pigeon hole principle

The **Pigeonhole Principle** is a simple yet powerful concept in combinatorics. It states that if you try to place more items into a set of containers than there are containers, at least one container must contain more than one item.

Formal Statement

If n items are put into m containers, and n >m, then at least one container must contain more than one item.

Examples of the Pigeonhole Principle

Example 1: Socks in a Drawer

Suppose you have 10 pairs of socks, each pair a different color, and you randomly pick 11 socks from the drawer. The Pigeonhole Principle guarantees that you will have at least one matching pair. Here, the items are the socks, and the containers are the pairs of colors. Since you have 11 socks and only 10 pairs (containers), at least one pair (container) must have more than one sock.

Generalized Example

If you have 12 apples and 5 baskets, by the Pigeonhole Principle, at least one basket must contain at least $\lceil 12/5 \rceil = \lceil 2.4 \rceil = 3$ apples.

Applications

The Pigeonhole Principle is a fundamental concept with applications in many areas of mathematics and computer science, such as proving the existence of solutions, combinatorial arguments, and even in algorithms related to hashing and data structures. It's often used as a starting point to prove more complex results.

proof by contradiction:

Proof by contradiction is a common technique used in Theory of Computation (ToC) to prove the correctness or properties of algorithms, automata, or systems.

The method involves assuming the opposite of what you want to prove, and then demonstrating that this assumption leads to a contradiction.

A basic outline of how proof by contradiction can be applied in ToC:

1. **Assume the opposite of what you want to prove:** Start by assuming that the statement you want to prove (let’s call it P) is false.

2. **Derive a contradiction:** Use this assumption to derive a contradiction with known facts, definitions, or properties within the context of ToC. This typically involves logical reasoning or applying known results.
3. **Conclude the original statement:** Since assuming ¬P (not P) led to a contradiction, it must be that P is actually true.

Example:

- Proving for not being regular language using pumping lemma theory for regular language.
- Proving for not being CFL using pumping lemma theory.
- Proving the Halting problem.

Induction

Induction proof is another fundamental technique in Theory of Computation (ToC) used to prove statements about recursively defined structures, such as algorithms, data structures, or properties of formal languages.

Mathematical Induction

1. **Base Case:** Start by proving the statement (let's call it P(n)) for the base case, typically the smallest value or base instance of the recursive definition. Show that P(1) (or P(0), depending on the context) holds true.
2. **Inductive Step:** Assume that P(k) holds true for some arbitrary $k \geq 1$. This is the induction hypothesis (IH).
3. **Prove P(k+1):** Using the induction hypothesis P(k), prove that P(k+1) also holds true. This step typically involves showing that if P(k) is true, then P(k+1) must also be true.
4. **Conclusion:** By the principle of mathematical induction, P(n) holds true for all $n \geq 1$.

Example 1:

Statement to prove: The sum of the first n natural numbers $1+2+\ldots+n=\frac{n(n+1)}{2}$.

Step 1: Base Case

For $n=1$:

$$1=\frac{1(1+1)}{2}=\frac{2}{2}=1$$

So, $P(1)$ is true.

Step 2: Inductive Step

Assume $P(k)$ is true for some arbitrary k. That is, assume:

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

Now, we need to prove that $P(k+1)$ is true:

$$\boxed{1+2+3+\cdots+k+(k+1)=\frac{(k+1)(k+2)}{2}}$$

Start from the inductive hypothesis: Adding (K+1) on both the sides:

$$1+2+3+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

Factor $(k+1)$ out of the right-hand side:

$$=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}=\frac{k(k+1)+2(k+1)}{2} \qquad =\frac{(k+1)(k+2)}{2}$$

This matches the formula for $P(k+1)$. Thus, $P(k+1)$ is true if $P(k)$ is true.

Step 3: Conclusion

Since both the base case and the inductive step have been proven, by the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n.

This completes the proof.

Example 2: Proving the Formula for the Sum of the First n Powers of 2

Statement: Prove that the sum of the first n powers of 2 is given by:

$$P(n): \quad 1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$$

Step 1: Base Case

For $n = 1$, the sum of the first 1 power of 2 is:

$$2^{1-1} = 2^0 = 1$$

And the right-hand side of the formula is:

$$2^1 - 1 = 2 - 1 = 1$$

So, $P(1)$ is true.

Step 2: Inductive Step

Assume $P(k)$ is true for some arbitrary k. That is, assume:

$$1 + 2 + 4 + \cdots + 2^{k-1} = 2^k - 1$$

Now, we need to prove that $P(k+1)$ is true. We need to show:

$$\boxed{1 + 2 + 4 + \cdots + 2^{k-1} + 2^k = 2^{k+1} - 1}$$

Start from the inductive hypothesis:

$$1 + 2 + 4 + \cdots + 2^{k-1} = 2^k - 1$$

Add 2^k to both sides:

$$1 + 2 + 4 + \cdots + 2^{k-1} + 2^k = (2^k - 1) + 2^k$$

Simplify the right-hand side:

$$= 2^k - 1 + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1$$

This matches the formula for $P(k+1)$. Thus, $P(k+1)$ is true if $P(k)$ is true.

Step 3: Conclusion

Since both the base case and the inductive step have been proven, by the principle of mathematical induction, the statement $P(n)$ is true for all natural numbers n.

This completes the proof.

Example 3: Prove that the sum of the first n odd positive integers is equal to n^2

Step 1: **Base Case**

Verify the statement for $n = 1$.

The first odd positive integer is 1, so the sum is:

$$S(1) = 1$$

The formula gives:

$$1^2 = 1$$

Since both sides are equal, the base case holds true.

Step 2: **Inductive Hypothesis**

Assume that the statement is true for some arbitrary positive integer k.
That is, assume:

$$S(k) = 1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Step 3: **Inductive Step**

We need to prove that if the statement is true for $n = k$, then it must also be true for $n = k + 1$.

Consider the sum of the first $k + 1$ odd positive integers:

$$S(k+1) = 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1)$$

Using the inductive hypothesis, substitute $S(k)$:

$$S(k+1) = k^2 + (2(k + 1) - 1)$$

Simplify the expression:

$$S(k+1) = k^2 + (2k + 2 - 1) = k^2 + (2k + 1)$$

Notice that:

$$S(k+1) = (k+1)^2$$

This shows that the formula holds true for $n = k + 1$.

Conclusion:

By the principle of mathematical induction, the formula

$$S(n) = n^2$$

is true for all natural numbers n. The sum of the first n odd positive integers is n^2.