Internal questions

1a. Show that the function $u = \cos x \cosh y$ is harmonic.

→ Given,

$$u = \cos x \cosh y$$

$u_x = -\sin x \cosh y$ $\quad\quad$ $u_y = \cos x \sinh y$

$u_{xx} = -\cos x \cosh y$ $\quad\quad$ $u_{yy} = \cos x \cosh y$

$\therefore u_{xx} + u_{yy} = 0$

$-\cos x \cosh y + \cos x \cosh y = 0$

$0 = 0$

Therefore, the given function is harmonic.

b) State the cauchy Integral formula.

→ The cauchy Integral formula states that, "If $f(z)$ is analytic within and on a closed curve c and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} dz$$

c) Derive the formula of $z[a^n]$.

→ By the defination of z-transform,

$$z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$$

So, $z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$

or, $Z(a^n) = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$

or, $Z(a^n) = 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$

or, $Z(a^n) = \frac{1}{1 - \frac{a}{z}}$

$\therefore Z(a^n) = \frac{z}{z-a}$ #

d) Write the fourier cosine and sine integral formula for the function $f(x)$.

→ The fourier cosine integral of $f(x)$ is defined by

$$\hat{f}_c\{f(x)\}w = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx$$

The fourier sine integral of $f(x)$ is defined by

$$\hat{f}_s\{f(x)\}w = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx$$

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2a. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a laurent series for the region $0 < |z+1| < 2$.

→ $0 < |z+1| < 2$

$-\frac{1}{2} < |z| < \frac{1}{2}$

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1/2}{z+1} + \frac{-1/2}{(z+3)}$$

$$f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

Let $z+1 = \omega$

$z = \omega - 1$

$$f(z) = \frac{1}{2}\left[\frac{1}{\omega - \cancel{1} + \cancel{1}} - \frac{1}{\omega - 1 + 3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega} - \frac{1}{\omega+2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega} - \frac{1}{2(1+\omega/2)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega} - \frac{1}{2}\left(1 + \frac{\omega}{2}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\omega} - \frac{1}{2}\left(1 - \frac{\omega}{2} + \frac{\omega^2}{4} - \frac{\omega^3}{8} + \cdots\right)\right]$$

$$= \frac{1}{2\omega} - \frac{1}{4} + \frac{\omega}{8} - \frac{\omega^2}{16} + \frac{\omega^3}{32}$$

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$\therefore f(z) = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{(z+1)^2}{8} - \frac{(z+1)^2}{16} + \frac{(z+1)^3}{32} \dots$

26. Find the poles and residues of $f(z) = \frac{z^2}{(z+2)(z-1)^2}$

→ Here, $f(z) = \frac{z^2}{(z+2)(z-1)^2}$

The poles of $f(z)$ is

$(z+2)(z-1)^2 = 0$

$\therefore z = -2, 1, 1$

Clearly $z = -2$, is a simple pole and and $z = 1$ is the pole of order 2.

Now,

$\operatorname{Res}_{z \to z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$

$= \lim_{z \to -2} \cancel{(z+2)} \frac{z^2}{\cancel{(z+2)}(z-1)^2}$

$\operatorname{Res}_{z \to -2} f(z) = \lim_{z \to -2} \frac{z^2}{(z-1)^2}$

$= \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}$

Now, for $z_0 = 1$

$\operatorname{Res}_{z \to z_0} f(z) = \lim_{z \to z_0} \left[\frac{d}{dz} (z - z_0)^2 f(z) \right]$

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$$= \lim_{z\to 1}\left[\frac{d}{dz}\cancel{(z-1)^2}\frac{z^2}{(z+2)\cancel{(z-1)^2}}\right]$$

$$= \lim_{z\to 1}\left[\frac{d}{dz}\left(\frac{z^2}{z+2}\right)\right]$$

$$= \lim_{z\to 1}\left[\frac{(z+2)\cdot 2z - z^2\cdot 1}{(z+2)^2}\right]$$

$$= \lim_{z\to 1}\left[\frac{2z^2+4z-z^2}{(z+2)^2}\right]$$

$$= \lim_{z\to 1}\frac{z^2+4z}{(z+2)}$$

$$= \frac{1^2+4}{1+2}$$

$$= \frac{5}{3}$$

2c. find the fixed point and the normal form of the bilinear transformation $w = \frac{z-1}{z+1}$

→ The fixed points are determined by

$$z = \frac{z-1}{z+1}$$

$$z^2+\cancel{z}-\cancel{z}+1=0$$

or, $z^2+1=0$

$\therefore z = \pm i$ are the two fixed points.

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for, Normal form,

we have, $\omega = \frac{z-1}{z+1}$

$\omega - i = \frac{z-1}{z+1} + i$ and $\omega + i = \frac{z-1}{z+1} + i$

$\frac{\omega - i}{\omega + i} = \frac{z-1-iz-i}{z-1+iz+i} = \frac{(1-i)(z-i)}{(1+i)(z+i)}$

$= \frac{(-i^2-i)(z-i)}{(1+i)(z+i)} = \frac{-i(i+1)(z-i)}{(1+i)(z+i)}$

$\therefore \frac{\omega-i}{\omega+i} = -i\,\frac{z-i}{z+i} = k\left(\frac{z-i}{z+1}\right)$ where $k = 1$

$\therefore$ The transformation is elliptic.

3a) Obtain the Z- transform of $\frac{z}{z^2+9z+20}$.

$\rightarrow$ $f(z) = \frac{z}{z^2+9z+20}$

$f(z) = \frac{z}{(z+4)(z+5)}$

Let $f(z) = \frac{Az}{z+4} + \frac{Bz}{z+5}$ ——— (1)

or, $z = Az(z+5) + Bz(z+4)$

or, $z = Az^2 + 5Az + Bz^2 + 4Bz$

or, $z = z^2(A+B) + z(5A+4B)$

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Equating the coefficient z^2 and z, we get

$A + B = 0$ $\quad\quad$ $5A + 4B = 1$

$A = -B$ $\quad\quad$ $-5B + 4B = 1$

$\therefore A = 1$ $\quad\quad$ $-B = 1$

$\therefore B = -1$

Now, from eqⁿ (1),

$$f(z) = \frac{z \times 1}{(z+4)} + \frac{z \times (-1)}{z+5}$$

$$f(z) = \frac{z}{z+4} - \frac{z}{z+5}$$

Taking inverse z-transform on both sides, we get

$$z^{-1}\{f(z)\} = z^{-1}\left\{\frac{z}{z+4}\right\} - z^{-1}\left\{\frac{z}{z+5}\right\}$$

or, $f(n) = (-4)^n - (-5)^n$

$\therefore f(n) = (-4)^n - (-5)^n$ #

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3 b) Solve the partial differential $u_{xy} - u = 0$ by seperating the variable.

→ We have $u_{xy} - u = 0$ — (i)

Let $U = X(x) \cdot Y(y)$ — (ii) be the solution of given partial differential Equation,

Diff. partially wrt x and y, we get

$$\frac{\partial u}{\partial x} = Y \frac{\partial X}{\partial x}$$

$$\frac{\partial u}{\partial y} = X \frac{\partial Y}{\partial y}$$

Again, diff. partially wrt y,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial Y}{\partial y} \frac{\partial X}{\partial x}$$

substituting values in eq^n (i)

$$\frac{\partial X}{\partial x} \cdot \frac{\partial Y}{\partial y} - XY = 0$$

$$\frac{1}{X}\frac{\partial X}{\partial x} = Y \frac{\partial y}{\partial Y} = K (say$$

$$\frac{\partial X}{\partial x} \cdot \frac{1}{X} = K \quad \text{and} \quad Y \frac{\partial y}{\partial Y} = K$$

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or, $\frac{1}{X}\partial X = K\,dx$ and $\frac{1}{Y}\partial Y = \frac{1}{K}dy$

integrating both sides,

or, $\ln(X) = Kx + c_1$, $\ln(Y) = \frac{y}{K} + c_2$

$\therefore X = e^{Kx+c_1}$ $\therefore Y = e^{y/K + c_2}$

from eq^n (ii)

$$u(x,y) = e^{Kx+c_1} \cdot e^{y/K + c_2}$$

$$= e^{Kx + y/K} \cdot e^{c_1+c_2}$$

$$= A\,e^{(Kx + y/K)}$$ where $c_1 + c_2 = A$ is required solution.

Q3c. find fourier cosine transform of $f(x) = e^{-mx}$, where $m > 0$

→ Given function $f(x) = e^{-mx}$

The fourier cosine transform is

$$\hat{f}_c\{f(x)\}\omega = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\omega x\,dx$$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-mx}\cos\omega x\,dx$$

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or, $\hat{f}_c(\omega) = \left[\frac{e^{-mx}}{(-m)^2+\omega^2}(-m\cos\omega x + \omega\sin\omega x)\right]_0^\infty$

or, $\hat{f}_c(\omega) = \left[0 - \frac{1}{m^2+\omega^2}(-m)\times 1 + 0\right]$

or, $\hat{f}_c(\omega) = \frac{-(-m)}{m^2+\omega^2}$

$\therefore \hat{f}_c(\omega) = \frac{m}{m^2+\omega^2}$

Q4.

a) Evaluate the integral $\oint_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where C is the circle $|z| = \frac{3}{2}$.

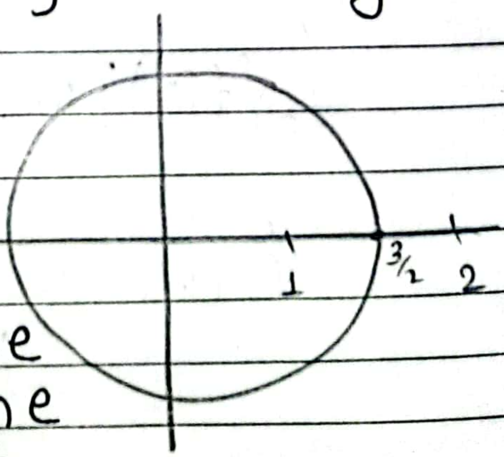
→ The given integral is

$$\oint_C \frac{4-3z}{z(z-1)(z-2)}$$

The poles of the integral are given by putting

$$z(z-1)(z-2) = 0$$

$$\therefore z = 0, 1, 2$$



Clearly the poles 0 and 1 lies inside the circle and $z = 2$ lie outside the circle $|z| = 3/2$

$$\oint_C \frac{4-3z}{z(z-1)(z-2)}\, dz = \oint_C \frac{\left(\frac{4-3z}{z-2}\right)}{z(z-1)}\, dz$$

$$= \oint_C \left(\frac{4-3z}{z-2}\right)\left[\frac{1}{z-1} - \frac{1}{z}\right] dz$$

$$= \int_C \frac{(4-3z)/(z-2)}{z-1} - \int_C \frac{\frac{4-3z}{z-2}}{z}\, dz$$

Here, $f(z) = \frac{4-3z}{z-2}$ is analytic within the circle curve c. so, by using cauchy integral formula,

$$\int_C \frac{4-3z}{z(z-1)(z-2)} = 2\pi i f(1) - 2\pi i f(0)$$

$$= 2\pi i \left(\frac{4-3}{1-2}\right) - 2\pi i \left(\frac{4-0}{0-2}\right)$$

$$= 2\pi i\,[-1+2]$$

$$= 2\pi i \times 1.$$

$$= 2\pi i$$

$\therefore \oint_C \frac{4-3z}{z(z-1)(z-2)} = 2\pi i$ #

4b) What do you mean by analyticity of function $f(z)$. State cauchy Riemann Equation and show that it is the necessary condition for the function to be analytic.

let $f(z)$ be a complex valued function defind on Domain D.

→ A function $f(z)$ is said to be analytic if $f(z)$ is differentiable at all the points of D.

→ Let $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D, so, $f'(z)$ exists for all z, and

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Since,

$f(z) = u(x,y) + iv(x,y)$ so

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

Therefore,

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + i[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y} \quad \text{--- (1)}$$

first suppose that Δz is purely real, so that $\Delta y = 0$. Therefore, eq (1) reduces to

$$f'(z) = \lim_{\Delta x \to 0} \frac{\{u(x+\Delta x, y) - u(x,y)\} + i\{v(x+\Delta x, y) - v(x,y)\}}{\Delta x}$$

$$f'(z) = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right]$$

$$\therefore f'(z) = u_x + iv_x \quad \text{--- (2)}$$

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further suppose that Δz is purely imaginary so that $\Delta x = 0$. And (1) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y) + i[v(x, y+\Delta y) - v(x,y)]}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \left[\frac{1}{i} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y} \right]$$

$$\therefore f'(z) = v_y - i u_y \quad \text{—} \ (3)$$

from (2) and (3) we have

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{—} \ (4)$$

The equation (4) are called cauchy-Riemann equation.

Q5a) State and prove first shifting theorem for Z-transform using it to find the value of Z (coshat sinbt).

→ first shifting theorem states that, "If $Z\{f(t)\} = F(z)$, then $Z\{e^{-at} f(t)\} =$

$$F(ze^{aT}) = |F(z)|_{z \to ze^{aT}}$$

Proof: By the defination of z-transform, we have

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$$z\{f(t)\} = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

So, $$z\{e^{-at} f(t)\} = \sum_{n=0}^{\infty} e^{-anT} f(nT)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) \cdot (ze^{aT})^{-n}$$

$$= f(ze^{aT})$$

$\therefore$ $$z\{e^{-at} f(t)\} = f(ze^{aT}) = \left| F(z) \right|_{z \to ze^{aT}}$$

This completes the proof.

$\rightarrow$ $$z(\cosh at \sin bt) = z\left[\left(\frac{e^{at} + e^{-at}}{2}\right) \sin bt\right]$$

$$= \frac{1}{2}\left[z(e^{at} \sin bt) + z(e^{-at} \sin bt)\right]$$

$$= \frac{1}{2}\left[z(\sin bt)\right]_{z \to ze^{-aT}} + \frac{1}{2}\left[z(\sin bt)\right]_{z \to ze^{aT}}$$

$$= \frac{1}{2}\left[\frac{z \sin bT}{z^2 - 2z\cos bT + 1}\right]_{z \to ze^{-aT}} + \frac{1}{2}\left[\frac{z \sin bT}{z^2 - 2z\cos bT + 1}\right]_{z \to ze^{aT}}$$

$$= \frac{1}{2}\left[\frac{ze^{-aT} \sin bT}{z^2 - 2z\cos bT + 1}\right]_{z \to ze^{-aT}} + \frac{1}{2}\left[\frac{z \sin bT}{z^2 - 2z\cos bT + 1}\right]_{z \to ze^{aT}}$$

$$= \frac{1}{2}\left[\left\{\frac{ze^{-aT} \sin bT}{(ze^{-aT})^2 - 2(ze^{-aT})\cos bT + 1}\right\} + \left\{\frac{ze^{aT} \sin bT}{(ze^{aT})^2 - 2(ze^{aT})\cos bT + 1}\right\}\right]$$

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$$= \frac{1}{2}\left[\frac{ze^{-aT}\sin bT}{z^2e^{-2aT} - 2ze^{-aT}\cos bT + 1} + \frac{ze^{aT}\sin bT}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}\right]$$

Q5 b) Solve the differential equation by using z-transform. $y_{n+2} - y_n = 2^n$ with $y_0 = 0, y_1 = 1$

→ Given differential Equation

$$y_{n+2} - y_n = 2^n \text{ with } y_0 = 0, y_1 = 1$$

Taking z-transform on both sides, we get

$$z\{y_{n+2}\} - z\{y_n\} = z\{2^n\}$$

or, $z^2\bar{y} - z^2y_0 - zy_1 - \bar{y} = \frac{z}{z-2}$

or, $z^2\bar{y} - z^2 \times 0 - z \times 1 - \bar{y} = \frac{z}{z-2}$ $[\because y_0 = 0, y_1 = 1]$

or, $\bar{y}(z^2 - 1) - z = \frac{z}{z-2}$

or, $\bar{y} = \frac{\frac{z}{z-2} + z}{z^2 - 1}$

or, $\bar{y} = \frac{z + z^2 - 2z}{(z+1)(z-1)(z-2)}$

or, $\bar{y} = \frac{z\cancel{(z-1)}}{(z+1)\cancel{(z-1)}(z-2)}$

or, $\bar{y} = \frac{z}{(z+1)(z-2)}$

Let $\frac{f(z)}{z} = \frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$

or, $1 = A(z-2) + B(z+1)$
or, $1 = Az - 2A + Bz + B$
Equating the coefficient z & constant terms,

$A + B = 0$
$A = -B$
$\therefore A = -\frac{1}{3}$

$-2A + B = 1$
$-2(-B) + B = 1$
$2B + B = 1$
$\therefore B = \frac{1}{3}$

Now,
$$\frac{f(z)}{z} = \frac{1}{3}\left(\frac{1}{z+1}\right) + \frac{1}{3}\left(\frac{1}{z-2}\right)$$

$$f(z) = -\frac{1}{3}\left(\frac{z}{z+1}\right) + \frac{1}{3}\left(\frac{z}{z-2}\right)$$

Taking inverse z-transform on both sides,

$$z^{-1}\{f(z)\} = -\frac{1}{3} z^{-1}\left\{\frac{z}{z+1}\right\} + \frac{1}{3} z^{-1}\left\{\frac{z}{z-2}\right\}$$

or, $f(n) = -\frac{1}{3}(1)^n + \frac{1}{3}(2^n)$
$= \frac{1}{3}\left[(2^n - (-1)^n\right]$

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6 a) Examine the suitable function show that:

$$\int_0^\infty \left[\frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\right] d\omega = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

→ We know, the fourier integral representation of $f(x)$ is

$$f(x) = \int_0^\infty \left(A(\omega)\cos\omega x + B(\omega)\sin\omega x\right) d\omega \quad \text{— ①}$$

where,

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\, dx$$

choosing the function

$$f(x) = \begin{cases} 0 & \text{if } x<0 \\ \pi e^{-x} & \text{if } x>0 \end{cases} \quad \text{— ②}$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{0} f(x)\cos\omega x\, dx + \frac{1}{\pi}\int_0^\infty f(x)\cos\omega x\, dx$$

$$= 0 + \frac{1}{\pi}\int_0^\infty \pi e^{-x}\cos\omega x\, dx$$

$$= \int_0^\infty e^{-x}\cos\omega x\, dx$$

$$= \left[\frac{e^{-x}}{(-1)^2+\omega^2}\left((-1)\cos\omega x + \omega\sin\omega x\right)\right]_0^\infty$$

$$= 0 - \frac{1}{\omega^2+1}(-1+0)$$

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$$\therefore A(\omega) = \frac{1}{\omega^2+1}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \sin\omega x \, dx$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{0} 0 \sin\omega x \, dx + \frac{1}{\pi}\int_{0}^{\infty} \pi e^{-x} \sin\omega x \, dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\infty} \pi e^{-x} \sin\omega x \, dx$$

$$= \int_{0}^{\infty} e^{-x} \sin\omega x \, dx$$

$$= \left\{ \frac{e^{-x}}{(-1)^2+\omega^2} \left[(-1) \sin\omega x - \omega\cos\omega x \right] \right\}_0^{\infty}$$

$$= \frac{e^{-\infty}}{1+\omega^2} \left[-\sin\omega\infty - \omega\cos\omega\infty \right] - \frac{e^{0}}{1+\omega^2}$$

$$\left[-\sin\omega 0 - \omega\cos 0 \right]$$

$$= -\frac{1}{1+\omega^2} \left[0 - \omega \right]$$

$$= \frac{\omega}{1+\omega^2}$$

from eqn ①

$$f(x) = \int_0^\infty \left[\frac{1}{1+\omega^2} \cos\omega x + \frac{\omega}{1+\omega^2} \sin\omega x\right] d\omega$$

$$f(x) = \int_0^\infty \frac{\cos\omega x + \omega \sin\omega x}{1+\omega^2}\, d\omega$$

or, $$\int_0^\infty \frac{\cos\omega x + \omega \sin\omega x}{1+\omega^2}\, d\omega = \begin{cases} 0 & \text{if } x<0 \\ \pi e^{-x} & \text{if } x>0 \end{cases} \quad -(3)$$

the function given by eq^n (2) is not defined at $x=0$, so it is discontinuous at $x=0$. At $x=0$, the value of the fourier integral is equal to the average value of the left and right hand limits of $f(x)$ at $x=0$.

$$LHL = \lim_{x\to 0^-} f(x) = 0$$

$$RHL = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \pi e^{-x} = \pi$$

$$\text{Average of the limit} = \frac{0+\pi}{2} = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\cos\omega x + \omega \sin x}{1+\omega^2}\, d\omega = \begin{cases} 0 & \text{if } x<0 \\ \pi/2 & \text{if } x=0 \\ \pi e^{-x} & \text{if } x>0 \end{cases}$$

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6b) find the temperature in a laterally insulated bar of length L, whose ends are kept at temperature 0, assuming that the initial temperature is

$$u(x,t) = \begin{cases} x; & 0 \le x \le L/2 \\ L-x; & L/2 \le x \le L \end{cases}$$

→ Given,

$$f(x) = u(x,0) = \begin{cases} x; & 0 \le x \le L/2 \\ L-x; & L/2 \le x \le L \end{cases}$$

One dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ — (i)

We have feasible solution of eqⁿ (i) is

$$u = (c_1 \cos px + c_2 \cancel{\cos} \sin px)\, c_3 e^{-c^2p^2t} \quad \text{— (ii)}$$

Using the boundry condition $u(0,t) = 0$

$$u(0,t) = (c_1 + 0)\, c_3 e^{-c^2p^2t}$$

$$0 = c_1 c_3 e^{-c^2p^2t}$$

$$\therefore c_3 e^{-c^2p^2t} \neq 0$$

$\boxed{\therefore c_1 = 0}$

Equation (ii) reduces to

$$u = c_2 \sin pl \cdot c_3 e^{-c^2p^2t}$$

$$0 = c_2 \sin pl \cdot c_3 e^{-c^2p^2t}$$

$\sin pl = 0$ $\qquad \therefore c_2 c_3 e^{-c^2p^2t} \neq 0$

or $\sin pl = \sin n\pi$

or $pl = n\pi$

$\therefore$ $p = \frac{n\pi}{l}$

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Therefore, $u = c_2 c_3 \sin \frac{n\pi}{l} x \; e^{-\frac{c^2 \pi^2 n^2 t}{l^2}}$

Most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{l} x \; e^{-\frac{c^2\pi^2 n^2 t}{l^2}} \quad \text{--- (iii)}$$

Where, $A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$

or, $$A_n = \frac{2}{l} \left[\int_0^{l/2} x \sin \frac{n\pi x}{l} \, dx + \int_{l/2}^{l} (l-x) \sin \frac{n\pi x}{l} \, dx \right]$$

1 or, $$A_n = \frac{2}{l} \left[x \frac{-\cos \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} + \frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right]_0^{l/2} + \left[(l-x) \frac{\cos \frac{n\pi x}{l}}{-\left(\frac{n\pi}{l}\right)} - \frac{\sin \frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \right]_{l/2}^{l}$$

or, $$A_n = \frac{2}{l} \left[\cancel{\frac{-l^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right)} + \frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) + \cancel{\frac{-l^2}{n^2\pi^2} \sin(n\pi)} + \cancel{\frac{l^2}{2n\pi} \cos\left(\frac{n\pi}{2}\right)} + \frac{l^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right]$$

or, $$A_n = \frac{2}{\cancel{l}} \times \frac{2 l^{\cancel{2}}}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

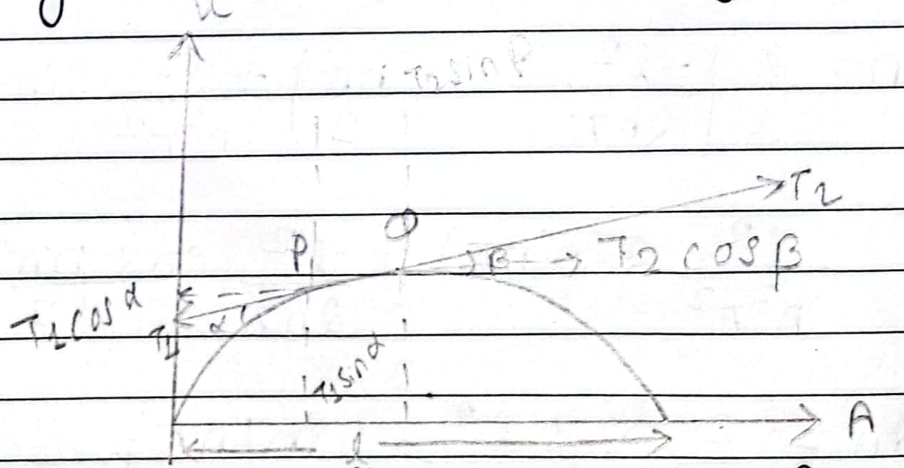
$\therefore$ $$A_n = \frac{4l}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4l}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{l}\right) x \, e^{-\frac{c^2\pi^2 n^2 t}{l^2}}$$

$$\therefore \; u(x,t) = \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{c^2\pi^2 n^2 t}{l^2}}$$ //

Q7a) Derive one dimensional wave equation of a string of length l ~~whose ends are kept at temperature 0, assuming~~ which is fixed in two end points with required assumptions.

→ Considering a tightly stretched elastic string of length l and fixed it at two points. Let one end of string at origin and another end at A along x-axis. Now, we allow it to vibrates and when the string vibrating each point of the string moves parallel to y-axis.



Let P & Q are the neighbouring points of string. Let T_1 & T_2 be the tensions at points P and Q making angle α and β

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with the horizontal. Since, the point of the string moves vertically, there is no motion in horizontal direction. Thus, the horizontal components of tension must be constant. Resolve the tension along horizontal and vertical direction. Then

$$T_2 \cos\beta - T_1 \cos\alpha = 0$$
$$T_2 \cos\beta = T_1 \cos\alpha = T \text{ (constant)} \quad \text{—(1)}$$

In vertical motion, the components are $-T_1 \sin\alpha$ and $T_2 \sin\beta$. Hence, the resultant of tension is

$$T_2 \sin\beta - T_1 \sin\alpha = 0$$

Now, by newton second law, we have

$$T_2 \sin\beta - T_1 \sin\alpha = \text{mass} \times \text{acceleration}$$
$$\rho \Delta x \frac{\partial^2 u}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha$$

or, $$\frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} = \frac{T_2 \sin\beta}{T} - \frac{T_1 \sin\alpha}{T}$$

or, $$\frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} = \frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha}$$

or, $$\frac{\rho \Delta x}{T} \frac{\partial^2 u}{\partial t^2} = \tan\beta - \tan\alpha$$

or, $$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho \Delta x} (\tan\beta - \tan\alpha)$$

or, $$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}$$

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Taking limit on $Q \to P$ i.e $\Delta x \to 0$

or, $$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \lim_{\Delta x \to 0} \left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x} \right]$$

or, $$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2}$$

or, $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{\rho}$$

This is called one dimensional wave equation.

7b) Express Laplacian $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in polar co-ordinate system.

Soln:- The Laplacian of U is $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$

i.e $U_{xx} + U_{yy}$ —— (1)

We have,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x}$$

or, $U_x = U_r r_x + U_\theta \theta_x$ —— (2)

Again differentiating (2) wrt x, we get

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$$U_{xx} = (U_r r_x)_x + (U_\theta \theta_x)_x$$

or, $U_{xx} = (U_r)_x r_x + U_r r_{xx} + (U_\theta)_x \theta_x + U_\theta \theta_{xx}$

or, $U_{xx} = (U_{rr} r_x + U_{r\theta} \theta_x) r_x + U_r r_{xx} + (U_{\theta r} r_x + U_{\theta\theta} \theta_x) \theta_x + U_\theta \theta_{xx}$ — (3)

To change it into polar form,
put $x = r\cos\theta$ and $y = r\sin\theta$
then, $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ — (4)

$\frac{\partial r}{\partial x}$ $\therefore r_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$

Again, diff wrt x, we get

$\therefore r_{xx} = \frac{r \cdot 1 - x r_x}{r^2}$

$$r_{xx} = \frac{1}{r} - \frac{x^2}{r^3}$$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{\partial\theta}{\partial x} = \theta_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right)$

$$= \frac{\cancel{x^2}}{x^2+y^2} \times \frac{-y}{\cancel{x^2}}$$

$$= \frac{-y}{r^2} = \frac{-\cancel{r}\sin\theta}{r^{\cancel{2}}} = \frac{-\sin\theta}{r}$$

Again,

$$\frac{\partial^2\theta}{\partial x^2} = \theta_{xx} = \frac{\left[-r\frac{\partial}{\partial x}\sin\theta - \sin\theta\frac{\partial r}{\partial x}\right]}{r^2}$$

or, $\theta_{xx} = \dfrac{-r\cos\theta\dfrac{\partial\theta}{\partial x} - \sin\theta\cdot\dfrac{x}{r}}{r^2}$

or, $\theta_{xx} = \dfrac{\dfrac{\cancel{r}\sin\theta\cos\theta}{\cancel{r}} + \dfrac{x}{r}\sin\theta}{r^2}$

or, $\theta_{xx} = \dfrac{r\sin\theta\cos\theta + r\sin\theta\cos\theta}{r^3}$

or, $\theta_{xx} = \dfrac{2\cancel{r}\sin\theta\cos\theta}{r^{\cancel{3}2}}$

$\therefore$ or, $\theta_{xx} = \dfrac{2\sin\theta\cos\theta}{r^2}$

Similary, partially differentiating wrt y, to (4) we get

$\frac{\partial r}{\partial y} = r_y = \frac{y}{r}$, $r_{yy} = \frac{1}{r} - \frac{y^2}{r^3}$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\theta_x = \frac{\cos\theta}{r}$, $\theta_{yy} = \frac{2\sin\theta\cos\theta}{r^2}$

Substituting these values in eqn (2)

$U_{xx} = \frac{\partial u}{\partial r}\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \frac{\partial^2 u}{\partial r^2}\cdot\frac{x^2}{r^2} + \frac{\partial u}{\partial \theta}$

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$$\left(\frac{2\sin\theta\cos\theta}{r^2}\right) + \frac{\partial^2 u}{\partial\theta^2}\left(\frac{\sin^2\theta}{r^2}\right)$$

Similarly,

$$u_{yy} = \frac{\partial u}{\partial r}\left(\frac{1}{r} - \frac{y^2}{r^3}\right) + \frac{\partial^2 u}{\partial r^2}\frac{y^2}{r^2} +$$

$$\frac{\partial u}{\partial\theta}\left(-\frac{2\sin\theta\cos\theta}{r^2}\right) + \frac{\partial^2 u}{\partial\theta^2}\left(\frac{\cos^2\theta}{r^2}\right)$$

So, laplace eqn (1) reduces to

$$u_{xx} + u_{yy} = \frac{\partial u}{\partial r}\left(\frac{1}{r} - \frac{x^2}{r^3} + \frac{1}{r} - \frac{y^2}{r^3}\right) +$$

$$\frac{\partial^2 u}{\partial r^2}\left(\frac{x^2+y^2}{r^2}\right) + \frac{\partial^2 u}{\partial\theta^2}\left(\frac{\sin^2\theta + \cos^2\theta}{r^2}\right)$$

or, $$u_{xx} + u_{yy} = \frac{\partial u}{\partial r}\left(\frac{2}{r} - \frac{r^2}{r^3}\right) + \frac{\partial^2 u}{\partial x^2}\left(\frac{r^2}{r^2}\right) +$$

$$\frac{\partial^2 u}{\partial\theta^2}\left(\frac{1}{r^2}\right)$$

$\therefore$ $\nabla^2 u = u_{xx} + u_{yy}$ //

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set-2

Q1 a) Define harmonic function. Check $u = y^3 - 3x^2y$ is harmonic or not? If yes, find corresponding harmonic conjugate v of u.

→ let function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D. Then, we say the function u is harmonic function if it satisfies its laplace equation i.e $u_{xx} + u_{yy} = 0$. Similar, the function v is harmonic function if it satisfies its laplace equation i.e $v_{xx} + v_{yy} = 0$.

We have, $u = y^3 - 3x^2y$

$u_x = \cancel{\;} - 6xy$

$u_y = \cancel{3y^2} + 3y^2 - 3x^2$

$u_{xx} = \cancel{\;} - 6y$

$u_{yy} = 6y$

Here,

$$u_{xx} + u_{yy} = 0$$
$$-6y + 6y = 0$$

∴ u is harmonic function.

By Cauchy-Riemann equation

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

So, $v_y = -6xy$

integrating w.r.t y, we get

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$$V = -6x\frac{y^2}{2} + h(x) \quad \text{--- (1)}$$

Differentiating wrt x, we get

$$V_x = -3y^2 + h'(x)$$

or, $-u_y = -3y^2 + h'(x)$

or, $3x^2 - \cancel{3y^2} = -\cancel{3y^2} + h'(x)$

$\therefore h'(x) = 3x^2$

integrating wrt x,

$$h(x) = \frac{\cancel{3}x^3}{\cancel{3}} + c$$

from eq^n (1)

$$V = -3xy^2 + x^3 + c$$

1b) State Cauchy Integral formula for derivative. Evaluate $\oint_c \frac{z^6}{(2z-1)^6}\,dz$, where c is the unit circle $|z| = 1$, counterclockwise.

→ Cauchy Integral formula states that, "let $f(z)$ be an analytic function inside and on a simple closed curve C enclosing a simply connected region R and if z_0 is a point in the interior of C, then

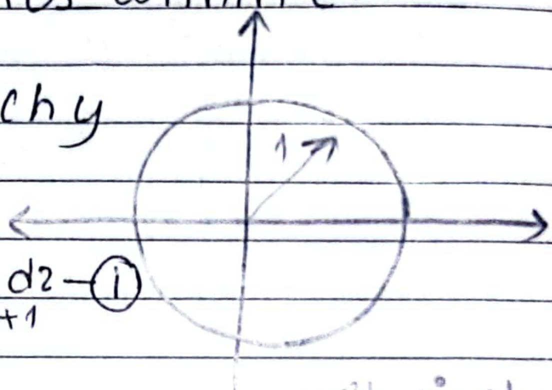
$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

→ We have $\oint_C \frac{z^6}{(2z-1)^6}$

Here $f(z) = \frac{z^6}{(2z-1)^6}$, which doesn't exist at $2z-1=0$

$\Rightarrow z = \frac{1}{2}$ which lies within c.

Then, we have Cauchy integral formula,



$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \quad \text{—(1)}$$

unit circle

Therefore,

$$f(z) = z^6$$
$$f'(z) = 6z^5$$
$$f''(z) = 30z^4$$
$$f'''(z) = 120z^3$$
$$f^{IV}(z) = 360z^2$$
$$f^{V}(z) = 720z^2$$

Now,

$$\oint_C \frac{z^6}{(2z-1)^6} dz = \frac{2\pi i \times 1}{5!\ 2^6} f^V(1/2)$$

$$= \frac{\cancel{2}\pi i}{5!} \times \frac{1}{\cancel{64}_{32}} \times 360$$

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$$\therefore \oint_C \frac{z^6}{(2z-1)^6}\,dz = \frac{3\pi i}{32} \quad \#$$

Q2 a) Find the image of triangular region of the z-plane bounded by the lines $x=0$, $y=0$, $x+y=1$ under the transformation of $w = ze^{i\pi/4}$ and show the sketch in the diagram.

→ Here, $w = ze^{\frac{i\pi}{4}}$

$$u + iv = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)(x+iy)$$

or, $u + iv = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(x+iy)$

or, $\sqrt{2}(u+iv) = x + ix + iy - y$

or, $\sqrt{2}u + i\sqrt{2}v = x - y + i(x+y)$

Comparing, we get

$$x - y = \sqrt{2}\,u \quad \text{——} \ (1)$$

$$x + y = \sqrt{2}\,v \quad \text{——} \ (2)$$

adding

$$2x = \sqrt{2}u + \sqrt{2}v$$

Subtracting (1) & (2), we get

$$2y = \sqrt{2}\,u - \sqrt{2}\,v$$

$$v - u = \sqrt{2}\,y$$

when $x=0$, then $u+v=0 \Rightarrow v=-u$

when $y=0$, then $v-u=0 \Rightarrow v=u$

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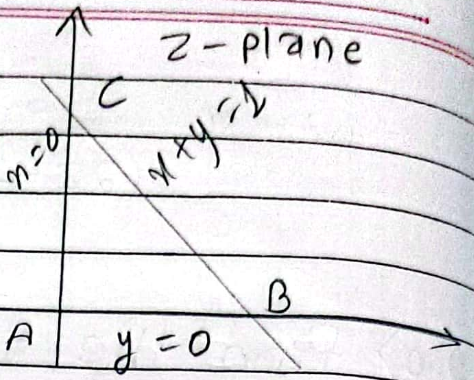
when $x+y=1$

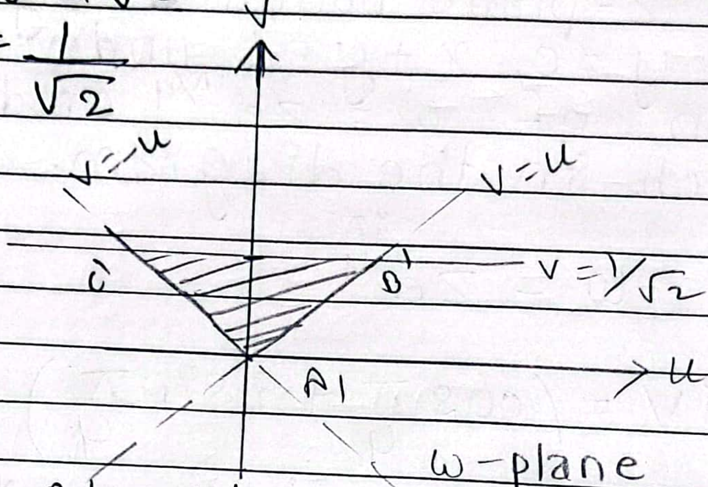


$$\frac{u+v}{\sqrt{2}} + \frac{v-u}{\sqrt{2}} = 1$$

or $u+v+v-u = \sqrt{2}$

or $2v = \sqrt{2}$

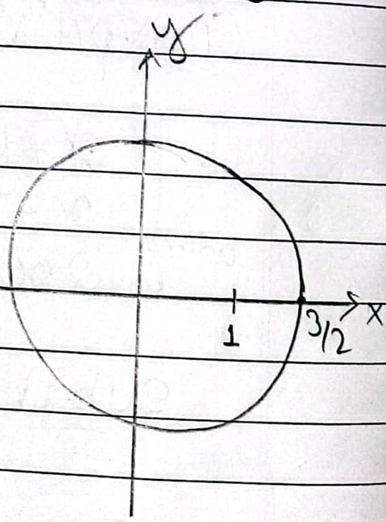
$\therefore$ $v = \frac{1}{\sqrt{2}}$



cauchy residue states that, $\int_C f(z)\,dz = 2\pi i\,(R_1+R_2+\cdots+R_n)$

Q6. State cauchy Residue Theorem. By apply cauchy residue theorem, evaluate $\oint_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where

$C: |z| = \frac{3}{2}$



Soln: $f(z) = \frac{4-3z}{z(z-1)(z-2)}$

The poles of $f(z)$ is

$z(z-1)(z-2)=0$

$\therefore$ $z = 0, 1, 2$

clearly $z=0$ and 1 is the simple pole and $z=2$ lies outside the circle.

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Now,

$$\operatorname*{Res}_{z\to z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z)$$

$$= \lim_{z\to 0} (\cancel{z}) \times \frac{4-3z}{\cancel{z}(z-1)(z-2)}$$

$$= \frac{4-3\times 0}{(0-1)(0-2)}$$

$$= \frac{4}{2} = 2$$

Now for $z_0 = 1$

$$\operatorname*{Res}_{z\to z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z)$$

$$= \lim_{z\to 1} (\cancel{z-1}) \times \frac{4-3z}{z(\cancel{z-1})(z-2)}$$

$$= \frac{4-3\times 1}{1(1-2)}$$

$$= \frac{4-3}{1-2}$$

$$= -1$$

Therefore by cauchy residue theorem,

$$\oint_C f(z)\,dz = 2\pi i (R_1 + R_2 + R_3 + \cdots + R_n)$$

$$= 2\pi i (2-1)$$

$$= 2\pi i \quad \#$$

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5a) State and prove first shifting theorem of z-transform. Using it evaluate the z-transform of $a^n \cos bt$ and $a^n \sin bt$.

Soln:- First shifting theorem states that "If $z\{f(t)\} = F(z)$ then $z\{e^{-at} f(t)\} =$

$$F(ze^{-aT}) = \left| F(z) \right|_{z \to ze^{-aT}}$$

Proof: By the defination of z-transform, we have

$$z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) \cdot z^{-n}$$

$$z\{e^{-at} f(t)\} = \sum_{n=0}^{\infty} e^{-anT} \cdot f(nT) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) \left(ze^{aT}\right)^{-n}$$

$$= f(ze^{aT})$$

$$\therefore \quad z\{e^{-at} f(t)\} = f(ze^{aT}) = \left| F(z) \right|_{z \to ze^{aT}}$$

$$\to \quad z(a^n e^{ibt}) = \left[z(e^{ibt}) \right]_{z \to z/a}$$

$$\text{or,} \quad z(a^n \cos bt + i a^n \sin bt) = \left(\frac{z}{z - e^{ibT}} \right)_{z \to z/a}$$

$$\text{or,} \quad z[a^n (\cos bt + i \sin bt)] = \left(\frac{z/a}{\frac{z}{a} - e^{ibt}} \right)$$

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or, $Z[a^n(\cos bt + i\sin bt)] = \dfrac{z}{z - a(\cos bT + i\sin bT)}$

or, $Z[a^n(\cos bt + i\sin bt)] = \dfrac{z}{(z - a\cos bT) - i(a\sin bT)}$

or, $Z[a^n(\cos bt + i\sin bT)] = \dfrac{z[(z-a\cos bT) + i(a\sin bt)]}{[(z-a\cos bT) - i(a\sin bT)][(z - a\cos bT) + i\, a\sin bt)]}$

or, $= \dfrac{z(z - a\cos bT) + iz(a\sin bT)}{(z - a\cos bT)^2 + (a\sin bT)^2}$

$= \dfrac{z(z - a\cos bT)}{(z - a\cos bT)^2 + (a\sin bT)^2} + \dfrac{i\, z a \sin bT}{(z - a\cos bT)^2 + (a\sin bT)^2}$

$= \dfrac{z(z - a\cos bT)}{z^2 - 2az\cos bT + a^2} + i\, \dfrac{az\sin bT}{z^2 - 2az\cos bT + a^2}$

compairing real and imaginary parts,

$Z(a^n\cos bt) = \dfrac{z(z - a\cos bT)}{z^2 - 2az\cos bT + a^2}$

$Z(a^n\sin bt) = \dfrac{az\sin bT}{z^2 - 2az\cos bT + a^2}$

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3b/. Solve the difference equation by using Z-transform:

$$y_{n+2} - 4y_{n+1} + 4y_n = 2^n \text{ with } y_0 = 0, y_1 = 1$$

Soln:- Given differential Equation is

$$y_{n+2} - 4y_{n+1} + 4y_n = 2^n \text{ with } y_0 = 0, y_1 = 1$$

Taking z-transform on both sides, we get

$$Z\{y_{n+2}\} - 4Z\{y_{n+1}\} + 4Z\{y_n\} = Z\{2^n\}$$

or, $$z^2\bar{y} - z^2 y_0 - zy_1 - 4z\bar{y} + 4zy_0 + 4\bar{y} = \frac{z}{z-2}$$

or, $$z^2\bar{y} - z^2 \times 0 - z - 4z\bar{y} + 4z \times 0 + 4\bar{y} = \frac{z}{z-2}$$

or, $$z^2\bar{y} - z - 4z\bar{y} + 4\bar{y} = \frac{z}{z-2}$$

or, $$\bar{y}(z^2 - 4z + 4) = \frac{z}{z-2} + z$$

or, $$\bar{y} = \frac{z + z^2 - 2z}{(z-2)(z^2 - 4z + 4)}$$

or, $$\bar{y} = \frac{z^2 - z}{(z-2)^3}$$

$$\bar{y} = \frac{z^2 - z}{(z-2)^2}$$

The poles are $z = 2, 2, 2$. It is the poles of order 3.

Now,

$$\operatorname*{Res}_{z \to z_0} f(z) = \frac{1}{2!} \lim_{z \to z_0} \left[\frac{d^2}{dz^2} (z - z_0)^3 f(z) \right]$$

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$$= \frac{1}{2!} \lim_{z \to 2} \left[\frac{d^2}{dz^2} \cancel{(z-2)^3} \times \frac{z^2 - z}{\cancel{(z-2)^3}} \right]$$

$$= \frac{1}{2} \lim_{z \to 2} \left[\frac{d}{dz} 2z - 1 \right]$$

$$= \frac{1}{2} \lim_{z \to 2} (2)$$

$$= \frac{1}{\cancel{2}} \times \cancel{2}$$

$$= 1$$

$\therefore \operatorname{Res}_{z \to 2} f(z) = 1$ #

4a) Show that $\int_0^\infty \frac{\sin \pi w \sin xw}{1 - w^2} \, dw = \begin{cases} \frac{\pi}{2} \sin x & \text{if } 0 \leq x \leq \pi \\ 0 & \text{if } x > \pi \end{cases}$

Soln: The fourier integral representation of $f(x)$ is

$$f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx] dx \quad \text{--- (1)}$$

where,

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos wx \, dx$$

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$$\cancel{\frac{1}{\pi} \times \frac{\pi}{2} \int_0^{\pi} \sin x \cos \omega x \, dx}$$

→ The given integral is sine integral

$$f(x) = \int_{-\infty}^{\infty} B(\omega) \sin \omega x \, d\omega \quad \text{--- (1)}$$

where,

$$B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(x) \sin \omega x \, dx$$

$$= \frac{\cancel{2}}{\pi} \int_0^{\infty} \frac{\pi}{\cancel{2}} \sin x \sin \omega x \, dx$$

$$= \int_0^{\infty} \sin x \sin \omega x \, dx$$

$$= \frac{1}{2} \int_0^{\infty} 2 \sin x \sin \omega x \, dx$$

$$= \frac{1}{2} \int_0^{2} [\cos(\omega x - x) - \cos(\omega x + x)] \, dx$$

$$= \frac{1}{2} \left[\frac{\sin(\omega-1)x}{\omega - 1} - \frac{\sin(\omega+1)x}{\omega+1} \right]_0^{\pi}$$

$$= \frac{1}{2} \left[\frac{\sin(\omega-1)\pi}{\omega-1} - \frac{\sin(\omega+1)\pi}{\omega+1} \right]$$

$$= \frac{1}{2} \left[\frac{-\sin \pi\omega}{\omega-1} + \frac{\sin \pi \omega}{\omega+1} \right]$$

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$$= \frac{\sin \pi\omega}{2}\left[\frac{1 + \cancel{\omega} + 1 - \cancel{\omega}}{1-\omega^2}\right]$$

$$= \frac{\sin \pi\omega}{1-\omega^2}$$

from eqn (1)

$$\therefore f(x) = \int_0^\infty \frac{\sin \pi\omega \sin x\omega}{1-\omega^2}\, d\omega$$

proved

4b) find the fourier sine transform of $f(x) = e^{-x}$ for $x>0$. Then prove that $\int_0^\infty \frac{\omega \sin \omega x}{1+\omega^2}\, dx = \frac{\pi}{2} e^{-x}$ for $m>0$

→ We know, fourier sine transform of $f(x)$ is

$$\hat{f}_s(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \omega x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{(-1)^2+\omega^2}\left(-\sin \omega x - \omega \cos \omega x\right)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{1}{1+\omega^2}(-\omega)\right]$$

$$\therefore \hat{f}_s[f(x)] = \sqrt{\frac{2}{\pi}} \times \left(\frac{\omega}{1+\omega^2}\right)$$

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Again, by inverse fourier signe transform,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s\,[f(x)]\, \sin\omega x\, dx$$

$$e^{-x} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{\omega}{1+\omega^2}\right) \sin\omega x\, dx$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\omega \sin\omega x}{1+\omega^2}\, dx$$

$$\therefore \frac{\pi}{2} e^{-x} \int_0^\infty \frac{\omega \sin\omega x}{1+\omega^2}\, dx$$

proved#

5a) Derive one dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ with necessary assumptions.

Soln:- Consider the flow of heat by conducting in a uniform bar:

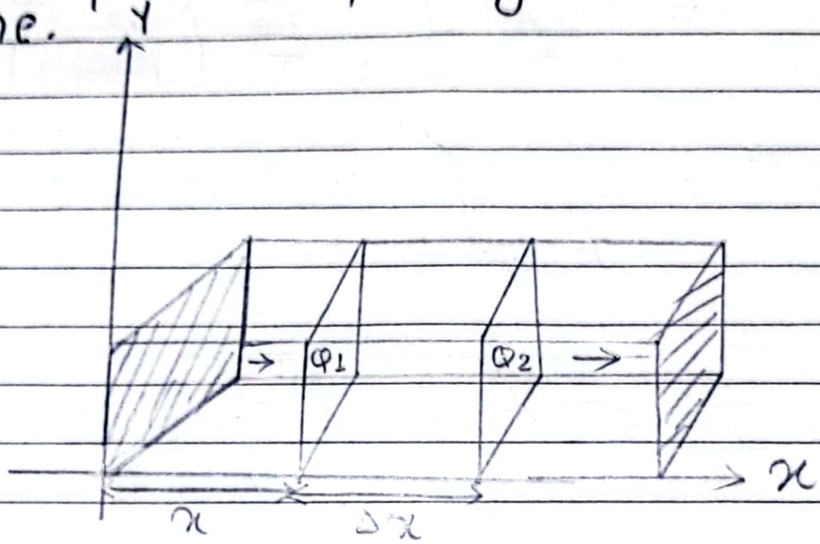
Assumption:

① It is assumed that the sides of the bar are insulated and the loss of heat from the sides by conduction or radation is negligible.

② Take one end of the bar as origin and the direction of flow as the positive x-axis.

Also heat flow from higher to lower temperature.

③ The temperature $u(x,t)$ at any points depends on the distance x of the point from one end and time 't'. Also, the temperature of all points of any cross section is same.



The amount of heat crossing any section of the bar per second depends on the area A of cross section, the conductivity K of the material of bar and the temperature of gradient $\frac{\partial u}{\partial x}$.

$\therefore$ Q_1, the quantity of heat flow into section at the distance x is given by

$$Q_1 = -KA\left(\frac{\partial u}{\partial x}\right)_x \text{ per second}$$

Again, Q_2, the quantity of heat flow out of the section at the distance $x+\Delta x$.

$$Q_2 = -KA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

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Hence, the total amount of heat ~~required~~ retained by the slap with thickness Δx is

$$Q_1 - Q_2 = KA\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - KA\left(\frac{\partial u}{\partial x}\right)_{x} \text{ per sec} \quad \text{—(1)}$$

But we have the rate of change of heat in slab of area A = mass x specific capacity x rate of change of temperature

$$= A\Delta x \rho \times s \times \frac{\partial u}{\partial t} \quad \text{—(2)}$$

Now, from eq^n (1) and (2)

$$\cancel{A}\Delta x \rho s \frac{\partial u}{\partial t} = K\cancel{A}\left[\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right]$$

or, $$\frac{\partial u}{\partial t} = \frac{K}{s\rho}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x}\right]$$

Taking limit as $\Delta x \to 0$ on both sides,

$$\frac{\partial u}{\partial t} = \frac{K}{s\rho}\lim_{\Delta x \to 0}\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}}{\Delta x}\right]$$

$$= \frac{K}{s\rho}\frac{\partial^2 u}{\partial x^2}$$

$\therefore$ $$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{K}{s\rho}$$ #

5b) Find $u(x,t)$ from one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial t^2}$ with boundary condition $u(0,t) = 0 = u(l,t)$, initial deflection $f(x)$ and initial velocity $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$.

→ We have wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ — ①

Since, the vibration of the string is periodic, therfore, the solution of (1) of the form.

$u = (c_1 \cos px + c_2 \sin px)(c_3 \cos pt + c_4 \sin cpt)$ — ②

using the boundary condition $u(0,t) = 0$,

$$u(0,t) = (c_1 + 0)(c_3 \cos cpt + c_4 \sin cpt)$$
$$0 = c_1(c_3 \cos cpt + c_4 \sin cpt)$$

$\therefore \boxed{c_1 = 0}$ but, $T(t) = c_3 \cos cpt + c_4 \sin cpt \neq 0$

So eqn ② reduces to

$$u(x,t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt) \text{ — ③}$$

Using another condition $u(l,t) = 0$, we get

$$u(l,t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$$

But $c_2(c_3 \cos cpt + c_4 \sin cpt) \neq 0$

$$\sin pl = 0$$

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$\sin pl = \sin n\pi$

$\therefore p = \frac{n\pi}{l}$

So, eqn (3) reduces to

$$u(x,t) = c_2 \sin\left(\frac{n\pi x}{l}\right)\left[c_3 \cos\frac{cn\pi t}{l} + c_4 \sin\frac{cn\pi t}{l}\right]$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\frac{cn\pi t}{l} + B_n \sin\frac{cn\pi t}{l}\right] \sin\frac{n\pi x}{l} \quad \text{--- (4)}$$

using the initial condition $u(x,0) = f(x)$ in (4)

$$u(x,0) = \sum_{n=1}^{\infty} [A_n + 0] \sin\frac{n\pi x}{l}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$$

This is half range fourier series of function $f(x)$ in the interval $(0, l)$. so, the fourier coefficient A_n is found by the Euler's formula.

$$\therefore A_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

partially differentiating wrt t in eqn (4)

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[-A_n \sin\frac{cn\pi t}{l} \times \frac{cn\pi}{l} + B_n \cos\frac{cn\pi t}{l} \times \frac{cn\pi}{l}\right] \sin\frac{n\pi x}{l} \quad \text{--- (5)}$$

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using the condition $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$ in ⑤

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{cn\pi}{l} \sin\frac{n\pi x}{l} \quad \text{—— (6)}$$

This is half range fourier series of function g(x) in the interval (0, l). So, the fourier coefficients $\left(\frac{cn\pi}{l} B_n\right)$ is found by the euler's formula.

$$\frac{cn\pi}{l} B_n = \frac{2}{l}\int_0^l g(x) \sin\frac{n\pi x}{l}\, dx \quad \text{—— (7)}$$

The solⁿ of wave equation is given by the eqⁿ ④ together with the values of An and Bn ~~from~~ which satisfies the boundary and initial condition.

Q6a) A homogenous rod of conducting material of length 100 cm has its ends kept at zero temperature and the initial temperature is given $u(x,t) = \begin{cases} x & ; 0 \le x \le 50 \\ 100-x & ; 50 \le x \le 100 \end{cases}$. find the temperature distribution on the rod at any time.

→ The length of rod (l) = 100 cm

Given $u(0,t) = 0 = u(l,t)$

Now,

The temperature at time (t) is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{100} x \, e^{-\frac{c^2 n^2 \pi^2 t}{l^2}} \quad \text{—— (1)}$$

where,

$$A_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$$

or, $$A_n = \frac{2}{l}\left[\int_0^{50} x \sin\frac{n\pi x}{l}\, dx + \int_{50}^{100} (100-x) \sin\frac{n\pi}{l}\, dx\right]$$

or, $$A_n = \frac{2}{l}\left[\frac{-x\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} + \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_0^{50} + \left[(100-x)\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} - \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right]_{50}^{100}$$

or, $$A_n = \frac{2}{l}\left[-\frac{50\,l}{n\pi}\cos\frac{n\pi}{100}\times 50 + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{100}\times 50\right.$$

$$- 0 - \sin\frac{n\pi}{100}\times 100 \times \frac{l^2}{n^2\pi^2} + \frac{50\,l}{n\pi}\cos\frac{n\pi}{100}\times 50$$

$$\left. + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{100}\times 50\right]$$

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or, $A_n = \frac{2}{\cancel{l}} \times \frac{2l^{\cancel{2}}}{n^2\pi^2} \sin\frac{n\pi}{2}$

or, $A_n = \frac{4 \times 100}{n^2\pi^2} \sin\frac{n\pi}{2}$

$\therefore$ $A_n = \frac{400}{n^2\pi^2} \sin\frac{n\pi}{2}$

from equation (1)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{400}{n^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi}{100}x \; e^{-\frac{c^2\pi^2 n^2 t}{100^2}}$$

$$\therefore \; u(x,t) = \frac{400}{\pi^2} \sum_{n=1}^{\infty} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{100} e^{-\frac{c^2 n^2\pi^2 t}{100^2}}$$ #

6b) Express Laplacian $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ in polar coordinate system.

Soln:- The laplacian of U is $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$ i.e $U_{xx} + U_{yy}$ — (1)

We have,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial x}$$

or, $U_x = U_r U_x + U_\theta \theta_x$ — (2)

Again, differentiating (2) wrt x, we get

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$$u_{xx} = (u_r r_x)_x + (u_\theta \theta_x)_x$$

or, $u_{xx} = (u_r)_x \cdot r_x + u_r r_{xx} + (u_\theta)_x \theta_x + u_\theta \theta_{xx}$

or, $u_{xx} = (u_{rr} r_x + u_{r\theta} \theta_x) r_x + u_r r_{xx} + (u_{\theta r} r_x + u_{\theta\theta} \theta_x)\theta_x + u_\theta \theta_{xx}$ — (3)

To change it into polar form,
put $x = r\cos\theta$ and $y = r\sin\theta$
then, $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ — (4)

$\frac{\partial r}{\partial x}$ $\therefore r_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$

Again, diff wrt x, we get

$\therefore r_{xx} = \frac{r \cdot 1 - x r_x}{r^2}$

$r_{xx} = \frac{1}{r} - \frac{x^2}{r^3}$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\frac{\partial \theta}{\partial x} = \theta_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right)$

$= \frac{\cancel{x^2}}{x^2+y^2} \times \frac{-y}{\cancel{x^2}}$

$= \frac{-y}{r^2} = \frac{-\cancel{r}\sin\theta}{r^{\cancel{2}}} = -\frac{\sin\theta}{r}$

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Again,

$$\frac{\partial^2\theta}{\partial x^2} = \theta_{xx} = \frac{\left[-r\frac{\partial}{\partial x}\sin\theta - \sin\theta\frac{\partial r}{\partial x}\right]}{r^2}$$

or, $$\theta_{xx} = \frac{-r\cos\theta\frac{\partial\theta}{\partial x} - \sin\theta\cdot\frac{x}{r}}{r^2}$$

or, $$\theta_{xx} = \frac{\frac{r\sin\theta\cos\theta}{r} + \frac{x}{r}\sin\theta}{r^2}$$

or, $$\theta_{xx} = \frac{r\sin\theta\cos\theta + r\sin\theta\cos\theta}{r^3}$$

or, $$\theta_{xx} = \frac{2r\sin\theta\cos\theta}{r^{\cancel{3}2}}$$

or, $$\theta_{xx} = \frac{2\sin\theta\cos\theta}{r^2}$$

Similary, partially differentiating wrt y, to (1) we get

$$\frac{\partial r}{\partial y} = r_y = \frac{y}{r}, \quad r_{yy} = \frac{1}{r} - \frac{y^2}{r^3}$$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\theta_x = \frac{\cos\theta}{r}, \quad \theta_{yy} = \frac{2\sin\theta\cos\theta}{r^2}$$

Substituting these values in eqn (2)

$$U_{xx} = \frac{\partial u}{\partial r}\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \frac{\partial^2 u}{\partial r^2}\cdot\frac{x^2}{r^2} + \frac{\partial u}{\partial\theta}$$

$$\left(\frac{2\sin\theta\cos\theta}{r^2}\right) + \frac{\partial^2 u}{\partial\theta^2}\left(\frac{\sin^2\theta}{r^2}\right)$$

Similarly,

$$U_{yy} = \frac{\partial u}{\partial r}\left(\frac{1}{r} - \frac{y^2}{r^3}\right) + \frac{\partial^2 u}{\partial r^2}\frac{y^2}{r^2} +$$

$$\frac{\partial u}{\partial\theta}\left(-\frac{2\sin\theta\cos\theta}{r^2}\right) + \frac{\partial^2 u}{\partial\theta^2}\left(\frac{\cos^2\theta}{r^2}\right)$$

So, laplace eqn (1) reduces to

$$U_{xx} + U_{yy} = \frac{\partial u}{\partial r}\left(\frac{1}{r} - \frac{x^2}{r^3} + \frac{1}{r} - \frac{y^2}{r^3}\right) +$$

$$\frac{\partial^2 u}{\partial r^2}\left(\frac{x^2+y^2}{r^2}\right) + \frac{\partial^2 u}{\partial\theta^2}\left(\frac{\sin^2\theta + \cos^2\theta}{r^2}\right)$$

or, $$U_{xx} + U_{yy} = \frac{\partial u}{\partial r}\left(\frac{2}{r} - \frac{r^2}{r^3}\right) + \frac{\partial^2 u}{\partial x^2}\left(\frac{r^2}{r^2}\right) +$$

$$\frac{\partial^2 u}{\partial\theta^2}\left(\frac{1}{r^2}\right)$$

$$\therefore \nabla^2 u = U_{xx} + U_{yy}$$ #

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Q7a) Find z-transform of $z(a^n)$.

→ By the defination of z-transform,

$$z(u^n) = \sum_{n=0}^{\infty} u_n z^{-n}$$

So,

$$z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$$

or, $$z(a^n) = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

or, $$z(a^n) = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

or, $$z(a^n) = \frac{1}{1 - \frac{a}{z}}$$

$$\therefore z(a^n) = \frac{z}{z-a}$$

b) Evaluate $\oint_c (\bar{z})^3 \, dz$, where c is unit circle.

→ We have $f(z) = (\bar{z})^3$ which does not exists at $\bar{z} = \infty$; which does not lies in in the given curve.
Thus, $f(z) = (\bar{z})^3$ is analytic in the unit circle. Then,

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$\oint_C (z^3)dz = 0$ by cauchy theorem.

c) Solve the partial differential equation $u_{yy} = u$.

→ We have

$$u_{yy} = u$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} - u = 0$$

It's auxilary eq^n is

$$m^2 - 1 = 0$$
$$m = \pm 1$$

Then its solution is

$$u = c_1 e^y + c_2 e^{-y} \quad \#$$

d) Find the Taylor Series of $f(z) = \frac{1}{z}$ at $z = i$

→ We have,

$$f(z) = \frac{1}{z} = \frac{1}{(z-i+i)} = \frac{1}{i\left[1 + \frac{z-i}{i}\right]}$$

$$= \frac{1}{i} \sum_{n=0}^{\infty} \left[\frac{z+i}{i}\right]^n$$

$$= \frac{1}{i} \sum_{n=0}^{\infty} \frac{(z+i)^n}{i^n}$$

$$\therefore f(z) = \sum_{n=0}^{\infty} \frac{(z+i)^n}{i^{n+1}} \quad \#$$

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