$

Solving eq^{n}s we get

$A = 2.8463$

$b = -0.1708$

So,

$\log a = A = 2.8463$

$\therefore a = 701.94$

So, $y = a x^b$

$= 701.94 x^{-0.1708}$

3) The pressure and volume of gas are related by the eq^n $pv^\gamma = C$ where γ & C being constant. Fit this eq^n to the following data

x kg/cm^2	P	0.5	1.0	1.5	2	2.5	3.0
y litres	V	1.62	1.00	0.75	0.62	0.52	0.46

Here

$PV^\gamma = C$

Taking log on both sides

$\log(PV^\gamma) = \log C$

or, $\log P + \gamma \log V = \log C$

or, $\gamma \log V = \log C - \log P$

or $\log V = \frac{1}{\gamma}\log C - \frac{1}{\gamma}\log P$

$Y = A + BX \longrightarrow (1)$

$Y = \log P$

where

$y = \log V$

$A = \frac{1}{\gamma}\log C$

$B = -\frac{1}{\gamma}$

$X = \log P$

Eq^n (1) is eq^n of straight line.

The normal equations are

$nA + B\Sigma X = \Sigma y \longrightarrow (i)$

$A\Sigma X + B\Sigma X^2 = \Sigma XY \longrightarrow (ii)$

Tabulating for $\Sigma X, \Sigma Y, \Sigma XY, \Sigma X^2$, we get

x(P)	y(V)	X=log P	Y=log V	XY	X^2
0.5	1.62	-0.3010	0.2095	-0.0631	0.0906
1	1	0	0	0	0
1.5	0.75	0.1761	-0.1249	-0.0220	0.0310
2	0.62	0.3010	-0.2076	-0.0625	0.0906
2.5	0.52	0.3979	-0.2840	-0.1130	0.1583
3.	0.46	0.4771	-0.3372	-0.1609	0.2276
Σx = 10.5		ΣX = 1.0511	ΣY = -0.7442	ΣXY = -0.4215	ΣX^2 = 0.5981

From normal eq^n

$6A + 1.0511 B = -0.7442$

$1.0511 A + 0.5981 B = -0.4215$

We get $A = -0.0008$

$B = -0.7033$

Now,
$-\frac{1}{\gamma} = B \Rightarrow \gamma = -\frac{1}{B} = -\frac{1}{0.7033}$
$= 1.4219$

Then
$\frac{1}{\gamma} \log c = A \Rightarrow \log c = A \times \gamma$
$\log c = (-0.0008 \times 1.4219)$
$\therefore c = 0.9974$

So,
$pv^{\gamma} = c$
$pv^{1.4219} = 0.9974$ //

4) Fit curve $y = e^{a+bx}$

x	-5	-4	-3	-1	0
y	12.96	6.94	4.63	2.11	0.09

Here $y = e^{a+bx}$
$\log y = (a+bx) \log e$
or $\log y = a \log e + b \log e \, x$
$Y = A + Bx$ ⟶ (i)
$Y = \log y$
$A = a \log e$
$B = b \log e$
$x = x$

$a = -1.3238$
$b = -0.8453$

Eq^n(i) is eq^n of straight line
Normal equations are:
$nA + B\Sigma x = \Sigma Y$
$A\Sigma x + B\Sigma x^2 = \Sigma xY$

x	y	$Y = \log y$	x^2	xY
-5	12.96	1.1126	25	-5.5630
-4	6.94	0.8414	16	-3.3656
-3	4.63	0.6656	9	-1.9968
-1	2.11	0.3243	1	-0.3243
0	0.09	-1.0458	0	0
$\Sigma x = 13$		$\Sigma Y =$ 1.8981	$\Sigma x^2 =$ 51	$\Sigma xY =$ -11.2497

$5A + 13B = 1.8981$ ⟶ (i)
$-13A + 51B = -11.2497$ ⟶ (ii)
We get $A = -0.5749$
$B = -0.3671$

Then,
$a = \frac{A}{\log e} = \frac{-0.5749}{\log e} = -1.3238$

$b = \frac{B}{\log e} = \frac{-0.3671}{\log e} = -0.8453$

Hence $y = e^{-1.3238 + 0.8453x}$ //

Chapter-Four

Numerical Differentiation and Integration

⇓

Numerical Differentiation

↳ Numerical differentiation is the process of calculating derivatives of a given tabulated function at any point. If (x_i, y_i) are the given set of values then the process of computing values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ etc is called numerical differentiation.

If the values of derivatives are required at a point near to the ~~bigging~~ beginning value we use Newton's forward difference formula for derivatives & if near to the ending values we use backward formula.

a) Newton's forward difference formula for derivatives.

We know Newton's forward difference interpolation formula

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\text{or } y = y_0 + p\Delta y_0 + \frac{p^2-p}{2!}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 y_0 + \frac{p^4-6p^3+11p^2-6p}{4!}\Delta^4 y_0 + \cdots$$

Diff both side w.r.to p

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \cdots$$

where $p = \frac{x-x_0}{h}$

$$\therefore \frac{dp}{dx} = \frac{1}{h}$$

Now, $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \cdots\right.$$

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Similarly

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 + \cdots\right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\frac{6}{3!}\Delta^3 y_0 + \frac{24p-36}{4!}\Delta^4 y_0 \cdots\right.$$

1) From the data compute $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x=1$

x	1	2	3	4	5	6
y	1	8	27	64	125	216

Here $x=1$ is the beginning value of given function, we use Newton forward difference formula for derivatives. The table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1	7	12		
2	8			6	0
		19	18		
3	27	37	24	6	
4	64	61	30	6	0
5	125	91			
6	216				

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From table

$x_0 = 1$

$x = 1 \quad \therefore p = \frac{x - x_0}{h} = 0$

$h = 1$

We know Forward difference interpolation formula.

$$y = \Delta y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$\text{or } y = y_0 + p\Delta y_0 + \frac{p^2-p}{2!}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 y_0 + \frac{p^4-6p^3+11p^2-6p}{4!}\Delta^4 y_0$$

Now, Diff both sides wrt to x we get,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0\right]$$

At $x=1$, $p=0$

$$\frac{dy}{dx} = 1\left[7 - \frac{1}{2!}\times 12 + \frac{2\times 6}{3!} - \frac{6}{4!}\times 0\right]$$

$$= \left[7 - 6 + 2\right]$$

$$= 3$$

$$\frac{d^2y}{dx^2} = \frac{2}{2!} \times 12 + \frac{6}{3!} \times 6 + \frac{22}{4!} \times 0$$

$$= 12 - \frac{36}{6}$$

$$= 12 - 6$$

$$= 6$$

Hence $\frac{dy}{dx} = 3$, $\frac{d^2y}{dx^2} = 6$.

2) The pop^n of a certain town is given below.

Year(x)	1931	1941	1951	1961	1971
pop^n (y)	40.6	60.8	79.9	103.6	132.7

Here 1961 is near to ending value So we use Newton backward difference formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.6				
		20.20			
1941	60.8		-1.1		
		19.1		5.57	
1951	79.9		4.6		-4.9
		23.70		0.8	
1961	103.6		5.4		
		29.1			
1971 x_0	132.7 y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	

From table

$x = 1961$

$x_0 = 1971$

$h = 10$

$$\therefore p = \frac{x - x_0}{h} = \frac{1961 - 1971}{10} = \frac{-10}{10} = -1$$

Then

$$y = y_0 + p \nabla y_0 + \frac{p(p+1)}{2!} \nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_0$$

$$y = y_0 + p \nabla y_0 + \frac{p^2 + p}{2!} \nabla^2 y_0 + \frac{p^3 + 3p^2 + 2p}{3!} \nabla^3 y_0 + \frac{p(p^3 + 6p^2 + 11p + 6)}{4!} \nabla^4 y_0$$

$$y = y_0 + p \nabla y_0 + \frac{p^2 + p}{2!} \nabla^2 y_0 + \frac{p^3 + 3p^2 + 2p}{3!} \nabla^3 y_0 + \frac{p^4 + 6p^3 + 11p^2 - 6p}{4!} \nabla^4 y_0$$

$$\frac{dy}{dp} = p + \frac{2p+1}{2!} \nabla^2 y_0 + \frac{3p^2 + 8p + 2p}{3!} \nabla^3 y_0 + \frac{4p^3 + 18p + 22p + 6}{4!} \nabla^4 y_0$$

Then

$$= \frac{1}{h}\left[\nabla y_0 + \frac{2p+1}{2!}\nabla^2 y_0 + \frac{3p^2+6p+2}{3!}\nabla^3 y_0 + \frac{4p^3+18p^2+22p+6}{4!}\nabla^4 y_0\right]$$

$$= \frac{1}{10}\left[29.1 + \frac{-8+1}{2!} \times 5.4 + \frac{3-6+2}{3!} \times 0.8 - \frac{4+18\cancel{+}22+6}{4!} \times (-4.9)\right]$$

$= 2.667$

3) PU 10 times

A rod is rotating a plane. The following table gives the angle θ radian through which the rod has turned for various values of the time t seconds.

t	0	0.2	0.4	0.6	0.8	1.0
θ	0	0.12	0.49	1.12	2.02	3.20

calculate angular velocity and angular acceleration at $t = 0.6$ s.

$\Rightarrow$ Here $t = 0.6$ is near to the ending value of given table. Newton's backward difference formula is used.

The difference table

$x=t$	$y=\theta$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	0				
		0.12			
0.2	0.12		0.25		
		0.37		0.01	
0.4	0.49		0.26		0
		0.63		0.01	
0.6	1.12		0.27		0
		0.9		0.01	
0.8	2.02		0.28		
		1.18		$\nabla^3 y_0$	$\nabla^4 y_0$
1.0	3.2	∇y_0	$\nabla^2 y_0$		
	y_0				

From table

$x_0 = 1$

$x = 0.6$

$h = 0.2$

$\therefore p = \frac{x - x_0}{h} = -2.$

Then we know.

$$y = y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_0$$

or $$y = y_0 + p\nabla y_0 + \frac{p^2+p}{2!}\nabla^2 y_0 + \frac{p^3+3p^2+2p}{3!}\nabla^3 y_0 + \frac{p^4+6p^3+11p^2+6p}{4!}\nabla^4 y_0$$

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$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_0 + \frac{2p+1}{2!}\nabla^2 y_0 + \frac{3p^2+6p+2}{3!}\right.$$

$$\left.\nabla^3 y_0 + \frac{4p^3+18p^2+22p+6}{4!}\nabla^4 y_0\right.$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\frac{2}{2!}\nabla^2 y_0 + \frac{6p+6}{3!}\nabla^3 y_0 + \right.$$

$$\frac{12p+36p+22}{4!}\nabla^4 y_0$$

Then

$$v = \frac{dy}{dx} = \frac{1}{0.2}\left[1.18 + \frac{-4+1}{2!} * 0.28 + \frac{12-12+2}{3!}\right.$$

$$\left. * 0.01 + \frac{4\times 8+18\times 4-22\times 2+6\times 0}{4!}\right.$$

$$= 3.8167$$

$$a = \frac{d^2y}{dx^2} = \frac{1}{0.2^2}\left[1 * 0.28 + \frac{6\times(-2)+6}{3!} * 0.01\right.$$

$$\left. + 0\right]$$

$$= 6.75$$

$v = 3.8167$ rad/s

$a = 6.75$ rad/sec^2

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4) Find $\frac{dy}{dx}$, $\frac{dy^2}{dx^2}$ & $\frac{d^3y}{dx^3}$ at $x = 1.6$

x	1	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.75	10.031

Soln:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	7.989					
		0.4140				
1.1	8.403		-0.0360			
		0.3780		0.0060		
1.2	8.781		-0.0300		-0.002	
		0.3480		0.004		
1.3	9.129		-0.0260		-0.001	
		0.3220		0.003		
1.4	9.451		-0.0230		0.002	
		0.2990		0.005		
1.5	9.75		-0.018			
		0.2810	$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$	
1.6 (x_0)	10.031 (y_0)	∇y_0				

From table

$x = 1.6$

$x_0 = 1.6$

$h = 0.1$

$\therefore p = \frac{x-x_0}{h} = \frac{1.6-1.6}{0.1} = 0$

$$y = y_0 + p\nabla y_0 + \frac{p(p+1)}{2!}\nabla^2 y_0 + \frac{p(p+1)}{3!}(p+2)\nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_0$$

$$\frac{dy}{dp} = y_0 + p\nabla y_0 + \frac{p^2+p}{2!}\nabla^2 y_0 + \frac{p^3+3p^2+2p}{3!}\nabla^3 y_0 + \frac{p^4+6p^3+11p^2+6p}{4!}\nabla^4 y_0$$

Now

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_0 + \frac{2p+1}{2!}\nabla^2 y_0 + \frac{3p^2+6p+2}{3!}\nabla^3 y_0 + \frac{4p^3+18p^2+22p+6}{4!}\nabla^4 y_0\right]$$

$$= \frac{1}{0.1}\left[0.281 + \frac{1}{2!}(-0.0180) + \frac{2}{3!}(0.005) + \frac{6}{4!} \times 0.02\right]$$

$$= 2.7867$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\frac{2}{2!}\nabla^2 y_0 + \frac{6p+6}{3!}\nabla^3 y_0 + \frac{12p^2+36p+22}{4!}\nabla^4 y_0\right]$$

$$= \frac{1}{0.1^2}\left[\frac{2}{2!}(-0.0180) + \frac{6}{3!} \times 0.005 + \frac{22}{4!} \times 0.002\right]$$

$$= \frac{1}{0.01}\left[-0.018 + 0.005 + 0.0018\right]$$

$$= \frac{1}{0.01} \times -0.0112$$

$$= -1.117$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_0 + \frac{24p+36}{4!}\nabla^4 y_0\right]$$

$$= \frac{1}{0.1^3}\left[0.005 + 0\right]$$

$$= 5$$

Numerical Differentiation

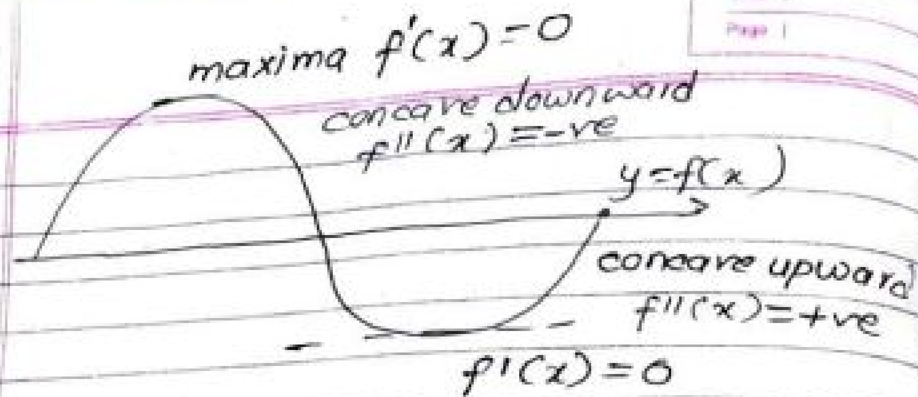
Maxima & minima of tabulated functⁿ

Note

i) For maxima & minima : $\frac{dy}{dx} = 0$

ii) For maxima $\frac{d^2y}{dx^2} = -ve$ ~~min..~~

For minima $\frac{d^2y}{dx^2} = +ve$



We know Newton's forward interpolation ~~Forward~~ Formula:

$$y=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$\text{or } y=y_0+p\Delta y_0+\frac{p^2-p}{2!}\Delta^2 y_0+\frac{p^3-3p^2+2p}{3!}\Delta^3 y_0+\cdots$$

Diff both sides w.r.to p we get

$$\frac{dy}{dp}=\Delta y_0+\frac{2p-1}{2!}\Delta^2 y_0+\frac{3p^2-6p+2}{3!}\Delta^3 y_0$$

$$\frac{d^2y}{dp^2}=\frac{2}{2!}\Delta^2 y_0+\frac{6p-6}{3!}\Delta^3 y_0+\cdots$$

For maxima & minima

$$\frac{dy}{dp}=0$$

$$\text{or } \Delta y_0+\frac{2p-1}{2!}\Delta^2 y_0+\frac{3p^2-6p+2}{3!}\Delta^3 y_0=0$$

Taking the value of Δy_0, $\Delta^2 y_0$ & $\Delta^3 y_0$ from difference table only up to third difference & solving we get value of p

If $\frac{d^2y}{dp^2}=+ve$ y has minima

If $\frac{d^2y}{dp^2}=-ve$ y has maxima

1) From the table below, for what value of x, y is maximum? Also this value of y

x	3	4	5	6	7	8
y	0.205	0.24	0.259	0.262	0.25	0.224

Here different table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
3 x_0	0.205 y_0	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
		0.035	-0.016	
4	0.240			0
		0.019	-0.016	
5	0.259			0.001
		0.003	-0.015	
6	0.262			0.001
		-0.012	-0.014	
7	0.25	-0.026		
8	0.224			

We know Newton's forwarded interpolation formula

$$y = y_0 + p\Delta y_0 + \frac{p^2-p}{2!}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 y_0$$

Diff both sides wr to p we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0$$

$$\frac{d^2y}{dp^2} = \frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0$$

For maxima & minima

$$\frac{dy}{dp} = 0$$

or $\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 = 0$

or $0.035 + \frac{2p-1}{2}(-0.016) + \frac{3p^2-6p+2}{6} \times 0 = 0$

or, $0.035 + (2p-1)(-0.008) = 0$

or $(2p-1) = \frac{-0.035}{-0.008}$

$\therefore p = 2.688$

When $p = 2.688$

$$\frac{d^2y}{dp^2} = \frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0$$

$= -0.016$ (-ve)

This shows that y has maxima at $p = 2.688$

Now,

$$p = \frac{x - x_0}{h}$$

or $2.688 = \frac{x-3}{1}$

or, $x = 5.688$

Then

$$y = y_0 + p\Delta y_0 + \frac{p^2-p}{2!}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 y_0$$

$= 0.205 + 2.688 \times 0.035 + \frac{2.688^2 - 2.688}{2!} \times (-0.016) + 0$

$= 0.2628$

Hence maximum value of $y = 0.263$ at $x = 5.688$

27) From following table, find x correct to 4 decimal places for which y is maximum & find this value of y

x	1.2	1.3	1.4	1.5	1.6
y	0.932	0.9636	0.985	0.9975	0.9996

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x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.2	0.9320			
		0.0316		
1.3	0.9636		-0.0097	
		0.0219		-0.0002
1.4	0.9855		-0.0099	
		0.0120		0
1.5	0.9975		-0.0099	
		0.0021		
1.6	0.9996			

Using Newton's forward interpolatn formula

$y = y_0 + p\Delta y_0 + \frac{p^2-p}{2!}\Delta^2 y_0 + \frac{p^3-3p^2+2p}{3!}\Delta^3 y_0$

Diff wrto p

$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0$

$\frac{d^2y}{dp^2} = \frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0$

For maxima & minima

$\frac{dy}{dp} = 0$

or, $\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 = 0$

or, $0.0316 + \frac{2p-1}{2} * -(0.0097) + \frac{3p^2-6p+2}{6} * -0.0002 = 0$

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or, $0.0316 - 0.0049(2p-1) - \frac{3p^2-6p+2}{6} * 0.0002 = 0$

or, $0.0316 - 0.0049 \times 2p + 0.0049 - \frac{0.0006p^2 - 0.0012p + 0.0004}{6} = 0$

or, $0.1896 - 0.0588p + 0.0049 - 0.0006p^2 + 0.0012p + 0.0004 = 0$

or, $0.0006p^2 + 0.0576 - 0.1949 = 0$

$\therefore p = 3.2721$

$p = -99.2721$

Now, $0.0316 - 0.0049 \times 2p + 0.0049 = 0$

or $0.0365 - 0.0098p = 0$

$\therefore p = 3.7245$

Now,

$\frac{d^2y}{dp^2} = \frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0$

$= 2 * (-0.0097) + \frac{2 \times 3.7245 - 6}{6} * (-0.0002)$

$= -0.0194$ -ve

So there is maxima

$p = \frac{x - x_0}{h}$

or, $3.7245 = \frac{x - 1.2}{0.1}$ $\therefore x = 1.5725$

$$y = y_0 + p\Delta y_0 + \frac{p^2 - p}{2!}\Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!}\Delta^3 y_0$$

$$= 0.9320 + 3.7245 \times 0.0316 + \frac{3.7245^2 - 3.7245}{2!} \times (-0.0097)$$

$$= 1.0005$$

Hence maximum value of $y = 1.0005$ at $x = 1.5725$

Trapezoidal and Simpson's ($\frac{1}{3}$ and $\frac{3}{8}$)

1) Trapezoidal rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

2) Simpson's $\frac{1}{3}$ rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + y_6 + 2y_2 + 4y_1 + 4y_3 + 2y_4 + 4y_5\right]$$

Simpson's $\frac{3}{8}$ rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[y_0 + y_6 + 3(y_1 + y_2 + y_4 + y_5) + 3(y_3\right]$$

Numerical Integration

Method IV> Romberg's Integration method

In Romberg's method we use trapezoidal rule to find

i) I_1 with $n=2$
ii) I_2 with $n=4$
iii) I_3 with $n=8$

then after

$$I_4 = I_2 + \frac{I_2 - I_1}{3}$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3}$$

$$I_6 = I_4 + \frac{I_4 - I_3}{3}$$

This process is continued to get two equal successive values of I

1) Evaluate

$\int_0^1 \frac{1}{1+x^2}\,dx$ by using Romberg's formula.

Here comparing $\int_0^1 \frac{1}{1+x^2}\,dx$ with $\int_a^b f(x)\,dx$

we get

$$f(x) = \frac{1}{1+x^2}$$

$$a = 0$$
$$b = 0$$

Using Trapezoidal formula we have to find $I_1, I_2 \& I_3$

i) When $n=2$

$h = \frac{b-a}{n} = \frac{1}{2}$

x	0	0.5	1
$f(x)$	1	0.8	0.5
	y_0	y_1	y_2

$I_1 = \frac{h}{2}[y_0 + y_2 + 2y_1]$

$= \frac{1}{4}[1 + 0.5 + 2 \times 0.8]$

$= 0.7750$

ii) When $n=4$

$h = \frac{b-a}{n} = \frac{1}{4}$

x	0	0.25	0.5	0.75	1
$f(x)$	1	0.9411	0.8	0.64	0.5

$I_2 = \frac{h}{2}[y_0 + y_4 + 2(y_1 + y_2 + y_3)]$

$= \frac{1}{8}[1 + 0.5 + 2(0.9411 + 0.8 + 0.64)]$

$= 0.7828$

iii) When $n=8$

$h = \frac{b-a}{n} =$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x)$	1	0.9846	0.9411	0.8767	0.8	0.7191	0.64	0.5664	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$I_3 = \frac{h}{2}[y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$

$= 0.7847$

Now,

Using Romberg's formula

$I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.7828 + \frac{0.7828 - 0.7750}{3}$

$= 0.7854$

$I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.7847 + \frac{0.7847 - 0.7828}{3}$

$= 0.7855$

Here $I_4 = I_5$ upto decimal places

$\int_0^1 \frac{1}{1+x^2} dx = 0.785$

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Apply Romberg's method to evaluate

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}}\,dx$$

Soln:

Here $a=0$

$b=\pi/2$

Using Trapezoidal formula we find I_1, I_2, I_3

i) When $n=2$

$$h=\frac{b-a}{2}=\frac{\pi}{4}$$

Tabulating:

x	0	0.7854	1.5708
$f(x)$	1	0.5412	0

Then

$$I_1=\frac{h}{2}\left[y_0+y_2+2y_1\right]$$

$$=\frac{\pi}{8}\left[1+0+2\times0.5412\right]$$

$$=0.8178$$

ii) When $n=4$

$$h=\frac{b-a}{n}=\frac{\pi}{8}$$

Tabulating

x	0	0.3927	0.7854	1.1781	1.5708
$f(x)$	1	0.7857	0.5412	0.2759	0 0
	y_0	y_1	y_2	y_3	y_4

$$I_2=\frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$=0.8258$$

When $n=8$

$$h=\frac{b-a}{n}=\frac{\pi}{16}$$

x	0	0.1963	0.3927	0.5890	0.7854	0.9817	1.1781	1.3744	1.5708
$f(x)$	1	0.8972	0.7857	0.6667	0.5412	0.4105	0.2759	0.1386	0

$$I_3=\frac{h}{2}\left[y_0+y_8+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$=0.8278$$

Then $I_4=I_2+\frac{I_2-I_1}{3}$

$$=0.8278+\frac{0.8278-0.8258}{3}$$

$$=0.8285$$

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$$I_5 = I_4 + \frac{I_4 - I_2}{3}$$

$$= 0.8278 + \frac{0.8278 - 0.8258}{3}$$

$$= 0.8285$$

Here $I_4 = I_5$

So $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}} = 0.8285$

Method V> Gauss - Legendre point method

Gauss Legendre integration is based on the concept that accuracy of numerical integration can be improved by choosing the sampling points than on the basis of equal intervals.

Formula: $\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} \omega_i f(x_i)$

i) Two point formula

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{2} \omega_i f(x_i)$$

$$= \omega_1 f(x_1) + \omega_2 f(x_2)$$

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where $\omega_1 = 1$ $x_1 = \frac{-1}{\sqrt{3}}$

$\omega_2 = 1$ $x_2 = \frac{1}{\sqrt{3}}$

ii) Three point formula

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{3} \omega_i f(x_i)$$

$$= \omega_1 f(x_1) + \omega_2 f(x_2) + \omega_3 f(x_3)$$

where

$\omega_1 = \frac{5}{9}$ $x_1 = -\sqrt{\frac{3}{5}}$

$\omega_2 = \frac{8}{9}$ $x_2 = 0$

$\omega_3 = \frac{5}{9}$ $x_3 = \sqrt{\frac{3}{5}}$

1) Evaluate $\int_{-1}^{1} e^{x}\,dx$ by using Gauss Legendre 2-point, 3-point formula.

Sol^n:

Here $a = -1$

$b = 1$

$f(x) = e^x$

We know Gauss Legendre

i) 2 point formula

$$\int_{-1}^{1} f(x)\,dx = \omega_1 f(x_1) + \omega_2 f(x_2)$$

$= 1e^{-\frac{1}{\sqrt{3}}} + 1e^{\frac{1}{\sqrt{3}}}$
$= 2.3427$

ii) 3-point formula

$\int_{-1}^{1} f(x)\,dx = \omega_1 f(x_1) + f_2(x_2)\omega_2 + \omega_3 f(x_3)$

$= \frac{5}{9} e^{-\sqrt{\frac{3}{5}}} + \frac{8}{9} e^{0} + \frac{5}{9} e^{\sqrt{\frac{3}{5}}}$

$= 2.3503$

Hence
$\int_{-1}^{1} e^x dx = 2.3427$ by 2point formula
$= 2.3503$ by 3-point formula

Changing limits of integration.

$\Downarrow$

Let $\int_a^b f(x)\,dx = c\int_{-1}^{1} g(z)\,dz$

Assume this transformation between x & new variable z satisfies the relation

$x = Az + B$

when $x = a$, $z = -1$
when $x = b$, $z = 1$

Then $a = -A + B$
$b = A + B$

solving we get
$A = \frac{b-a}{2}$, $B = \frac{a+b}{2}$

From eq^n (i)
$x = \frac{b-a}{2} z + \frac{a+b}{2}$

or $\frac{dx}{dz} = \frac{b-a}{2}$

or $dx = \frac{b-a}{2} dz \Rightarrow c = \frac{b-a}{2}$

Now,
$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} g(z)\,dz$

$= \frac{b-a}{2} \sum_{i=1}^{n} \omega_i g(z_i)$

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2) Evaluate the integral $\int_0^1 \frac{1}{1+x}\,dx$

Using Gauss Legendre 3-point formula
soln:

Here
$I=\int_0^1 \frac{1}{1+x}\,dx$

$a=0, b=1, f(x)=\frac{1}{1+x}$

changing limits of integration we get

$x=\frac{b-a}{2}z+\frac{a+b}{2}$

$x=\frac{z}{2}+\frac{1}{2}$

$\therefore f(x)=\frac{1}{1+x}$

$g(z)=\frac{1}{1+\left(\frac{z}{2}+\frac{1}{2}\right)}$

$=\frac{2}{z+3}$

We know, Gauss Legendre 3-point formula,

$\int_a^b f(x)dx=\frac{b-a}{2}\int_{-1}^{1} g(z)dz$

$=\frac{b-a}{2}\sum_{i=1}^{3}(w_i\, g(z_i)$

$=\frac{b-a}{2}\left[w_1 g(z_1)+w_2 g(z_2)+w_3 (g(z_3))\right]$

$=\frac{b-a}{2}\left[w_1\left(\frac{2}{z_1+3}\right)+w_2\left(\frac{2}{z_2+3}\right)+w_3\left(\frac{2}{z_3+3}\right)\right]$

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$\doteq\frac{1-0}{2}\left[\frac{5}{9}\left(\frac{2}{-\sqrt{\frac{3}{5}}+3}\right)+\frac{8}{9}\left(\frac{2}{0+3}\right)+\frac{5}{9}\left(\frac{2}{\sqrt{\frac{3}{5}}+3}\right)\right]$

$\doteq$ 0.6931

3) Using Gauss Legendre point formula evaluate

$I=\int_3^5 \frac{1}{x^6+5}\,dx$

Here $a=3, b=5, f(x)=\frac{1}{x^6+5}$

changing limits of integration

$x=\left(\frac{b-a}{2}\right)z+\frac{a+b}{2}$

$=\left(\frac{5-3}{2}\right)z+\frac{8}{2}$

$\therefore x = z+4$

$\therefore f(x)\doteq\frac{1}{x^6+5}$

$g(z)=\frac{1}{(z+4)^6+5}$

We know Gauss Legendre 2-point formula,

$\int_a^b f(x)dx=\frac{b-a}{2}\int_{-1}^{1} g(z)dz$

$=\frac{5-3}{2}\sum_{i=1}^{2}(w_i g(z_i)]$

$=[w_1 g(z_1)+w_2 g(z_2)]$

$$= \omega_1\left[\frac{1}{(z_1+4)^6+5}\right] + \omega_2\left[\frac{1}{(z_2+4)^6+5}\right]$$

$$= \frac{5}{9}\left[\frac{1}{\left(-\sqrt{\frac{3}{5}}+4\right)^6+5}\right] + \frac{8}{9}\left[\frac{1}{(0+4)^6+5}\right]$$

$= 0.0007$

Then For 3 point formula

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[\omega_1 g(z_1) + \omega_2 g(z_2) + \omega_3 g(z_3)\right]$$

$$= \frac{5-3}{2}\left[\frac{5}{9}\left[\frac{1}{(z_1+4)^6+5}\right] + \frac{8}{9}\left[\frac{1}{(z_2+4)^6+5}\right] + \frac{5}{9}\left[\frac{1}{(z_3+4)^6+5}\right]\right]$$

$$= \frac{5}{9}\left[\frac{1}{\left(-\sqrt{\frac{3}{5}}+4\right)^6+5}\right] + \frac{8}{9} * \frac{1}{(0+4)^6+5} + \frac{5}{9}\left(\frac{1}{\left(\sqrt{\frac{3}{5}}+4\right)^6+5}\right)$$

$= 0.0061$

CHAPTER-5

Solution of ordinary Differential Eq^{ns}

Solution of ordinary differential equations means finding the values of y in the equation $\frac{dy}{dx} = f(x,y)$ with initial condit^n $y(x_0) = y_0$ & step size h. In this method we apply step by step procedure where

$x_1 = x_0 + h$

$x_2 = x_0 + 2h$

$x_3 = x_0 + 3h$ & so on

Methods

i) Taylor's series method
ii) Picard's method
iii) Euler's method
iv) Runge-kutta method.
v) Heun's method
vi) Shooting method

I) Taylor's series method

Let $\frac{dy}{dx} = y' = f(x,y)$ be a differential eq^n with initial condition $y(x_0) = y_0 \Rightarrow x = x_0$, $y = y_0$ & step size h. Then

Taylor's expansion at point x_0 is given by $f(x)=f(x_0)+\frac{x-x_0}{1!}f'(x_0)$

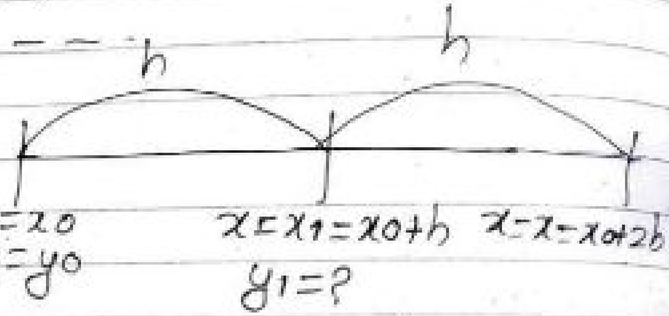
$+\frac{(x-x_0)^2}{2!}f''(x_0)+\frac{(x-x_0)^3}{3!}f'''(x_0)+$

$- - - - -$

This can be written as

$y=y_0+\frac{(x-x_0)}{1!}y_0'+\frac{(x-x_0)^2}{2!}y_0''+\frac{(x-x_0)^3}{3!}y_0'''$

$+ - - - -$



But $x=x_1=x_0+h \Rightarrow x-x_0=h$

$y_1=y_0+\frac{h}{1!}y_0+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\frac{h^4}{4!}y_0^{IV}$

$y_{n+1}=y_n+\frac{h}{1!}y_n'+\frac{h^2}{2!}y_n''+\frac{h^3}{3!}y_n'''+\frac{h^4}{4!}y_n^{IV}$

We have to evaluate up to 4th derivative

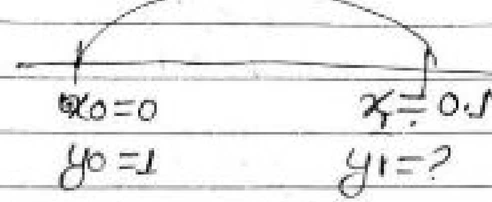
1) Using taylor's series evaluate $y(0.3)$ $\frac{dy}{dx}=x+y$ with initial condition. $y(0)=1$, Take $h=0.1$

Soln:

$y'=x+y$

$y(0)=1 \Rightarrow x_0=0$

$y_0=1$



$h=0.1$

$x_1=x_0+h=0+0.1=0.1$

$y_1=?$

Now

$y_1=y_0+\frac{h}{1!}y_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\frac{h^4}{4!}y_0^{IV} \rightarrow (1)$

where

$y_1'=x+y \Rightarrow y_0'=x_0+y_0=0+1=1$

$y''=1+y' \Rightarrow y_0''=1+y_0'=1+1=2$

$y'''=0+y'' \Rightarrow y_0'''=y_0''=2$

$y^{IV}=y''' \Rightarrow y_0^{IV}=y_0'''=2$

From eqn (1)

$y_1=y_0+\frac{h}{1!}y_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\frac{h^4}{4!}y_0^{IV}$

$=1+0.1\times1+\frac{0.1^2}{2}\times2+\frac{0.1^3}{3!}\times2+\frac{0.1^4}{4!}\times2$

$=1.1103$

Hence, $y(0.1)=1.1103$

Apply the taylor's series method to find the value of y(1.1) & y(1.2)

$\frac{dy}{dx} = xy^{1/3}, y(1)=1, h=0.1$

Here

$y' = xy^{1/3}$

$y(1)=1$ $\quad x_0=1$ $\quad x_1=1.1$ $\quad x_2=1.2$

$x_0=1$ $\quad y_0=1$

$y_0=1$

$h=0.1$

$y_1=?$

Now,

$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y^{iv}$

where

$y' = xy^{1/3} \Rightarrow y_0' = x_0 y_0^{1/3} = 1.1^{1/3} = 1$

$\cancel{y'' = y^{1/3} \Rightarrow y_0''}$

$\cancel{y'' = y^{1/3} + \frac{x}{3y^{2/3}}y' \Rightarrow y_0'' = y_0^{1/3} + x_0 y_0'}$

where

$y' = xy^{1/3} \Rightarrow y_0' = x_0 y_0^{1/3} = 1.1^{1/3} = 1$

$y'' = y^{1/3} + \frac{x}{3y^{2/3}}y' \Rightarrow y_0'' = y_0^{1/3} + \frac{x_0 y_0'}{3y_0^{2/3}}$

$= 1 + \frac{1}{3} = \frac{4}{3}$

$y^{1/3-1} \quad y^{-2/3} \qquad \frac{1}{3} - \frac{2}{3}$

$y'' = y^{1/3} + \frac{x}{3y^{2/3}}y' = y^{1/3} + \frac{x}{3y^{2/3}} * xy^{1/3}$

$= y^{1/3} + \frac{x^2 \cdot y^{-1/3}}{3}$

$y_0'' = 1 + \frac{1.1}{3} = \frac{4}{3}$

$y''' = \frac{1}{3}y^{-2/3} + \frac{1}{3}(y^{-1/3} \times 2x + x^2(\frac{-1}{3})y^{-4/3}y'$

$= \frac{1}{3y^{2/3}} + \frac{2x}{3y^{1/3}} - \frac{x^2}{3}y^{-4/3} \cdot xy^{1/3}$

$= \frac{1}{3}y^{-2/3} + \frac{2}{3}xy^{-1/3} - \frac{x^3}{3}y^{-1}$

$y_0''' = \frac{1}{3}y_0^{-2/3} + \frac{2}{3}x_0 y_0^{-1/3} - \frac{x_0^3}{3}y_0^{-1}$

$= \frac{1}{3} + \frac{2}{3} - \frac{1}{3}$

$= \frac{2}{3}$

$y^{iv} = -\frac{2}{9}y_0^{-5/3} + \frac{2}{3}[y_0^{-1/3} - \frac{1}{3}xy^{-4/3}] - \frac{1}{3}[3x^2y^{-1} - x^3y^{-2}]$

$y_0^{iv} = -\frac{2}{9}y_0^{-5/3} + \frac{2}{3}[y_0^{-1/3} - \frac{1}{3}x_0 y^{-4/3} xy^{1/3}] - \frac{1}{3}[3x_0^2 y_0^{-1} - x_0^3 y_0^{-2} xy^{1/3}]$

$= -\frac{2}{9} + 2[1 - \frac{1}{3} - \frac{1}{3}[3 - 1 - 1]$

$= -\frac{2}{9} + 2[1 - \frac{1}{3} - 1 + \frac{2}{3}] = \frac{4}{9}$

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Then

$y_1 = 1 + 0.1 \times 1 + \frac{0.1^2}{2!} \times \frac{4}{3} + \frac{0.1^3}{3!} \times \frac{2}{3} +$

$\frac{0.1^4}{4!} \times \frac{4}{9}$

$= 1.1068$

Also

$y_0' = 1.1 \times 1.1068^{1/3} = 1.1378$

$y_0'' = 1.1^{1/3} +$

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Picard's method of successive approximations

Suppose the given differential equation is

$$\frac{dy}{dx} = f(x,y) \longrightarrow (1)$$

Let us solve the given eq^n with the given initial condition $y = y_0$ when $x = x_0$. The given differential eq^n can be written as

$$dy = f(x,y)dx \longrightarrow (2)$$

Integrating eq^n (2) from x_0 to x wrt x, we get,

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\text{or } [y]_{y_0}^{y} = \int_{x_0}^{x} f(x,y)\,dx$$

$$\text{or } y - y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\text{or, } y = y_0 + \int_{x_0}^{x} f(x,y)\,dx \longrightarrow 3)$$

As a first approximation, we replace y by y_0 in $f(x,y)$ we get

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

Similarly replacing y by y_1 in $f(x,y)$ we get second approximation

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx \text{ \& so on}$$

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx$$

We continue the process we get two successive approximation are equal.

Q. No 1) Using picard's method, find y when $x = 0.2$ show that $y(0)=1$, $\frac{dy}{dx} = x - y$

soln:

Here, Given that

$$\frac{dy}{dx} = f(x,y) = x - y$$

$y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$

$x = 0.2, y = ?$

We know picard's formula,

$$y_{n+1} = y_0 + \int_{x_0}^{x} f(x, y_n)\,dx$$

The successive approximations are

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_{0}^{0.2} (x - y_0)\,dx$$

$= \cancel{1.02}$

$= 0.82$

2) $$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= y_0 + \int_{x_0}^{x} (x - y_1)\,dx$$

$$= 1 + \int_{0}^{0.2} (x - 0.82)\,dx$$

$= 0.8560$

3) $$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= y_0 + \int_{x_0}^{x} (x - y_2)\,dx$$

$$= 1 + \int_{0}^{0.2} (x - 0.8560)\,dx$$

$= 0.8488$

4) $$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3)\,dx$$

$$= 1 + \int_{0}^{0.2} (x - y_3)\,dx$$

$= 0.8502$

5) $y_5 = 0.8500$

$y(0.2) = 0.85$ Ans

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Q) Use picard's method to approximate the value of y when $x=0.1, x=0.2$
$x=0.4$
~~solⁿ~~
Given that $y=1$, when $x=0$ &
$\frac{dy}{dx}=1+x+xy$ correct to three decimal places (upto 3rd approximation)
solⁿ:
Here Given that $\frac{dy}{dx}=f(x,y)=1+xy$

$$y(0)=1 \Rightarrow x_0=0$$
$$y_0=1$$

when $x=0.1, y=?$
when $x=0.2, y=?$
when $x=0.4, y=?$

i) For $x=0.1$
We know picard's formula
$$y_{n+1}=y_0+\int_{x_0}^{x} f(x,y_n)dx$$

The successive approximations are
$$y_1=y_0+\int_{x_0}^{x} f(x,y_0)dx$$
$$=y_0+\int_{0}^{0.1}(1+xy_0)dx$$
$$=1+\int_{0}^{0.1}(1+x.1)dx$$
$$=1.1050$$

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$$y_2=1+\int_{0}^{0.1} 1+x*1.1050)dx=1.1055$$

$$y_3=1+\int_{0}^{0.1}(1+x*1.1055)dx=1.1055$$

So $y(0.1)=1.1055$

ii) for $x=0.2$
$$y_1=y_0+\int_{x_0}^{x}(1+xy_0)dx$$
$$=1+\int_{0}^{0.2}(1+x.1)dx$$
$$=1.2200$$

$$y_2=1+\int_{x_0}^{0.2} 1+x.1.22)dx$$
$$=1.2244$$

$y_3=1.2245$
So $y(0.2)=1.2245$

iii) For $x = 0.4$

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 1 + \int_0 (1 + x y_0)\,dx$$

$$= 1 + \int_0^{0.4} (1 + x \times 1)\,dx$$

$$= 1.48$$

$$y_2 = 1 + \int_0^{0.4} (1 + x \times 1.48)\,dx$$

$$= 1.5184$$

$$y_3 = 1 + \int_0^{0.4} (1 + x \times 1.5184)\,dx$$

$$= 1.5215$$

Solution of ordinary Differential Equation

⇓

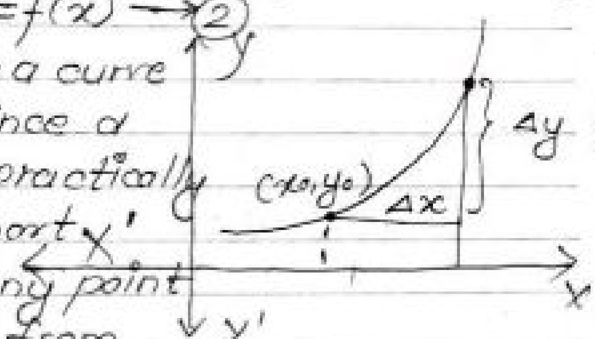
Method III > Euler's method

Consider the first order differential eq^n

$$\frac{dy}{dx} = f(x, y) \longrightarrow (1)$$

Let us solve this differential equation under the condition $y(x_0) = y_0$. The sol^n of (1) gives y as a function of x which may be written as $y = f(x) \longrightarrow (2)$

The graph of (2) is a curve in xy plane & since a smooth curve is practically straight for a short distance from any point on it. We have from figure.



$$\tan\theta = \frac{\Delta y}{\Delta x}$$

$$\text{or, } \Delta y = \Delta x \tan\theta$$

$$\text{or, } \Delta y = \Delta x \left(\frac{dy}{dx}\right)_0$$

$$\because \text{slope at } (x_0, y_0) = \left(\frac{dy}{dx}\right)_0$$

Now, $y_1 = y_0 + \Delta y$

or $y_1 = y_0 + \left(\frac{dy}{dx}\right)_0 \Delta x$

or $y_1 = y_0 + f(x_0, y_0) h$

$= y_0 + h y_0'$

where $h = \Delta x$

Similarly,

$y_2 = y_1 + h\left(\frac{dy}{dx}\right)$

$= y_1 + h y_1'$

$y_3 = y_2 + h y_2'$ & so on

In general

$$\boxed{y_{n+1} = y_n + h y_n'}$$

This is called Euler's formula.

1) Solve the equation $\frac{dy}{dx} = 1 - y$ with the initial condition $y(0) = 0$ using Euler's method & tabulate the solution at $x = 0.1, 0.2, 0.3$

soln:

Here $\frac{dy}{dx} = f(x, y) = 1 - y$

$y(0) = 0 \Rightarrow x_0 = 0$

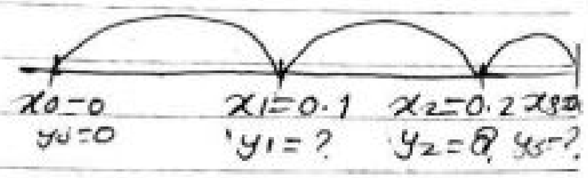
$y_0 = 0$

Let $h = 0.1$

$\therefore x_1 = x_0 + h = 0 + 0.1 = 0.1$, $y_1 = ?$

$x_2 = x_0 + 2h = 0 + 0.1 \times 2 = 0.2$, $y_2 = ?$

$x_3 = x_0 + 3h = 0 + 3 \times 0.1 = 0.3$, $y_3 = ?$



We know the Euler's formula,

$y_{n+1} = y_n + h y_n'$

Now, $y_1 = y_0 + h y_0'$

$= y_0 + h f(x_0, y_0)$

$= y_0 + h[1 - y_0]$

$= 0 + 0.1[1 - 0]$

$= 0.1$

$y_2 = y_1 + h y_1'$

$= 0.1 + h[1 - y_1]$

$= 0.1 + 0.1[1 - 0.1]$

$= 0.19$

$y_3 = y_2 + h y_2'$

$= 0.19 + h[1 - 0.19]$

$= 0.19 + 0.1[1 - 0.19]$

$= 0.271$

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उत्तर 2) Solve by Euler's method:

$\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0)=0.1$, $h=0.02$, for

$x=0.1$

$x_0=0$ $\quad x_1=0.02$ $\quad x_2=0.04$ $\quad x_3=0.06$ $\quad x_4=0.08$ $\quad y_5=0.1$

$y_0=1$ $\quad y_1=?$ $\quad y_3=?$ $\quad y_3=?$ $\quad y_4=?$ $\quad y_5=?$

We know $y_{n+1} = y_n + h y_n'$

$y(0)=0.1 \Rightarrow x_0=0,\ y_0=1$

(1) $y_1 = y_0 + h y_0'$

$= y_0 + h f(x_0, y_0)$

$= y_0 + h\left[\frac{y_0-x_0}{y_0+x_0}\right]$

$= 1 + 0.02\left[\frac{1-0}{1+0}\right]$

$= 1.02$

ii) $y_2 = y_1 + h y_1'$

$= 1.02 + 0.02\left[\frac{y_1-x_1}{y_1+x_1}\right]$

$= 1.02 + 0.02\left[\frac{1.02-0.02}{1.02+0.02}\right]$

$= 1.0392$

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iii) $y_3 = y_2 + h y_2'$

$= 1.0392 + 0.02\left[\frac{1.0392-0.04}{1.0392+0.04}\right]$

$= 1.0577$

iv) $y_4 = y_3 + h y_3'$

$= 1.0577 + 0.02\left[\frac{1.0577-0.06}{1.0577+0.06}\right]$

$= 1.0756$

v) $y_5 = y_4 + h y_4'$

$= 1.0756 + 0.02\left[\frac{1.0756-0.08}{1.0756+0.08}\right]$

$= \cancel{1.0785}$ 1.0928

vi) $y_6 = y_5 + h y_5'$

$= 1.0928 + 0.02\left[\frac{1.0928-0.1}{1.0928+0.1}\right]$

$= 1.1094$

3) $\frac{dy}{dx} = x+y$, $y(0)=1$, $h=0.1$ for $x=1$

soln:

$\frac{dy}{dx} = x+y$

$y(0)=1 \Rightarrow y_0 = 1,\ x_0 = 0$

$h = 0.1$

$x_0 = 0$ $x_1 = 0.1$ $x_2 = 0.2$ $x_3 = 0.3$ $x_4 = 0.4$ $x_5 = 0.5$ $x_6 = 0.6$ $x_7 = 0.7$ $x_8 = 0.8$

$y_0 = 1$ $y_1 = ?$ $y_2 = ?$ $y_3 = ?$ $y_4 = ?$ $y_5 = ?$ $y_6 = ?$ $y_7 = ?$ $y_8 = ?$

$x_9 = 0.9$

$y_9 = ?$

$x_{10} = 1$

$y_{10} = ?$

We know

$y_{n+1} = y_n + h y_n'$

Then

$y_1 = y_0 + h y_0'$

$= 1 + 0.1[x_0 + y_0]$

$= 1 + 0.1[0 + 1]$

$= 1.1$

$y_2 = y_1 + h y_1'$

$= 1.1 + 0.1[0.1 + 1.1]$

$= 1.22$

$y_3 = y_2 + h y_2'$

$= 1.22 + 0.1[0.2 + 1.22]$

$= 1.362$

$y_4 = y_3 + h y_3'$

$= 1.362 + 0.1[0.3 + 1.362]$

$= 1.5282$

$y_5 = y_4 + h y_4'$

$= 1.5282 + 0.1[0.4 + 1.5282]$

$= 1.7210$

$y_6 = y_5 + h y_5'$

$= 1.7210 + 0.1[0.5 + 1.7210]$

$= 1.9431$

$y_7 = y_6 + h y_6'$

$= 1.9431 + 0.1[0.6 + 1.9431]$

$= 2.1974$

$y_8 = y_7 + h y_7'$

$= 2.1974 + 0.1[2.1974 + 0.7]$

$= 2.4871$

$y_9 = y_8 + h y_8'$

$= 2.4871 + 0.1[0.8 + 2.4871]$

$= 2.8158$

$y_{10} = y_9 + hy_9'$
$= 2.8158 + 0.1[0.9 + 2.8158)$
$= 3.1874$

$y_{10} = y_9 + hy_9'$
$= 3.1874 + 0.1[1 + 3.1874]$
$= 3.6061$

Method IV

Runge-Kutta Method

This method was devised by Runge, about the year 1934 and extended by Kutta a few year later therefore we call this method as Runge-Kutta method.

I) Fourth order Runge-Kutta method.

This method is commonly used and is often referred to as Runge-Kutta method only or classical Fourth order R-K method.

Let $\frac{dy}{dx} = f(x,y)$ be a given differential Equation to be solved under the initial condition $y(x_0) = y_0$. If h be the length of interval between equidistant values. Then

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(x_0, y_0)$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Similarly

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(x_1, y_1)$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_1 + h, y_1 + k_3)$$

& so on.

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1) Apply Runge-Kutta 4th order method to find an approximate value of y when $x = 0.2$ Given that $\frac{dy}{dx} = x+y$, $y(0)=1$, $h=0.2$

soln:

$\frac{dy}{dx} = f(x,y) = x+y$

$y(0)=1 \Rightarrow x_0 = 0, y_0 = 1$

$h = 0.2$

$\therefore x_1 = x_0 + h = 0+0.2 = 0.2$

$y_1 = ?$

We know R-K 4th order formula

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) \longrightarrow (1)$$

where

$K_1 = hf(x_0, y_0)$

$= 0.2\, f(0,1)$

$= 0.2\,[0+1]$

$= 0.2$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$= 0.2\, f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$

$= 0.2\, f(0.1, 1.1)$

$= 0.2\,[0.1 + 1.1]$

$= 0.24$

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$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$

$= 0.2\, f\left[0 + \frac{0.2}{2}, 1 + \frac{0.24}{2}\right]$

$= 0.2\, f(0.1, 1.12)$

$= 0.2\,[0.1 + 1.12)$

$= 0.244$

$K_4 = hf(x_0 + h, y_0 + K_3)$

$= 0.2\, f(0 + 0.2, 1 + 0.244)$

$= 0.2\, f(0.2, 1.244)$

$= 0.2\,[0.2 + 1.244)$

$= 0.2888$

Now,

$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 1 + \frac{1}{6}(0.2 + 2 \times 0.24 + 2 \times 0.244 + 0.2888)$

$y(0.2) = 1.2428$

2) solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0)=1$ at $x=0.4$ when $x = 0.2$ Given $h = 0.2$

soln:

$\frac{dy}{dx} = f(x,y) = \frac{y^2 - x^2}{y^2 + x^2}$

Here $y(0)=1 \Rightarrow x_0 = 0$
$y_0 = 1$

$h = 0.2$

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$x_0 = 0$ $\quad x_1 = 0.2 \quad x_2 = 0.4$

$y_0 = 1$ $\quad y_1 = ? \quad y_2 = ?$

$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$

$x_2 = x_0 + 2h = 0 + 2*0.2 = 0.4$

We know

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \rightarrow (1)$$

where

$k_1 = hf(x_0, y_0)$

$= 0.2 f(0, 1)$

$= 0.2 * \left(\frac{1^2 - 0^2}{1^2 + 0^2}\right)$

$= 0.2$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= 0.2 f(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2})$

$= 0.2 f(0.1, 1.1)$

$= 0.2 \left(\frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2}\right)$

$= 0.1967$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$= 0.2 f(0 + \frac{0.2}{2}, 1 + \frac{0.1967}{2})$

$= 0.2 f(0.1, 1.0984)$

$= 0.2 \left(\frac{1.0984^2 - 0.1^2}{1.0984^2 + 0.1^2}\right)$

$= 0.1967$

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$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.2 f(0 + 0.2, 1 + 0.1967)$

$= 0.2 f(0.2, 1.1967)$

$= 0.2 \left(\frac{1.1967^2 - 0.2^2}{1.1967^2 + 0.2^2}\right)$

$= 0.1891$

Then

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(0.2 + 2*0.1967 + 2*0.1967 + 0.1891)$

$= 1.1960$

Again,

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where

$k_1 = hf(x_1, y_1)$

$= 0.2 f(0.2, 1.1960)$

$= 0.2 \left(\frac{1.1960^2 - 0.2^2}{1.1960^2 + 0.2^2}\right)$

$= 0.1891$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= 0.2 f(0.2 + 0.1, 1.1960 + \frac{0.1891}{2})$

$= 0.2 f(0.3, 1.2906)$

$= 0.2 \left[\frac{1.2906^2 - 0.3^2}{1.2906^2 + 0.3^2}\right]$

$= 0.1795$

$$k_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right)$$

$$= 0.2f\left(0.2+0.1, 1.1960+\frac{0.1795}{2}\right)$$

$$= 0.2f(0.3, 1.2858)$$

$$= 0.2\left(\frac{1.2858^2-0.3^2}{1.2858^2+0.3^2}\right)$$

$$= 0.1793$$

$$k_4 = hf(x_1+h, y_1+k_3)$$

$$= 0.2f(0.2+0.2, 1.1960+0.1793)$$

$$= 0.2f(0.4, 1.3753)$$

$$= 0.1688$$

Then

$$y_2 = y_1+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= 1.1960+\frac{1}{6}(0.1891+2*0.1795+2*0.1793+0.1688)$$

$$= 1.3753$$

$$\therefore y_2(0.4) = 1.3753$$

II Solution of simultaneous Differential eq^n

Let $\frac{dy}{dx} = f(x,y,z)$

$\frac{dz}{dx} = \phi(x,y,z)$

be simultaneous differential eq^{n}s with initial conditions $y(x_0)=y_0$ & $z(x_0)=z_0$ & interval size h Then

$$y_1 = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$z_1 = z_0+\frac{1}{6}(l_1+2l_2+2l_3+l_4)$$

where

$$k_1 = hf(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$$

$$k_4 = hf(x_0+h, y_0+k_3, z_0+l_3)$$

Similarly

$$l_1 = h\phi(x_0, y_0, z_0)$$

$$l_2 = h\phi\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right)$$

$$l_3 = h\phi\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$$

$$l_4 = h\phi(x_0+h, y_0+k_3, z_0+l_3)$$

Replacing x_0, y_0, z_0 by x, y, z we get y_2 & z_2 & so on,

3) Solve $\frac{dy}{dx} = yz + x$, $\frac{dz}{dx} = xz + y$ with initial conditions, $y(0) = 1$ & $z(0) = -1$, $h = 0.1$ for $x = 0.1$

Soln:

Here Given that,

$$\frac{dy}{dx} = f(x, y, z) = yz + x$$

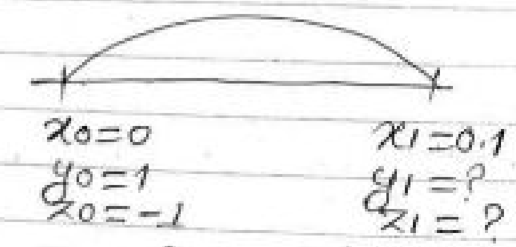
$$\frac{dz}{dx} = \phi(x, y, z) = xz + y$$

Initial conditions $y(0) = 1 \Rightarrow x = 0$, $y_0 = 1$

$z(0) = -1$, $x_0 = 0$, $z_0 = -1$

$h = 0.1$

$\therefore x_1 = x_0 + h = 0 + 0.1 = 0.1$



We know R-K formula,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

where

$k_1 = hf(x_0, y_0, z_0)$
$= 0.1 f(0, 1, -1)$
$= 0.1 [1 \times -1 + 0]$
$= -0.1$

$l_1 = h\phi(0, 1, -1)$
$= 0.1(0 \times -1 + 1)$
$= 0.1$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$
$= 0.1 f(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2}, -1 + \frac{0.1}{2})$
$= 0.1 f(0.05, 0.95, -0.95)$
$= 0.1(-0.95 \times 0.95 + 0.05)$
$= -0.0853$

$l_2 = h\phi(0.05, 0.95, -0.95)$
$= 0.1(0.05 \times -0.95 + 0.95)$
$= 0.0903$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$
$= 0.1 f(0.05, 0.9574, -0.9549)$
$= 0.1((-0.9549 \times 0.9574 + 0.05)$
$= -0.0864$

$l_3 = h\phi(0.05, 0.9574, -0.9549)$
$= 0.1[0.05 \times (-0.9549) + 0.9574)$
$= 0.0910$

$k_4 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}, z_0 + \frac{l_3}{2})$
$= 0.1 f(0 + 0.1, 1 - 0.0864, -1 + 0.091)$
$= 0.1(0.9136 \times (-0.9090) + 0.1)]$
$= -0.0730$

$l_4 = h\phi(0.1, 0.9136, -0.9090)$
$= 0.1[0.1 \times -0.9090 + 0.9136]$
$= 0.0823$

$y_1 = y_0 + \frac{1}{6}(k_1 - 2k_2 + 2k_3 + k_4)$

$= 1 + \frac{1}{6}(-0.1 + 2 * -0.0853 - 2 * 0.0864 - 0.0730)$

$= 0.9139$

$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= -1 + \frac{1}{6}(0.1 + 2 * 0.0903 + 2 * 0.0910 + 0.0823)$

$= -0.9092$

Hence $y(0.1) = 0.9139$

$z(0.1) = -0.9092$

Using R-K method solve the system $\frac{dy}{dx} = x+z, \frac{dz}{dx} = x-y, y(0)=0, z(0)=1$

taking, for $x = 0.2$

Soln

$\frac{dy}{dx} = f(x,y,z) = x+z$

$\frac{dz}{dx} = \phi(x,y,z) = x-y$

$y(0) = 0 \Rightarrow x_0 = 0, y_0 = 0$

$z(0) = 1 \Rightarrow x_0 = 0, z_0 = 1$

$h = 0.1$

$x_0 = 0$, $y_0 = 0$, $z_0 = 1$ $\quad$ $x_1 = 0.1$, $y_1 = ?$, $z_1 = ?$ $\quad$ $x_2 = 0.2$, $y_2 = ?$, $z_2 = ?$

$\therefore x_1 = x_0 + h = 0 + 0.1 = 0.1$

$x_2 = x_0 + 2h = 0 + 0.2 = 0.2$

$y_2 = ?$

$z_2 = ?$

R-K method

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \longrightarrow (1)$

$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \longrightarrow (2)$

where

$k_1 = hf(x_0, y_0, z_0)$	$l_1 = h\phi(x_0, y_0, z_0)$
$= 0.1 f(0, 0, 1)$	$= 0.1\phi(0, 0, 1)$
$= 0.1 [0+1]$	$= 0.1(0-0)$
$= 0.1$	$= 0$
$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2})$	$l_2 = h\phi(0.05, 0.05, 1)$
$= 0.1 f(0.05, 0.05, 1)$	$= 0.1(0.05 - 0.05)$
$= 0.1(0.05 + 1)$	$= 0$
$= 0.1050$	
$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2})$	$l_3 = h\phi(0.05, 0.0525, 1)$
$= 0.1 f(0.05, 0.0525, 1)$	$= 0.1(0.05 - 0.0525)$
$= 0.1(0.05 + 1)$	$= -0.0003$
$= 0.1050$	

$K_4 = hf(x_0+h, y_0+k_3, z_0+l_3)$ $\quad l_4 = h\phi(0.1, 0.1050, 0.9997)$

$= hf(0.1, 0.1050, 0.9997)$ $\quad = 0.1(0.1000 - 0.1050)$

$= 0.1(0.1 + 0.9997)$ $\quad = 0.0005$

$= 0.1100$

Then

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0 + \frac{1}{6}(0.1 + 2 \times 0.1050 + 2 \times 0.1050 + 0.11)$

$= 0.1050$

$z_1 = z_1 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= 1 + \frac{1}{6}(0 + 2 \times 0 + 2 \times (-0.0008) + 0.0005)$

$= 0.9998$

Again

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.1050 + \frac{1}{6}(0.1 + 2 \times 0.1050 + 2 \times 0.1050 + 0.11)$

$= 0.2100$

$z_2 = z_1 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= 0.9998 + \frac{1}{6}(0 + 2 \times 0 - 0.0003 \times 2 + 0.0005)$

$= 0.9996$//

$k_1 = hf(x_1, y_1, z_1)$

$= 0.1 f(0.1, 0.1050, 0.9998)$

$= 0.1(0.1 + 0.9998)$

$= 0.11$

$l_1 = h\phi(0.1, 0.1050, 0.9998)$

$= 0.1 \phi(0.1 - 0.1050)$

$= -0.0005$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2})$

$= 0.1 f(0.15, 0.16, 0.9996)$

$= 0.1(0.15 + 0.9996)$

$= 0.1150$

$l_2 = h\phi(0.15, 0.16, 0.9996)$

$= 0.1(0.15 - 0.16)$

$= -0.001$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2})$

$= 0.1 f(0.15, 0.1625, 0.9993)$

$= 0.1(0.15 + 0.9993)$

$= 0.1149$

$l_3 = h\phi(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2})$

$= 0.1\phi(0.15, 0.1625, 0.9993)$

$= 0.1(0.15 - 0.1625)$

$= -0.0013$

$k_4 = hf(x_1+h, y_1+k_3, z_1+l_3)$
$= 0.1 f(0.2, 0.2199, 0.9985)$
$= 0.1(0.2+0.9985)$
$= 0.1199$
$l_4 = h\phi(0.2, 0.2199, 0.9985)$
$= 0.1(0.2-0.2199)$
$= -0.0020$

So,
$$y_2 = y_1 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$
$$= 0.1050 + \frac{1}{6}[0.11+2*0.1150+2* 0.1149+0.1199]$$
$$= 0.2200$$

$$z_2 = z_1 + \frac{1}{6}(l_1+2l_2+2l_3+l_4)$$
$$= 0.9998 + \frac{1}{6}(-0.0005-2\times 0.001 -2\times 0.0013 - 0.002)$$
$$= 0.9986$$

Model III >
Solution of Second order differential eq^n by Runge Kutta method.

Let us consider the second order differential equations
$$\frac{d^2y}{dx^2} = \phi\left(x, y, \frac{dy}{dx}\right) \longrightarrow (1)$$
If we put $\frac{dy}{dx} = z \longrightarrow (2)$ Then
$$\frac{d^2y}{dx^2} = \frac{dz}{dx} \longrightarrow (3)$$
From eq^n (1) $\frac{dz}{dx} = \phi(x, y, z)$.
Now, we have two simultaneous differential eq^n
$$\frac{dy}{dx} = f(x, y, z)$$
$$\& \frac{dz}{dx} = \phi(x, y, z)$$
with initial conditions $y(x_0) = y_0$ & $z(x_0) = z_0$
solving these eq^{ns} we get,
$$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$
$$z_1 = z_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4)$$

where

$k_1 = hf(x_0, y_0, z_0)$

$l_1 = h\phi(x_0, y_0, z_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$l_2 = h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$l_3 = h\phi\left(\text{''} \quad \text{''} \quad \text{''}\right)$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$

$l_4 = h\phi(\text{''} \quad \text{''} \quad \text{''})$

Replacing x_0, y_0, z_0 by x_1, y_1, z_1 we get y_2, z_2 & so on.

5) Using R-K method of 4th order, solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$ for $x = 0.2$ with initial condn $y(0) = 1, y'(0) = 0$ take $h = 0.2$

soln:

Here, Given that

$$\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$$

Let $\frac{dy}{dx} = z$ then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$$\frac{dy}{dx} = z$$

Now, we have following simultaneous eqn

$$\frac{dy}{dx} = f(x, y, z) = z$$

$$\frac{dz}{dx} = \phi(x, y, z) = xz^2 - y^2$$

with $y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$

$y'(0) = 0 \Rightarrow z(0) = 0 \Rightarrow x_0 = 0$

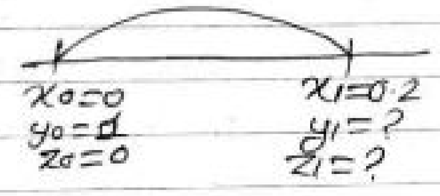
$z_0 = 0$

$h = 0.2$

$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$

$y_1 = ?$

$z_1 = ?$



we know R-K formula,

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ — ①

$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$ — ②

where

$k_1 = hf(x_0, y_0, z_0)$
$= hf(0, 1, 0)$
$= 0.2 \times 0$
$= 0$

$l_1 = h\phi(x_0, y_0, z_0)$
$= 0.2\phi(0, 1, 0)$
$= 0.2(0.0^2 - 1^2)$
$= -0.2$

$$k_2 = h*f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{\cancel{k_1}}{2}\right)$$

$$= 0.2 f(0.1, 1, -0.1)$$
$$= 0.2(-0.1)$$
$$= -0.02$$

$$l_2 = h\phi \cancel{f}\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2}\right)$$

$$= 0.2\,\phi(0.1, 1, -0.1)$$
$$= 0.2\,(0.1*(-0.1)^2-1)$$
$$= -0.1998$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2}\right)$$

$$= 0.2 f(0.1, -0.99, -0.0999)$$
$$= 0.2(-0.0999)$$
$$= -0.02$$

$$l_3 = h\phi(0.1, -0.99, -0.0999)$$
$$= 0.2\,(0.1(-0.0999)^2 - 0.99^2)$$
$$= \cancel{0.0002} - 0.1958$$

$$k_4 = hf(x_0+h, y_0+k_3, z_0+l_3)$$
$$= 0.2 f(0.2, 0.9800, -0.1958)$$
$$= -0.0392$$

$$l_4 = h\phi(0.2, 0.98, -0.1958)$$
$$= 0.2\,(0.2\,(0.1958)^2 - 0.98^2)$$
$$= -0.1905$$

$$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= 1+\frac{1}{6}(0 - 2*0.2 - 2*0.2 - 0.0392)$$

$$= 0.9801$$

$$z_1 = z_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4)$$

$$= 0+\frac{1}{6}(-0.2 - 2*0.1998 - 2*0.1958 - 0.1905)$$

$$= -0.1970$$

6> Solve

$y'' = xy' - y$, $y(0)=3$, $y'(0)=0$ to $y(0.1)$

Here

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$$

Let $\frac{dy}{dx} = z$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

From eq^n ①

$$\frac{dz}{dx} = xz - y$$

Now we have two simultanecus eq^ns

$$\frac{dy}{dx} = f((x,y,z) = z,$$

$$\frac{dz}{dx} = \phi(x,y,z) = xz - y$$

with initial condⁿ

$y(0)=3 \Rightarrow x_0=0$
$\qquad y_0=3$

$y'(0)=0 \Rightarrow x_0=0$
$\qquad z_0=0$

Taking $h=0.1$

$x_1 = x_0+h = 0+0.1 = 0.1$

$y_1 = ?$

$z_1 = ?$

We know R-K formula

$$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$z_1 = z_0 + \frac{1}{6}(l_1+2l_2+2l_3+l_4)$$

Now,

$k_1 = hf(x_0,y_0,z_0)$
$= 0.1 f(0,3,0)$
$= 0.1 \times 0$
$= 0$

$l_1 = h\phi(x_0,y_0,z_0)$
$= 0.1\,\phi(0,3,0)$
$= 0.1\,(0.0-3)$
$= -0.3.$

$k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, z_0+\frac{l_1}{2})$

$= 0.1 f(0.05, 3, -0.15)$
$= 0.1(-0.15) = -0.015$

$l_2 = h\phi(0.05, 3, -0.15)$
$= 0.1\,[0.05\times(-0.15-3)$
$= -0.3008$

$k_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, z_0+\frac{l_2}{2})$

$= 0.1 f(0.05, 2.9250, -0.1504)$
$= 0.1(-0.1504)$
$= -0.015$

$l_3 = h\phi(0.05, 2.9250 - 0.1504)$
$= 0.1\,(-0.05\times 0.1504 - 2.9250)$
$= -0.3$

$k_4 = hf(x_0+h, y_0+k_3, z_0+l_3)$
$= 0.1f(0.1, 2.9850, -0.3000)$
$= 0.1(-0.3000)$
$= \cancel{-0.293}$ $\qquad l_4 = -0.3015$
$= -0.03$

Then $y_1 = y_0 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$

$= 3 + \frac{1}{6}[0+2\times(-0.015)-2\times 0.015 - 0.03]$

$= 2.9850$

$z_1 = z_0 + \frac{1}{6}[l_1+2l_2+2l_3+l_4]$

$= 0 + \frac{1}{6}[-0.3 - 2\times 0.3008 - 2\times 0.3 - 0.3015]$

$= -0.3005$

7) $20\frac{d^2y}{dx^2}+2\frac{dy}{dx}-4y=5$

$y(0)=0,\ y'(0)=0,\ h=0.25$
for $0\le x\le 0.5$)

Soln:

$20\frac{d^2y}{dx^2}+2\frac{dy}{dx}-4y=5$

Let $\frac{dy}{dx}=z$

So, $\frac{d^2y}{dx^2}=\left[5+4y-2\frac{dy}{dx}\right]\frac{1}{20}$

$\frac{dz}{dx}=\left[5+4y-2z\right]\frac{1}{20}$

Now,

$\frac{dy}{dx}=f(x,y,z)=z$

$\frac{dz}{dx}=\phi(x,y,z)=\frac{1}{20}[5+4y-2z]$

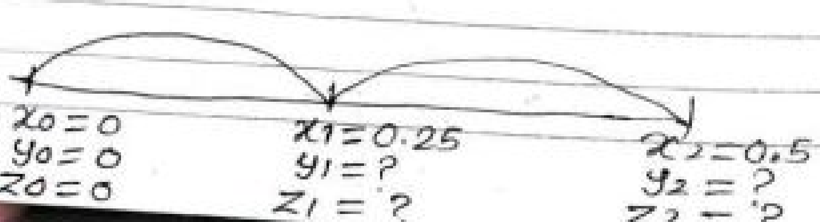
Here $y(0)=0 \Rightarrow x_0=0$

$y_0=0$

$y'(0)=0 \Rightarrow x_0=0$

$z_0=0$

Then,



Now, From R-K formula

$y_1=y_0+\frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$z_1=z_0+\frac{1}{6}[l_1+2l_2+2l_3+l_4]$

Then

$K_1=hf(x_0,y_0,z_0)$
$=0.25f(0,0,0)$
$=0.25*0$
$=0$

$l_1=h\phi(x_0,y_0,z_0)$
$=0.25\phi(0,0,0)$
$=0$

$K_2=hf(x_0+\frac{h}{2},y_0+\frac{K_1}{2},z_0+\frac{h}{2})$
$=0.25f(0.125,0,0)$
$=0$

$l_2=h\phi(0.125,0,0)$
$=0.25[\frac{5}{20}]$
$=0.0625$

$K_3=hf(x_0+\frac{h}{2},y_0+\frac{K_2}{2},z_0+\frac{l_2}{2})$

$=0.25(0.125,0,0.0313)$
$=0.25(0.0313)$
$=0.0078$

$l_3=h\phi(x_0+\frac{h}{2},y_0+\frac{K_2}{2},z_0+\frac{l_2}{2})$

$=0.25\phi(0.125,0,0.0313)$
$=0.25[\frac{1}{20}(5+0-2*0.0313]$
$=0.0617$

$$K_4 = hf(x_0+h, y_0+k_3, z_0+l_3)$$
$$= 0.25 f(0.25, 0.0078, 0.0617)$$
$$= 0.25 \times 0.0617$$
$$= 0.0154$$
$$l_4 = h\phi(0.25, 0.0078, 0.0617)$$
$$= 0.25\left[\frac{1}{20}(5+4\times 0.0078 - 2\times 0.0617)\right]$$
$$= 0.0613$$

Then

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$
$$= 0 + \frac{1}{6}[0 + 2\times 0 + 2\times 0.0078 + 0.0154]$$
$$= 0.0052$$
$$z_1 = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$
$$= 0 + \frac{1}{6}[0 + 2\times 0.0625 + 2\times 0.0617 + 0.0613]$$
$$= 0.0516$$

again $k_1 = hf(x_1, y_1, z_1)$
$$= 0.25 f(0.25, 0.0052, 0.0516)$$
$$= 0.25 \times 0.0516$$
$$= 0.0129$$

$$l_1 = h\phi(x_1, y_1, z_1)$$
$$= 0.25\phi(0.25, 0.0052, 0.0516)$$
$$= 0.25\left[\frac{1}{20}(5+4\times 0.0052 - 2\times 0.0516)\right]$$
$$= 0.0615$$
$$K_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}, z_1+\frac{l_1}{2}\right)$$
$$= 0.25 f(0.3750, 0.0117, 0.0824)$$
$$= 0.25 \times 0.0824$$
$$= 0.0206$$
$$l_2 = h\phi(0.3750, 0.0117, 0.0824)$$
$$= 0.25\left[\frac{1}{20}(5+4\times 0.0117 - 2\times 0.0824)\right]$$
$$= 0.0610$$
$$K_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}, z_1+\frac{l_2}{2}\right)$$
$$= 0.25(0.3750, 0.0155, 0.0821)$$
$$= 0.25(0.0821)$$
$$= 0.0205$$
$$l_3 = h\phi(0.3750, 0.0155, 0.0821)$$
$$= 0.25\left(\frac{1}{20}(5+4\times 0.0155 - 2\times 0.0821)\right)$$
$$= 0.0612$$
$$K_4 = hf(x_1+h, y_1+k_3, z_1+l_3)$$
$$= 0.25(0.5, 0.0257, 0.1128)$$
$$= 0.25\times 0.1128 = 0.0282$$

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$l_4 = h\,\phi(0.5, 0.0257, 0.1128)$

$= 0.25\left(\frac{d}{20}\left[5+4*0.0257-2*0.1128\right]\right)$

$= 0.0610$

Then,

$y_2 = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 0.0052 + \frac{1}{6}[0.0129 + 2*0.0206 + 2*0.0205 + 0.0282]$

$= 0.0258$

$z_2 = z_1 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4]$

$= 0.0516 + \frac{1}{6}(0.0615 + 2*0.0610 + 2*0.0612 + 0.0610)$

$= 0.1128$

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Method V) Heun's Method :-
(R-K 2nd order method)
or modified Euler's Method).

Formula:

If $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$ & step size h. Then

$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0+h, y_0+k_1)$

Replacing x_0, y_0 by x, y we get y_2 & so on.

1) Using Heun's method, find an appropri-
-ate value of y for $x = 0.3$. Given $\frac{dy}{dx} = x+y$; $y(0) = 1$, $h = 0.1$

Soln:

Given $\frac{dy}{dx} = f(x,y) = x+y$

$y(0) = 1 \Rightarrow x_0 = 0$

$y_0 = 1$

$h = 0.1$

$\therefore x_1 = x_0 + h = 0 + 0.1 = 0.1$

$x_2 = x_0 + 2h = 0 + 2*0.1 = 0.2$

$x_3 = x_0 + 3h = 0 + 3*0.1 = 0.3$

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We know Heun's formula

$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

For y_1 where $k_1 = hf(x_0, y_0)$

$= 0.1\, f(0,1)$

$= 0.1\,[0+1]$

$= 0.1$

$k_2 = hf(x_0+h, y_0+k_1)$

$= 0.1 f(0.1, 1.1)$

$= 0.1\,(0.1+1.1)$

$= 0.12$

$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$

$= 1 + \frac{1}{2}(0.1+0.12)$

$= 1.11$

Then For y_2

$y_2 = y_1 + \frac{1}{2}(k_1+k_2)$

where

$k_1 = hf(x_1, y_1)$

$= 0.1 f(0.1, 1.11)$

$= 0.1\,(0.1+1.11)$

$= 0.1210$

$k_2 = hf(x_1+h, y_1+k_1)$

$= 0.1 f(0.2, 1.11+0.121)$

$= 0.1 f(0.2, 1.232)$

$= 0.1\,(0.2+1.232)$

$= 0.1431$

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$y_2 = y_1 + \frac{1}{2}(k_1+k_2)$

$= 1.2421$

For y_3

$y_3 = y_2 + \frac{1}{2}(k_1+k_2)$

where

$k_1 = hf(x_2, y_2)$

$= 0.1 f(0.2, 1.2421)$

$= 0.1 \times (0.2+1.2421)$

$= 0.1442$

$k_2 = hf(x_2+h, y_2+k_2)$

$= 0.1 f(0.3, 1.3863)$

$= 0.1686$

Then

$y_3 = y_2 + \frac{1}{2}(k_1+k_2)$

$= 1.2421 + \frac{1}{2}(0.1442+0.1686)$

$= 13985$

Hence $\boxed{y(0.3) = 1.3985}$

Heun's method for second order differential

Solve for $y(0.2)$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x, \quad y(0)=0, \; y'(0)=1$$

$h = 0.2$

soln:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x$$

Put $\frac{dy}{dx} = z$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

From eqn (1)

$$\frac{dz}{dx} + 2z - 3y = 6x$$

or, $\frac{dz}{dx} = 6x + 3y - 2z$

Now we have simultaneous diff eqn

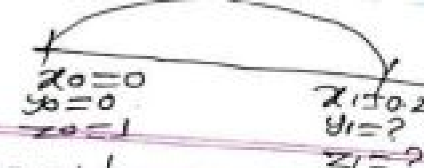
$$\frac{dy}{dx} = f(x,y,z) = z$$

$$\frac{dz}{dx} = \phi(x,y,z) = 6x + 3y - 2z$$

with initial conditions

$y(0) = 0 \Rightarrow x_0 = 0$, $y_0 = 0$

$y'(0) = 1 \Rightarrow z_0 = 1$, $x_0 = 0$



$\therefore x_1 = x_0 + h = 0 + 0.2$, $y_1 = ?$
$z_1 = ?$

we know Heun's formula

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

where

$k_1 = hf(x_0, y_0, z_0)$
$= 0.2 f(0,0,1)$
$= 0.2(1)$
$= 0.2$

$l_1 = h\phi(0,0,1)$
$= 0.2\phi(0,0,1)$
$= 0.2[6(0) + 3 \cdot 0 - 2 \times 1]$
$= -0.4$

$k_2 = hf(x_0+h, y_0+k, z_0+l_1)$
$= 0.2 f(0.2, 0.2, +0.6)$
$= 0.2 \times 0.6$
$= 0.12$

$l_2 = 0.2(0.2, 0.2, 0.6)$
$= 0.2(6 \cdot 0.2 + 3 \times 0.2 - 2 \times 0.6)$
$= 0.12$

$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$
$= 0 + \frac{1}{2}(0.2 + 0.12)$
$= 0.16$

$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$
$= 0 + \frac{1}{2}(-0.4 + 0.12)$
$= -0.14$

2013)

\# Solve $\frac{dy}{dx}=1+xz$, $\frac{dz}{dx}=-xy$ for $y(0.6)$, $z(0.6)$, Given $y=0, z=1$ at $x=0$ by using Heun's method Take $h=0.3$

Soln:

Here Given

$\frac{dy}{dx}=1+xz$

$\frac{dz}{dx}=\phi(x,y,z)=-xy$

$y(0)=0$ & $z(0)=1 \Rightarrow x_0=0$

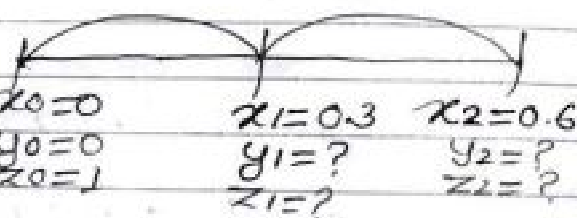
Now, $x_1=x_0+h=0+0.3 = 0.3$ $\quad y_0=0$, $z_0=1$

$x_2=x_0+2h=0+2\times0.3=0.6$

$y_2=?$

$z_2=?$

Now,



For y_1

$k_1=hf(x_0,y_0,z_0)$
$=0.3(0,0,1)$
$=0.3(1+0\times1)$
$=0.3$

$l_1=h\phi(x_0,y_0,z_0)$
$=0.3\,\phi(0,0,1)$
$=0.3(-0\times0)$
$=0$

$k_2=hf(x_0+h,y_0+k_1,z_0+l_1)$
$=0.3f(0.3,0.3,1)$
$=0.3(1+0.3\times1)$
$=0.39$

$l_2=h\phi(0.3,0.3,1)$
$=0.3(-0.3\times0.3)$
$=-0.027$

Then

$y_1=y_0+\frac{1}{2}(k_1+k_2)$
$=0+\frac{1}{2}(0.3+0.39)$

$=0.3450$

$z_1=z_0+\frac{1}{2}(l_1+l_2)$

$=1+\frac{1}{2}(0-0.027)$
$=0.9865$

Again For y_2

$k_1=hf(x_1,y_1,z_1)$
$=0.3f(0.3,0.345,0.9865)$
$=0.3(1+0.3\times0.9865)$
$=0.3888$

$l_1=h\phi(0.3,0.345,0.9865)$
$=0.3(-0.3\times0.345)$
$=-0.0311$

$k_2=hf(x_1+h,y_1+k_1,z_1+l_1)$
$=0.3f(0.6,0.7338,0.9554)$
$=0.3(1+0.6\times0.9554)$
$=0.4720$

$l_2=0.3(-0.6\times0.7338)$
$=-0.1321$

$y_2 \doteq y_1 + \frac{1}{2}(k_1+k_2)$

$= 0.3450 + \frac{1}{2}(0.3888+0.4720)$

$= 0.7754$

$z_2 = z_1 + \frac{1}{2}(l_1 + l_2)$

$= 0.9865 + \frac{1}{2}(-0.0311 - 0.1321)$

$= \cancel{0.8}$

$= 0.9049$

Hence $y(0.6) = 0.7754$

$z(0.6) = 0.9049$ //

Method VI **Shooting Method**

Consider the eqn

$$\frac{d^2y}{dx^2} = \phi\left(x, y, \frac{dy}{dx}\right) \text{ or, } y'' = \phi(x, y, y')$$

$y(a) = A$ & $y(b) = B$

Let $\frac{dy}{dx} = z$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$$\therefore \frac{dz}{dx} = \phi(x, y, z)$$

In order to ~~this~~ solve this set as an initial value problem, we need two condns at $x=a$. We have one condition for $f(a)=A$ and therefore require another condition for z at $x=a$.

Let us assume that $z(a)=M$, where M_1 is the guess. M_1 represents the slope y' at $x=a$. Thus $y'=z$, $y(a)=A$

$z' = \phi(x,y,z)$ $z(a) = M_1$ —(1)

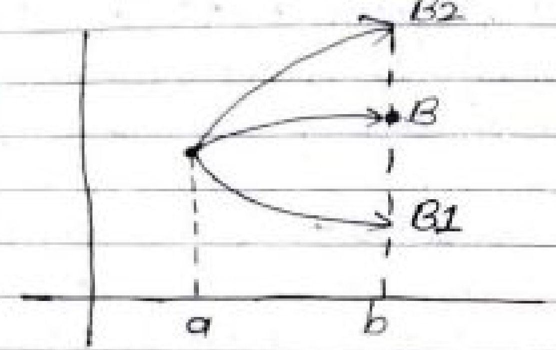
Eqn (1) can be solve for y & z using Heun's method using steps of h until $x=b$ be B_1. If $B_1 = B$ then we have obtained the required solution. If $B_1 \neq B$ then we obtain the solution with another guess. Say $z(a)=M_2$. Let the new estimate of y at $x=b$ be B_2. If $B_2 \neq B$ then process may be continued till we obtain the correct estimate of y. However the procedure can be accelerated by using an improved guess for $z(a)$ after estimates of B_1 & B_2. Let us consider that $z(a)=M_3$ leads to the value $y(b)=B$. If we assume that the values of M & B are linearly related then

$$\frac{M_3 - M_1}{B - B_1} = \frac{M_2 - M_1}{B_2 - B_1}$$

or, $M_3 = M_2 + \frac{B - B_2}{B_2 - B_1} * (M_2 - M_1)$

$$= M_2 - \frac{B_2 - B}{B_2 - B_1} * (M_2 - M_1)$$

Now with $z(a) = M_3$, we can again obtain solution of y.



1) Using shooting method, solve the eq^n $\frac{d^2y}{dx^2} = 6x$, $y(1)=2$, $y(2)=9$ in interval (1,2)

soln:

Here $\frac{d^2y}{dx^2} = 6x \longrightarrow (1)$

Let $\frac{dy}{dx} = z$ then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

From eq^n (1) $\frac{dz}{dx} = 6x$

$\therefore \frac{dz}{dx} = \phi(x,y,z) = 6x$

$\frac{dy}{dx} = f(x,y,z) = z$

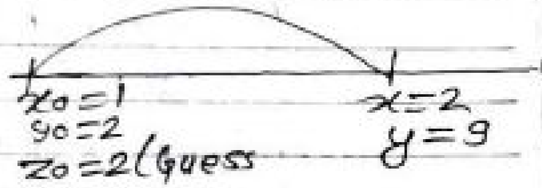
Initial cond^n

$y(a) = A$ or $y(1) = 2 \Rightarrow a = 1,\ A = 2$

$y(b) = B$ or $y(2) = 9 \Rightarrow b = 2,\ B = 9$

$y(1) = 2 \Rightarrow x_0 = 1$

$y_0 = 2$



Let $z_0 = 2$ (M_1)

Then $x_0 = 1$

$y_0 = 2$

$z_0 = 2$

let, $h = 1$ $x_1 = x_0 + h = 1 + 1 = 2$, $y_1 = ?$ $z = ?$

Using Heun's method we get

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

Where $K_1 = hf(x_0, y_0, z_0)$

$= 1.f(1,2,2)$

$= 2$

$l_1 = h\phi(x_0, y_0, z_0)$

$= h\phi(1,2,2)$

$= 6 \times 1 \times 1$

$= 6$

$k_2 = hf(x_0+h, y_0+k_1, z_0+k_2)$
$= 1f(2, 4, 8)$
$= 8$
$l_2 = 1\phi(2, 4, 8)$
$= 12$

Then
$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$ $\quad z_1 = 2 + \frac{1}{2}(6+12)$
$= 2 + \frac{1}{2}(2+8)$ $\quad = 11$
$= 7$

This gives $B_1 = 7$ which is less than g.
Now assume $z(1) = 4 = M_2$
Then $x_0 = 1$
$y_0 = 2$
$z_0 = 4$

$y_1 = y_0 + \frac{1}{2}(k_1+k_2) = \cancel{2+\frac{1}{2}}\,6$

$z_1 = z_0 + \frac{1}{2}(l_1+l_2)$

where
$k_1 = hf(x_0, y_0, z_0)$ $\quad l_1 = h\phi(1,2,4)$
$= 1f(1,2,4)$ $\quad = 1(6*1)$
$= 4$ $\quad = 6$
$k_2 = hf(x_0+h, y_0+k, z_0+l_1)$
$= 1f(2, 6, 10)$
$= 10$
$l_2 = h\phi(2, 6, 10)$
$= 1(6*2)$
$= 12$

So, $y_1 = y_0 + \frac{1}{2}(k_1+k_2)$ $\quad z_1 = z_0 + \frac{1}{2}(l_1+l_2)$
$= 2 + \frac{1}{2}(4+10)$ $\quad = 4 + \frac{1}{2}(6+12)$
$= 9 = 13$ $\quad = 13$

So $y(1) = 2$, $y(2) = 9$ $\quad z(1) = 4$

Hence $y(1) = 2$, $z(1) = 4$ Ans

2) Solve
$\frac{d^2y}{dx^2} = 6x$, $y(1) = 2$, $y(2) = 9$, $h = 0.5$

Soln:
$\frac{d^2y}{dx^2} = 6x$

Here Let $\frac{dy}{dx} = z$

So, $\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Then $\frac{dz}{dx} = 6x$

So Simultaneous eqn is
$\frac{dy}{dx} = f(x,y,z) = z$

$\frac{dz}{dx} = \phi(x,y,z) = 6x$

Initial condⁿ

$y(1)=2 \Rightarrow x_0=1, y_0=2, h=0.5$

$\therefore x_1 = x_0+h = 1+0.5 = 1.5$

$x_2 = x_0+2h = 1+2\times0.5 = 2$

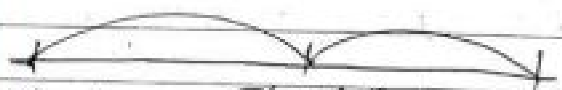
$y_2 = ?$

$z_2 = ?$

$\because y(2)=9 \Rightarrow y=9$ when $x=2$

$b=2, B=9.$

let $z(1)=2=M$



$x_0=1$ $x_1=1.5$

$y_0=2$

$z_0=2$

Using Heun's formula

$y_1 = y_0+\frac{1}{2}(k_1+k_2)$

$z_1 = z_0+\frac{1}{2}(l_1+l_2)$

where

$k_1 = hf(x_0,y_0,z_0)$

$=0.5f(1,2,2)$

$=1$

$l_1 = h\phi(1,2,2)$

$=0.5(6\times1]$

$=3$

$k_2 = hf(x_0+h, y_0+k, z_0+l_1)$

$=0.5f(1.5,3,5]$

$=0.5\times5$

$=2.5$

$l_2 = h\phi(1.5,3,5)$

$=0.5\times6\times1.5$

$=4.5$

Then $y_1 = y_0+\frac{1}{2}(k_1+k_2)$

$=2+\frac{1}{2}(1+2.5)$

$=3.75$

$z_1 = z_0+\frac{1}{2}(l_1+l_2)$

$=2+\frac{1}{2}(3+4.5)$

$=5.75$

II step

where $k_1 = hf(x_1,y_1,z_1)$

$=0.5f(1.5, 3.75, 5.75)$

$=0.5\times5.75$

$=2.8750$

$l_1 = h\phi(1.5, 3.75, 5.75)$

$=0.5\times6\times1.5$

$=4.5$

$k_2 = hf(x_1+h, y_1+k_1, z_1+l_1)$

$=0.5f(2, 6.6250, 10.25)$

$=0.5(10.25)$

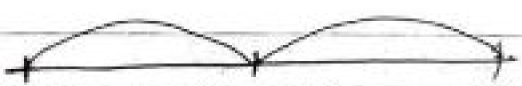
$=5.125$

$l_2 = h\phi(2, 6.625, 10.25)$
$= 0.5(6*2)$
$= 6$

So,
$y_2 = y_1 + \frac{1}{2}(k_1+k_2)$
$= 3.75 + \frac{1}{2}(2.875+5.125)$

$= 7.75 = B_1$

This given $B_1 = 7.75$ which is less than $B = 9$. Let us assume another guess $z(1) = 4 = M_2$



$x_0 = 1$ $x_1 = 1.5$ $x_2 = 2$
$y_0 = 2$ $y_2 = 9$
$z_0 = 4$

$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$

$z_1 = z_0 + \frac{1}{2}(l_1+l_2)$

where

$k_1 = hf(x_0, y_0, z_0)$ $l_1 = 0.5\phi(1,2,4)$
$= 0.5f(1,2,4)$ $= 0.5*6*1$
$= 0.5*4$ $= 3$
$= 2$

$k_2 = hf(x_0+h, y_0+k_1, z_0+l_1)$
$= 0.5f(1.5, 4, 7)$
$= 0.5*7$
$= 3.5$

$l_2 = h\phi(1.5, 4, 7)$
$= 0.5*6*1.5$
$= 4.5$

Then
$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$

$= 2 + \frac{1}{2}(2+3.5)$
$= 4.75$

$z_1 = z_0 + \frac{1}{2}(l_1+l_2)$
$= 4 + \frac{1}{2}(3+4.5)$

$= 7.75$

Again $y_2 = y_1 + \frac{1}{2}(k_1+k_2)$
$z_2 = z_1 + \frac{1}{2}(l_1+l_2)$

where
$k_1 = hf(x_1, y_1, z_1)$
$= 0.5f(1.5, 4.75, 7.75)$
$= 0.5*7.75$
$= 3.8750$

$l_1 = h\phi(1.5, 4.75, 7.75)$
$= 0.5*6*1.5$
$= 4.5$

$k_2 = hf(2, 8.6250, 12.25)$
$= \cancel{0.5 \times 6 \times 12.25}\ 2\ 0.5 \times 12.25$
$= \cancel{3.875}\ 6.1250$

Then
$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$
$= 4.75 + \frac{1}{2}(3.875 + 6.125)$
$= 9.\cancel{8}875 = B_2 \neq B$

Then
$$\frac{M_3 - M_2}{B - B_2} = \frac{M_2 - M_1}{B_2 - B_1}$$
or, $$\frac{M_3 - 4}{9 - 9.75} = \frac{4 - 2}{9.75 - 7.75}$$
or $$\frac{M_3 - 4}{-0.75} = \frac{2}{2}$$
or, $M_3 - 4 = -0.75$
$\therefore M_3 = 3.25$ //

Chapter - Six

Solution of partial Differential Eq^n:-

The second order equation involving two independent variable in general form as
$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = f\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \quad \rightarrow (1)$$

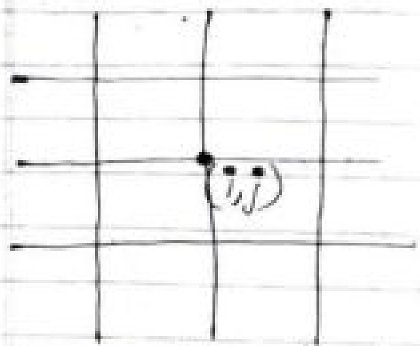
where coefficients, a, b, c may be constant or function of x, y. Depending on the value these coefficients, this equations may be classified as.

i) Elliptic if $b^2 - 4ac < 0$
ii) parabolic if $b^2 - 4ac = 0$
iii) Hyperbolic if $b^2 - 4ac > 0$

Elliptic Equation:-

A differential equation is the elliptic in a region R if $b^2 - 4ac < 0$ at all points of the region. The boundary condition of this type of equation specify the function u at every point of the closed boundary of the region R with in which the solution $u(x, y)$ is to be determined

Boundary conditions are given at every or some point of the boundary.



Method I> Laplace Equations

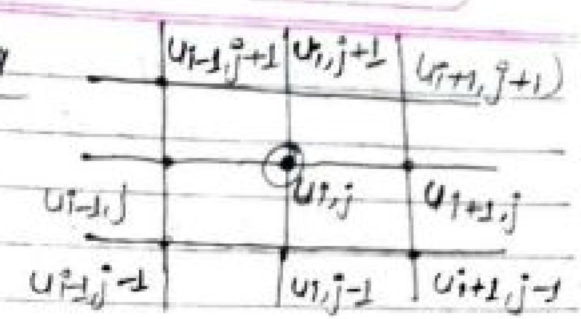
If $a=1, b=0, c=1$ & $F(x,y,f,f_x,f_y)=0$ in eq^n $a\frac{\partial^2 f}{\partial x^2}+b\frac{\partial^2 f}{\partial y^2}+c\frac{\partial^2 f}{\partial z^2}=F\left(x,y,f,\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right) \longrightarrow (1)$

Then $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$

or, $\nabla^2 f=0$

or, $f_{xx}+f_{yy}=0$ is called laplace equation.

Laplace formula



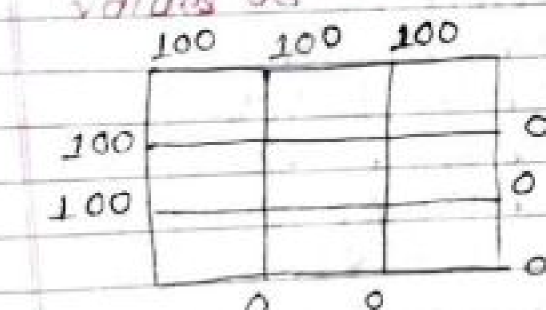
Standard 5-point formula

$$u_{i,j}=\frac{1}{4}\left[u_{i+1,j}+u_{i,j+1}+u_{i-1,j}+u_{i,j-1}\right]$$

Diagonal 5-point formula

$$u_{i,j}=\frac{1}{4}\left[u_{i+1,j+1}+u_{i-1,j+1}+u_{i-1,j-1}+u_{i+1,j-1}\right]$$

1> Solve the elliptic eq^n $u_{xx}+u_{yy}=0$ for following square mesh with boundary values as



Here Using standard 5-point formula

$u_1 = \frac{1}{4}(u_2 + 100 + 100 + u_3) = \frac{1}{4}(u_2 + u_3 + 200)$

$u_2 = \frac{1}{4}(0 + 100 + u_1 + u_4) = \frac{1}{4}(u_1 + u_4 + 100)$

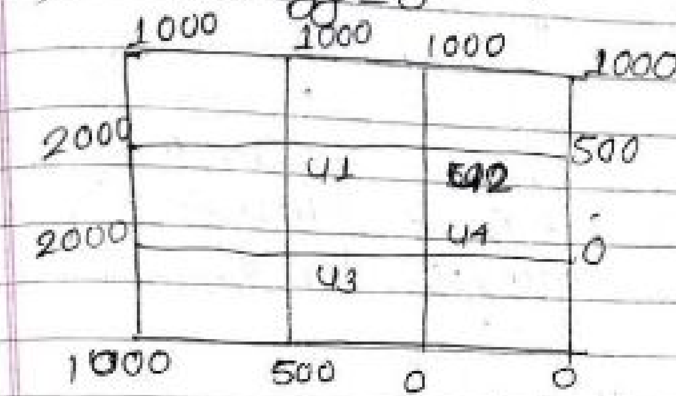
$u_3 = \frac{1}{4}(u_4 + u_1 + 100 + 0) = \frac{1}{4}(u_1 + u_4 + 100)$

$u_4 = \frac{1}{4}(0 + u_2 + u_3 + 0) = \frac{1}{4}(u_2 + u_3)$

Solving eq^ns by Gauss seidel method, we get

itn	u_1	u_2	u_3	u_4
Initial	0	0	0	0
	50	37.5	37.5	18.75
	68.75	46.875	46.8750	23.4375
	73.4375	49.2188	49.2188	24.6094
	74.6094	49.8047	49.8047	24.9023
	74.9023	49.9512	49.9512	24.9756
	74.9756	49.9878	49.9878	24.9939
	74.9939	49.9969	49.9969	24.9985
	74.9985	49.9992	49.9992	24.9996
	74.9996	49.9998	49.9998	24.9999
	74.9999	50.000	50.000	25.000
	75	50	50	25
	75	50	50	25

2) $u_{xx} + u_{yy} = 0$



Here using standard 5-point formula

$u_1 = \frac{1}{4}(1000 + u_3 + 2000 + u_2) = \frac{1}{4}(3000 + u_3 + u_2)$

$u_2 = \frac{1}{4}(1000 + u_1 + u_4 + 500) = \frac{1}{4}(u_1 + u_4 + 1500)$

$u_3 = \frac{1}{4}(u_1 + 500 + 2000 + u_4) = \frac{1}{4}(2500 + u_1 + u_4)$

$u_4 = \frac{1}{4}(u_2 + u_3 + 0 + 0) = \frac{1}{4}(u_2 + u_3)$

Then

itn	u_1	u_2	u_3	u_4
Intn	0	0	0	0
1	750	562.5	812.5	343.75
2	1093.75	734.375	984.375	429.6875
3	1179.6875	777.3438	1027.3938	451.17
4.	1201.17	777.34	451.17	1201.17
5)	1206.54	788.09	1038.09	456.54
6)	1206.54	790.77	1040.77	457.89

7)	1207.89	791.44	1041.44	458.22
8)	1208.22	791.61	1041.61	458.31
9)	1208.31	791.65	1041.65	458.33
10	1208.33	791.66	1041.66	458.33
11	1208.33	791.67	1041.67	458.33
12	1208.33	791.67	1041.67	458.33

Hence $u_1 = 1208.33$

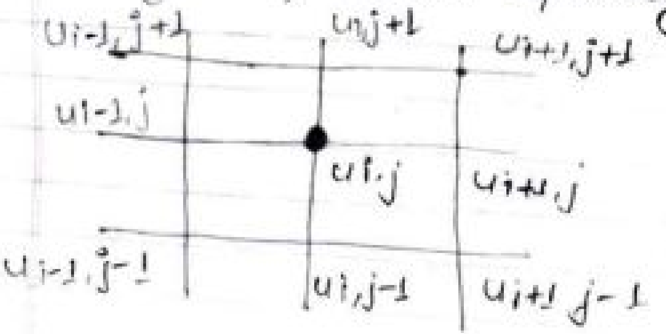
$u_2 = 791.67$

$u_3 = 1041.67$

$u_4 = 458.33$ //

Method II > Poisson Equations

The partial differential equation of the form $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$ is called poisson equation where $f(x,y)$ is a given function of x & y



Formula

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

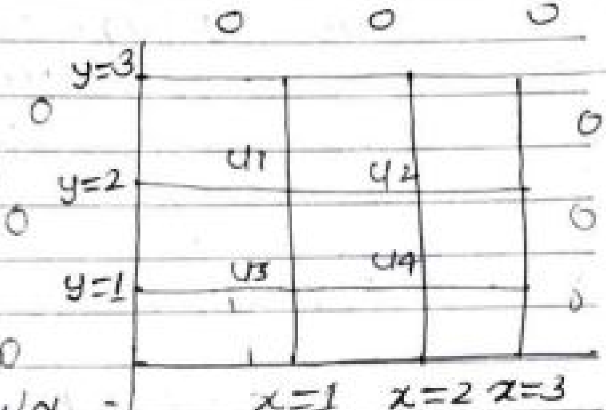
1) Solve the equation $\nabla^2 u = -10(x^2+y^2+10)$ over the square with sides $x=0=y$ & $x=3=y$ with $u=0$ on the boundary and mesh length $=1$ unit

Here

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -10(x^2+y^2+10)$

It is poission equation.

We know, poission standard formula



$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i-1,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

For u_1, $i = x = 1$

$j = y = 2$

$\therefore u_2 + 0 + 0 + u_3 - 4u_1 = 1^2 f(1,2)$

or, $u_2 + u_3 - 4u_1 = -10(1+4+10)$

or, $u_2 + u_3 - 4u_1 = -150$

$$u_1 = \frac{u_2 + u_3 + 150}{4} \quad \text{(1)}$$

For u_2 : $i=2, j=2$

$$0+0+u_1+u_4-4u_2 = 1^2 f(2,2)$$
$$\text{or, } u_1+u_4-4u_2 = -10(4+4+10)$$
$$\text{or, } u_1+u_4-4u_2 = -180$$
$$\therefore u_2 = \frac{u_1+u_4+180}{4} \quad \text{(ii)}$$

For u_3 : $i=1, j=1$

$$u_1+u_4+0+0-4u_3 = 1^2 f(1,1)$$
$$u_1+u_4-4u_3 = -10(12)$$
$$\therefore u_3 = \frac{u_1+u_4+120}{4} \quad \text{(iii)}$$

For u_4 : $i=2, j=1$

$$0+0+u_2+u_3-4u_4 = 1^2 f(2,1)$$
$$\text{or, } u_2+u_3-4u_4 = -10(4+1+10)$$
$$\text{or, } u_2+u_3-4u_4 = -150$$
$$\therefore u_4 = \frac{u_2+u_3+150}{4} \quad \text{(iv)}$$

Using Gauss seidel method

It^n	u_1	u_2	u_3	u_4
Int^n	0	0	0	0
1	37.50	54.38	39.38	60.94
2	60.94	75.47	60.47	71.48
3.	71.48	80.74	65.74	74.12
4.	74.12	82.06	67.06	74.78
5.	74.78	82.39	67.39	74.95
6	74.95	82.47	67.47	74.99
7	74.99	82.49	67.49	75.00
8	75	82.5	67.5	75.00
9	75	82.5	67.5	75.00

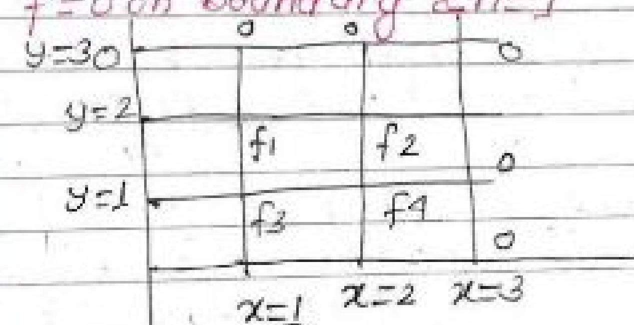
Hence $u_1 = 75$

$u_2 = 82.5$

$u_3 = 67.5$

$u_4 = 75$

2) Solve the poisson eq^n $\nabla^2 f = 2x^2y^2$ over square domain $0 \le x \le 3$, $0 \le y \le 3$ with $f=0$ on boundary & $h=1$



Here

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2x^2y^2$$

This is poisson eq^n

By using poisson formula

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{ij} = h^2 f(ih, jh)$$

For f_1 $i=1, j=2$

Then $0+0+f_2+f_3-4f_1 = 1^2 f(1,2)$

or $f_2+f_3-4f_1 = 2(1 \times 4)$

or, $f_2+f_3-4f_1 = 8$

$$\therefore f_1 = \frac{f_2+f_3-8}{4} \quad \text{—(i)}$$

Again For f_2, $i=2, j=2$

$0+0+f_1+f_4-4f_2 = 1 f(2,2)$

or, $f_1+f_4-4f_2 = 2 \times 4 \times 4$

or, $$f_2 = \frac{f_1+f_4-32}{4} \quad \text{—(ii)}$$

For f_3 $i=1, j=1$

$0+f_1+0+f_4-4f_3 = 1^2 f(1,1)$

or, $f_1+f_4-4f_3 = 2$

$$\therefore f_3 = \frac{f_1+f_4-2}{4} \quad \text{—(iii)}$$

For f_4: $i=2, j=1$

$0+f_2+0+f_3-4f_4 = 1^2 f(2,1)$

$f_2+f_3-4f_4 = 2 \times 4$

$\therefore f_4 = f_1+f_3-8$

Using Gauss seidel method.

It^n	u_1	u_2	u_3	u_4
Initial	0	0	0	0
1	-2	-8.5	-1	-2.75
2	-4.38	-9.78	-2.28	-3.66
3.	-5.02	-10.17	-2.67	-3.92
4.	-5.21	-10.28	-2.78	-4.00
5	-5.27	-10.32	-2.82	-4.02
6	-5.28	-10.33	-2.83	-4.03
7	-5.29	-10.33	-2.83	-4.03
8.	-5.29	-10.33	-2.83	-4.03

Hence $u_1 = -5.29$

$u_2 = -10.33$

$u_3 = -2.83$

$u_4 = -4.03$ //

17/20

Solution of System of Linear Equations.

Method VI

Relaxation method

$20x + y - 2z = 17$

$3x + 20y - z = 18$

$2x - 3y + 20z = 25$

soln:

Here residuals are given by

$R_x = 17 - 20x - y + 2z$

$R_y = 18 - 3x - 20y + z$

$R_z = 25 - 2x + 3y - 20z$

The operation table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-20	-1	2
δy	-3	-20	1
δz	-2	3	-20

The relation table is

	R_x	R_y	R_z
$x=y=z=0$	17	18	25
$\delta z = 1$	19	19	5
$\delta x = 1$	-1	16	3
$\delta y = 1$	-2	-4	6
$\delta z = 0.3$	-1.4	-3.7	0
$\delta y = -0.185$	-1.215	0	-0.555
$\delta x = -0.061$	0.005	0.183	-0.433

$\Sigma \delta x = 1 - 0.061 = 0.939$

$\Sigma \delta y = 1 - 0.185 = 0.823$

$\Sigma \delta z = 1 + 0.3 = 1.3$

Ans