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DERIVATION OF THE SCORES AND THE SECOND DERIVATIVES

For subjects i=1,…,n, let

> t_i represent the assay value for subject i,
> Y_i=1 represent the event that subject i develops disease,
> Y_i=0 represent the event that they do not.

Then

$$\begin{aligned}
\mathrm{P}\left(Y_i = y_i\right) &= \left[\lambda\left(1-\frac{\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)}\right)\right]^{y_i}\left[1-\lambda\left(1-\frac{\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)}\right)\right]^{(1-y_i)} \\
&= \left[\lambda\left(\frac{1}{1+\exp(\alpha+\beta t_i)}\right)\right]^{y_i}\left[\frac{1+\exp(\alpha+\beta t_i)-\lambda}{1+\exp(\alpha+\beta t_i)}\right]^{(1-y_i)} \\
\mathrm{L}\left(\lambda,\alpha,\beta\right) &= \prod_i\left[\lambda\left(\frac{1}{1+\exp(\alpha+\beta t_i)}\right)\right]^{y_i}\left[\frac{1+\exp(\alpha+\beta t_i)-\lambda}{1+\exp(\alpha+\beta t_i)}\right]^{(1-y_i)} \\
\mathrm{l}\left(\lambda,\alpha,\beta\right) &= \sum_i \begin{array}{l} y_i\log\lambda - y_i\log\left(1+\exp(\alpha+\beta t_i)\right)+(1-y_i)\log\left(1+\exp(\alpha+\beta t_i)-\lambda\right) \\ -(1-y_i)\log\left(1+\exp(\alpha+\beta t_i)\right)\end{array} \\
&= \sum_i y_i\log\lambda + (1-y_i)\log\left(1+\exp(\alpha+\beta t_i)-\lambda\right)-\log\left(1+\exp(\alpha+\beta t_i)\right)
\end{aligned}$$

$$\frac{d\,\mathrm{l}}{d\lambda} = \sum_i y_i\frac{1}{\lambda}-\left(1-y_i\right)\frac{1}{1+\exp(\alpha+\beta t_i)-\lambda}$$

$$\frac{d\,\mathrm{l}}{d\alpha} = \sum_i\left(1-y_i\right)\frac{\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)-\lambda}-\frac{\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)}$$

$$\frac{d\,\mathrm{l}}{d\beta} = \sum_i\left(1-y_i\right)\frac{t_i\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)-\lambda}-\frac{t_i\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)}$$

$$\frac{d^2\,\mathrm{l}}{d\lambda^2} = \sum_i -y_i\frac{1}{\lambda^2}-\left(1-y_i\right)\frac{1}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}$$

$$\frac{d^2\,\mathrm{l}}{d\lambda d\alpha} = \sum_i\left(1-y_i\right)\frac{\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}$$

$$\frac{d^2\,\mathrm{l}}{d\lambda d\beta} = \sum_i\left(1-y_i\right)\frac{t_i\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}$$

$$\frac{d^2 \mathrm{l}}{d\alpha d\lambda} = \sum_i (1-y_i)\frac{\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}$$

$$\frac{d^2 \mathrm{l}}{d\alpha^2} = \sum_i \begin{array}{l} (1-y_i)\left(-\dfrac{\left(\exp(\alpha+\beta t_i)\right)^2}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}+\dfrac{\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)-\lambda}\right) \\ -\left(-\dfrac{\left(\exp(\alpha+\beta t_i)\right)^2}{\left(1+\exp(\alpha+\beta t_i)\right)^2}+\dfrac{\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)}\right) \end{array}$$

$$= \sum_i (1-y_i)\left(\frac{(1-\lambda)\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}\right)-\left(\frac{\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)\right)^2}\right)$$

$$\frac{d^2 \mathrm{l}}{d\alpha d\beta} = \sum_i \begin{array}{l} (1-y_i)\left(\dfrac{t_i\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)-\lambda}-\dfrac{t_i\left(\exp(\alpha+\beta t_i)\right)^2}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}\right) \\ -\left(\dfrac{t_i\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)}-\dfrac{t_i\left(\exp(\alpha+\beta t_i)\right)^2}{\left(1+\exp(\alpha+\beta t_i)\right)^2}\right) \end{array}$$

$$= \sum_i (1-y_i)\left(\frac{t_i(1-\lambda)\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}\right)-\left(\frac{t_i\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)\right)^2}\right)$$

$$\frac{d^2 \mathrm{l}}{d\beta d\lambda} = \sum_i (1-y_i)\frac{t_i\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}$$

$$\frac{d^2 \mathrm{l}}{d\beta d\alpha} = \sum_i \begin{array}{l} (1-y_i)\left(\dfrac{t_i\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)-\lambda}-\dfrac{t_i\left(\exp(\alpha+\beta t_i)\right)^2}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}\right) \\ -\left(\dfrac{t_i\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)}-\dfrac{t_i\left(\exp(\alpha+\beta t_i)\right)^2}{\left(1+\exp(\alpha+\beta t_i)\right)^2}\right) \end{array}$$

$$= \sum_i (1-y_i)\left(\frac{t_i(1-\lambda)\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}\right)-\left(\frac{t_i\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)\right)^2}\right)$$

$$\frac{d^2 \mathrm{l}}{d\beta^2} = \sum_i \begin{array}{l} (1-y_i)\left(\dfrac{t_i^2\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)-\lambda}-\dfrac{t_i^2\left(\exp(\alpha+\beta t_i)\right)^2}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}\right) \\ -\left(\dfrac{t_i^2\exp(\alpha+\beta t_i)}{1+\exp(\alpha+\beta t_i)}-\dfrac{t_i^2\left(\exp(\alpha+\beta t_i)\right)^2}{\left(1+\exp(\alpha+\beta t_i)\right)^2}\right) \end{array}$$

$$= \sum_i (1-y_i)\left(\frac{t_i^2(1-\lambda)\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)-\lambda\right)^2}\right)-\left(\frac{t_i^2\exp(\alpha+\beta t_i)}{\left(1+\exp(\alpha+\beta t_i)\right)^2}\right)$$