Surfaces of constant principal–curvatures ratio in isotropic geometry: auxiliary computations.

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Here we check auxiliary computations in the proofs.

1. Checking of Theorem 21

Isotropic Gaussian Curvature

```
K[u_] := (u^3 * f''[u] * f'[u] - 1) / u^4
```

```
K[u]
```

$$\frac{-1 + u^3 f'[u]\, f''[u]}{u^4}$$

Isotropic Mean Curvature

```
H[u_] := (f'[u] + u * f''[u]) / (2 * u)
```

```
H[u]
```

$$\frac{f'[u] + u\, f''[u]}{2\, u}$$

The equation H ^ 2 / K – (a + 1) ^ 2 / (4 a) = 0

```
eq := H[u]^2 / K[u] - (a + 1)^2 / (4 * a)
```

```
eq
```

$$-\frac{(1+a)^2}{4\,a}+\frac{u^2\,(f'[u]+u\,f''[u])^2}{4\,\left(-1+u^3\,f'[u]\,f''[u]\right)}$$

1.1. Checking that equations (13) and (14) are equivalent

```
eq1 :=
 ((a - 1) * ((u^2 * f''[u] + u * f'[u]) / (2 * (a + 1))))^2 - ((u^2 * f''[u] - u * f'[u]) / 2)^2 - 1
```

```
eq1
```

$$-1-\frac{1}{4}\left(-u\,f'[u]+u^2\,f''[u]\right)^2+\frac{(-1+a)^2\,\left(u\,f'[u]+u^2\,f''[u]\right)^2}{4\,(1+a)^2}$$

```
FullSimplify[eq / eq1]
```

$$\frac{1+2\,a+a^2}{4\,a-4\,a\,u^3\,f'[u]\,f''[u]}$$

1.2. Checking of equations (15) and (16)

$$\text{FullSimplify}\left[\text{Solve}\left[\text{Csc}[2*s[u]]==\frac{(-1+a)\,\left(u\,f'[u]+u^2\,f''[u]\right)}{2*(1+a)}\ \&\&\ \text{Cot}[2*s[u]]==\frac{1}{2}\left(u\,f'[u]-u^2\,f''[u]\right),\ \{f'[u],\ f''[u]\}\right]\right]$$

$$\left\{\left\{f'[u]\to\frac{a\,\text{Cot}[s[u]]+\text{Tan}[s[u]]}{(-1+a)\,u},\ f''[u]\to\frac{\text{Cot}[s[u]]+a\,\text{Tan}[s[u]]}{(-1+a)\,u^2}\right\}\right\}$$

1.3. Checking of equation (17)

$$\text{FullSimplify}\left[\text{Solve}\left[\partial_u\left(\frac{a\,\text{Cot}[s[u]]+\text{Tan}[s[u]]}{(-1+a)\,u}\right)==\frac{\text{Cot}[s[u]]+a\,\text{Tan}[s[u]]}{(-1+a)\,u^2},\ \{s'[u]\}\right]\right]$$

$$\left\{\left\{s'[u]\to\frac{2\,(1+a)\,\text{Csc}[2\,s[u]]}{u\,\left(-a\,\text{Csc}[s[u]]^2+\text{Sec}[s[u]]^2\right)}\right\}\right\}$$

$$\text{FullSimplify}\left[\frac{2\,(1+a)\,\text{Csc}[2\,s[u]]}{u\,\left(-a\,\text{Csc}[s[u]]^2+\text{Sec}[s[u]]^2\right)}*(\text{Tan}[s[u]]-a*\text{Cot}[s[u]])\,/\,(a+1)\right]$$

$$\frac{1}{u}$$

1.4 . Checking of equation (18)

```
f[s_] := s + Cot[2 * s] + (a^2 + 1) Csc[2 * s] / (a^2 - 1) + c3
```

```
Simplify[∂s (f[s]) - (Tan[s] + a * Cot[s]) * (Tan[s] - a * Cot[s]) / ((a - 1) * (a + 1))]
```

0

2. Checking of Lemma 25

Isotropic Gaussian curvature

```
gp (v) = g' (v)
gpp (v) = g'' (v)
```

```
b := (a + 1) / (a - 1)
```

```
K[u_, v_] := f'[u] * f''[u] * gpp[v]
```

```
K[u, v]
```

$\text{gpp}[v]\, f'[u]\, f''[u]$

Isotropic Mean curvature

```
H[u_, v_] := -(f'[u] * gpp[v] + (1 + gp[v] * gp[v]) * f''[u]) / 2
```

```
H[u, v]
```

$$\frac{1}{2}\left(-\text{gpp}[v]\, f'[u] - \left(1 + \text{gp}[v]^2\right) f''[u]\right)$$

2.1 Checking of equation (29)

```
f[u_] := Exp[λ * u]
```

```
gpp := λ * ((a + 1) + Sqrt[(a - 1)^2 - 4 * a * g'[v]^2])^2 / (4 * a)
```

```
gpp
```

$$\frac{\lambda\left(1 + a + \sqrt{(-1 + a)^2 - 4\, a\, g'[v]^2}\right)^2}{4\, a}$$

```
FullSimplify[((1 + g'[v]^2) * f''[u] / f'[u] + gpp)^2 - (a + 1)^2 * f''[u] * gpp / (a * f'[u])]
```

0

2.2 Checking of equations (32) and (33)

```
FullSimplify[∂p(-Log[a + 1 + Sqrt[(a - 1) ^2 - 4 a p^2]] - (a + 1)/(a + 1 + Sqrt[(a - 1) ^2 - 4 a p^2]) + C1)]
```

$$\frac{4\,a\,p}{\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)^2}$$

```
FullSimplify[∂p(-Log[a + 1 - Sqrt[(a - 1) ^2 - 4 a p^2]] - (a + 1)/(a + 1 - Sqrt[(a - 1) ^2 - 4 a p^2]) + C1)]
```

$$\frac{4\,a\,p}{\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)^2}$$

```
FullSimplify[
 ∂p((-1 + a)^2 (ArcTan[p] + ArcTan[(1+a) p/Sqrt[(-1+a)^2-4 a p^2]])/(4 a) + p (1 + a)/(a + 1 - Sqrt[(a - 1) ^2 - 4 a p^2]) + C2)]
```

$$\frac{4\,a}{\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)^2}$$

```
FullSimplify[
 ∂p((-1 + a)^2 (ArcTan[p] - ArcTan[(1+a) p/Sqrt[(-1+a)^2-4 a p^2]])/(4 a) + p (1 + a)/(a + 1 + Sqrt[(a - 1) ^2 - 4 a p^2]) + C2)]
```

$$\frac{4\,a}{\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)^2}$$

2.3 Checking of equations (34) and (35)

Case “-“ between tangents.

```
b := (a + 1) / (a - 1)
w := ArcTan[p] - ArcTan[(1 + a) p/Sqrt[(-1 + a)^2 - 4 a p^2]] + Pi / 2 * Sign[a - 1]
```

When a >1

```
w := ArcTan[p] - ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]] + Pi / 2
```

Finding sin(w)

```
FullSimplify[TrigExpand[Sin[w]]]
```

$$\frac{(1+a)\,p^2+\sqrt{(-1+a)^2-4\,a\,p^2}}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

Finding cos(w)

```
FullSimplify[TrigExpand[Cos[w]]]
```

$$\frac{p\left(1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right)}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

```
sinw := ((1 + a) p^2 + Sqrt[(-1 + a)^2 - 4 a p^2]) / ((a - 1) * (1 + p^2))

cosw := (p (1 + a - Sqrt[(-1 + a)^2 - 4 a p^2])) / ((a - 1) * (1 + p^2))
```

Checking (34)

```
FullSimplify[-Log[(1 + a + Sqrt[(-1 + a)^2 - 4 a p^2])] -
  (a + 1) / (1 + a + Sqrt[(-1 + a)^2 - 4 a p^2]) - ((b^2 - 1) * Log[b - sinw] + b * sinw) / (b^2 - 1)]
```

$$-\frac{(1+a)^2}{4\,a}-\text{Log}\left[\frac{1+a-\sqrt{(-1+a)^2-4\,a\,p^2}}{(-1+a)\,\left(1+p^2\right)}\right]-\text{Log}\left[1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right]$$

```
FullSimplify[(1 + a + Sqrt[(-1 + a)^2 - 4 a p^2]) * (1 + a - Sqrt[(-1 + a)^2 - 4 a p^2]) / ((-1 + a) (1 + p^2))]
```

$$\frac{4\,a}{-1+a}$$

Thus it is

$$-\frac{(1+a)^2}{4\,a}-\text{Log}\left[\frac{4\,a}{-1+a}\right]$$

$$-\frac{(1+a)^2}{4\,a}-\text{Log}\left[\frac{4\,a}{-1+a}\right]$$

Checking (35)

```
FullSimplify[(a - 1)^2 / (4 a) * (ArcTan[p] - ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]]) +
  (a + 1) * p / ((1 + a + Sqrt[(-1 + a)^2 - 4 a p^2])) - (w + b * cosw) / (b^2 - 1)]
```

$$-\frac{(-1+a)^2\,\pi}{8\,a}$$

When a < 1

```
b := (a + 1) / (a - 1)

w := ArcTan[p] - ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]] - Pi / 2
```

Finding sin(w)

```
FullSimplify[TrigExpand[Sin[w]]]
```

$$\frac{-\left((1+a)\,p^2\right)-\sqrt{(-1+a)^2-4\,a\,p^2}}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

Finding cos(w)

```
FullSimplify[TrigExpand[Cos[w]]]
```

$$\frac{p\left(-1-a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

```
sinw := ((1 + a) p^2 + Sqrt[(-1 + a)^2 - 4 a p^2]) / ((a - 1) * (1 + p^2))

cosw := (p (1 + a - Sqrt[(-1 + a)^2 - 4 a p^2])) / ((a - 1) * (1 + p^2))
```

Checking (34)

```
FullSimplify[-Log[(1 + a + Sqrt[(-1 + a)^2 - 4 a p^2])] -
  (a + 1) / (1 + a + Sqrt[(-1 + a)^2 - 4 a p^2]) - ((b^2 - 1) * Log[b - sinw] + b * sinw) / (b^2 - 1)]
```

$$-\frac{(1+a)^2}{4\,a}-\mathrm{Log}\left[\frac{1+a-\sqrt{(-1+a)^2-4\,a\,p^2}}{(-1+a)\,\left(1+p^2\right)}\right]-\mathrm{Log}\left[1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right]$$

```
FullSimplify[(1 + a + Sqrt[(-1 + a)^2 - 4 a p^2]) * (1 + a - Sqrt[(-1 + a)^2 - 4 a p^2]) / ((-1 + a) (1 + p^2))]
```

$$\frac{4\,a}{-1+a}$$

Thus it is

$$-\frac{(1+a)^2}{4\,a} - \mathrm{Log}\left[\mathrm{ABS}\left[\frac{4\,a}{-1+a}\right]\right]$$

$$-\frac{(1+a)^2}{4\,a} - \mathrm{Log}\left[\mathrm{ABS}\left[\frac{4\,a}{-1+a}\right]\right]$$

Checking (35)

```
FullSimplify[(a - 1)^2 / (4 a) * (ArcTan[p] - ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]]) +
  (a + 1) * p / ((1 + a + Sqrt[(-1 + a)^2 - 4 a p^2])) - (w + b * cosw) / (b^2 - 1)]
```

$$\frac{(-1+a)^2\,\pi}{8\,a}$$

Case “+“ between tangents.

```
b := (a + 1) / (a - 1)

w := ArcTan[p] + ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]] - Pi / 2 * Sign[a - 1]
```

When a >1

```
w := ArcTan[p] + ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]] - Pi / 2
```

Finding sin(w)

```
FullSimplify[TrigExpand[Sin[w]]]
```

$$\frac{(1+a)\,p^2 - \sqrt{(-1+a)^2 - 4\,a\,p^2}}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2 - 4\,a\,p^2}}$$

Finding cos(w)

```
FullSimplify[TrigExpand[Cos[w]]]
```

$$\frac{p\left(1+a+\sqrt{(-1+a)^2-4 a p^2}\right)}{\sqrt{1+p^2}\sqrt{\frac{(-1+a)^2\left(1+p^2\right)}{(-1+a)^2-4 a p^2}}\sqrt{(-1+a)^2-4 a p^2}}$$

```
sinw := ((1 + a) p^2 - Sqrt[(-1 + a)^2 - 4 a p^2]) / ((a - 1) * (1 + p^2))

cosw := (p (1 + a + Sqrt[(-1 + a)^2 - 4 a p^2])) / ((a - 1) * (1 + p^2))
```

Checking (34)

```
FullSimplify[-Log[(1 + a - Sqrt[(-1 + a)^2 - 4 a p^2])] -
  (a + 1) / (1 + a - Sqrt[(-1 + a)^2 - 4 a p^2]) - ((b^2 - 1) * Log[b - sinw] + b * sinw) / (b^2 - 1)]
```

$$-\frac{(1+a)^2}{4 a}-\text{Log}\left[1+a-\sqrt{(-1+a)^2-4 a p^2}\right]-\text{Log}\left[\frac{1+a+\sqrt{(-1+a)^2-4 a p^2}}{(-1+a)\left(1+p^2\right)}\right]$$

```
FullSimplify[(1 + a + Sqrt[(-1 + a)^2 - 4 a p^2]) * (1 + a - Sqrt[(-1 + a)^2 - 4 a p^2]) / ((-1 + a) (1 + p^2))]
```

$$\frac{4 a}{-1+a}$$

Thus it is

```
-(1 + a)^2/(4 a) - Log[4 a/(-1 + a)]
```

$$-\frac{(1+a)^2}{4 a}-\text{Log}\left[\frac{4 a}{-1+a}\right]$$

Checking (35)

```
FullSimplify[(a - 1)^2 / (4 a) * (ArcTan[p] + ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]]) +
  (a + 1) * p / ((1 + a - Sqrt[(-1 + a)^2 - 4 a p^2])) - (w + b * cosw) / (b^2 - 1)]
```

$$\frac{(-1+a)^2 \pi}{8 a}$$

When a < 1

```
b := (a + 1) / (a - 1)
```

```
w := ArcTan[p] + ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]] + Pi / 2
```

Finding sin(w)

```
FullSimplify[TrigExpand[Sin[w]]]
```

$$\frac{-\left((1+a)\,p^2\right)+\sqrt{(-1+a)^2-4\,a\,p^2}}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

Finding cos(w)

```
FullSimplify[TrigExpand[Cos[w]]]
```

$$-\frac{p\left(1+a+\sqrt{(-1+a)^2-4\,a\,p^2}\right)}{\sqrt{1+p^2}\,\sqrt{\frac{(-1+a)^2\,(1+p^2)}{(-1+a)^2-4\,a\,p^2}}\,\sqrt{(-1+a)^2-4\,a\,p^2}}$$

```
sinw := ((1 + a) p^2 - Sqrt[(-1 + a)^2 - 4 a p^2]) / ((a - 1) * (1 + p^2))

cosw := (p (1 + a + Sqrt[(-1 + a)^2 - 4 a p^2])) / ((a - 1) * (1 + p^2))
```

Checking (34)

```
FullSimplify[-Log[(1 + a - Sqrt[(-1 + a)^2 - 4 a p^2])] -
  (a + 1) / (1 + a - Sqrt[(-1 + a)^2 - 4 a p^2]) - ((b^2 - 1) * Log[b - sinw] + b * sinw) / (b^2 - 1)]
```

$$-\frac{(1+a)^2}{4\,a}-\text{Log}\left[1+a-\sqrt{(-1+a)^2-4\,a\,p^2}\right]-\text{Log}\left[\frac{1+a+\sqrt{(-1+a)^2-4\,a\,p^2}}{(-1+a)\,(1+p^2)}\right]$$

```
FullSimplify[(1 + a + Sqrt[(-1 + a)^2 - 4 a p^2]) * (1 + a - Sqrt[(-1 + a)^2 - 4 a p^2]) / ((-1 + a) (1 + p^2))]
```

$$\frac{4\,a}{-1+a}$$

Thus it is

```
-(1 + a)^2 / (4 a) - Log[ABS[4 a / (-1 + a)]]
```

$$-\frac{(1+a)^2}{4\,a}-\text{Log}\left[\text{ABS}\left[\frac{4\,a}{-1+a}\right]\right]$$

Checking (35)

```
FullSimplify[(a - 1) ^2 / (4 a) * (ArcTan[p] + ArcTan[(1 + a) p / Sqrt[(-1 + a)^2 - 4 a p^2]]) +
  (a + 1) * p / ((1 + a - Sqrt[(-1 + a)^2 - 4 a p^2])) - (w + b * cosw) / (b^2 - 1)]
```

$$-\frac{(-1+a)^2\,\pi}{8\,a}$$

2.4 Derivation of the determinant of (39)

```
a * (1 + g'[v]^2) * X^2 + (a + 1) * Sqrt[g''[v]] * X * Y +
  g''[v] * Y^2 - (a + 1) * Sqrt[g''[v]] * g'[v] * X + g''[v] == 0
```

$$a X^2 \left(1+g'[v]^2\right) + (1+a)\, X Y \sqrt{g''[v]} - (1+a)\, X g'[v] \sqrt{g''[v]} + g''[v] + Y^2 g''[v] == 0$$

```
FullSimplify[
 Det[{{a * (1 + g'[v]^2), (a + 1) * Sqrt[g''[v]] / 2, - (a + 1) * Sqrt[g''[v]] * g'[v] / 2},
   {(a + 1) * Sqrt[g''[v]] / 2, g''[v], 0}, {- (a + 1) * Sqrt[g''[v]] * g'[v] / 2, 0, g''[v]}}]]
```

$$-\frac{1}{4}\,(-1+a)^2\left(1+g'[v]^2\right) g''[v]^2$$

2.5 Checking of (44)

```
FullSimplify[∂u((-f'[u] g''[v] + h''[v]) / f''[u]) -
  (g''[v] * (f'''[u] * f'[u] - f''[u]^2) - h''[v] * f'''[u]) / f''[u]^2]
```

0

2.6 Checking of (50)

```
L[Y_] := g''[v] - h''[v] * (g'[v] - Y) / h'[v]
```

```
FullSimplify[
 Discriminant[a * ((1 + g'[v]^2) * X + (Y^2 + 1) * L[Y])^2 - (a + 1)^2 * X * (Y - g'[v])^2 * L[Y],
   X] - (a + 1)^2 * (Y - g'[v])^2 * L[Y]^2 *
   ((a - 1)^2 * g'[v]^2 - 4 * a - 2 * (a + 1)^2 * g'[v] * Y + ((a - 1)^2 - 4 * a * g'[v]^2) * Y^2)]
```

0

2.7 Derivation of the discriminant of the square free term of (50)

```
FullSimplify[Discriminant[
  (a - 1) ^2 * g'[v] ^2 - 4 * a - 2 * (a + 1) ^2 * g'[v] * Y + ((a - 1) ^2 - 4 * a * g'[v] ^2) * Y^2, Y]]
```

$$16\,(-1+a)^2\, a\,\left(1+g'[v]^2\right)^2$$

3. Euclidean rotational CRPC surfaces

```
Simplify[
 D[Hypergeometric2F1[1 / 2, 1 / 2 + 1 / (2 a), 3 / 2 + 1 / (2 a), r^(2 a)] * r^(a + 1) / (a + 1), r] -
  r^a / Sqrt[1 - r^(2 a)]]
```

0

4. Dual – translational surfaces

Here we provide the derivation of (53) and (54).

4.1. Derivation of (53)

```
x[u_, v_] := v + b * Cos[v]
y[u_, v_]  := b * Sin[v] + (b * b - 1) * Log[b - Sin[v]] + (1 - b^2) * u
z[u_, v_]  := Exp[u]
Solve[∂u z[u, v] == zx * ∂u x[u, v] + zy * ∂u y[u, v] &&
  ∂v z[u, v] == zx * ∂v x[u, v] + zy * ∂v y[u, v], {zx, zy}]
```

$$\left\{\left\{zx \to \frac{e^u \,\mathrm{Cos}[v]}{\left(-1+b^2\right)\,(b-\mathrm{Sin}[v])},\; zy \to -\frac{e^u}{-1+b^2}\right\}\right\}$$

$$\{zx, zy\} = \left\{\frac{e^u \,\mathrm{Cos}[v]}{\left(-1+b^2\right)\,(b-\mathrm{Sin}[v])},\; -\frac{e^u}{-1+b^2}\right\}$$

$$\left\{\frac{e^u \,\mathrm{Cos}[v]}{\left(-1+b^2\right)\,(b-\mathrm{Sin}[v])},\; -\frac{e^u}{-1+b^2}\right\}$$

```
zstar[u_, v_] := FullSimplify[ x[u, v] * zx + y[u, v] * zy - z[u, v]]
zstar[u, v]
```

$$e^u\left(-1+u-\mathrm{Log}[b-\mathrm{Sin}[v]]+\frac{b+v\,\mathrm{Cos}[v]-b^2\,\mathrm{Sin}[v]}{\left(-1+b^2\right)\,(b-\mathrm{Sin}[v])}\right)$$

4.2. Derivation of (54)

```
x[u_, v_] := u + v
y[u_, v_] := Log[Cos[u]] - Log[Cos[v]]
z[u_, v_] := u
```

```
Solve[∂u z[u, v] == zx * ∂u x[u, v] + zy * ∂u y[u, v] &&
  ∂v z[u, v] == zx * ∂v x[u, v] + zy * ∂v y[u, v], {zx, zy}]
```

$$\left\{\left\{zx \to \frac{\text{Tan}[v]}{\text{Tan}[u] + \text{Tan}[v]},\ zy \to -\frac{\text{Cot}[u]}{1 + \text{Cot}[u]\ \text{Tan}[v]}\right\}\right\}$$

$$\{zx, zy\} = \left\{\frac{\text{Tan}[v]}{\text{Tan}[u] + \text{Tan}[v]},\ -\frac{\text{Cot}[u]}{1 + \text{Cot}[u]\ \text{Tan}[v]}\right\}$$

$$\left\{\frac{\text{Tan}[v]}{\text{Tan}[u] + \text{Tan}[v]},\ -\frac{\text{Cot}[u]}{1 + \text{Cot}[u]\ \text{Tan}[v]}\right\}$$

```
zstar[u_, v_] := FullSimplify[x[u, v] * zx + y[u, v] * zy - z[u, v]]
```

```
zstar[u, v]
```

$$\frac{-\text{Log}[\text{Cos}[u]] + \text{Log}[\text{Cos}[v]] - u\,\text{Tan}[u] + v\,\text{Tan}[v]}{\text{Tan}[u] + \text{Tan}[v]}$$