Surfaces of constant principal–curvatures ratio in isotropic geometry

Khusrav Yorov, Mikhail Skopenkov, Helmut Pottmann

Here we check auxiliary computations in the proofs.

1. Checking for Theorem 21

Isotropic Gaussian Curvature

```
K[u_] := (u^3 * f''[u] * f'[u] - 1) / u^4
```

```
K[u]
```

$$\frac{-1 + u^3 f'[u]\, f''[u]}{u^4}$$

Isotropic Mean Curvature

```
H[u_] := (f'[u] + u * f''[u]) / (2 * u)
```

```
H[u]
```

$$\frac{f'[u] + u\, f''[u]}{2 u}$$

The equation H ^ 2 / K – (a + 1) ^ 2 / (4 a) = 0

```
eq := H[u]^2 / K[u] - (a + 1)^2 / (4 * a)
```

```
eq
```

$$-\frac{(1+a)^2}{4 a} + \frac{u^2 \left(f'[u] + u\, f''[u]\right)^2}{4 \left(-1 + u^3 f'[u]\, f''[u]\right)}$$

1.1. Checking that equation (13) and (14) are equivalent

In[∘]:= eq1 :=
((a - 1) * ((u^2 * f''[u] + u * f'[u]) / (2 * (a + 1))))^2 - ((u^2 * f''[u] - u * f'[u]) / 2)^2 - 1

In[∘]:= eq1

Out[∘]= $-1-\frac{1}{4}\left(-u f'[u]+u^2 f''[u]\right)^2+\frac{(-1+a)^2\left(u f'[u]+u^2 f''[u]\right)^2}{4(1+a)^2}$

In[∘]:= FullSimplify[eq / eq1]

Out[∘]= $\frac{1+2a+a^2}{4a-4a u^3 f'[u] f''[u]}$

1.2. Checking equations (15) and (16)

In[∘]:= $\text{FullSimplify}\left[\text{Solve}\left[\text{Csc}[2*s[u]] == \frac{(-1+a)\left(u f'[u]+u^2 f''[u]\right)}{2*(1+a)}\ \&\&\ \text{Cot}[2*s[u]] == \frac{1}{2}\left(u f'[u]-u^2 f''[u]\right),\ \{f'[u], f''[u]\}\right]\right]$

Out[∘]= $\left\{\left\{f'[u]\to\frac{a\,\text{Cot}[s[u]]+\text{Tan}[s[u]]}{(-1+a)\,u},\ f''[u]\to\frac{\text{Cot}[s[u]]+a\,\text{Tan}[s[u]]}{(-1+a)\,u^2}\right\}\right\}$

1.3. Checking equations (17)

In[∘]:= $\text{FullSimplify}\left[\text{Solve}\left[\partial_u\left(\frac{a\,\text{Cot}[s[u]]+\text{Tan}[s[u]]}{(-1+a)\,u}\right) == \frac{\text{Cot}[s[u]]+a\,\text{Tan}[s[u]]}{(-1+a)\,u^2},\ \{s'[u]\}\right]\right]$

Out[∘]= $\left\{\left\{s'[u]\to\frac{2(1+a)\,\text{Csc}[2s[u]]}{u\left(-a\,\text{Csc}[s[u]]^2+\text{Sec}[s[u]]^2\right)}\right\}\right\}$

In[∘]:= $\text{FullSimplify}\left[\frac{2(1+a)\,\text{Csc}[2s[u]]}{u\left(-a\,\text{Csc}[s[u]]^2+\text{Sec}[s[u]]^2\right)}*(\text{Tan}[s[u]]-a*\text{Cot}[s[u]])/(a+1)\right]$

Out[∘]= $\frac{1}{u}$

1.4 . Checking equation (18)

In[∘]:= f[s_] := s + Cot[2 * s] + (a^2 + 1) Csc[2 * s] / (a^2 - 1) + c3

In[ ]:=
```
Simplify[∂s (f[s]) - (Tan[s] + a * Cot[s]) * (Tan[s] - a * Cot[s]) / ((a - 1) * (a + 1))]
```
Out[ ]= 0

2. Checking for Lemma 25

Isotropic Gaussian curvature

```
gp (v)  = g' (v)
gpp (v) = g'' (v)
```

In[ ]:=
```
b := (a + 1) / (a - 1)
```

In[ ]:=
```
K[u_, v_] := f'[u] * f''[u] * gpp[v]
```

In[ ]:=
```
K[u, v]
```
Out[ ]= $\mathrm{gpp}[v]\, f'[u]\, f''[u]$

Isotropic Mean curvature

In[ ]:=
```
H[u_, v_] := -(f'[u] * gpp[v] + (1 + gp[v] * gp[v]) * f''[u]) / 2
```

In[ ]:=
```
H[u, v]
```
Out[ ]= $\frac{1}{2}\left(-\mathrm{gpp}[v]\, f'[u] - \left(1 + \mathrm{gp}[v]^2\right) f''[u]\right)$

2.1 Checking the equation (29)

In[ ]:=
```
f[u_] := Exp[λ * u]
```

In[ ]:=
```
gpp := λ * ((a + 1) + Sqrt[(a - 1)^2 - 4 * a * g'[v]^2])^2 / (4 * a)
```

In[ ]:=
```
gpp
```
Out[ ]= $\frac{\lambda\left(1 + a + \sqrt{(-1 + a)^2 - 4\, a\, g'[v]^2}\right)^2}{4\, a}$

In[ ]:=
```
FullSimplify[((1 + g'[v]^2) * f''[u] / f'[u] + gpp)^2 - (a + 1)^2 * f''[u] * gpp / (a * f'[u])]
```
Out[ ]= 0

In[ ]:=

2.2 Checking the equation (32) and (33)

In[ ]:= $\mathrm{FullSimplify}\left[\partial_p\left(-\mathrm{Log}\left[a + 1 + \sqrt{(a - 1)\,\hat{}\,2 - 4\, a\, p^2}\,\right] - \frac{a + 1}{a + 1 + \sqrt{(a - 1)\,\hat{}\,2 - 4\, a\, p^2}} + \mathrm{C1}\right)\right]$

Out[ ]= $\frac{4\, a\, p}{\left(1 + a + \sqrt{(-1 + a)^2 - 4\, a\, p^2}\right)^2}$

In[ ]:=
$$\text{FullSimplify}\left[\partial_p\left(-\text{Log}\left[a+1-\sqrt{(a-1)^\wedge 2-4\,a\,p^2}\right]-\frac{a+1}{a+1-\sqrt{(a-1)^\wedge 2-4\,a\,p^2}}+\text{C1}\right)\right]$$

Out[ ]=
$$\frac{4\,a\,p}{\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)^2}$$

In[ ]:=
$$\text{FullSimplify}\left[\partial_p\left(\frac{(-1+a)^2\left(\text{ArcTan}[p]+\text{ArcTan}\left[\frac{(1+a)\,p}{\sqrt{(-1+a)^2-4\,a\,p^2}}\right]\right)}{4\,a}+\frac{p\,(1+a)}{a+1-\sqrt{(a-1)^\wedge 2-4\,a\,p^2}}+\text{C2}\right)\right]$$

Out[ ]=
$$\frac{4\,a}{\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)^2}$$

In[ ]:=
$$\text{FullSimplify}\left[\partial_p\left(\frac{(-1+a)^2\left(\text{ArcTan}[p]-\text{ArcTan}\left[\frac{(1+a)\,p}{\sqrt{(-1+a)^2-4\,a\,p^2}}\right]\right)}{4\,a}+\frac{p\,(1+a)}{a+1+\sqrt{(a-1)^\wedge 2-4\,a\,p^2}}+\text{C2}\right)\right]$$

Out[ ]=
$$\frac{4\,a}{\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)^2}$$

2.3 Checking the equation (34) and (35)

Case “-“ between tangents.

In[ ]:=
```
b := (a + 1) / (a - 1)
```

In[ ]:=
$$w := \text{ArcTan}[p]-\text{ArcTan}\left[\frac{(1+a)\,p}{\sqrt{(-1+a)^2-4\,a\,p^2}}\right]+\text{Pi}/2*\text{Sign}[a-1]$$

When a >1

In[ ]:=
$$w := \text{ArcTan}[p]-\text{ArcTan}\left[\frac{(1+a)\,p}{\sqrt{(-1+a)^2-4\,a\,p^2}}\right]+\text{Pi}/2$$

Finding sin(w)

In[ ]:=
```
FullSimplify[TrigExpand[Sin[w]]]
```

Out[ ]=
$$\frac{(1+a)\,p^2+\sqrt{(-1+a)^2-4\,a\,p^2}}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

Finding cos(w)

In[ ]:=
```
FullSimplify[TrigExpand[Cos[w]]]
```

Out[ ]=
$$\frac{p\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

In[ ]:=
$$\text{sinw} := \left((1+a)\,p^2+\sqrt{(-1+a)^2-4\,a\,p^2}\right)\Big/\ ((a-1)*(1+p\text{^}2))$$

In[ ]:=
$$\text{cosw} := \left(p\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)\right)\Big/\ ((a-1)*(1+p\text{^}2))$$

Checking (34)

In[ ]:=
$$\text{FullSimplify}\Big[-\text{Log}\Big[\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)\Big] - (a+1)\Big/\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right) - ((b\text{^}2-1)*\text{Log}[b-\text{sinw}]+b*\text{sinw})\,/\,(b\text{^}2-1)\Big]$$

Out[ ]=
$$-\frac{(1+a)^2}{4\,a}-\text{Log}\left[\frac{1+a-\sqrt{(-1+a)^2-4\,a\,p^2}}{(-1+a)\,(1+p^2)}\right]-\text{Log}\left[1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right]$$

In[ ]:=
$$\text{FullSimplify}\Big[\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)*\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)\Big/\left((-1+a)\,(1+p^2)\right)\Big]$$

Out[ ]=
$$\frac{4\,a}{-1+a}$$

Thus it is

In[ ]:=
$$-\frac{(1+a)^2}{4\,a} - \text{Log}\left[\frac{4\,a}{-1+a}\right]$$

Out[ ]=
$$-\frac{(1+a)^2}{4\,a}-\text{Log}\left[\frac{4\,a}{-1+a}\right]$$

Checking (35)

In[ ]:=
$$\text{FullSimplify}\Big[(a-1)\text{^}2\,/\,(4\,a)*\left(\text{ArcTan}[p]-\text{ArcTan}\left[\frac{(1+a)\,p}{\sqrt{(-1+a)^2-4\,a\,p^2}}\right]\right) + (a+1)*p\Big/\left(\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)\right) - (w+b*\text{cosw})\,/\,(b\text{^}2-1)\Big]$$

Out[ ]=
$$-\frac{(-1+a)^2\,\pi}{8\,a}$$

When a < 1

In[∘]:=
```
b := (a + 1) / (a - 1)
```

In[∘]:= $w := \text{ArcTan}[p] - \text{ArcTan}\left[\frac{(1+a)\,p}{\sqrt{(-1+a)^2 - 4\,a\,p^2}}\right] - \text{Pi}/2$

Finding sin(w)

In[∘]:=
```
FullSimplify[TrigExpand[Sin[w]]]
```

Out[∘]= $$\frac{-\left((1+a)\,p^2\right) - \sqrt{(-1+a)^2 - 4\,a\,p^2}}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2 - 4\,a\,p^2}}\,\sqrt{(-1+a)^2 - 4\,a\,p^2}}$$

Finding cos(w)

In[∘]:=
```
FullSimplify[TrigExpand[Cos[w]]]
```

Out[∘]= $$\frac{p\left(-1 - a + \sqrt{(-1+a)^2 - 4\,a\,p^2}\right)}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2 - 4\,a\,p^2}}\,\sqrt{(-1+a)^2 - 4\,a\,p^2}}$$

In[∘]:= $\text{sinw} := \left((1+a)\,p^2 + \sqrt{(-1+a)^2 - 4\,a\,p^2}\right) \Big/ ((a-1) * (1+p\text{^}2))$

In[∘]:= $\text{cosw} := \left(p\left(1 + a - \sqrt{(-1+a)^2 - 4\,a\,p^2}\right)\right) \Big/ ((a-1) * (1+p\text{^}2))$

Checking (34)

In[∘]:= $\text{FullSimplify}\Big[-\text{Log}\Big[\Big(1 + a + \sqrt{(-1+a)^2 - 4\,a\,p^2}\Big)\Big] - (a+1) \Big/ \Big(1 + a + \sqrt{(-1+a)^2 - 4\,a\,p^2}\Big) - ((b\text{^}2 - 1) * \text{Log}[b - \text{sinw}] + b * \text{sinw}) / (b\text{^}2 - 1)\Big]$

Out[∘]= $$-\frac{(1+a)^2}{4\,a} - \text{Log}\left[\frac{1 + a - \sqrt{(-1+a)^2 - 4\,a\,p^2}}{(-1+a)\,(1+p^2)}\right] - \text{Log}\left[1 + a + \sqrt{(-1+a)^2 - 4\,a\,p^2}\right]$$

In[∘]:= $\text{FullSimplify}\Big[\Big(1 + a + \sqrt{(-1+a)^2 - 4\,a\,p^2}\Big) * \Big(1 + a - \sqrt{(-1+a)^2 - 4\,a\,p^2}\Big) \Big/ \left((-1+a)\,(1+p^2)\right)\Big]$

Out[∘]= $$\frac{4\,a}{-1+a}$$

Thus it is

In[∘]:= $-\frac{(1+a)^2}{4\,a} - \text{Log}\left[\text{ABS}\left[\frac{4\,a}{-1+a}\right]\right]$

Out[∘]= $$-\frac{(1+a)^2}{4\,a} - \text{Log}\left[\text{ABS}\left[\frac{4\,a}{-1+a}\right]\right]$$

Checking (35)

In[ ]:=
```
FullSimplify[(a - 1)^2 / (4 a) * (ArcTan[p] - ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]]) +
   (a + 1) * p / ((1 + a + Sqrt[(-1 + a)^2 - 4 a p^2])) - (w + b * cosw) / (b^2 - 1)]
```

Out[ ]= $$\frac{(-1+a)^2\,\pi}{8\,a}$$

Case “+“ between tangents.

In[ ]:=
```
b := (a + 1) / (a - 1)
```

In[ ]:=
```
w := ArcTan[p] + ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]] - Pi / 2 * Sign[a - 1]
```

When a >1

In[ ]:=
```
w := ArcTan[p] + ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]] - Pi / 2
```

Finding sin(w)

In[ ]:=
```
FullSimplify[TrigExpand[Sin[w]]]
```

Out[ ]= $$\frac{(1+a)\,p^2-\sqrt{(-1+a)^2-4\,a\,p^2}}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

Finding cos(w)

In[ ]:=
```
FullSimplify[TrigExpand[Cos[w]]]
```

Out[ ]= $$\frac{p\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

In[ ]:=
```
sinw := ((1 + a) p^2 - Sqrt[(-1 + a)^2 - 4 a p^2]) / ((a - 1) * (1 + p^2))
```

In[ ]:=
```
cosw := (p (1 + a + Sqrt[(-1 + a)^2 - 4 a p^2])) / ((a - 1) * (1 + p^2))
```

Checking (34)

In[∘]:= $$\texttt{FullSimplify}\Big[-\texttt{Log}\Big[\Big(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\Big)\Big]-(a+1)\Big/\Big(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\Big)-((b\wedge 2-1)*\texttt{Log}[b-\texttt{sinw}]+b*\texttt{sinw})/(b\wedge 2-1)\Big]$$

Out[∘]= $$-\frac{(1+a)^2}{4\,a}-\text{Log}\Big[1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\Big]-\text{Log}\Big[\frac{1+a+\sqrt{(-1+a)^2-4\,a\,p^2}}{(-1+a)\,(1+p^2)}\Big]$$

In[∘]:= $$\texttt{FullSimplify}\Big[\Big(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\Big)*\Big(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\Big)\Big/\big((-1+a)\,(1+p^2)\big)\Big]$$

Out[∘]= $$\frac{4\,a}{-1+a}$$

Thus it is

In[∘]:= $$-\frac{(1+a)^2}{4\,a}-\text{Log}\Big[\frac{4\,a}{-1+a}\Big]$$

Out[∘]= $$-\frac{(1+a)^2}{4\,a}-\text{Log}\Big[\frac{4\,a}{-1+a}\Big]$$

Checking (35)

In[∘]:= $$\texttt{FullSimplify}\Big[(a-1)\wedge 2/(4\,a)*\Big(\texttt{ArcTan}[p]+\texttt{ArcTan}\Big[\frac{(1+a)\,p}{\sqrt{(-1+a)^2-4\,a\,p^2}}\Big]\Big)+(a+1)*p\Big/\Big(\Big(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\Big)\Big)-(w+b*\texttt{cosw})/(b\wedge 2-1)\Big]$$

Out[∘]= $$\frac{(-1+a)^2\,\pi}{8\,a}$$

When a < 1

In[∘]:= `b := (a + 1) / (a - 1)`

In[∘]:= $$w:=\texttt{ArcTan}[p]+\texttt{ArcTan}\Big[\frac{(1+a)\,p}{\sqrt{(-1+a)^2-4\,a\,p^2}}\Big]+\texttt{Pi}/2$$

Finding sin(w)

In[∘]:= `FullSimplify[TrigExpand[Sin[w]]]`

Out[∘]= $$\frac{-\big((1+a)\,p^2\big)+\sqrt{(-1+a)^2-4\,a\,p^2}}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

Finding cos(w)

In[ ]:= FullSimplify[TrigExpand[Cos[w]]]

$$\text{Out[ ]=}\ -\frac{p\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)}{\sqrt{1+p^2}\ \sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\ \sqrt{(-1+a)^2-4\,a\,p^2}}$$

$$\text{In[ ]:=}\ \text{sinw} := \left((1+a)\,p^2-\sqrt{(-1+a)^2-4\,a\,p^2}\right)\Big/\ ((a-1)*(1+p\verb|^|2))$$

$$\text{In[ ]:=}\ \text{cosw} := \left(p\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)\right)\Big/\ ((a-1)*(1+p\verb|^|2))$$

Checking (34)

$$\text{In[ ]:=}\ \text{FullSimplify}\left[-\text{Log}\left[\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)\right]-(a+1)\Big/\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)-((b\verb|^|2-1)*\text{Log}[b-\text{sinw}]+b*\text{sinw})/(b\verb|^|2-1)\right]$$

$$\text{Out[ ]=}\ -\frac{(1+a)^2}{4\,a}-\text{Log}\left[1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right]-\text{Log}\left[\frac{1+a+\sqrt{(-1+a)^2-4\,a\,p^2}}{(-1+a)\,(1+p^2)}\right]$$

$$\text{In[ ]:=}\ \text{FullSimplify}\left[\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)*\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)\Big/\ ((-1+a)\,(1+p^2))\right]$$

$$\text{Out[ ]=}\ \frac{4\,a}{-1+a}$$

Thus it is

$$\text{In[ ]:=}\ -\frac{(1+a)^2}{4\,a}-\text{Log}\left[\text{ABS}\left[\frac{4\,a}{-1+a}\right]\right]$$

$$\text{Out[ ]=}\ -\frac{(1+a)^2}{4\,a}-\text{Log}\left[\text{ABS}\left[\frac{4\,a}{-1+a}\right]\right]$$

Checking (35)

$$\text{In[ ]:=}\ \text{FullSimplify}\left[(a-1)\verb|^|2/(4\,a)*\left(\text{ArcTan}[p]+\text{ArcTan}\left[\frac{(1+a)\,p}{\sqrt{(-1+a)^2-4\,a\,p^2}}\right]\right)+(a+1)*p\Big/\left(\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)\right)-(w+b*\text{cosw})/(b\verb|^|2-1)\right]$$

$$\text{Out[ ]=}\ -\frac{(-1+a)^2\,\pi}{8\,a}$$

2.4 Derivation the determinant of the (39)

```
In[ ]:= a * (1 + g'[v]^2) * X^2 + (a + 1) * Sqrt[g''[v]] * X * Y +
  g''[v] * Y^2 - (a + 1) * Sqrt[g''[v]] * g'[v] * X + g''[v] == 0
```

$$\text{Out[ ]=}\ a\,X^2\left(1+g'[v]^2\right)+(1+a)\,X\,Y\,\sqrt{g''[v]}-(1+a)\,X\,g'[v]\,\sqrt{g''[v]}+g''[v]+Y^2\,g''[v]=0$$

```
In[•]:= FullSimplify[
         Det[{{a * (1 + g'[v]^2), (a + 1) * Sqrt[g''[v]] / 2, - (a + 1) * Sqrt[g''[v]] * g'[v] / 2},
           {(a + 1) * Sqrt[g''[v]] / 2, g''[v], 0}, {- (a + 1) * Sqrt[g''[v]] * g'[v] / 2, 0, g''[v]}}]]
```

Out[•]= $-\frac{1}{4}(-1+a)^2\left(1+g'[v]^2\right)g''[v]^2$

2.5 Checking (44)

In[•]:= FullSimplify$\left[\partial_u\left(\frac{-f'[u]\,g''[v]+h''[v]}{f''[u]}\right)-\right.$

```
    (g''[v] * (f'''[u] * f'[u] - f''[u]^2) - h''[v] * f'''[u]) / f''[u]^2]
```

Out[•]= 0

2.6 Checking (50)

```
In[•]:= L[Y_] := g''[v] - h''[v] * (g'[v] - Y) / h'[v]

In[•]:= FullSimplify[
         Discriminant[a * ((1 + g'[v]^2) * X + (Y^2 + 1) * L[Y])^2 - (a + 1)^2 * X * (Y - g'[v])^2 * L[Y],
           X] - (a + 1)^2 * (Y - g'[v])^2 * L[Y]^2 *
           ((a - 1)^2 * g'[v]^2 - 4 * a - 2 * (a + 1)^2 * g'[v] * Y + ((a - 1)^2 - 4 * a * g'[v]^2) * Y^2)]
```

Out[•]= 0

2.7 Derivation the discriminant of the free square term of (50)

```
In[•]:= FullSimplify[Discriminant[
          (a - 1)^2 * g'[v]^2 - 4 * a - 2 * (a + 1)^2 * g'[v] * Y + ((a - 1)^2 - 4 * a * g'[v]^2) * Y^2, Y]]
```

Out[•]= $16(-1+a)^2\,a\left(1+g'[v]^2\right)^2$

3. Euclidean rotational CRPC surfaces

```
In[•]:= Simplify[
         D[Hypergeometric2F1[1 / 2, 1 / 2 + 1 / (2 a), 3 / 2 + 1 / (2 a), r^(2 a)] * r^(a + 1) / (a + 1), r] -
          r^a / Sqrt[1 - r^(2 a)]]
```

Out[•]= 0

4. Dual to translational surfaces

Here we provide derivation of the (53) and (54).

4.1 Derivation of (53)

In[∘]:=
```
x[u_, v_] := v + b * Cos[v]
```

In[∘]:=
```
y[u_, v_] := b * Sin[v] + (b * b - 1) * Log[b - Sin[v]] + (1 - b^2) * u
z[u_, v_] := Exp[u]
```

In[∘]:=
```
Solve[∂u z[u, v] == zx * ∂u x[u, v] + zy * ∂u y[u, v] &&
  ∂v z[u, v] == zx * ∂v x[u, v] + zy * ∂v y[u, v], {zx, zy}]
```

Out[∘]=
$$\left\{\left\{zx \to \frac{e^u \operatorname{Cos}[v]}{\left(-1+b^2\right)\left(b-\operatorname{Sin}[v]\right)},\ zy \to -\frac{e^u}{-1+b^2}\right\}\right\}$$

In[∘]:=
$$\{zx, zy\} = \left\{\frac{e^u \operatorname{Cos}[v]}{\left(-1+b^2\right)\left(b-\operatorname{Sin}[v]\right)},\ -\frac{e^u}{-1+b^2}\right\}$$

Out[∘]=
$$\left\{\frac{e^u \operatorname{Cos}[v]}{\left(-1+b^2\right)\left(b-\operatorname{Sin}[v]\right)},\ -\frac{e^u}{-1+b^2}\right\}$$

In[∘]:=
```
zstar[u_, v_] := FullSimplify[ x[u, v] * zx + y[u, v] * zy - z[u, v]]
```

In[∘]:=
```
zstar[u, v]
```

Out[∘]=
$$e^u \left(-1+u-\operatorname{Log}[b-\operatorname{Sin}[v]]+\frac{b+v\operatorname{Cos}[v]-b^2\operatorname{Sin}[v]}{\left(-1+b^2\right)\left(b-\operatorname{Sin}[v]\right)}\right)$$

4.2 Derivation of (54)

In[∘]:=
```
x[u_, v_] := u + v
y[u_, v_] := Log[Cos[u]] - Log[Cos[v]]
z[u_, v_] := u
```

In[∘]:=
```
Solve[∂u z[u, v] == zx * ∂u x[u, v] + zy * ∂u y[u, v] &&
  ∂v z[u, v] == zx * ∂v x[u, v] + zy * ∂v y[u, v], {zx, zy}]
```

Out[∘]=
$$\left\{\left\{zx \to \frac{\operatorname{Tan}[v]}{\operatorname{Tan}[u]+\operatorname{Tan}[v]},\ zy \to -\frac{\operatorname{Cot}[u]}{1+\operatorname{Cot}[u]\operatorname{Tan}[v]}\right\}\right\}$$

In[ ]:=
```
{zx, zy} = {Tan[v]/(Tan[u] + Tan[v]), -Cot[u]/(1 + Cot[u] Tan[v])}
```

Out[ ]=
$$\left\{\frac{\text{Tan}[v]}{\text{Tan}[u]+\text{Tan}[v]},\ -\frac{\text{Cot}[u]}{1+\text{Cot}[u]\ \text{Tan}[v]}\right\}$$

In[ ]:=
```
zstar[u_, v_] := FullSimplify[x[u, v] * zx + y[u, v] * zy - z[u, v]]
```

In[ ]:=
```
zstar[u, v]
```

Out[ ]=
$$\frac{-\text{Log}[\text{Cos}[u]]+\text{Log}[\text{Cos}[v]]-u\,\text{Tan}[u]+v\,\text{Tan}[v]}{\text{Tan}[u]+\text{Tan}[v]}$$