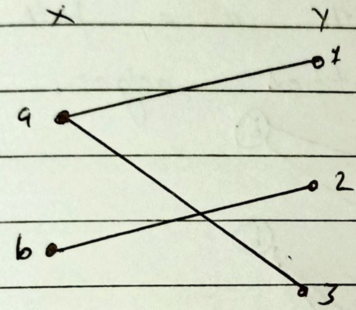
UNIT - 6

OPTIMIZATION OF GRAPHS

(Matching & Network Flow)

Matching —

A matching M in a graph G is the set of edges such that every vertex of G is incident to atmost one edge in M.



$G = (X \cup Y, E)$

$M = \phi$ (is also matching)

$M_0 = \{ (a,3) \}, \cancel{(a,1)\}}$ ✓

$M_1 = \{(a,3), (a,1)\}$ ✗ → Not a matching

Perfect matching from X to Y ← $M_2 = \{ (a,3), (b,2) \}$ ✓ , $|M_2| = 2$

↳ X-saturates. , not Y-saturates.

→ The size of a matching is the number of edge in M

→ The match M is said to be maximum if has largest possible size.

→ Matching having all vertex of X, then, M is X-saturates.

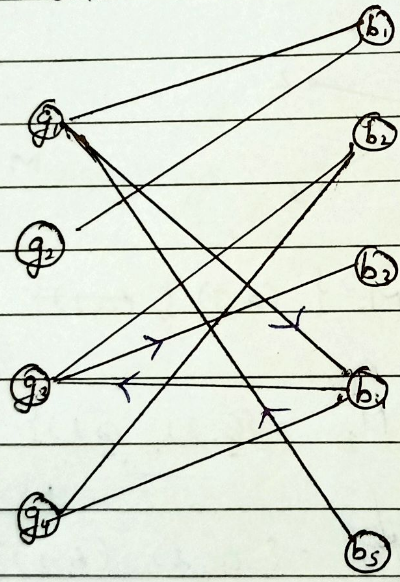
Perfect Matching

A matching contains all the vertices of G.

Alternating Path

An alternating path is a path that alternates b/w matched and unmatched edges. of ~~a~~

ex –



$M = \{(g_1, b_4), (g_2, b_1), (g_3, b_3), (g_4, b_2)\}$

Path –

(b_5, g_1) (g_1, b_4) (b_4, g_3) (g_3, b_3)

U.M. M U.M. M

↓

Not augmenting path

Augmenting Path

An alternating path is said to be augmenting path starts & ends on unmatched vertices.

Augmenting Path Algorithm

It is used to find maximum matching.

Input : $G = (X \cup Y, E)$

Output : Maximum matching.

Starts with $M = \phi$

while (there exists an augmenting path w.r.t. M)

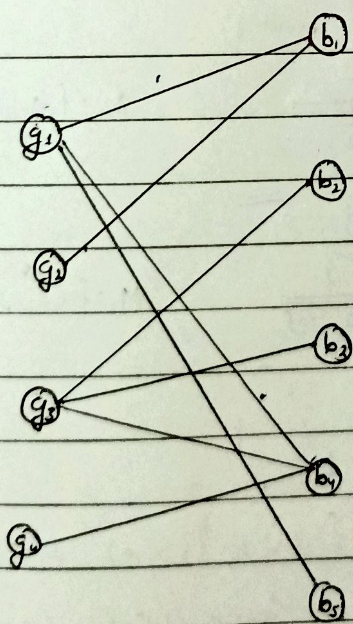
{

$$M = M \Delta P = (M - P) \cup (P - M)$$

}

Return M , // Here M is maximum matching.

Q



Iterations—

I. $M = \Phi$

Let, $P = (g_1, b_1)$

$M = (M \Delta P)$

w.r.t M $M = \{(g_1, b_2)\}$

II. $P = (b_4, g_1)(g_1, b_1)(b_1, g_2)$

$M = M \Delta P = (M - P) \cup (P - M)$

$M = \{(b_4, g_1), (b_1, g_2)\}$

III $P = \underset{U}{(b_5, g_1)}\underset{M}{(g_1, b_4)}\underset{U}{(b_4, g_3)}$

$M = M \Delta P = \{(b_5, g_1), (b_4, g_3)\}\{b_1, g_2\}\}$

→ Not necessary to take all the matching edges to find augmenting path.

(IV) $P = (b_2, g_3)(g_3, b_4)(b_4, g_4)$

$M = M \Delta P = \{ \underbrace{(b_2 g_3), (b_4, g_4)}_{P-M} \underbrace{(b_1, g_2), (g_1, b_5)}_{M-P} \}$

$|M| = 4$

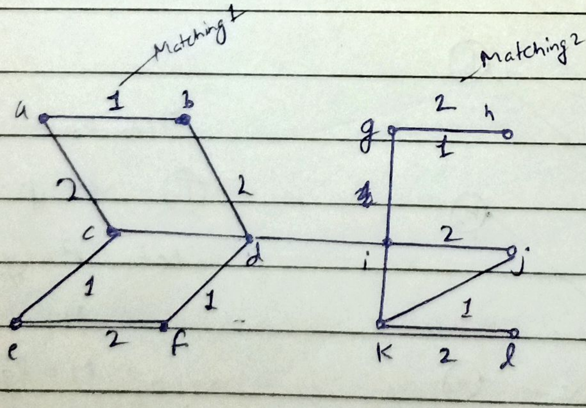
(V) P does not exist, There is no ~~alternating~~ augmenting Path.

Stop

$\therefore$ Current M is the maximum matching

– Every component of the symmetric difference of two matching is an alternating path or an even cycle.

ex –

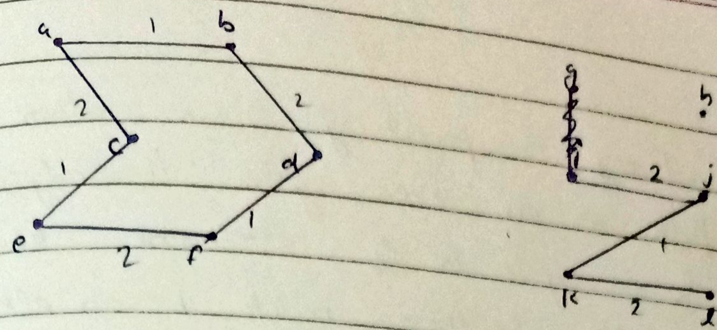


$M_1 = \{ (a, b), (c, e), (d, f), (i, g), (k, j) \}$

$M_2 = \{ (a, c), (e, f), (b, d), (g, h), (i, j), (k, l) \}$

$M_1 \Delta M_2 = F$

$F = (M_1 - M_2) \cup (M_2 - M_1) = \{ (a, b), (c, e), (d, f), (i, g), (k, j), (a, c), (e, f), (b, d), \cancel{(g, h)}, (i, j), (k, l) \}$



Theorem – (Berge, 1957) $P \leftrightarrow Q$

A matching M is maximum iff there is no augmenting path w.r.t. M.

Proof – By Contradication – ~~Suppose M is not maximum.~~
$(P \to Q)$ Given that M is maximum matching.
~~Suppose M is not maximum M is not~~
there exist an augumenting path P w.r.t. M.

So, By Augmenting path Algorithm, there exist a matching M,

$$M' = M \Delta P$$
$$\Rightarrow |M'| = |M| + 1$$

So, M is not maximum, which is contradiction.
There does not exist Augumenting path.

$(Q \to P) \Rightarrow$ There is no augumenting path P wrt M.
Suppose M is not maximum. Let M^* be a maximum matching then $|M^*| > |M|$

Let, $Q = M \Delta M^*$

Q has more edges from M^*.

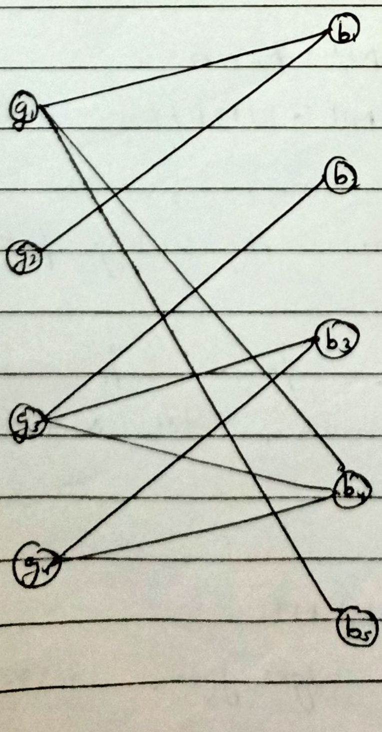
~~$Q = M \Delta M^*$~~

Every component of Q is either alternate path or an even cycle the edges of every path & cycle in Q alternates b/w edges in M & M*. So there must be an alternating path with more edges from M*, So ~~there~~ this is contradiction.

<u>Neighbours of a Set in Bipartite Graph</u>

Let $G = (X \cup Y, E)$, if
$S \subseteq X$, then
$N(S) = \{ y \in Y \mid (x, y) \in E \text{ for all } x \in X \}$
$\downarrow$
Neighbour of S

ex-



$X = \{ g_1, g_2, g_3, g_4 \}$
$Y = \{ b_1, b_2, b_3, b_4, b_5 \}$
Let,
$S = \{ g_1, g_2 \} \subseteq X$

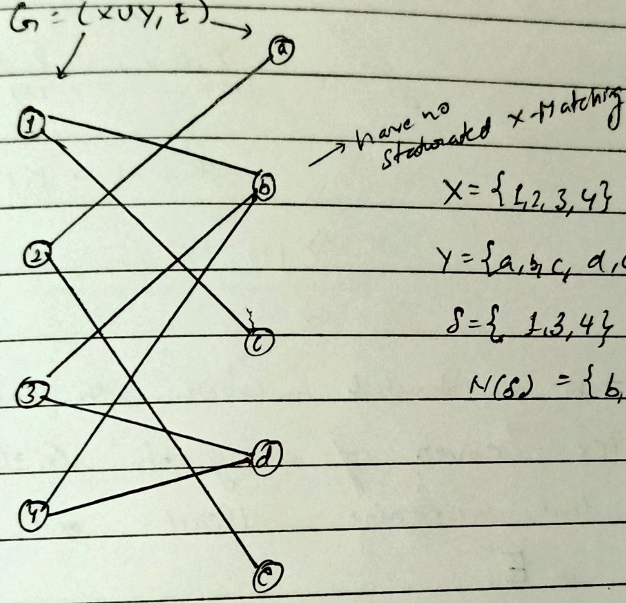
$N(S) = \{ b_1, b_4, b_5 \}$

$|N(S)| \leq |Y|$

Hulle's Theorem

A bipartile graph $G = (X \cup Y, E)$ has a matching that saturate-X iff $|S| \le |N(S)|$, $\forall S \subseteq X$

$\#$ subset $= 2^{|X|}$



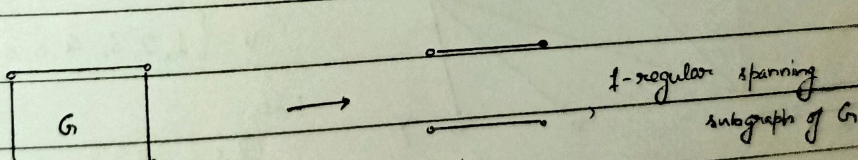
$X = \{1, 2, 3, 4\}$

$Y = \{a, b, c, d, e\}$

$S = \{1, 3, 4\}$

$N(S) = \{b, d\}$

→ A K-regular spanning subgraph is called K-factor.

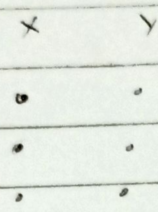


$H \subseteq G$

So, H is 1-regular and it is 1-factor subgraph

→ A subgraph $H \subseteq G$ is a 1-factor of G iff $E(H)$ is a perfect matching of G

→ If $G = (X \cup Y, E)$ is K-regular, with $K > 1$, then G has a 1-factor / perfect matching.
(matching which have all the vertices)



$|X|$ & $|Y|$

Since G is K-regular, then,

$$\sum_{v \in X} d(v) = \sum_{v \in Y} d(v)$$

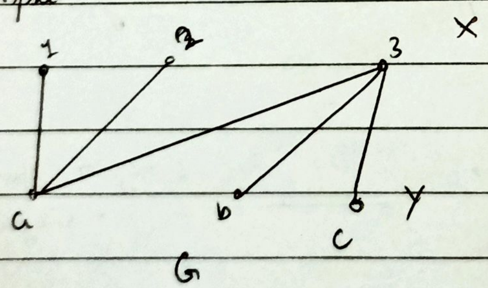
$$K|X| = K|Y|$$

$$\Rightarrow |X| = |Y|$$

<u>Vertex Cover</u> → We are interested in cover with minimum size.

A vertex cover of a graph $G = (V, E)$ is a set $Q \subseteq V$ that contains atleast one end point of every edge in E.

example –



$V = \{1, 2, 3, a, b, c\}$

Let,

$Q_1 = \{1, 2, 3\}$

$Q_2 = \{a, 3\}$ → minimum vertex cover

Matching $M = \{(1, a), (3, c)\}$ → maximum matching.

It has no x-saturated matching.??

Let, $S = \{1, 2\}$, $|S| = 2$

$N(S) = \{a\}$

$|S| > |N(S)|$

→ Hall's Theorem doesn't satisfy.

Hence the given graph have no x-saturation.

→ If $G' \subseteq G$ and G' contains the edges $(x,y) \in E$

$G' = (V', E')$

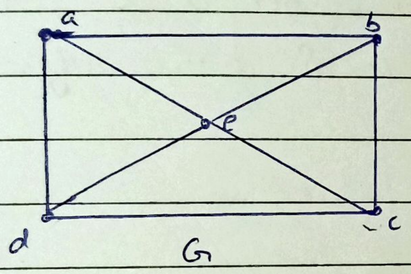
$G = (V, E)$ $\quad V' \subseteq V$, $E' \subseteq E$

with $x, y \in V'$, then G' is an induced subgraph of G.

↓ depends on the vertex set

$E' = E \cap (V' \times V')$ → all the edges in induced subgraph

example —

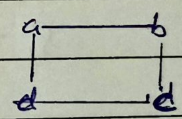


$V = \{a, b, c, d, e\}$

$V_1' = \{a, b, c, d\}$

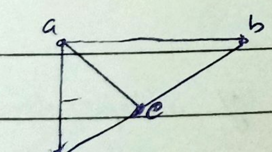
then,

$G' \Rightarrow$



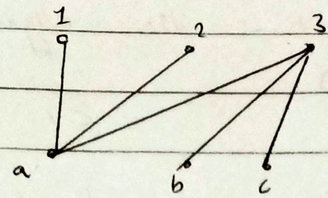
$V_2' = \{a, b, e, d\}$

$G_2' \Rightarrow$



Kong's Theorem

If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover of G.



$Q = \{a, 3\}$, is minimum size vertex cover.

max-matching : $M = \{(2,a), (3,c)\}$

$|Q| = 2$

$|M| = 2$

Independent Set → we are interested in maximum size.

An independent set is a set $S \subseteq V$ such that no two vertices in S are adjacent.

In Above graph –

$S_1 = \{1, 2\}$, $S_2 = \{2, 3\}$, $S_3 = \{c, b\}$

$S = \{1, 2, b, c\}$ → Maximum size, $|S| = 4$

$\bar{S} = \{3, a\}$ → complement of ~~indeperdest~~ independent set

$|\bar{S}| = 2$

vertex ~~open~~ cover

So, complement of an independent set is a vertex cover.

em– In a graph $G = (V, E)$, $S \subseteq V$ is an ~~independepend~~ independent set iff $\bar{S}$ is a vertex cover of G

maximum (independent set); minimum (vertex cover)

→ Also, then $\alpha(G) + \beta(G) = |V|$ → true for all graph.

where $\alpha(G)$ = max. size of Ind. set, $\beta(G)$ = min. size of vertex cover.

Proof:- If S is an independent set, every edge is incident to atmost one vertex in S. This implies that every edge is incident to atleast one vertex in $\bar{S}$.

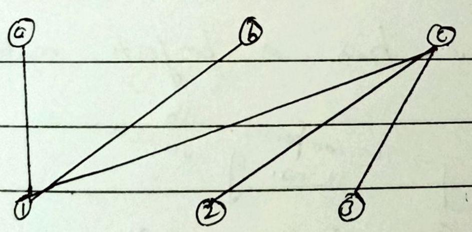
So, $\bar{S}$ covers all the edges of G. So $\bar{S}$ is a vertex cover.

Let $\bar{S}$ be a vertex cover of G, then $\bar{S}$ covers all the edges of G, then there are no edge joining vertices of S.

So, S is an independent set.

Edge cover (minimum size)

An edge cover of G is a set of edges, $L \subseteq E$, such that ~~su~~ every vertex of G is incident to some edge of L.



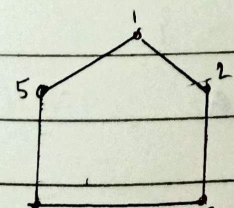
$L = \{a1, b1, c2, c3\}$ is an edge cover

$\alpha'(G)$

max matching $M = \{a1, c2\}$

$\beta'(G)$

$$\alpha'(G) + \beta'(G) = |V|$$



Edge cover

$L = \{15, 43, 12\}$

Max matching

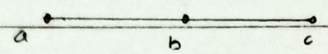
$M = \{54, 23\}$

$\alpha'(G) = 3$

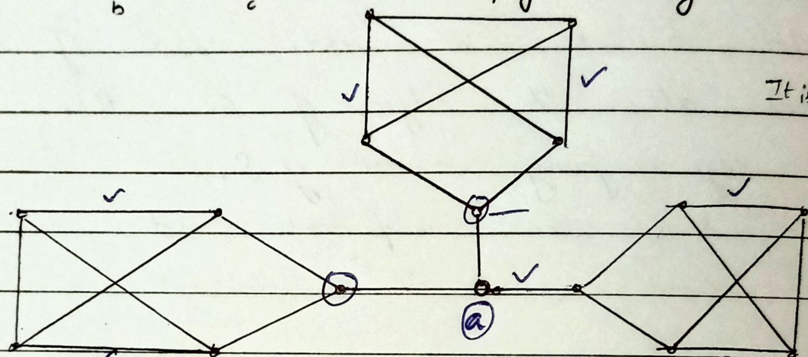
$\beta'(G) = 2$, $3 + 2 = 5 = |V|$

Matching in General Graph:

A perfect match is matching, which matches all the vertices of graph



→ No perfect matching



It is a 3-regular graph

$|V| = 16$

Sylvester Graph

↓

No perfect matching

Tutte's Theorem

A graph $G = (V, E)$ has a perfect matching iff for every $S \subseteq V$,

→ The component with odd no. of vertices

$$\# \text{odd component} = C_{odd}(G - S) \leq |S|$$

P_3

$V = \{a, b, c\}$

Let $S = \{a\}$, $|S| = 1$

$G - S$: b —— c

$C_{odd} = 0$

$\therefore 0 \leq 1$

$S = \{b\}$

$G - S$: a c

$C_{odd} = 2$, $C_{odd} > |S|$

In Sylvester Graph,

$S = \{a\}$

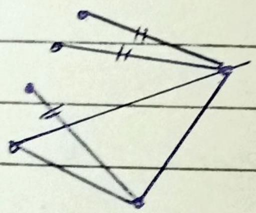
$G - S$:

$C_{odd} = 3 \to$ components

$|S| = 1$

$C_{odd} > 3$, $\therefore$ it has no perfect matching

→ A bridge is an edge of a graph G, whose removal increase the no. of components.



→ 3 - bridge edges.

Theorem :- An edge of a connected graph is a bridge edge iff it does not lie on any cycle.

→ A graph containing no bridges is called bridgeless graph.

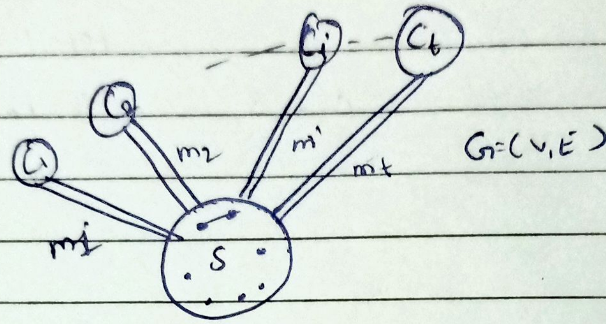
Theorem - (Peterson Theorem, 1891)

If G is a bridgeless ~~graph~~ 3-regular graph, then it has a perfect matching.

oof-

$G=(V,E)$

Let $S \subseteq V$, let $C_1, C_2, \ldots, C_t$ be the odd components of $G-S$.



Let m_i be the no of edges with one end in C_i & other end in S.

then, $\sum_{v\in C_i} d(v) = 3|S|$

$\sum_{v\in C_i} d(v) = 3C_i$, where V_{C_i} are number of vertices in C_i.

So, $m_i = \overbrace{3V_{C_i}}^{odd} - \overbrace{2e_{C_i}}^{even}$, $\{(odd - even) = odd\}$

No. of vertex & edges in component C_i

where $m_i \neq 1$, ∵ it is 3-regular graph.

$m_i \geq 3$

$\Rightarrow m_i$ is odd no. $\Uparrow$

$\Rightarrow \sum_{i=1}^{t} m_i \geq 3t$

$t \leq \dfrac{\sum_{i=1}^{t} m_i}{3}$

Since, $\cancel{\sum_{i=1}^{t} m_i \leq \sum}$ $\cancel{t \leq}$

Since, $\sum_{i=1}^{t} m_i \leq \sum_{v \in G} d(v)$

$$t \leq \frac{\sum_{v \in G} d(v)}{3} \geq 3|S|$$

$$\cancel{t \leq \frac{|S|}{3}} \qquad t \leq |S| \Rightarrow \text{By Tutte's Theorem}$$

Graph G have a perfect matching.

Stable Matching

Problem :- The Stable matching Problem

Input : List of men, women & their preferences

Output : A stable marriage.

men	Preferences (high ← low)				Women	Preferences (high ← low)			
1	2	4	1	3	1	2	1	4	3
2	3	1	4	2	2	4	3	1	2
3	2	3	1	4	3	1	4	3	2
4	4	1	3	2	4	2	1	4	3

→ A matching is a 1-1 corresponding b/w men & women

ex- $M_1 = \{ \underset{M\,W}{(1,2)}, (2,3), (3,1), (4,4) \}$

If $(m,w) \in M$, then

$m = P_m(w)$, &

$w = P_m(m)$

→ m is partener of w.

If there is no blocking ~~pair~~ pair, then matching is stable.

~~m is the partener of ω.~~

→ A pair $(m, \omega) \notin M$ is said to be a blocking pair for matching M if m prefers ω to $P_M(\omega)$

$\overset{m}{(3}, \overset{\omega}{2)}$ is a blocking pair of M_2

The man 3 prefers woman 2 to $1 = P_M(3)$

woman 2 prefers man 3 to $1 = P_M(2)$

So, $(3,2) \notin M$, is a blocking pair ~~of~~ in M_2

$M_2 = \{(1,1), (2,3), (3,2), (4,4)\}$

Is $(1,2)$ a blocking pair.??

man 1 prefers woman 2 to 1

woman 2 prefers 3 to 1

So $(1,2) \notin M_2$, is not a blocking

→ A matching for which, there is atleast one blocking pair is called unstable matching, otherwise stable.

Q The matching $M = \{(1,4), (2,3), (3,2), (4,1)\}$ is stable matching. How?

∵ 2, 3, 4 have a first preference, but not 1

$(1, 2)$ is NOT blocking.

man 1 prefer 2 ~~into~~ 4,

woman 2 prefers 3 to 1.

Gale - Shapley Algorithm

Initially, all $m \in M$ & $w \in W$ are free.

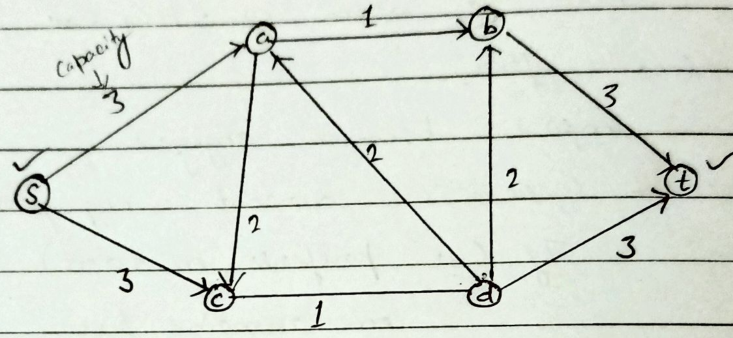
while (there is a man m who is free)
{ w = highest ranked woman in m's preference list to whom m has not yet proposed.
If (w is free)
(m,w) becomes engaged
else if (w is current engaged with man m')
{ If (w prefers m' to m)
m remains free
else (w prefers m to m')
{ (m,w) becomes ~~m to m')~~ engaged
m' becomes free
}
}
}

I. (1,2) engaged — ×
II. (1,3) engaged
III. (3, 2) 2 — 3 ✓, 1 × - (free)

×
IV. (4,4) engaged.
V. (1,4) 4 — 4, 1
VI. (4,1)

Flow Network

A flow network $G=(V,E)$ is a directed graph where each edge $(u,v) \in E$ has a capacity $c(u,v) \geq 0$, a source s and a sink t.



Flow in Network

Def^n → A flow f in G is a real valued function $f: V \times V \longrightarrow \mathbb{R}$ such that

I. $f(u,v) \leq c(u,v)$: capacity rule, that is flow on an edge cannot exceed the capacity of the edge.

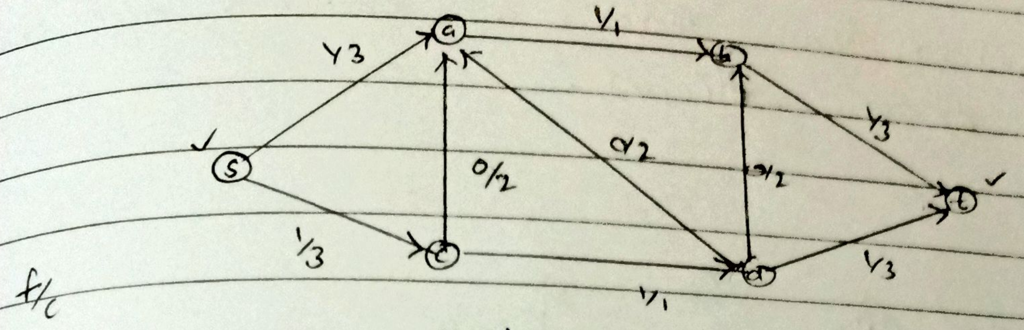
II. Skew Symmetricity : $f(u,v) = -f(v,u)$

III. Flow conservation for every vertex :

$$v \notin \{s,t\} \Rightarrow \underbrace{\sum_{u \in V} f(u,v)}_{In} = \underbrace{\sum_{w \in V} f(v,w)}_{out}$$

↓

Amount of flow into a node equals the amount of flow out of it.



The flow is valid

Given a flow network, find that maximum flow ~~out of it.~~ from s ~~to~~ t.

In the previous valid flow network,
Total outgoing flow from s ~~to t~~. = 1+1 = 2
Total incoming flow ~~from s~~ to t = 1+1 = 2

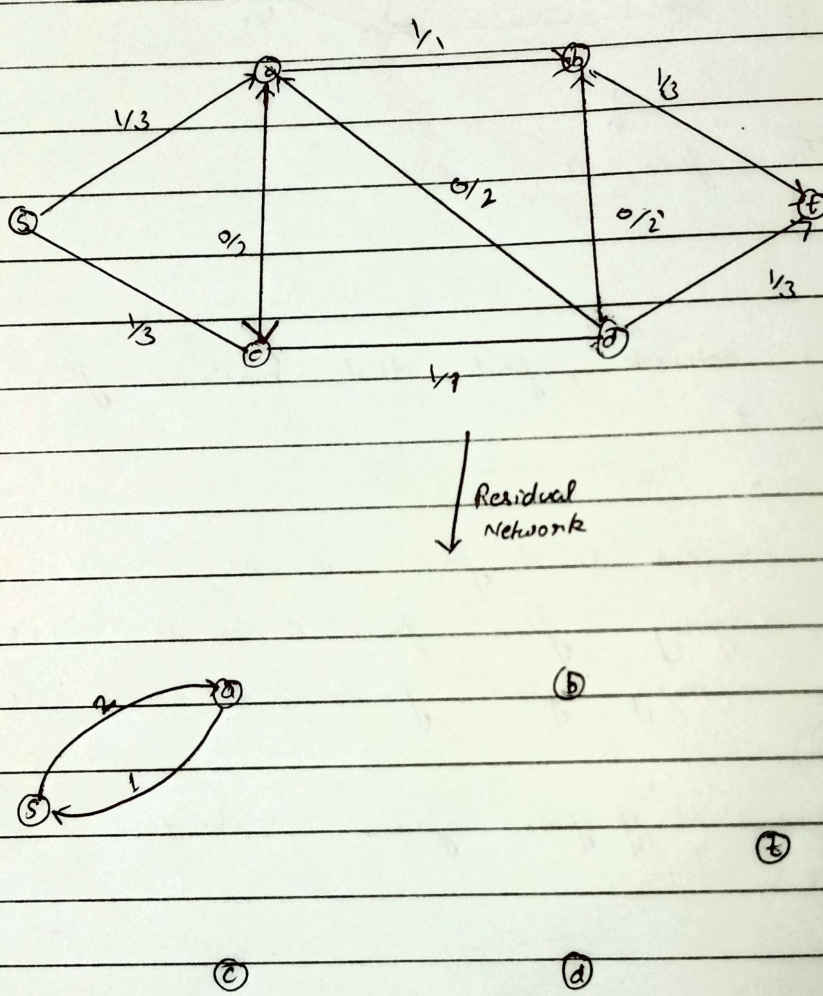
Total amount of flow from s to t = 2 (maximum)

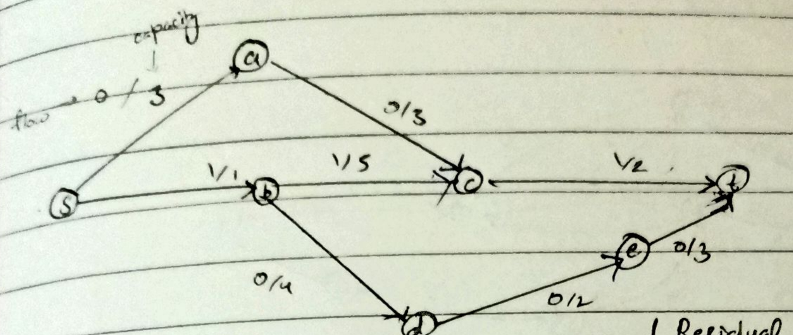
Residual Network & Augmenting Path –

Consider an arbitrary flow f in a network G. The residual network G_f has the same number of vertices as the original network and one or two edges for each edge (u, v) in the original network by using following :–

If $f(u,v) < c(u,v)$, then there is a forward edge (u,v) with flow capacity $c_f = c(u,v) - f(u,v)$

If $f(u,v) > 0$, then there is a backward edge (v,u) with capacity $c_f(v,u) = f(u,v)$.

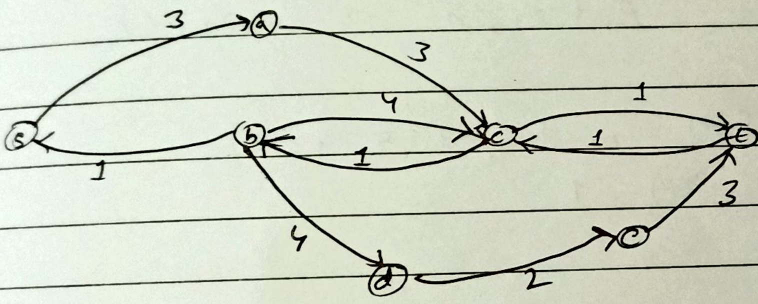




G

|f|=1

↓ Residual Network



→ Augmenting Path — An augmenting path is simply a path in the residual network from source 's' to sink 't', whose purpose is to increase the flow.

In Above,

Let $P = \{sa, ac, ct\}$ be an augmenting path, the path capacity

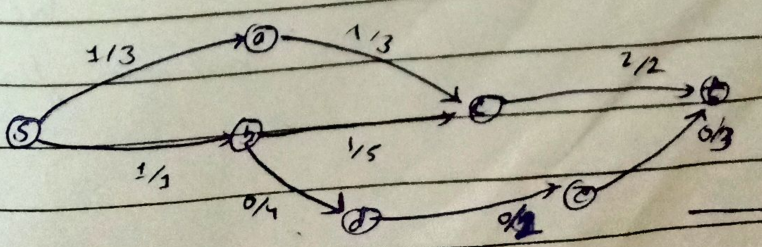
$$\delta(P) = \min\{c(s,a), c(a,c), c(c,t)\}$$

$$= \min(3, 3, 1) = 1$$

$$\delta(P) = 1.$$

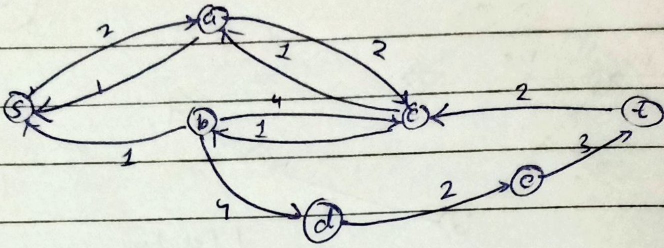
Increase the flow along augmenting path by $\delta(P)$ unit.

|f|=2



Again find → Residual Network

Again Residual Network —

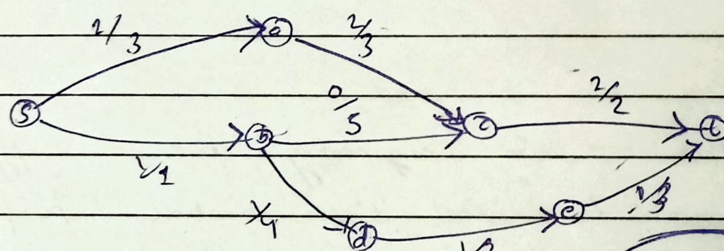


$P = \{ s, a, c, b, d, e, t \}$

Augmenting Path, $P = \{ sa, ac, cb, bd, de, et \}$

$\delta(P) = \min\{2, 2, 1, \cancel{4}, 4, 2, 3\}$

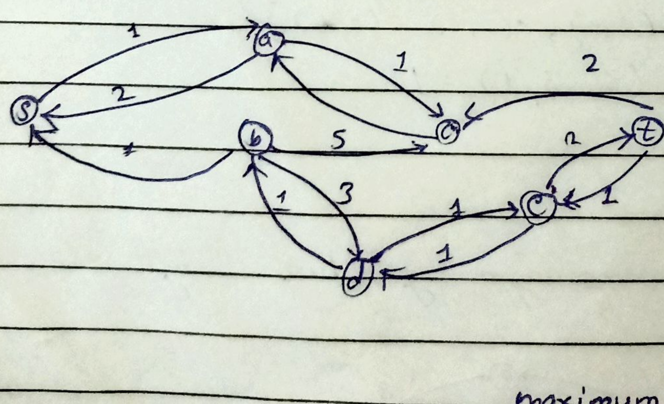
$\delta(P) = 1$



Direction from original Network G

$|f| = 3 = 2+1$

↓ Residual Network



There is no augmenting Path.

We will STOP now.

maximum flow is $|f| = 3$

→ Augmenting Path Algorithm (Ford & Fulkerson Algorithm, 1956)
↳ for maximum flow

[Input: Flow Network G

Output: Maximum Flow]

{Initially $f=0$, arbitrary flow will be taken zero initially

while (G_f contains an augmenting path P)
{ Identify an augmenting path p in G_f;
$\delta(P) = \min \{ c_f(u,v) / (u,v) \}$

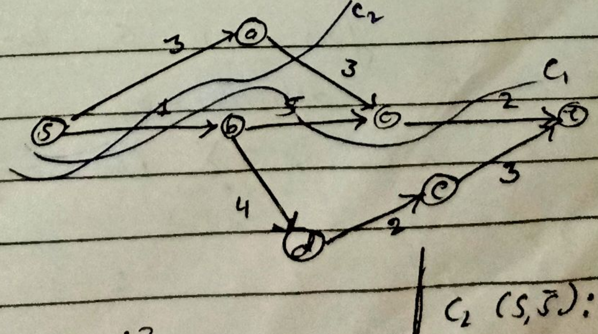
augment $\delta(P)$ unit flow along the augmenting path in G_f

Return (maxflow)
}

$V = S \cup \bar{S}$, $S \cap \bar{S} = \phi$

→ A Cut $(S, \bar{S})$ partitions, the vertex V into two subsets S & $\bar{S}$, $S \subseteq V$ & $\bar{S} = V - S$ & it contains the edges with one end point in S ~~with one end~~ ~~point~~ in S and other in $\bar{S}$.

$V = \{s, a, b, c, d, e, t\}$



Let $S = \{s, a, c\}$
then $\bar{S} = \{b, d, e, t\}$

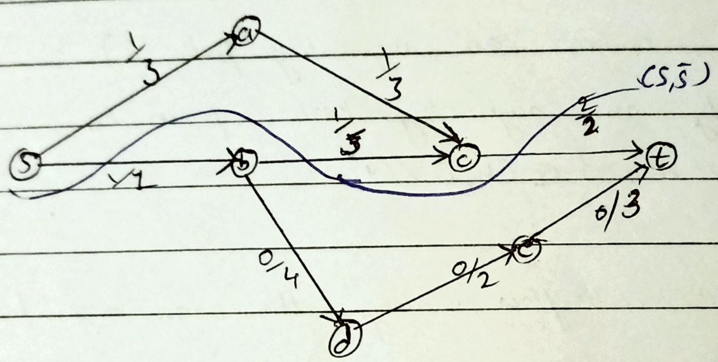
$c_2 \; (S, \bar{S}) := \{(s,b), (a,c)\}$
s-t cut

$c_1 \; (S, \bar{S}) := \{(s,b), (b,c), (c,t)\}$

The cut $(S,\bar{S})$ is an ~~an~~ s-t cut if $s \in S$ ~~& t ∉ c~~ & $t \in \bar{S}$.

The capacity of s-t cut is given by

$$c(S,\bar{S}) = \sum_{\substack{u \in S \\ v \in \bar{S}}} c(u,v)$$



$(S,\bar{S}) = \{(S,b), (b,c), (c,t)\}$

$= c(S,b) + c(c,t)$ → Capacity of Edges start from S & end at $\bar{S}$.

$= 1 + 2 = 3$

$(b,c) \rightarrow \bar{S} \times S \neq S \times \bar{S}$

A min cut is an s-t cut having min capacity.

If f is a flow in G, then the net flow across the cut $(S,\bar{S})$ is defined by

$$f(S,\bar{S}) = \sum_{\substack{e \text{ out} \\ \text{from } S}} f(e) - \sum_{\substack{e \text{ into} \\ S}} f(e) \quad , e \rightarrow \text{edge}$$

$= 1 + 2 - 1 = 2$ → Total flow across the cut $(S,\bar{S})$

Lemma I: Let f be a flow in a flow network G with source S & sink t and let $(S, \bar{S})$ be a ~~cut~~ s-t cut on G, the net flow across $(S, \bar{S})$ is $|f|$.

II: Let f be any flow and ~~$(S, \bar{V}$~~ $(S, \bar{S})$ be a s-t cut, then $|f| \le c(S, \bar{S})$

III. If $|f| = c(S, \bar{S})$, $2<n$, then f is max-flow and $(S, \bar{S})$ is min cut.

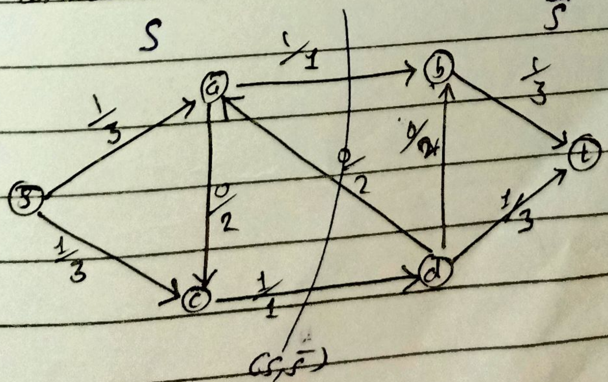
Theorem: If f is a flow in a flow network G with source s & sink t, then the following are equivalent:

I. f is a max flow in G.

II. The residual network have no augmenting path.

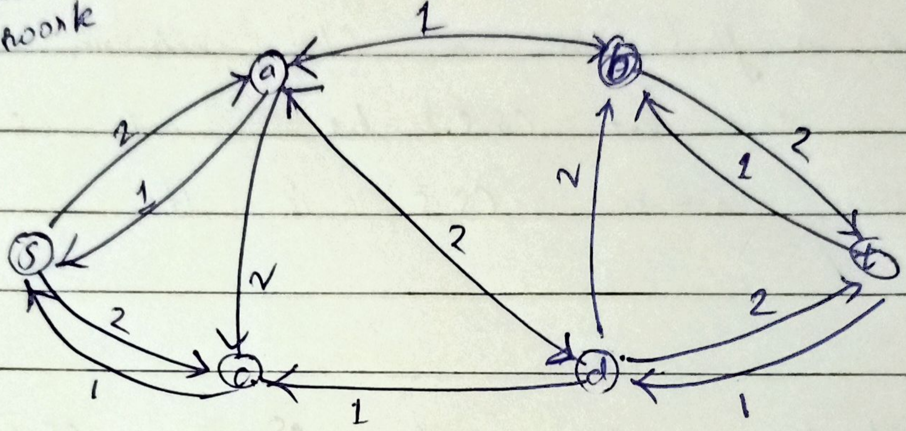
III. $|f| = c(S, \bar{S})$

Q Let us consider the network. Find max-flow of.



Initially $|f| = 2$.

Residual Network



$P = \{ s, c \ldots$ → No augmenting path

∴ $|f| = 2$ is maximum flow.

$(S, \bar{S}) = \{(a,b), (c,d), (d,a)\}$

$C(S, \bar{S}) = c(a,b) + c(c,d)$ → edges capacity which start from S & end at $\bar{S}$

$= 1 + 1$

$= 2$