Maths

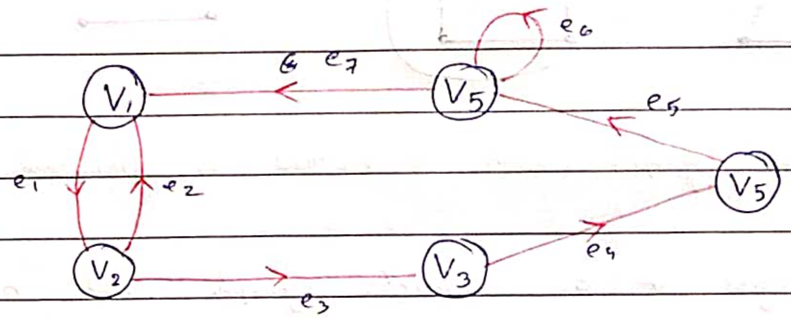
5/3/24

Unit 4 : Graph Theory

Graph : A graph G is a pair (V, E) of sets satisfying $E \subseteq V \times V$ where

E : Set of edges

V : Set of vertex



$V = \{v_1 \; v_2 \; v_3 \; v_4 \; v_5\}$ $\qquad |V| = 5$

$E = \{e_1 \; e_2 \; e_3 \; e_4 \; e_5 \; e_6 \; e_7\}$ $\qquad |E| = 7$

→ If there is an edge joining V_i & V_j then V_i & V_j are called adjacent vertices, otherwise they are non-adjacent.

Here $(v_1 \; v_2)$ are adjacent

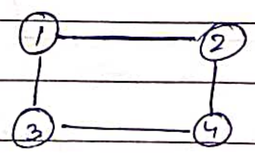
$(v_1 \; v_4)$ are ~~not~~ non-adjacent.

→ Two or more edges with ~~Some~~ Same vertices / end points, are called parallel edges.

Ex:- $(e_1 \; e_2)$ are two parallel edges.

→ A graph is said to be a simple if it has ~~no~~ NO parallel edges & self loops.

Ex:-

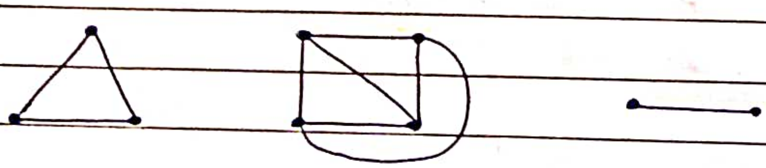


Simple graph.

__/__/__

→ A graph is said to be complete if every pair of ~~vertex~~ vertices are adjacent

Ex:-



→ A complete graph with n-vertices is denoted by K_n.

→ No. of edges in a complete graph with "n" vertices are ${}^nC_2$.

Degree : The degree of a vertex V in a graph G, is denoted by d(V) is the no. of edges on G incident to V.

In previous graph : $d(v_1) = 3$ $d(V_2) = 3$ $d(V_3) = 2$ $d(V_4) = 2$ $d(V_5) = 4$

→ A graph G is said to be k-regular if degree of all the vertices are 'k'.

→ A complete graph with n-vertices is said to be ~~(freely)~~ (n-1)-regular graph.

Q) If a graph G with 'e' edges has vertices $v_1, v_2, \ldots, v_n$ ~~then~~ then

$$\boxed{\sum_{i=1}^{n} d(V_i) = 2 \cdot e}$$

⇒ Since ~~each~~ each edge in a graph contributes twice in a degree of a vertex.

So, $$\boxed{\sum_{i=1}^{n} d(v_i) = 2 \cdot e}$$

(Also called hand-shaking Theorem)

__/__/__

→ ΔG : Maximum degree $= 4$ (Previous Example)

δG : Minimum degree $= 2$

⇒ If $\frac{2 \cdot e}{n}$ is the average degree then $\delta G \leq \frac{2e}{n} \leq \Delta G$

Q> In any graph G, no. of odd-vertices are always even.

⇒ Odd: If degree is odd

Even: If degree is even.

Let $|V| = n$, $|E| = e$

So we know that $\sum_{i=1}^{n} (v_i) = 2 \cdot e$

Let us assume that $V_1\ V_2 \ldots V_t$ are odd vertices, remaining $V_{t+1}, V_{t+2} \ldots V_n$ are even vertices.

$\therefore$

$$d(v_1) + d(v_2) \ldots + d(V_t) = 2 \cdot e - \left[d(V_{t+1}) + d(V_{t+2}) \ldots d(V_n) \right]$$

$$= 2 \cdot e - 2 \cdot m$$

$$d(V_1) + d(V_2) + \ldots d(V_t) = 2k \quad [\because e - m = k]$$

Since $d(V_i)$ are odd for $i=1$ to t.

So their sum is even only if t is even.

Hence, No. of odd-vertices are always even.

__/__/__

Walk : In a graph, it is an alternating sequence of vertices & edges $V_0 e_1 V_1 e_2 V_2 \dots V_{n-1} e_n V_n$, beginning & ending with vertices such that

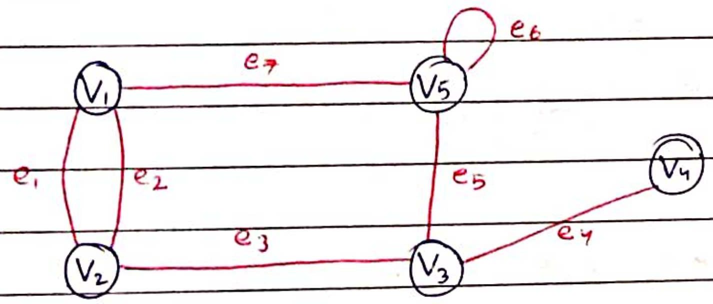
$$e_i = (V_{i-1}, V_i)$$

→ The walk joins V_0 & V_n is denoted by (V_0, V_n) walk.

→ A walk is closed if $V_0 \equiv V_n$

→ It is a trail if the edges are distinct.

→ It is a path if the vertex are distinct. (Edges can NOT be repeated.)



$(V_1 V_5)$ walk : $V_1 e_1 V_2 e_3 V_3 e_4 V_4 e_4 V_3 e_5 V_5$

$(V_1 V_5)$ trail : $V_1 e_1 V_2 e_3 V_3 e_5 V_5$

$(V_1 V_1)$ cycle : $V_1 e V_2 e_3 V_3 e_5 V_5 e_7 V_1$ (Initial & Final vertex is same.)

Connected Graph : A graph G is said to be connected if there is a path between any pair of vertices u & $v \in V(G)$, otherwise it is dis-connected.

Maths

5/3/24

Sub-graph : $V' \subseteq V$ & $E' \subseteq E$, $G' = (V', E')$ is a sub-graph of G.

→ Maximal connected sub-graphs of a graph are called components of graph.

(After removing one edge, if graph be connected / disconnected.

→ A disconnected graph will have atleast two components.

Theorem : Let G be a graph of order n. If $d(u) + d(v) \geq n-1$, for every pair of vertices $u, v \in V$, then G is connected graph.

Proof : We need to prove that every two vertices on G are connected.

Case I : If $(u, v) \in E$, then G is connected

Case II : If $(u, v) \notin E$, then

$$d(u) + d(v) \geq n-1$$

By Piegonhole Principle, '$n+1$' piegons are put into 'n' holes then at least one hole will occupy by two or more piegon.

It implies that there must be a vertex V_i that is adjacent u & v both. So, u & v are connected.

Maths

5/3/24

(1) If G is a graph of order 'n' with $\delta(G) \geq \frac{n-1}{2}$, then P.T G is connected.

$\Rightarrow$ $\delta(G)$ = Min. of all the degree

For every pair of non-adjacent vertices u & v, we know that

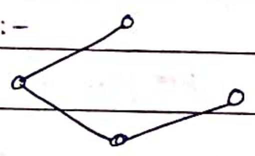
$$d(u) + d(v) \geq \frac{n-1}{2} + \frac{n+1}{2} = n-1$$

So, by previous question, given graph is connected.

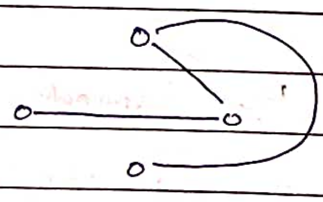
Complement of a graph

It is denoted by $\bar{G}$, which has the same set of vertices on G but two vertices are adjacent in $\bar{G}$ iff they are non-adjacent in G.

Ex: G:-



$\bar{G}$:



Theorem : If G is disconnected graph, then $\bar{G}$ is connected. (& vice versa).

Proof :

Maths

6/3/24

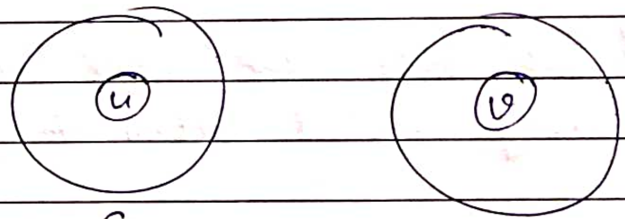
Q) If G is disconnected, then $\bar{G}$ is always ~~&~~ connected.

$\Rightarrow$ Here $\bar{G}$ is a complemented graph of graph G.

Proof : For any pair $(u, v) \in V(G)$, we need to show that there exists a path btw u & v in $\bar{G}$.

Since G is disconnected, so it have at least two components

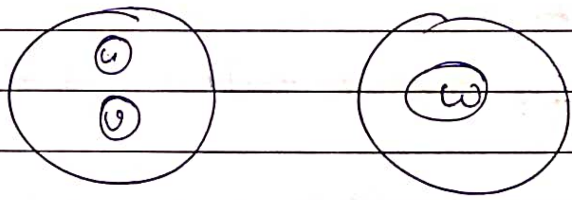
Case I : u & v are in different components of G.



G_1 G_2

So, u & v will be adjacent in $\bar{G}_1$. It means, there exists a path btw u & v in $\bar{G}$.

Case II : u & v are in same components.



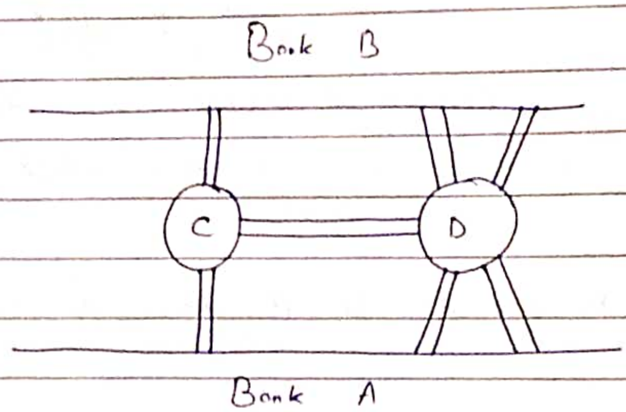
a) u & v are adjacent, in $\bar{G}$, there exists an edge btw u & w, v & w $\Rightarrow$ ~~then~~ there exists a path

$uwv \Rightarrow u$ & v are connected.

b) u & v are not adjacent, in complement of G, both u & v will be adjacent.

_/_/_

Konigsberg Seven Bridge Problem



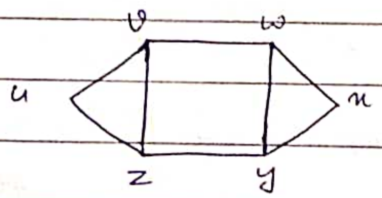
=> Here all vertices are odd vertices.

There is ~~no~~ NO such route that would take them over each bridge exactly once & return to the starting point.

Eulerian Path : A path which contains each edge of the graph exactly once or a trail is called an eulerian path.

Eulerian Circuit or Trail : A close trail is said to be eulerian circuit.

A graph is said to be eulerian graph if it contains at least on eulerian circuit.



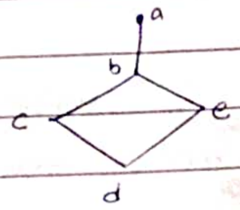
Trail : $xyzuv$ of length 5

$xyzvw$ is a path of length 4

Here, there is no eulerian path.

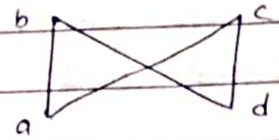
Hence ~~Here~~, no eulerian circuit.

Ex>



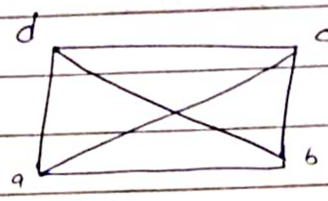
abcdeb → E.P
but not E.C.

Ex>



E.P → bacdb
& it is E.C

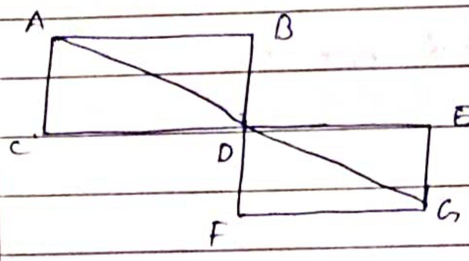
Ex>



b a cd b/a ~~NO EP~~

NO E.P & E.C

Ex>



GDABDEGFDCA : EP
& is not E.C

Results :-

1) A connected graph or multi-graph is an eulerion graph if every vertex is of even degree.

Even $\Longrightarrow$ Eulerion
Not Even $\Longrightarrow$ Not eulerian
↓
at least one even

2) A non-empty connected graph is an eulerion if it has no vertices of odd degree.

3) E.C $\Longrightarrow$ E.P but Not EP $\Longrightarrow$ Not E.C

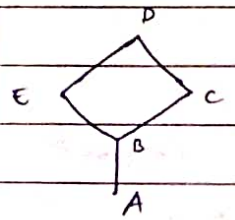
_/_/_

Hamiltonian Graph

Hamiltonian Path: A path is said to be hamiltonian path if it contains all the vertices exactly once.

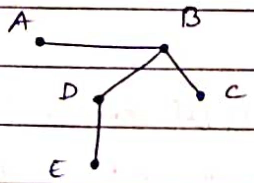
Hamiltonian ckt: If first & last vertices are same, then hamiltonian path is called H.C.

Ex>



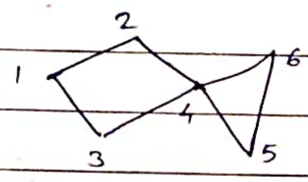
H.P: ABCDE , Not H.C

Ex>



Not HP & Not H.C

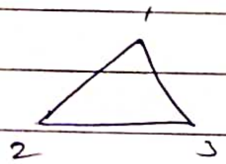
Ex>



H. Path: 2 1 3 4 5 6

Not H.C

Ex>



~~H.P & HC ⟹ 1 2 3 1~~

A graph contains atleast one H.C is said to be a hamiltonian graph.

Note: A complete graph: K_n for $n \geq 3$ are always hamiltonian graph.

__/__/__

Results

1) If G has a hamiltonian circuit then $d(v) \geq 2,\ \forall v \in V(G)$.

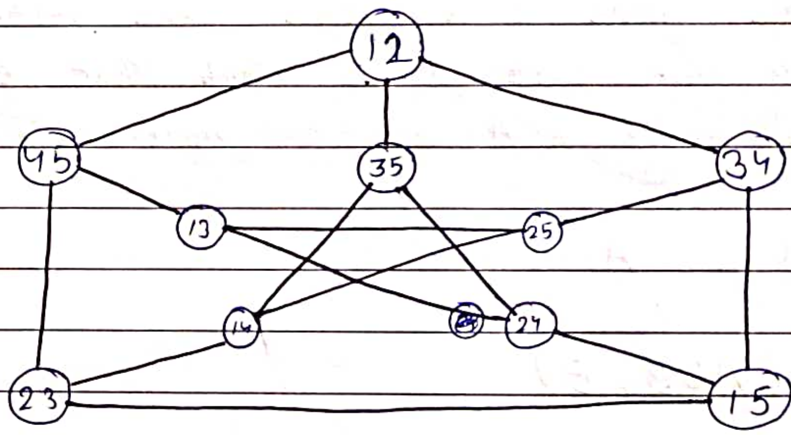
2) If G with 'n' vertices has a H.C, then G must have at least 'n' edges.

3) If G has no loops, & no parallels edges, & if $|V| = n\ (\geq 3)$ & $d(u) \geq n/2,\ \forall u \in V(G)$, then G is hamiltonian.

Peterson Graph

$S = \{1, 2, 3, 4, 5\}$ $|V| = 10$

$V = \{12 \cdot 13\ 14\ 15\ 23\ 24\ 25\ 34\ 35\ 45\}$



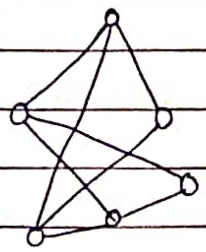
$\Rightarrow$ H.P & Not H.C

Closure of a graph:

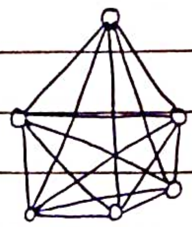
The closure $C(G)$ of a graph G of order n is obtained from G by recursively joining the pair of non-adjacent vertices where degree sum is at-least 'n' until no such pair exists.

__/__/__

Ex>



$\Longrightarrow$



$\underline{\underline{n = 6}}$

K_6

G $\qquad\qquad$ C(G)

G is a hamiltonian iff C(G) is Hamiltonian

Two vertices are adjacent if both are disjoint.

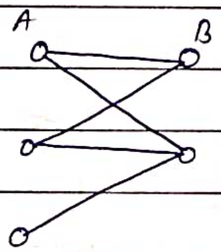
$\boxed{\text{Bipartite Graph}}$

A bipartite graph is a graph whose vertices can be divided into two parts A & B such that every edge connects a vertex in A to a vertex in B.

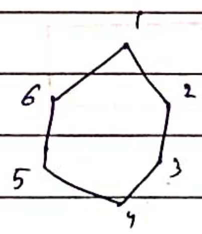
$V = A \cup B \quad , \quad A \cap B = \phi$

$G = (A \cup B, E)$

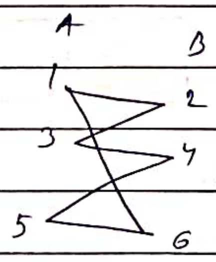
Ex> 1>



2>



$\equiv$



__/__/____

A graph is bipartite, if it has No odd-cycle

$\boxed{\text{Diameter of a graph}}$

$$\text{diam}(G) = \max \text{dis}(u,v), \quad u,v \in V(G)$$

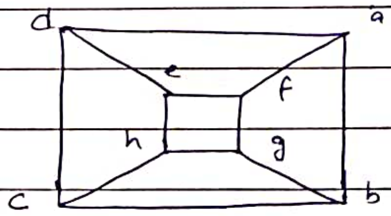
where dis (u, v) is the length of shortest path btw u & v.

For Petersen Graph : $\text{dis}(u,v) = 1$ if u & v are adjacent

$\text{dis}(u,v) = 2$ if u & v are ~~not~~ non-adjacent.

So, $\boxed{\text{diam}(\text{Petersen}) = 2}$

For cube :



$\text{dis}(a,b) = 1$

$\text{dis}(b,f) = 2$

$\text{dis}(a,h) = 3$

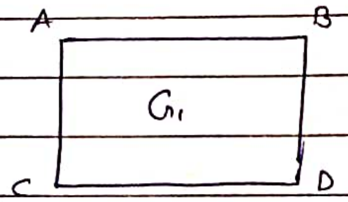
$\text{diam}(\text{cube}) = 3$

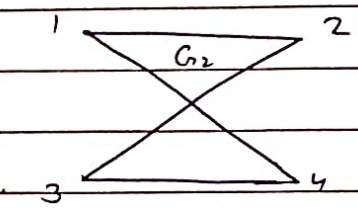
Maths

_/_/_

Unit - 4

Isomorphic Graph





$V = \{A, B, C, D\}$ $\quad V' = \{1, 2, 3, 4\}$

G_1 & G_2 are isomorphic ?

$\Rightarrow$ <u>Isomorphic Graph</u> : Let $G_1 = (V, E)$ & $G_2(V', E)$ be two graphs, then G_1 & G_2 are said to be isomorphic. $G_1 \equiv G_2$, if there exist a bijection map.

$f: V \to V'$ with $(u, v) \in E \Rightarrow (u', v') \in E'$ for all $u, v \in V$ such a map is called isomorphism.

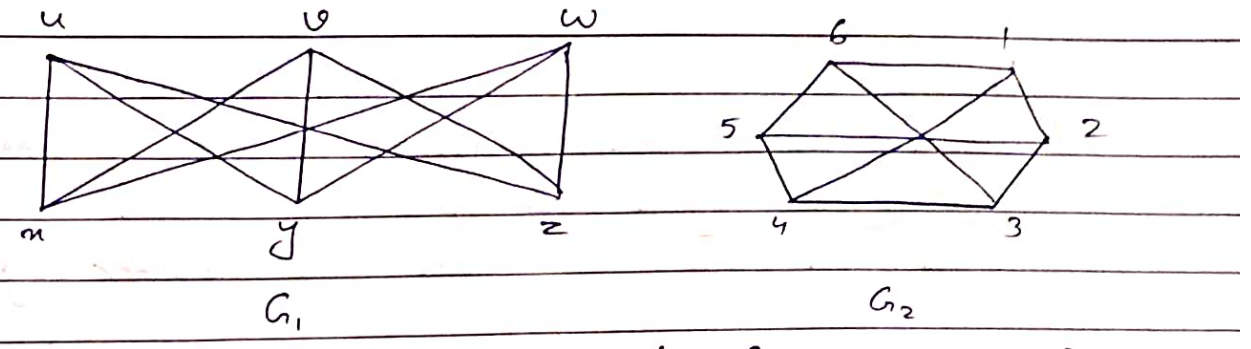
For the above graph, Let $f: V \to V'$

s.t $f(A) = 1$ $\quad f(B) = 2$ $\quad f(C) = 4$ $\quad f(D) = 3$

$(A, B) \in E \Leftrightarrow (1, 2) \in E'$
$(B, D) \in E \Leftrightarrow (2, 3) \in E'$
$(A, C) \in E \Leftrightarrow (1, 4) \in E'$
$(D, C) \in E \Leftrightarrow (3, 4) \in E'$

So, $G_1 \equiv G_2$

Q)



$V = \{x, y, z, u, v, w\}$ $\qquad$ $V' = \{1, 2, 3, 4, 5, 6\}$

=> $f : V \to V'$ such that

$f(u) = 1$	$f(x) = 2$
$f(v) = \cancel{2}\, 3$	$f(y) = 4$
$f(w) = 5$	$f(z) = 6$

$(u, x) \in E \iff (1, 2) \in E'$

$(v, y) \in E \iff (3, 4) \in E'$

$(w, z) \in E \iff (5, 6) \in E'$

$\vdots \qquad \vdots$ $\Big\}$ Similarly remaining edges.

$\therefore \; G_1 \equiv G_2$

Remarks:-

i) If two graphs are isomorphic then, they have same no. of
   1) vertices
   2) edges
   3) Same degree sequence
   4) # Components
   5) Diameter
   6) Length of largest path

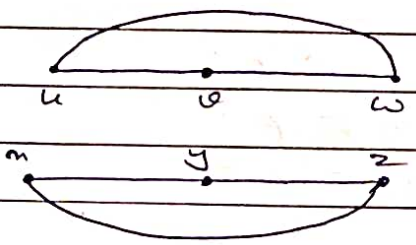
ii) If two graphs are different in any of the above, then they are not isomorphic.

Remarks:-

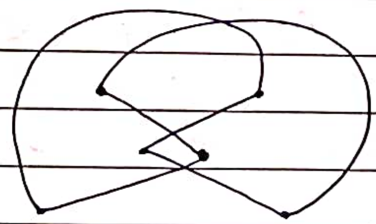
iii) If two graphs are isomorphic & one of them contains a cycle of particular length then, same will also be true for another graph.

Theorem : G & G' are isomorphic iff their complements are isomorphic.

Q) Using above theorem, P.T G_1 & G_3 are not isomorphic.



G_1

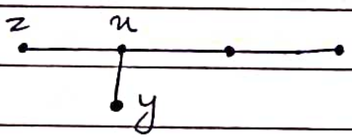


G_3

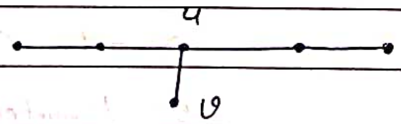
Since $\bar{G_1}$ have two complement but $\bar{G_3}$ have one complement.

$$\therefore G_1 \not\cong G_3$$

Q) G_1:



G_2 :



Check whether it is isomorphic.

__/__/__

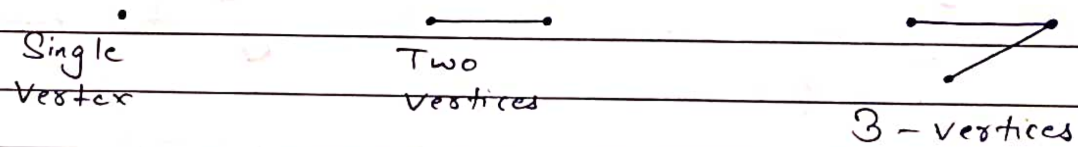
$\Rightarrow$ In G_1, the adjacent vertices of vertex 'x' with degree 3 are z & y where degree are 2.

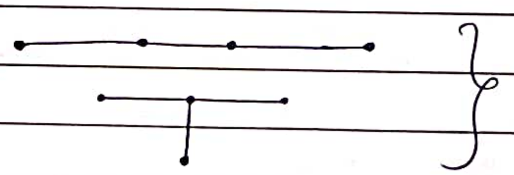
but in G_2, it is NOT possible, the vertex u, where degree is 3, have unique pendent vertex of v.

$$G_1 \neq G_2.$$

$\boxed{\text{Tree}}$

A connected and acyclic graph is called a tree.





Leaf : A vertex of a tree is said to be a leaf if it's degree is 1.

or

A $\underbrace{\text{pendent vertex}}_{\text{single degree}}$ of a tree are called leaves.

then, A tree with $n (\geq 2)$ vertices have at least two vertices.

Maths

Let T be a tree with $n (\geq 2)$ vertices.
Further, Let $p: V_0 V_1 V_2 \dots V_k$ be a largest path btw V_0 & V_n in T.

Since p is largest path, then every neighbour of V_0 lies on p.

Since T is acyclic, so v is only neighbour of V_0 in p.

So $d(V_0) = 1$ Similarly, $d(V_k) = 1$.

Theorem : A graph G is a tree if & only if there is exactly one simple path btw any two pairs of vertices of G.

$\Rightarrow$ Suppose T is a tree, then we need to show there exist unique simple path.

Let us assume that there exist two simple path btw any two vertices, then union of these two simple path creates a cycle, which is the contradiction.

Conversely, Let ~~as~~ btw any two vertices of there exists unique simple path, this implies that G is connected, since for ~~as~~ any cycle, we need two diff. simple path which is not possible as there is only unique simple path.

So, given graph is a cyclic.

Maths

4/4/24

(Assignment: Submit the soln of Mid-Sem papers by 12/4)

Theorem: Every Tree with n-vertices have n-1 edges.

Proof: By induction method

A tree with 1 vertices have 0 edges.

Suppose a tree with k vertices have k-1 edges. (Assumption)

Then for k+1 vertices, we need to prove that it contains k+1-1 edges.

Let T be a tree with k+1 vertices

Since $k+1 \geq 2$, so T have a leaf x

So $T-x$ is a tree with k vertices which have k-1 edges which implies that T have k edges.

By induction, it is proved.

Theorem: A forest with k components and v vertices have v-k edges.

$\Rightarrow$ Let $v_1\ v_2\ v_3 \dots v_k$ be the number of vertices of k components.

Since each components of a forest is a tree.

Total no. of edges $\Rightarrow \sum_{i=1}^{k} (v_i - 1) \Rightarrow \left(\sum_{i=1}^{k} v_i\right) - k$

$\#$ edges $= v - k$ where $v_i - 1$ is the edges in i^{th} component.

__/__/__

Result

A graph with n vertices and n-1 edges need not be a tree.

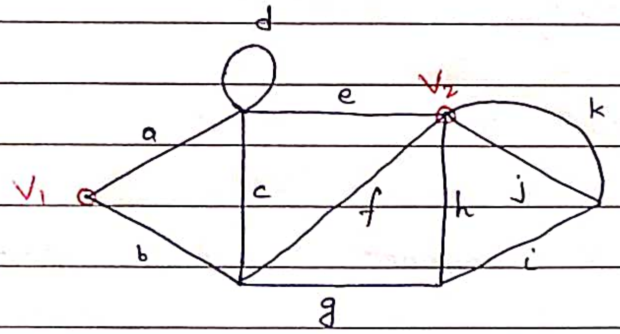
Every connected graph with n vertices & n-1 edges is a tree.

A graph with n vertices, n-1 edges and no circuit is a tree.

A connected graph G is minimally connected if removal of an edge ~~e~~ from G ~~o~~. disconnects the graph G.

Distance & Centers in Tree

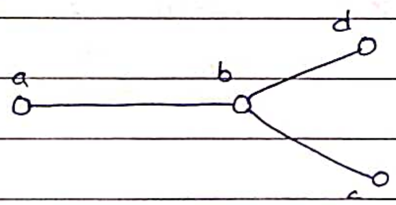
In any connected graph G, the distance $d(v_i\ v_j)$ btw two vertices v_i & v_j is the length of shortest path (i.e., the no. of edges in the shortest path).



$d(v_1, v_2) = 2$

In a tree, there exists unique path btw any pair of vertices.

Ecentricity of a vertex in graph



$d(a, d) = 2$
$d(a, b) = 1$
$d(a, c) = 2$
$d(c, b) = 1$

Ecentricity of a vertex v in a graph G is the distance of v to the vertex farthest from v in G.

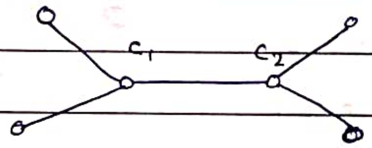
i.e
$$\boxed{E(v) = \max d(v, v_i) \quad , \quad v_i \in G}$$

$E(a) = E(c) = E(d) = 2$ & $E(b) = 1$.

Center of a graph

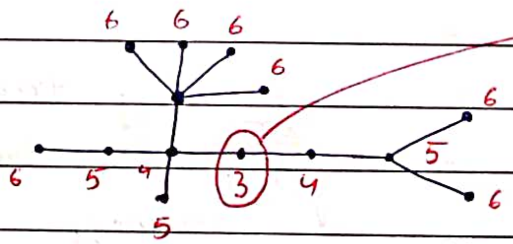
The vertex with minimum ~~ee~~ ecentricity is called center of graph.

Ex:-



$c_1, c_2 \Rightarrow$ Center.

Ex:-



Centre.

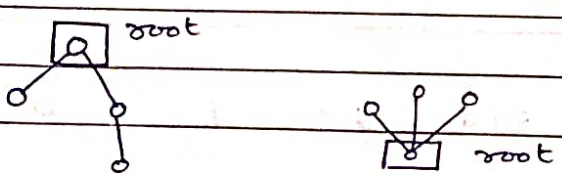
Remark:

i) Every tree have either one or two ~~centres~~ centers.

Rooted & Binary Trees

4/4/24

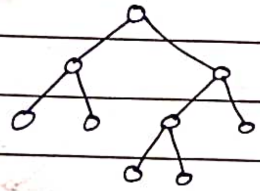
A tree in which one vertex (called root) is distinguished from others is called rooted tree.



Binary Tree

A special class of rooted tree is called binary tree.

A binary tree is defined as a tree in which there is exactly one vertex of degree 2 and remaining vertices have either degree 1 or 3.



Q) The number of vertices "n" in any binary tree is ~~even~~ odd.

$\Rightarrow$ Since there is only one vertex in a binary tree which have degree 2 and remaining vertices have degree ~~value~~ either one or three.

Remaining $(n-1)$ vertices are odd.

Since total no. of odd vertices is always even.

$\Rightarrow$ $n-1$ is even $\Rightarrow$ $n-1 = 2k$

$\Rightarrow$ $n \rightarrow$ odd

Maths

4/4/24

Q) Find the total no. of pendant vertices in a binary tree with n vertices.

⟹ Let p be the number of pendant vertices in a Tree with n vertices.

Total number of edges = n - 1

& no. of odd vertices which are not pendant = n - 1 - p

By handshaking theorem,

$$p + 3(n-p-1) + 2 = 2\cdot(n-1)$$

$$p + 3n - 3p - 3 + 2 = 2n - 2$$

$$-2p + 3n - 1 = 2n - 2$$

$$n + 1 = 2p$$

$$\boxed{p = \frac{n+1}{2}}$$

Internal Vertex in a binary tree

Non-pendant vertex is called internal vertex.

$$n - p = n - \left(\frac{n+1}{2}\right)$$

$$n - p = \frac{n-1}{2}$$

$$\boxed{\begin{aligned} n-p &= \frac{n+1}{2} - 1 \\ n-p &= p-1 \end{aligned}}$$ # internal vertex

Q) What is the total no. of vertices in a binary tree with 20 leaves?

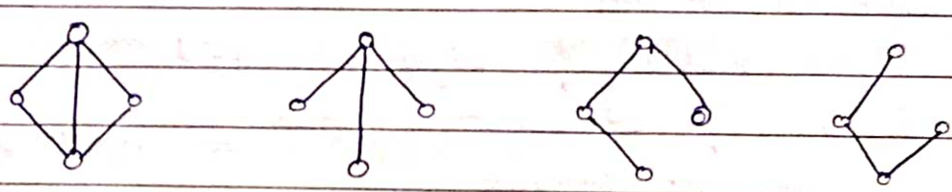
⟹ $p = 20$ then $\frac{n+1}{2} = p = 20$

$$\boxed{n = 39}$$

Spanning Tree of a graph

__/__/__

A spanning tree of a graph G is a subgraph which is a tree that contains all the vertices of a graph G.



G T₁ T₂ T₃

T_1 T_2 T_3 → Spanning tree of G.

Minimum Spanning Tree (MST)
(Weighted graphs)

A minimum spanning tree ~~of~~ if a graph G is a spanning tree of G for which the sum of edges is minimum.

→ A graph may have one or more minimum spanning tree.

Algorithms to find MST

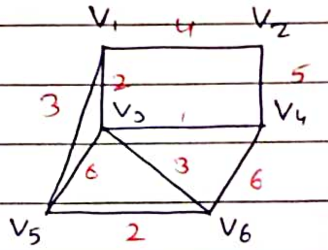
(I) Krushkal's Algorithm (uses edges)
(II) Prim's Algorithm (uses vertices)

4/4/24

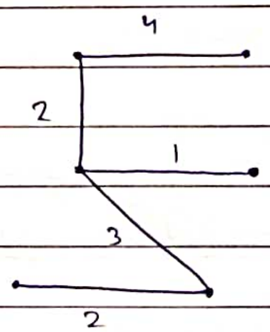
Procedure for Krushkal Algorithm

1) List all the edges in non- ~~des~~ decreasing order
2) Select a smallest edge
3) Select a new ~~way~~ edge of smallest possible order (weight) that does not form a cycle / circuit with previous selected edge.
4) Continue step 3 until $n-1$ edges are selected.

①



⟹



✓ 1 $(V_3\ V_4)$	✓ 4 $(V_1\ V_2)$
✓ 2 $(V_5\ V_6)$	5 $(V_2\ V_4)$
✓ 2 $(V_1\ V_3)$	6 $(V_4\ V_6)$
✓ 3 $(V_3\ V_6)$	6 $(V_3\ V_5)$
✗ 3 $(V_1\ V_5)$	

$\boxed{\text{Ans : } 12}$

~~Procedure for Prim Algorithm~~

~~1) Write a weighted~~ matrix

Procedure for Prim Algorithm

1> Write a weighted matrix
2> Start from vertex V_1 and connected with vertex with the smallest weight in row 1.
3>

Maths

4/4/24

Q) Use Prim algorithm to solve previous question.

=>

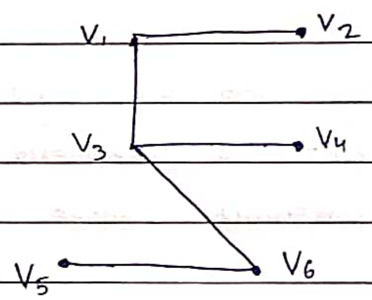
	1	2	3	4	5	6
1	—	4	$\boxed{2}$	∞	$\boxed{3}$	∞
2	4	—	∞	5	∞	∞
3	2	∞	—	$\boxed{1}$	6	3
4	∞	5	1	—	∞	6
5	3	∞	6	∞	—	$\boxed{2}$
6	∞	∞	3	6	2	—

(1, 3) × 4 => (1,4) (4,1) (3,4) (4,3)

(1, 3, 4) × 5 => (1,5) (5,1) (3,5) (5,3) (4,5) (5,4).

(1, 3, 4, 5) × 6 =>

(1, 3, 4, 5, 6) × 2 =>



$\boxed{\text{Ans} = 12}$

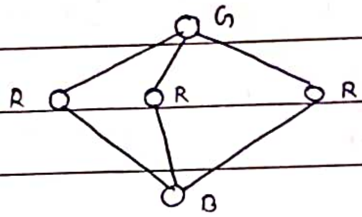
For full matrix, open only row.
For upper triangular matrix, open both row & columns.

Maths

Graph Colouring

8/4/24

→ If G is a graph without loops, then G is said to be k-colourable if its vertices can be coloured with k colours, so that the adjacent vertices have different colours.



3-colourable, $\chi(G) = 3$

Remarks

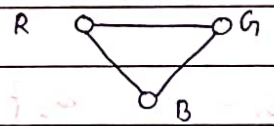
1) If G is k-colourable but not $(k-1)$-colourable, we say that the chromatic number of G is k.

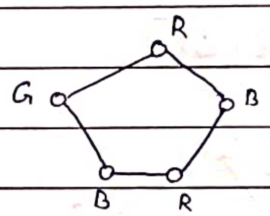
$$\boxed{\chi(G) = k}$$

2) $\chi(G) = 1$ iff G is null graph or the graph has no edges.

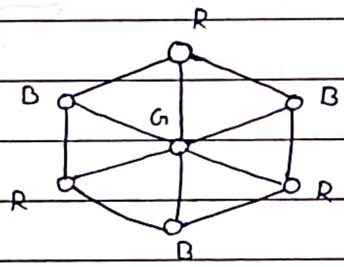
3) $\chi(G) = 2$ iff G is non-null bipartite graph.

Ex) For $\chi(G) = 3$:-





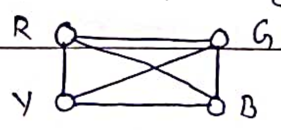
Ex) Wheels with odd number of vertices



4) If H is a subgraph of G, then $\chi(H) \leq \chi(G)$

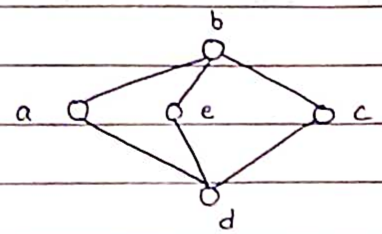
5) $\chi(G) = \max\{\chi(C) ; C \text{ is a component of } G\}$ where G is disconnected.

6) K_n is a ~~Compete~~ complete graph with n chromatic number.



__/__/____

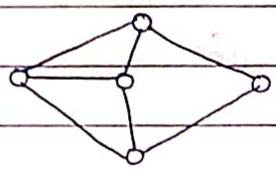
Clique : A clique is a subset of vertices of a undirected graph such that every two distinct vertices in the clique are ~~distinct~~ adjacent.



$C(G) = \{a, b, d\} = \{e, b, d\} = \{b, d, e\}$

Size of max clique $= \omega(G) = 3$

For



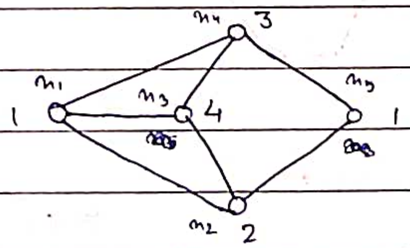
Size of max clique $= \omega(G) = 4$

$$\boxed{\text{Lower bounds for } \chi(G)}$$

i) $\chi(G) \geq \omega(G)$

ii) $\chi(G) \geq \dfrac{|V(G)|}{\alpha(G)}$ where $\alpha(G)$ is size of max. independent set (S_i)

Ex:-



$S_1 = \{x_1, x_5\}$

$S_2 = \{x_2\}$

$S_3 = \{x_4\}$

$S_4 = \{x_3\}$

independent set $\Rightarrow$ A collection of ~~edge~~ vertices which are not adjacent.

Here $\alpha(G) = 2 = |S_1|$

So, $\alpha(G) \geq |S_i|$

Proof of 2nd statement :-

Let G be coloured with colours $1, 2, 3, \ldots \chi(G)$

Let S_i = set of vertices coloured with i = colour class.

then $|S_i| \leq \alpha(G)$, where $\alpha(G)$ is the size of max. independent set

$$\sum_{i=1}^{\chi(G)} |S_i| \leq \sum_{i=1}^{n} \alpha(G)$$

$$|V(G)| \leq \chi(G)\, \alpha(G)$$

$$\Rightarrow \boxed{\chi(G) \geq \frac{|V(G)|}{\alpha(G)}}$$ Hence proved.

Q) $\underbrace{K_1 \quad K_1 \quad K_1 \quad K_1}$ $\qquad K_k$ → Complete graph with k-vertices

We have n-1 complete graphs with 1 vertex.

$|V(G)| = n - 1 + k$ Find $\alpha(G)$.

$$\Rightarrow \chi(G) \geq \frac{|V(G)|}{\alpha(G)} = \frac{n-1+k}{n}$$ ~~(for large k)~~.

$$= 1 + \frac{k-1}{n}$$

So, $\frac{n-1+k}{n} > 1$

$\chi(G) = \max \{ \chi(C) \mid C \text{ is component of } G \}$

$$\boxed{\chi(G) = k}$$

~~Upper boun~~

Chromatic number & Max degree

$\Delta(G) =$ Max. degree & $\delta(G) =$ Min. degree

Greedy Algorithm

The colours : 1, 2, 3, 4, ...

The greedy colouring relative to the vertex ordering $v_1 < v_2 < v_3 < \ldots < v_n$ of

$V(G)$ is obtained by the colouring of vertices in the order $v_1\ v_2\ v_3 \ldots v_n$

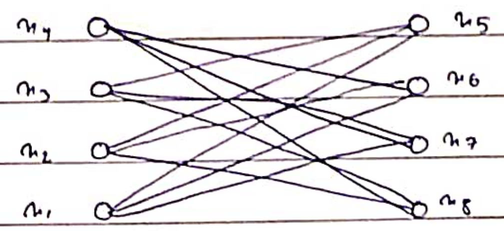
Assign to v_i, the smallest indexed colour not already used for its neighbours among $v_1, v_2, \ldots, v_{i-1}$.

In a vertex ordering, each vertex has at most $\Delta(G)$ earlier neighbours. So greedy algo can not force to use more than "$\Delta(G) + 1$" colours

~~So, $\chi(G) \le \Delta(G)$~~

So, $\chi(G) \le \Delta(G) + 1$ $\boxed{\text{Upper bounds for } \chi(G)}$

Ex:-



1	1	1	1	2	2	2	2
x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8

Here $\chi(G) = 2$

x_1	x_8	x_2	x_7	x_3	x_6	x_4	x_5
1	1	2	2	3	3	4	4

Maths

8/4/24

→ **Propositions**

i) $\chi(G) \leq \Delta(G) + 1$.

→ **Theorem** : (Brooks, 1941)

If G contains a vertex of a degree $d(v) < \Delta(G)$ then $\chi(G) \leq \Delta(G)$

Note: This theorem is not valid for complete graph & a cycle of length $2n+1$.

Ex:- $\chi(K_n) = n$, $\Delta(K_n) = n-1$ (Invalid)

$\chi(C_{2n+1}) = 3$ $\Delta(C_{2n+1}) = 2$ (Invalid)

→ **Theorem** : (Brooks, 1941)

If G is not K_n or C_{2n+1} for some n, then

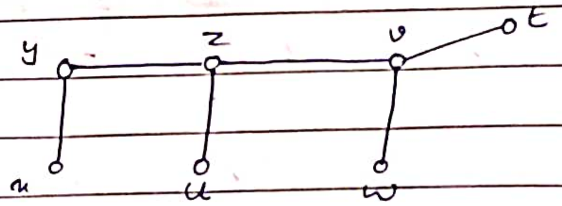
$$\chi(G) \leq \Delta(G)$$

Maths

9/4/24

Matching in a graph

A matching M in a graph G is a set of edges such that every vertex of G is incident to at most one edge in M.



$M_1 = \{(x, y)\}$

$M_2 = \{(x, y), (z, v)\}$

$M_3 = \{(y, z)(v, w)\}$

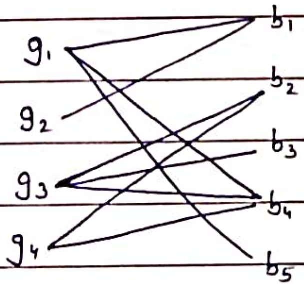
$M_4 = \{(x, y)(u, z)(w, v)\}$

$M_5 = \{(x, y)(u, z)(v, t)\}$

M_4 & M_5 are maximum matching.

Ex> Bipartile graph

There are four girls g_1 g_2 g_3 g_4 & 5 boys b_1 b_2 b_3 b_4 b_5.



$M_1 = \{(g_1 b_4)(g_2 b_1)(g_3 b_2)\}$

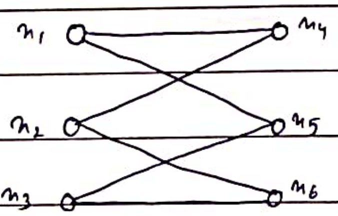
$M_2 = \{(g_1 b_5)(g_2 b_1)(g_3 b_2)(g_4 b_4)\}$

$|M_1| = 3$ $|M_2| = 4$ (No perfect matching)

1> The size of matching is the no. of edges is in that matching.

2> A matching is said to be maximum when it has the longest possible size.

3> A perfect matching in a graph is a matching that contains all the vertices of G.



$M = \{(x_1 x_4)(x_2 x_6)(x_3 x_5)\}$

Perfect & Maximum Matching

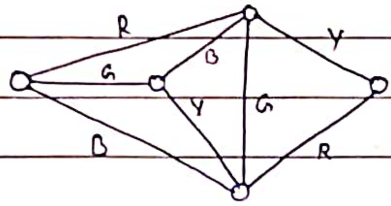
4> If perfect matching does not exist, then we try to find maximum matching.

Perfect matching $\Longrightarrow$ Maximum matching

Converse not true.

Edge - Colouring

A graph $G = (V, E)$ is k-edge colourable if its edges can be coloured with k colours so that no two adjacent edges have same colour.



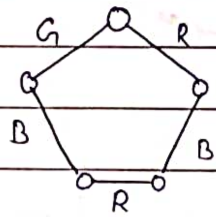
$$\chi'(G) = 4$$

Remark :

1) If G is k-edge colourable but not $(k-1)$-edge-colourable, then we say that the chromatic index of G is k, i.e

$$\boxed{\chi'(G) = k}$$

2) Since edges sharing an end vertex need ~~dif~~ different colours, so $\boxed{\cancel{\chi'(G) \leq \Delta(G)}}$ $\boxed{\chi'(G) \geq \Delta(G)}$



$\Delta = 2$ ~~(Exceptional example)~~.

$\chi'(G) = 3$

3) Theorem : (Vizing, Gupta, 1964)

If G is a simple graph, then $\boxed{\chi'(G) \leq \Delta(G) + 1}$

So, $\boxed{\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1}$

Therefore, we can define two classes :-

class I : If $\chi'(G) = \Delta(G)$

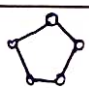
class II : If $\chi'(G) = \Delta(G) + 1$

Theorem : Let G_n = set of graphs of order n.

G'_n = set of graphs of order n and of class - I.

then $\lim \frac{|G'_n|}{|G_n|} = 1 \Rightarrow |G'_n| \equiv |G_n|$

$\Rightarrow$ Almost every graph is in the class - I.



$\Delta = 2$ $\chi'(G) = 3$

__/__/____

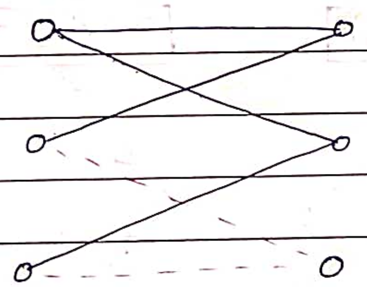
Regular graphs of odd order are class – II graph.

Ex :– Bipartile graph is of class – I.

Theorem : (Konig's 1916)

If G is bipartile, then $\chi'(G) = \Delta(G)$.

Proof : G can be embedded into Δ – regular bipartile graph by adding dummy vertices & edges.



→ 2 – regular bipartile graph.

We show that every Δ – regular bipartile graph is Δ – edge – colourable.

Since G is Δ – regular bipartile graph, then it will have a perfect matching.

Colour all the edges in the perfect matching with some colours.
Remove all the edges obtained in the perfect.
Repeat this process Δ times, until we don't get ~~not~~ null graph.

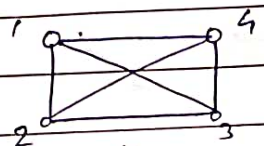
$$\therefore \chi'(G) = \Delta(G).$$

Maths

Planor Graph

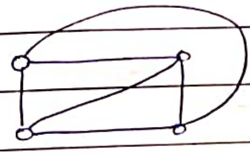
9/4/24

It is a graph that can be drawn on the plane so that its ed
meet only at vertices.



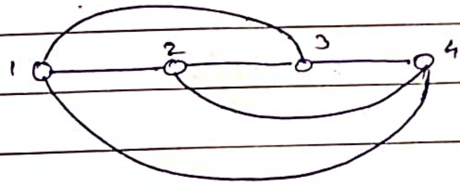
K_4

$\Longrightarrow$ 1)

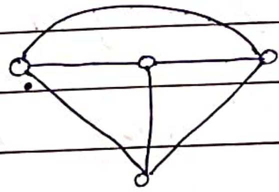


It is a planor graph

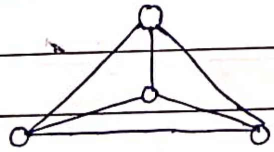
2)



4)

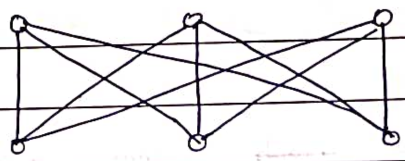


3)



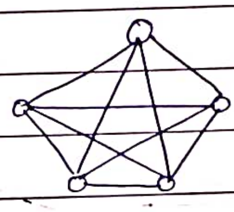
Ex:- Two graphs which are not plannor.

1)

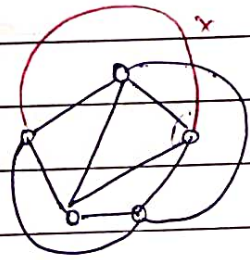


Not plannor

2)

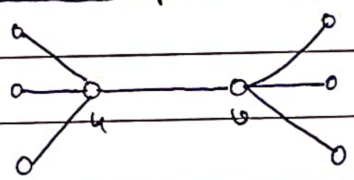


Not plannor.

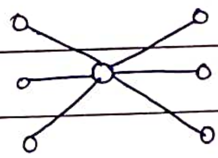


Two Operations :-

I) Contraction ~~Deletion~~ of an edge



$\xrightarrow[(u,v)]{\text{Contract } \cancel{\text{Delete}}}$

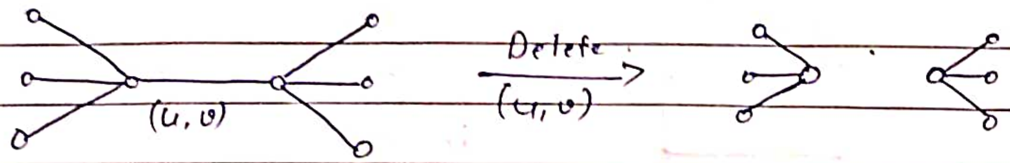


V = 8   E = 7   F = 1

V = 7   E = 6

Both V, E decrease by 1.

II) Deletion ~~Contraction~~ of a edge

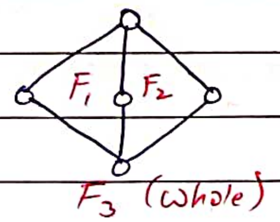


$V = 8$, $E = 7$

$F = 1$

$V' = 8$ $E = 6$

Definition : Let G be a planar graph & consider the regions bounded by the edges of G. These regions are called faces.



$\#$ faces $= 3$

Theorem : If a connected ~~graph~~ planar graph G has exactly "v" vertices "e" edges & "f" faces then

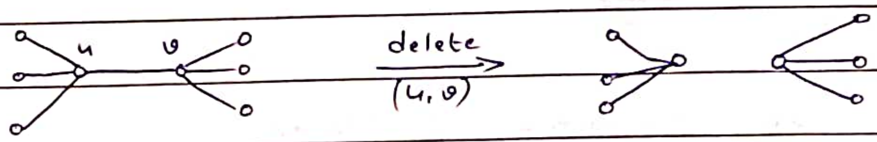
$$\boxed{v - e + f = 2}$$

Maths

15/4/24

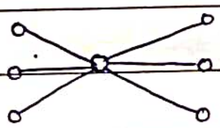
Unit V: Planar Graphs

Deletion of an edge & Contraction of an edge



$(v=8,\ e=6,\ f=1)$

contract $\downarrow (u,v)$



$(v=7,\ e=6,\ f=1)$

Euler Theorem : If G be a connected planar graph with 'v' vertices 'e' edges & 'f' faces then

$$\boxed{v-e+f=2}$$

In general : $v-e+f = \#\text{ component} + 1$

(where # comp = No. of sub part of the graph)

From above ~~del.~~ deleted edge graph

$\Rightarrow v-e+f = \#C + 1$

$\Rightarrow 8-6+f = 2+1$

$$\boxed{f=1}$$

Proof : If $v=1$

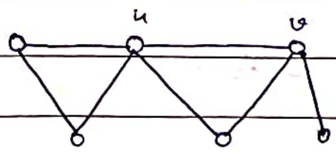
If $e=0$, there will be only one face, formula hold

$e=1$, there will be two faces, formula hold

$e=2$, " " " three faces, formula hold

So, each added loop passes through a face & cuts it into two faces.

Let $v>1$



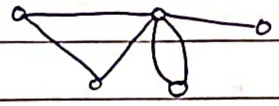
G

When we contract an edge, we obtain a planar graph with 'v' vertices & 'e' edges, 'f' faces.

__/__/__

Now we observe that the contraction does not change the no. of faces.

So by contraction of (u, v).



G'

Further, in the contraction, the no. of vertices & edges are reduced by 1.

So $v' = v - 1$ & $e' = e - 1$

$$\therefore \quad v - e + f = (v'+1) - (e'+1) + f'$$

$$v - e + f = v' - e' + f'$$

$$2 = v' - e' + f' \quad (\because \text{Given: } v - e + f = 2)$$

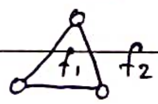
(Hence proved)

So, it is valid for G' (G' is contracted graph).

Degree of a face

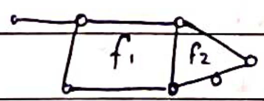
If f is any face then the degree of f is the no. of edges encountered in a walk around the boundary of face.

Ex:-



$d(f_1) = 3$ $\quad$ $d(f_2) = 3$

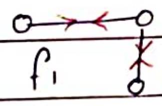
Ex:-



$d(f_1) = 4$ $\quad$ $d(f_2) = 4$

$d(f_3) = 8$

The initial & final position must be same in a walk. You can repeat the edges.

Ex:-  $d(f_1) = 4$

Second handshaking Theorem

Sum of faces degrees is equal to ~~2×e~~ $2 \cdot e$

Lemma (I)

Let G be a connected plannar & simple graph with the no. of vertices greater than 3 $(v \geq 3)$ and 'e' edges then $e \leq 3v - 6$ (It is necessary condⁿ but not sufficient condⁿ to check whether a graph is plannar or not)

Proof: Since a graph G is connected & simple, then degree of each face will be greater than or equal to 3.

$$d(f) \geq 3, \ \forall f$$

By 2nd handshaking theorem,

$$2e = \sum d(f) \geq 3f$$ where f is no. of faces.

$\Rightarrow \ 2e \geq 3f$

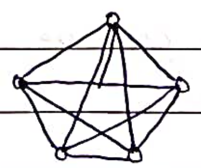
$\Rightarrow \ 2e \geq 3(2 + e - v)$

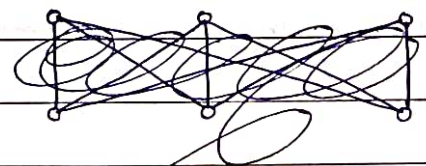
$2e \geq 6 + 3e - 3v$

$3v - 6 \geq e \implies e \leq 3v - 6$ (Hence proved)

Q) Prove or disprove that K_5 & ~~$K_{3,3}$~~ are ~~not~~ planar.

$\Rightarrow$





$V = 5$

$e = 10$

$3v - 6 = 15 - 6 = 9$

$e \nleq 3v - 6$ So, K_5 is not plannar.

~~$V = 6$ $e = 9$~~

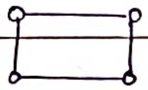
~~$3v - 6 = 18 - 6 = 12$~~

~~Here $e \leq 3v - 6$~~

—/—/—

Lemma (II)

Let G be a connected plannar simple graph with 'v' vertices, 'e' edges & no triangle then $e \leq 2v-4$
where $v \neq 0, 1, 2$ $(v \geq 3)$

Ex:- 

Since G is connected & triangle free graph, then
$d(f) \geq 4$

Q) ~~Prove $K_{3,3}$ is not plannar~~

By handshaking theorem

$2e = \sum d(f) \geq 4.f$

$2e \geq 4f$

$2e \geq 4(2-v+e)$

$2e \geq 8-4v+4e$

$\boxed{e \leq 2v-4}$

Q) Prove $K_{3,3}$ is not plannar.

→ $v=6$, $e=9$,

$2v-4 = 12-4 = 8$ $\therefore$ $e \nleq 2v-4$

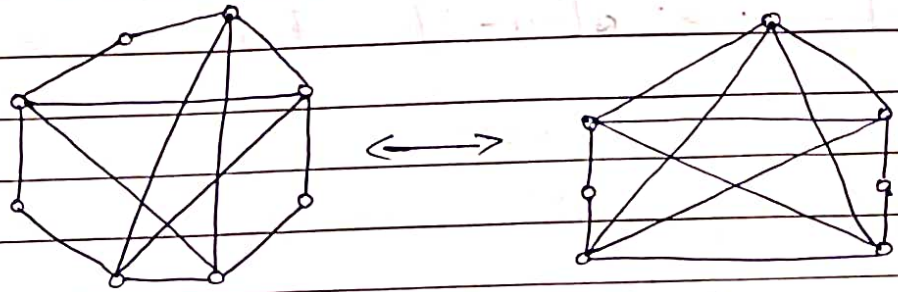
So, it is not plannar.

Defn :- Two graph are homomorphic if one can be obtained from other graph by edge – subdivision.

(Subdivision → Addition of vertices & edges such that the graph configuration & structure do not get changed)

Maths

15/4/24



K_5

Kuratowski Theorem (1930)

A finite graph G is plannar iff it has no sub graph which is homomorphic to K_5 or $K_{3,3}$.

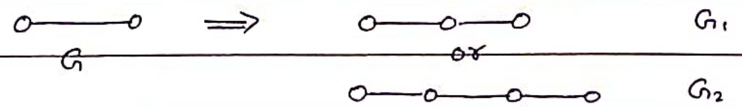
Ex> Peterson graph is non-plannor.

Maths

16/4/24

Edge - Subdivision

Introduce one or more vertices on some edges.



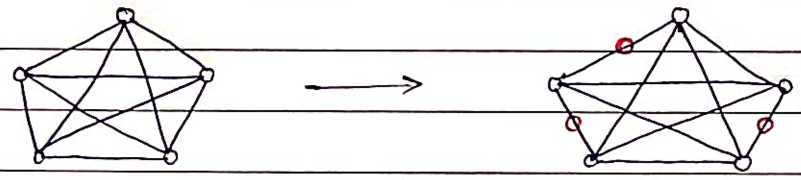
Homeographic graphs

Two graph are said to be homeographic if one ~~or~~ can be obtained from other by edge - subdivision.

G & G1
G & G2 } Homeographic.

Kuratowski's Theorem

A finite graph G is plannar iff it has no subgraph which is homeographic to K_5 or $K_{3,3}$.



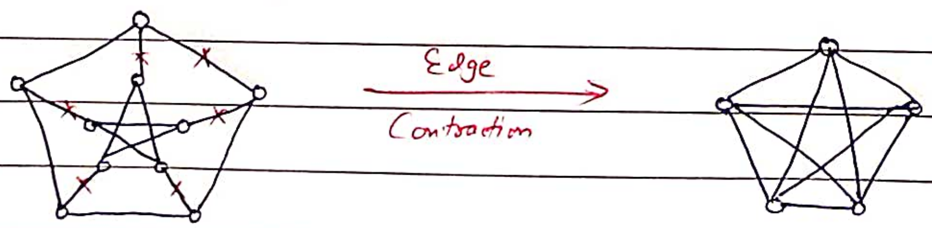
K_5 G

G has a subgraph which is homeographic to K_5.

Wagner Theorem (1937)

A finite graph G is plannar iff it contains no subgraphs that is edge - contraction to K_5 or $K_{3,3}$.

Ex :- Peterson graph is ~~non-plannar~~. non-planar.



K_5

__/__/__

Lemma (III)

Every planar graph has a vertex of degree ≤ 5.

Proof Suppose, every vertex of G have degree ≥ 6

By 1st handshaking theorem

$$2e = \sum_{v_i \in V} d(v_i) \geq 6v$$

where v is total no. of vertex.

So, $e \geq 3v$

But if G is planar graph then $e \leq 3v - 6$

which is the contradiction.

So, every planar graph has a vertex of degree ≤ 5.

k- degenerate graph

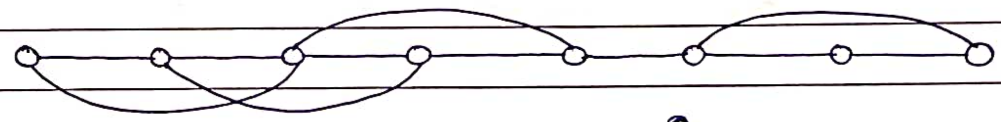
A graph is k- degenerate if each of its subgraph has a vertex of degree at most k.

$\rightarrow$ A graph is k- degenerate iff there is an ordering $v_1 < v_2 < v_3 \ldots\ldots < v_n$ of the vertices such that

$\rightarrow$ It is just a way to write, it is not "greater than" sign. Here the vertices are their right side only.

each vertex v_i has atmost k - neighbours to its left.

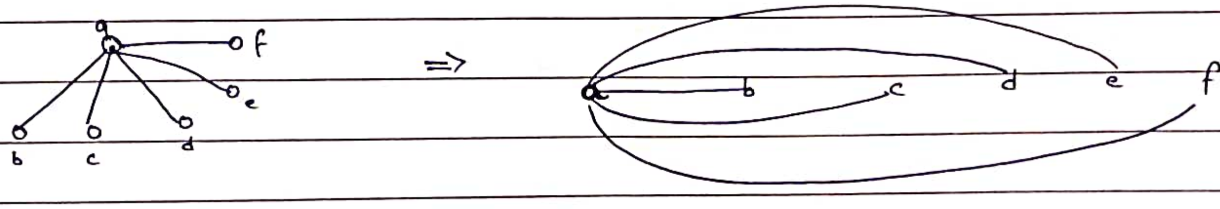
(i)



2

$\Rightarrow$ It is 2- degenerate graph.

(ii)



1 - degenerate graph.

_/_/_

Result: The degeneracies of G, $\delta^*(G)$ is the smallest k such that G is k-degenerate.

Proposition: $\chi(G) \leq \delta^*(G) + 1$

Theorem: Every planar graph is 6-colourable.

Proof: Let G be a planar graph. Every ~~planar graph~~ sub graph of G has a vertex of degree ≤ 5.

Hence G is 5-degenerate. So by above proposition.

$$\chi(G) \leq \delta^*(G) + 1 = 5+1$$

$$\Rightarrow \chi(G) \leq 6$$

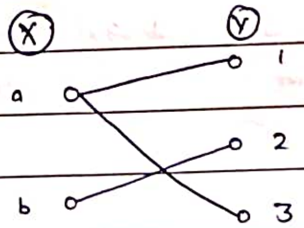
Maths

Unit VI : Optimization of graphs

16/4/24

Matching & Network - Flow

Matching : A matching M in a graph G is the set of edges such that every vertex of G is incident to at most one edge in M.



$M_0 = \{(a,3)\}$ ✓

$M = \phi$ (Matching can be null) ✓

$M_1 = \{(a,3)\ (a,1)\}$ ✗

$M_2 = \{(a,3)\ (b,2)\}$ ✓

→ Size of a matching is the no. of edges in M. Ex:- $|M_2| = 2$

→ Matching M is said to be maximum if it has largest possible size.

→ X - saturates Matching : A matching containing all the vertices of X.

Here M_2 is X - saturates matching.

→ Y - Saturates Matching : A matching containing all the vertices of Y

Here No Y - saturates matching.

→ Perfect Matching : A matching contains all the vertices of 'G'.

Alternating Path

An alternating path is a path that alternate between matched and unmatched edges.