The number of quartic D_4 -fields ordered by conductor

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Abstract

We consider families of number fields of degree 4 whose normal closures over \mathbb{Q} have Galois group isomorphic to D_4 , the symmetries of a square. To any such field L, one can associate the Artin conductor of the corresponding 2-dimensional irreducible Galois representation with image D_4 . We determine the asymptotic number of such quartic D_4 -fields ordered by conductor, and compute the leading term explicitly as a mass formula, verifying heuristics of Kedlaya and Wood. Additionally, we are able to impose any local splitting conditions at any finite number of primes (sometimes, at an infinite number of primes), and as a consequence, we also compute the asymptotic number of order 4 elements in class groups and narrow class groups of quadratic fields ordered by discriminant.

Traditionally, there have been two approaches to counting quartic fields, using arithmetic invariant theory in combination with geometry-of-number techniques, and applying Kummer theory together with L-function methods. Both of these strategies fall short in the case of D_4 -fields ordered by conductor since counting quartic fields containing a quadratic subfield with large discriminant is difficult. However, when ordering by conductor, we utilize additional algebraic structure arising from the outer automorphism of D_4 combined with both approaches mentioned above to obtain exact asymptotics.

1 Introduction

The main purpose of this article is to determine the asymptotic number of quartic dihedral fields with bounded conductor. If L denotes a quartic field whose normal closure M over \mathbb{Q} has Galois group $Gal(M/\mathbb{Q})$ isomorphic to the group of symmetries of a square, we refer to L as a D_4 -field. Furthermore, there is a unique (up to conjugacy) irreducible 2-dimensional Galois representation

$$\rho_M : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathbb{C})$$

that factors through $Gal(M/\mathbb{Q}) \cong D_4$. We define the *conductor* of L to be equal to the Artin conductor of ρ_M (see [6, Pg. 158-159]).

Theorem 1. Let $N_{D_4}^{(r_2)}(X)$ denote the number of isomorphism classes of D_4 -quartic fields with $4-2r_2$ real embeddings and conductor bounded by X. Then

$$\begin{split} N_{\mathrm{D_4}}^{(0)}(X) &= \frac{1}{4} \cdot \prod_p \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4} \right) \cdot X \log X + O(X \log \log X); \\ N_{\mathrm{D_4}}^{(1)}(X) &= \frac{3}{8} \cdot \prod_p \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4} \right) \cdot X \log X + O(X \log \log X); \\ N_{\mathrm{D_4}}^{(2)}(X) &= \frac{1}{8} \cdot \prod_p \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4} \right) \cdot X \log X + O(X \log \log X). \end{split}$$

Understanding the distribution of number fields with fixed signature and Galois group is a fundamental question in number theory with several significant applications. For example, the inverse Galois problem follows from understanding the main terms for the asymptotic number of field extensions of each fixed degree and Galois closure over a given base field. Furthermore, if the results are refined enough to determine the

asymptotic number of field extensions satisfying certain local specifications, then another application of counting number fields is towards understanding the distribution of torsion in class groups of number fields of fixed degree, i.e., to proving cases of the Cohen-Lenstra heuristics [10] as well as the extensions given by Gerth [16], Cohen-Martinet [11], and Malle [21].

There are heuristics (see Conjecture 1.2 of [7]) for the order of growth for the number of field extensions of each fixed degree and Galois closure over a given base field when the extensions are bounded by their (norms of the relative) discriminants, due to Linnik, Malle, and Turkelli. Linnik predicted that the number $N_{S_n}(X)$ of S_n -number fields of degree n with discriminant bounded by X satisfies $N_{S_n}(X) \sim c_n X$, for some constant c_n as $X \to \infty$. Additionally, the heuristics of Malle [20] imply that the proportion of degree-n fields with Galois closure S_n amongst all degree-n fields is expected to be 100% only when n is a prime. Cohen-Diaz y Diaz-Olivier [8] verified a case of Malle's full conjecture in the quartic dihedral case by proving that the number of D_4 -fields with discriminant bounded by X is asymptotically equal to cX, where $c \approx .052326$.

Results of Gauss, Davenport-Heilbronn [13] and Bhargava [2, 4] established that $N_{S_n}(X) \sim c_n X$ for n=2, 3, 4, and 5, where c_n is determined to be given by a mass formula, i.e., the constants c_n take the form of an Euler product of local masses. In [3], Bhargava predicted the constants c_n for all n, explicitly describing them in terms of Euler products of local masses derived from the heuristic assumption that the completions of S_n -number fields at different places behave independently of one another. However, if one computes the analogous product of local masses for D_4 -fields ordered by discriminant, the resulting constant differs in magnitude from that obtained in the asymptotics of Cohen-Diaz y Diaz-Olivier!

In particular, Corollary 1.4 of [8] implies that the number of totally real D_4 -fields with absolute discriminant bounded by X is asymptotically equal to cX, where

$$c = \frac{3}{\pi^2} \cdot \left(\sum_{\substack{[K:\mathbb{Q}]=2\\0<\operatorname{Disc}(K)<\infty}} \frac{1}{\operatorname{Disc}(K)^2} \cdot \frac{L(1,K/\mathbb{Q})}{L(2,K/\mathbb{Q})} \right), \tag{1}$$

but whether this constant c can be determined by a mass formula was not addressed. Our results imply that the number of totally real D_4 -fields with conductor bounded by X is asymptotically equal to a similar sum:

$$N_{\mathrm{D_4}}^{(0)}(X) \sim \frac{3}{\pi^2} \cdot \left(\sum_{\substack{[K:\mathbb{Q}]=2\\0<\mathrm{Disc}(K)\leq X}} \frac{1}{\mathrm{Disc}(K)} \cdot \frac{L(1,K/\mathbb{Q})}{L(2,K/\mathbb{Q})} \right) \cdot X. \tag{2}$$

Note that the discriminant of a D_4 -field is equal in magnitude to the product of its conductor and the discriminant of its quadratic subfield, and the extra power of $\operatorname{Disc}(K)$ in (1) can essentially be attributed to this difference in the choice of invariants. Additionally, we are able to prove that the right hand side of (2) does indeed satisfy a mass formula:

Theorem 2. We have the following:

$$\sum_{\substack{[K:\mathbb{Q}]=2\\0<\mathrm{Disc}(K)\leq X}}\frac{1}{\mathrm{Disc}(K)}\cdot\frac{L(1,K/\mathbb{Q})}{L(2,K/\mathbb{Q})}\sim\frac{\zeta(2)}{2}\cdot\prod_{p}\left(1-\frac{1}{p^2}-\frac{2}{p^3}+\frac{2}{p^4}\right)\cdot\log(X);$$

$$\sum_{\substack{[K:\mathbb{Q}]=2\\-X\leq\mathrm{Disc}(K)<0}}\frac{1}{|\mathrm{Disc}(K)|}\cdot\frac{L(1,K/\mathbb{Q})}{L(2,K/\mathbb{Q})}\sim\frac{\zeta(2)}{2}\cdot\prod_{p}\left(1-\frac{1}{p^2}-\frac{2}{p^3}+\frac{2}{p^4}\right)\cdot\log(X).$$

The existence of mass formulae when ordering by invariants other than the discriminant has been predicted by heuristics of Kedlaya [18] and Wood [26]. Theorem 1 verifies these conjectures in the quartic dihedral case when ordering by conductor. In particular, we show that Question 5.1 of [26] is answered in the affirmative for D_4 -fields ordered by conductor (see Equation 10), but it is not true for D_4 -fields ordered by discriminant (see Equation 11). Thus, the choice of invariant proves to be a subtle issue when determining the asymptotics for families of number fields whose Galois closures have a fixed Galois group other than S_n .

Additionally, we are able to determine refined asymptotics for families of D_4 -fields with certain prescribed local specifications, but to describe these results, we must first introduce some notation. We say that

 $\Sigma = (\Sigma_v)_v$ is a collection of local specifications, if for each place v of \mathbb{Q} , Σ_v contains pairs (L_v, K_v) consisting of an étale algebra L_v of \mathbb{Q}_v of degree 4 along with a quadratic subalgebra K_v . We say that such a collection Σ is acceptable if for all but finitely many primes p, the set Σ_p contains all pairs (L_p, K_p) with conductor indivisible by p^2 . Here, the conductor C of such a pair is equal to

$$C(L_p, K_p) := Disc(L_p)/Disc(K_p),$$

and we also let C_p denote the *p*-part of C. If $\mathcal{L}(\Sigma)$ denotes all D_4 -fields L such that $L \otimes \mathbb{Q}_v \in \Sigma_v$ for all v, and $N_{D_4}(\Sigma, X)$ denotes the number of isomorphism classes of D_4 -fields in $\mathcal{L}(\Sigma)$ whose conductor is bounded by X, we then have:

Theorem 3. If $\Sigma = (\Sigma_v)_v$ is an acceptable collection of local specifications such that Σ_2 contains every degree 4 étale algebra of \mathbb{Q}_2 containing a quadratic subalgebra, then

$$N_{\mathrm{D}_4}(\Sigma, X) \sim \frac{1}{2} \cdot \left(\sum_{(L, K) \in \Sigma_{\infty}} \frac{1}{\# \operatorname{Aut}(L, K)} \right) \cdot \prod_{p} \left(\sum_{(L_p, K_p) \in \Sigma_p} \frac{1}{\# \operatorname{Aut}(L_p, K_p)} \cdot \frac{1}{\operatorname{C}_p(L_p, K_p)} \right) \left(1 - \frac{1}{p} \right)^2 \cdot X \log(X),$$

where for all v, $Aut(L_v, K_v)$ consists of the automorphisms of L_v which send K_v to itself.

The constant determined in Theorem 3 is completely analogous to the constants $c_{n,\Sigma}$ predicted in Equation 4.2 of [3] for the number of S_n -fields of degree n satisfying a collection of local specifications with bounded discriminant. The extra factor of $\frac{p-1}{p}$ that occurs here can be explained by a double pole for the relevant counting functions, which also accounts for the factor of $\log(X)$. It is also noteworthy that Theorem 3 establishes Sato-Tate equidistribution for the family of D_4 -fields ordered by conductor, in the sense of [24].

Theorem 3 allows us to compute the asymptotic number of order 4 elements in class groups and narrow class groups of quadratic fields ordered by discriminant. Such elements in the class groups of a quadratic field K determine D_4 -fields L whose normal closures over \mathbb{Q} contain K as the fixed field of $C_4 \subset D_4$. We obtain the following theorem by determining asymptotics for the acceptable collection of D_4 -fields that arise in this manner, even when we restrict the set of quadratic fields by imposing local specifications at a finite set of primes. We remark that it is crucial for the below result that we order D_4 -fields by conductor and furthermore, that we can impose acceptable local specifications at an infinite number of primes.

Theorem 4. For a quadratic field K, let $\operatorname{Cl}_{2^k}(K)$ (resp. $\operatorname{Cl}_{2^k}^+(K)$) denote the 2^k -torsion subgroup in its ideal class group $\operatorname{Cl}(K)$ (resp. narrow class group $\operatorname{Cl}^+(K)$). Let K denote a family of quadratic fields with prescribed splitting types at a finite set S consisting of odd primes. We then have:

(a)
$$\sum_{\substack{K \in \mathcal{K} \\ 0 < \text{Disc}(K) \le X}} (\# \operatorname{Cl}_4(K) - \# \operatorname{Cl}_2(K)) \sim \frac{1}{16} \cdot \prod_{p \in S} m_{\operatorname{Cl}}(p) \cdot \prod_p \left(1 + \frac{2}{p}\right) \left(1 - \frac{1}{p}\right)^2 \cdot X \log(X),$$

(b)
$$\sum_{\substack{K \in \mathcal{K} \\ -X \le \mathrm{Disc}(K) < 0}} (\# \operatorname{Cl}_4(K) - \# \operatorname{Cl}_2(K)) \sim \frac{1}{4} \cdot \prod_{p \in S} m_{\operatorname{Cl}}(p) \cdot \prod_p \left(1 + \frac{2}{p}\right) \left(1 - \frac{1}{p}\right)^2 \cdot X \log(X), \text{ and }$$

(c)
$$\sum_{\substack{K \in \mathcal{K} \\ 0 < \text{Disc}(K) \le X}} (\# \operatorname{Cl}_{4}^{+}(K) - \# \operatorname{Cl}_{2}(K)) \sim \frac{1}{8} \cdot \prod_{p \in S} m_{\operatorname{Cl}}(p) \cdot \prod_{p} \left(1 + \frac{2}{p}\right) \left(1 - \frac{1}{p}\right)^{2} \cdot X \log(X).$$

Here, $m_{Cl}(p)$ is determined in terms of the prescribed splitting type for $p \in S$:

$$m_{\mathrm{Cl}}(p) := \frac{2}{p+2}$$
 if p ramifies, and $m_{\mathrm{Cl}}(p) := \frac{p}{2p+2}$ otherwise.

The above result is a generalization of work of Fouvry-Klüners [15] that is derived from their own previous results [14] completely verifying Gerth's extension [16] of the Cohen-Lenstra heuristics to the 4-rank of the narrow class group of quadratic fields. In [14], Fouvry-Klüners compute all moments for the 4-ranks of narrow class groups of quadratic fields ordered by discriminant. In conjuction with those results, Theorem

4 gives evidence towards the belief that the 4-ranks and the sizes of 2-torsion subgroups in class groups and narrow class groups of quadratic fields behave independently (see Remark 9.4).

As a byproduct of the methods used to obtain (2), we also prove a refinement of Theorem 2 that allows for imposing local specifications at a finite number of primes.

Theorem 5. Let K denote a set of quadratic fields with prescribed splitting types at a finite set S of odd primes. We then have:

(a)
$$\sum_{\substack{K \in \mathcal{K} \\ 0 < \mathrm{Disc}(K) \leq X}} \frac{1}{\mathrm{Disc}(K)} \cdot \frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} \sim \frac{\zeta(2)}{2} \cdot \prod_{p \in S} \frac{m(p)}{2p^2 + 4p + 4} \cdot \prod_{p} \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4}\right) \cdot \log(X),$$

(b)
$$\sum_{\substack{K \in \mathcal{K} \\ -X < \mathrm{Disc}(K) \leq 0}} \frac{1}{|\mathrm{Disc}(K)|} \cdot \frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} \sim \frac{\zeta(2)}{2} \cdot \prod_{p \in S} \frac{m(p)}{2p^2 + 4p + 4} \cdot \prod_{p} \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4}\right) \cdot \log(X),$$

where m(p) is determined in terms of the prescribed splitting type for $p \in S$:

$$m(p) := \begin{cases} p^2 + 2p + 1 & \text{if } p \text{ splits;} \\ p^2 + 1 & \text{if } p \text{ is inert;} \\ 2(p+1) & \text{if } p \text{ is ramified.} \end{cases}$$

It is interesting to note that the proof of Theorem 2 is entirely contained within Section 5 while the proof of Theorem 5 requires the whole article. We first summarize the argument for Theorem 2. First, for a quadratic field K, $\frac{L(1,K/\mathbb{Q})}{L(2,K/\mathbb{Q})}$ can be written as a product of infinite sums simply using Möbius inversion:

$$\frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} = \left(\sum_{n=1}^{\infty} \frac{\chi_K(n)}{n}\right) \cdot \left(\sum_{m=1}^{\infty} \mu(m) \cdot \frac{\chi_K(m)}{m^2}\right),\tag{3}$$

where χ_K is the quadratic character associated to K. The proof then relies on the following observation: weighting this product by $\operatorname{Disc}(K)^{-1}$, the main contribution when summing over quadratic fields K with bounded discriminant as in (2) comes from certain diagonal terms of the right hand side of (3), i.e., terms where mn is a square. For each K, the sum of these diagonal terms is expressible as an Euler product (see Equation 21), which then yields Theorem 2. If we were to instead weight the product in (3) by $\operatorname{Disc}(K)^{-2}$ when summing over all quadratic fields K as in (1), then the non-diagonal terms are no longer negligible. It is these terms that cause the constant c to not follow the heuristics of [3, 18, 26]. We go further to show that the sum over quadratic fields with discriminant bounded by K of the diagonal terms in the right hand side of (3) weighted by $\operatorname{Disc}(K)^{-2}$ is asymptotically equal to

$$\frac{3 \cdot 11^2}{2^6 \cdot 17} \cdot \prod_{p} \left(1 + \frac{1}{p^2} - \frac{1}{p^3} - \frac{1}{p^4} \right) \cdot X,$$

which is equal to the mass formula predicted for the number of D_4 -fields of bounded discriminant (see Theorem 5.4).

Before describing the proof of Theorem 1, we compare our argument for obtaining (2) to that of Cohen-Diaz y Diaz-Olivier implying (1). Because of the convergence of $\sum_K \text{Disc}(K)^{-2}$ when K runs over all quadratic fields, the main contribution to the sum in (1) comes from the terms indexed by K with $\text{Disc}(K) < X^{1/2}$, i.e., quadratic fields with small discriminant relative to X. However, this is not the case for the sum in (2); furthermore, the combination of Kummer theory and L-function methods utilized in [8] to prove (1) can only be used to determine the asymptotic number of D_4 -fields with conductor bounded by X whose quadratic subfield has small discriminant relative to X (see Theorem 4.3). Additionally, adapting the geometry-of-numbers techniques of [2] in combination with Wood's parametrization of quartic rings with a quadratic subring [27] is also limited to counting D_4 -fields whose quadratic subfields have small discriminant.

Nevertheless, we obtain Theorem 1 and subsequently (2) by employing algebraic properties of the conductor C(L) of a D_4 -field L, namely that it is invariant under the outer automorphism ϕ of D_4 . More precisely,

 ϕ acts on the Galois group of the normal closure M over \mathbb{Q} of L, and the fixed field of $\phi(\operatorname{Gal}(M/L))$ is another D_4 -field $\phi(L)$ in M which is not isomorphic to L. The conductors of L and $\phi(L)$ coincide (even though their discriminants do not). Moreover, we relate the discriminants of K and $\phi(K)$ using the central inertia of L (see Definition 2.3). We prove (see Proposition 2.6)

$$|\operatorname{Disc}(K)| > \operatorname{C}(L)^{1/2} \quad \Rightarrow \quad |\operatorname{Disc}(\phi(K))| < \operatorname{C}(\phi(L))^{1/2},$$

and we use this phenomenon to obtain $N_{\mathrm{D}_4}^{(r_2)}(X)$ from the asymptotic number of D_4 -fields ordered by conductor whose quadratic subfield has small discriminant by employing a simple sieve.

The proof of Theorem 1 does ultimately rely on both the analytic techniques of [8] used to count D_4 -fields by discriminant as well as the geometry-of-numbers methods used to count S_4 -fields as in [2]. Either can be used to obtain asymptotics for D_4 -fields of bounded conductor whose quadratic subfield has small discriminant, but the sieve used to determine the asymptotics of $N_{D_4}^{(r_2)}(X)$ from counting such D_4 -fields requires two ingredients: first, uniform error estimates on the number of such D_4 -fields having large central inertia, and second, asymptotics for the number of such D_4 -fields with prescribed ramification conditions. We are able to obtain the former using analytic methods and the latter using geometry-of-numbers techniques. In fact, this method of proof allows us to count D_4 -fields with prescribed splitting and ramification conditions yielding Theorems 3 and 4. Additionally, Theorem 3 in conjunction with Theorem 4.3 and the computations of p-adic densities (see Proposition 6.7) implies Theorem 5.

We remark that we obtain three different interpretations for the Euler product

$$\prod_{p} \left(1 - \frac{1}{p^2} + \frac{2}{p^3} + \frac{2}{p^4} \right).$$

It arises in the residue of a double Dirichlet series defined in Section 3, the contribution from diagonal terms in the sum over quadratic fields K of $\frac{L(1,K/\mathbb{Q})}{L(2,K/\mathbb{Q})}$ weighted by $\mathrm{Disc}(K)^{-1}$ in Section 5, and the p-adic volumes of a coregular representation of a non-reductive group in Section 6.

This paper is organized as follows. First, we summarize basic arithmetic properties of D_4 -fields and their invariants in Section 2, including a table of the splitting behavior of primes in D_4 -fields depending on their inertia and decomposition group. We explicitly relate the conductors and quadratic discriminants of D_4 -fields L and $\phi(L)$. Next in Section 3, we further develop the method of [3, 18, 26] and simultaneously obtain heuristics for families of D_4 -fields ordered in multiple different ways. We begin counting D_4 -fields with bounded invariants in Section 4. Using the analytic methods of [8], we obtain asymptotics for the number of such fields with small quadratic discriminant in terms of a sum of ratios of L-values. By isolating the diagonal terms, we prove Theorem 2 in Section 5. In Section 6, we recall Wood's parametrization of quartic rings with a quadratic subrings and adapt it to obtain a modified parametrization of D_4 -fields in terms of certain integral orbits of a coregular representation V for a non-reductive group G. We study the p-adic properties of this representation, including those arising from the outer automorphism ϕ . We then use geometry-of-numbers methods in Section 7 to count integral orbits of G on V having bounded invariants. Using the results and methods from Sections 4, 6, and 7, we obtain crucial uniformity estimates in Section 8 that will be necessary to carry out the various requisite sieves. Finally, in Section 9, we prove the main theorems by using the analytic results of Sections 4, 7, and 8, in conjunction with the algebraic properties of the outer automorphism ϕ proved in Sections 2 and 6.

2 General properties of D_4 -fields

Recall that D_4 denotes the order-8 group of symmetries of a square, and a (quartic) D_4 -field is a degree-4 field extension of \mathbb{Q} whose normal closure has Galois group D_4 over \mathbb{Q} . We let σ denote a 90°-rotation of a square and τ denote a reflection of a square so that

$$D_4 = \langle \sigma, \tau \mid \sigma^4 = 1, \tau^2 = 1, \tau^{-1} \sigma \tau = \sigma^3 \rangle.$$

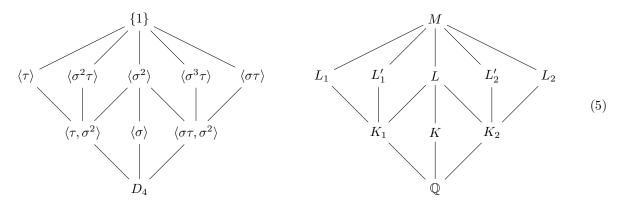
The group D_4 has nontrivial center $Z(D_4) = \{1, \sigma^2\}$.

2.1 Automorphisms of D_4 and the Galois theory of D_4 -fields

We first describe important group-theoretic properties of D_4 as well as their applications to D_4 -fields via Galois theory. Recall that the inner automorphism group $D_4/Z(D_4)$ is isomorphic to the Klein four group V_4 , but the full automorphism group is isomorphic to D_4 (see pgs. 83–85 of [23]). The non-trivial outer automorphism ϕ of D_4 has order 4 and can be described explicitly by

$$\phi(\sigma) = \sigma \quad \text{and} \quad \phi(\tau) = \sigma \tau.$$
 (4)

Let L_1 be a D_4 -field, and denote its normal closure by M so that $Gal(M/\mathbb{Q}) = D_4$. Below, we describe a subfield diagram of M corresponding to the subgroup lattice of D_4 :



Here, a subgroup G of D_4 and a subfield F of M in the same position are related by $\operatorname{Gal}(M/F) = G$. The fields L_i' are the (unique) Galois conjugates of L_i for i = 1 or 2, and L is the unique quartic Galois subfield of M. While L_1 and L_2 are not conjugate, the outer automorphism ϕ maps $\operatorname{Gal}(M/L_1) \to \operatorname{Gal}(M/L_2)$ and $\operatorname{Gal}(M/L_1) \to \operatorname{Gal}(M/L_2) \to \operatorname{Gal}(M/L_1) \to \operatorname{Gal}(M/L_1)$. It also interchanges $\operatorname{Gal}(M/K_1)$ and $\operatorname{Gal}(M/K_2)$ while leaving $\operatorname{Gal}(M/K)$ fixed.

Definition 2.1. If L_1 is a D_4 -field with Galois closure M over \mathbb{Q} , then we denote by $\phi(L_1)$ the quartic subfield of M fixed by $\phi(\operatorname{Gal}(M/L_1))$. If K_1 denotes the quadratic subfield of L_1 , then we denote by $\phi(K_1)$ the quadratic subfield of M fixed by $\phi(\operatorname{Gal}(M/K_1))$.

In the notation of (5), we have $\phi(L_1) = L_2$ and $\phi(K_1) = K_2$.

2.2 Arithmetic of D_4 -fields

We now describe the splitting behavior of primes in octic D_4 -fields M and their subfields.

Definition 2.2. If F is a number field, then the splitting type $\varsigma_n(F)$ at p of F satisfies

$$\varsigma_p(F) = (f_1^{e_1} f_2^{e_2} \dots) \quad \Leftrightarrow \quad \mathcal{O}_F / p \mathcal{O}_F \cong \mathbb{F}_{p^{f_1}}[t_1] / (t_1^{e_1}) \oplus \mathbb{F}_{p^{f_2}}[t_2] / (t^{e_2}) \oplus \dots$$

Similarly, if R is a ring, then the splitting type $\varsigma_p(R)$ at p is equal to $(f_1^{e_1}f_2^{e_2}\ldots)$ if and only if

$$R/pR \cong \mathbb{F}_{p^{f_1}}[t_1]/(t_1^{e_1}) \oplus \mathbb{F}_{p^{f_2}}[t_2]/(t^{e_2}) \oplus \dots$$

Let D_p denote the decomposition group of p in $Gal(M/\mathbb{Q})$, and let I_p denote the inertia subgroup of D_p . For an arbitrary prime p, we determine the splitting type of M and all of its subfields using the notation described in (5) depending on the choices for D_p and I_p in the table below.

In Table 1, the first group consists of unramified splitting types, the second and third groups consist of tamely ramified splitting types, and the fourth group consists of wildly ramified splitting types. In particular, the splitting type of an odd prime p must appear in the first three groups of Table 1. We distinguish between the tamely ramified splitting types depending on whether the center $\langle \sigma^2 \rangle$ of D_4 is contained in I_p .

I_p	D_p	$\varsigma_p(M)$	$ \varsigma_p(L_1) $	$\varsigma_p(K_1)$	$ \varsigma_p(L_2) $	$\varsigma_p(K_2)$	$\varsigma_p(L)$	$\varsigma_p(K)$		
{1} {1} {1} {1} {1} {1}	$ \begin{cases} 1 \\ \langle \sigma^2 \rangle \\ \langle \sigma \tau \rangle \\ \langle \tau \rangle \\ \langle \sigma \rangle \end{cases} $	(11111111) (2222) (2222) (2222) (2222) (44)	(1111) (22) (22) (112) (4)	(11) (11) (2) (11) (2)	(1111) (22) (112) (22) (4)	(11) (11) (11) (2) (2)	(1111) (1111) (22) (22) (22)	(11) (11) (2) (2) (11)	Unramified	Lacks central inertia
$\begin{array}{c} \langle \tau \rangle \\ \langle \tau \rangle \\ \langle \sigma \tau \rangle \\ \langle \sigma \tau \rangle \end{array}$	$ \begin{array}{c} \langle \tau \rangle \\ \langle \tau, \sigma^2 \rangle \\ \langle \sigma \tau \rangle \\ \langle \sigma \tau, \sigma^2 \rangle \end{array} $	$ \begin{array}{c} (1^21^21^21^2) \\ (2^22^2) \\ (1^21^21^21^2) \\ (2^22^2) \end{array} $	$ \begin{array}{c} (1^211) \\ (1^22) \\ (1^21^2) \\ (2^2) \end{array} $	$ \begin{array}{c} (11) \\ (11) \\ (1^2) \\ (1^2) \end{array} $	$ \begin{array}{c} (1^21^2) \\ (2^2) \\ (1^211) \\ (1^22) \end{array} $	$ \begin{array}{c} (1^2) \\ (1^2) \\ (11) \\ (11) \end{array} $	$ \begin{array}{c} (1^21^2) \\ (1^21^2) \\ (1^21^2) \\ (1^21^2) \end{array} $	$ \begin{array}{c} (1^2) \\ (1^2) \\ (1^2) \\ (1^2) \end{array} $	Tame	al inertia
$ \begin{array}{c} \langle \sigma \rangle \\ \langle \sigma \rangle \\ \langle \sigma^2 \rangle \end{array} $	$ \begin{array}{c} \langle \sigma \rangle \\ D_4 \\ \langle \sigma^2 \rangle \\ \langle \tau, \sigma^2 \rangle \\ \langle \sigma \tau, \sigma^2 \rangle \\ \langle \sigma \sigma, \sigma^2 \rangle \\ \langle \sigma \rangle \end{array} $	$ \begin{array}{c} (1^41^4) \\ (2^4) \\ (1^21^21^21^2) \\ (2^22^2) \\ (2^22^2) \\ (2^22^2) \end{array} $	$ \begin{array}{c} (1^4) \\ (1^4) \\ (1^21^2) \\ (1^21^2) \\ (2^2) \\ (2^2) \end{array} $	$ \begin{array}{c} (1^2) \\ (1^2) \\ (11) \\ (11) \\ (2) \\ (2) \end{array} $	$ \begin{array}{c} (1^4) \\ (1^4) \\ (1^21^2) \\ (2^2) \\ (1^21^2) \\ (2^2) \end{array} $	(1 ²) (1 ²) (11) (2) (11) (2)	$ \begin{array}{c} (1^21^2) \\ (2^2) \\ (1111) \\ (22) \\ (22) \\ (22) \end{array} $	(11) (1 ²) (11) (2) (2) (11)	Tame	Has central inertia
$ \begin{array}{c} \langle \tau, \sigma^2 \rangle \\ \langle \sigma \tau, \sigma^2 \rangle \\ D_4 \end{array} $	$ \begin{array}{c} \langle \tau, \sigma^2 \rangle \\ \langle \tau, \sigma^2 \rangle \\ D_4 \end{array} $	$ \begin{array}{c} (1^41^4) \\ (1^41^4) \\ (1^8) \end{array} $	$ \begin{array}{c} (1^21^2) \\ (1^4) \\ (1^4) \end{array} $	(11) (1^2) (1^2)	$ \begin{array}{c} (1^4) \\ (1^21^2) \\ (1^4) \end{array} $	(1^2) (11) (1^2)	$ \begin{array}{c} (1^21^2) \\ (1^21^2) \\ (1^4) \end{array} $	$ \begin{array}{c} (1^2) \\ (1^2) \\ (1^2) \end{array} $	Wild	ıertia

Table 1: Splitting type for a given decomposition and inertia group.

Definition 2.3. We say that a D_4 -field L_1 (or a pair (L_1, K_1) consisting of a D_4 -field L_1 and its quadratic subfield K_1) has central inertia at p when I_p contains σ^2 or equivalently, when the pair $(\varsigma_p(L_1), \varsigma_p(K_1))$ is $((1^21^2), (11)), ((2^2), (2)), \text{ or } ((1^4), (1^2)).$

Note that if M and K are as in (5), the extension M/K is ramified at a prime p if and only if (L_1, K_1) has central inertia at p.

By analogy, we define splitting types at ∞ . If K is any quadratic field and L is any quartic field, we write

$$\varsigma_{\infty}(K) := \begin{cases}
(11) & \text{if } K \otimes \mathbb{R} = \mathbb{R}^2, \\
(2) & \text{if } K \otimes \mathbb{R} = \mathbb{C};
\end{cases}$$

$$\varsigma_{\infty}(L) := \begin{cases}
(1111) & \text{if } L \otimes \mathbb{R} = \mathbb{R}^4, \\
(112) & \text{if } L \otimes \mathbb{R} = \mathbb{R}^2 \oplus \mathbb{C}, \\
(22) & \text{if } L \otimes \mathbb{R} = \mathbb{C}^2.
\end{cases}$$
(6)

2.3 Invariants of D_4 -fields

Next, we compare the Artin conductor of a D_4 -field L_1 to the discriminant of L_1 as well as the products of the discriminants of certain subfields of the normal closure of L_1 . Additionally, we define two fundamental invariants and a slightly refined conductor that partially recovers the splitting type of L_1 at ∞ .

If $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ denotes the absolute Galois group of \mathbb{Q} , and M is the normal closure of L_1 as in (5), then there is a (unique up to conjugacy) irreducible 2-dimensional Galois representation

$$\rho_M: \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{GL}_2(\mathbb{C})$$

that factors through $\operatorname{Gal}(M/\mathbb{Q})$. It arises as the composition of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \operatorname{Gal}(M/\mathbb{Q})$ and the unique 2-dimensional irreducible representation of $D_4 \cong \operatorname{Gal}(M/\mathbb{Q})$. We let $\operatorname{Cond}(\rho_M)$ denote the Artin conductor of ρ_M . This invariant can be described in terms of the discriminant of the quadratic subfield K_1 and $\operatorname{Nm}_{K_1}(\operatorname{Disc}(L_1/K_1))$, the image under the norm map of K_1 of the relative discriminant of L_1 over K_1 :

Proposition 2.4. Let L_1 denote a D_4 -field with normal closure M, and let K_1 be its quadratic subfield. We then have:

$$\operatorname{Cond}(\rho_M) = |\operatorname{Disc}(K_1) \cdot \operatorname{Nm}_{K_1}(\operatorname{Disc}(L_1/K_1))|.$$

Proof. The proposition will follow from the fact that the representation $\operatorname{Ind}_{\langle \tau \rangle}^{D_4} \mathbb{1}$ of D_4 decomposes into a direct sum of $\operatorname{Ind}_{\langle \tau, \sigma^2 \rangle}^{D_4} \mathbb{1}$ and the irreducible 2-dimensional representation of D_4 . To prove this fact, first note that each coset of $\langle \tau \rangle$ in D_4 contains a unique power of σ , so we can represent $\operatorname{Ind}_{\langle \tau \rangle}^{D_4}$ in terms of the basis $\langle [1], [\sigma], [\sigma^2], [\sigma^3] \rangle$. We can then decompose $\operatorname{Ind}_{\langle \tau \rangle}^{D_4} \mathbb{1} = V_1 \oplus V_2$ where

$$V_1 = \left\langle [1] + [\sigma^2], [\sigma] + [\sigma^3] \right\rangle; \qquad V_2 = \left\langle [1] - [\sigma^2], [\sigma] - [\sigma^3] \right\rangle.$$

Since σ swaps the two basis elements of V_1 while τ and σ^2 act trivially, V_1 can be identified with $\operatorname{Ind}_{\langle \tau, \sigma^2 \rangle}^{D_4} \mathbb{1}$. Furthermore, one can see that V_2 is irreducible, and it is well-known that there is a unique irreducible 2-dimensional representation of D_4 .

Now, if M is the normal closure of L_1 as in (5) and ρ denotes the Galois representation constructed by composing $\operatorname{Ind}_{\langle \tau \rangle}^{D_4} \mathbbm{1}$ with $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \operatorname{Gal}(M/\mathbb{Q})$, then its Artin conductor satisfies

$$\operatorname{Cond}(\rho) = |\operatorname{Disc}(L_1)|.$$

We can compute $\operatorname{Cond}(\rho)$ as a product of the conductors of its subrepresentations: the Galois representation $\operatorname{Ind}_{\langle \tau, \sigma^2 \rangle}^{D_4} \mathbb{1} \circ \left(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \operatorname{Gal}(M/\mathbb{Q}) \right)$ has conductor $\operatorname{Disc}(K_1)$, so we obtain

$$|\operatorname{Disc}(L_1)| = |\operatorname{Disc}(K_1)| \cdot \operatorname{Cond}(\rho_M).$$

The relative discriminant formula implies that $|\operatorname{Disc}(L_1)| = \operatorname{Disc}(K_1)^2 \operatorname{Nm}_{K_1}(\operatorname{Disc}(L_1/K_1))$, and so we conclude the proposition.

Definition 2.5. The (signed) conductor $C(L_1)$ of a D_4 -field L_1 whose quadratic subfield is denoted by K_1 is defined as

$$C(L_1) := \frac{\operatorname{Disc}(L_1)}{\operatorname{Disc}(K_1)}.$$

From the definition of the conductor and Proposition 2.4, it follows immediately that two D_4 -fields L_1 and L_2 with the same normal closure M have the same conductor. Furthermore, if L_1 has no central inertia, then $C(L_1) = Disc(K_1) \cdot Disc(\phi(K_1))$. More precisely, if L is a number field and p is a prime number, let $Disc_p(L)$ denote the p-part of the discriminant, and let $C_p(L)$ be the p-part of the conductor. We then have:

Proposition 2.6. If L_1 is a D_4 -field with quadratic subfield K_1 , then for all odd primes p:

$$C_p(L_1) = \begin{cases} p^2 \cdot \operatorname{Disc}_p(K_1) \cdot \operatorname{Disc}_p(\phi(K_1)) & \text{if } I_p = \langle \sigma^2 \rangle; \\ \operatorname{Disc}_p(K_1) \cdot \operatorname{Disc}_p(\phi(K_1)) & \text{otherwise.} \end{cases}$$

Proof. We refer to the notation described in (5), where $\phi(K_1) = K_2$. Table 1 shows that if $I_p \neq \langle \sigma^2 \rangle$, then $\operatorname{Disc}_p(K_2) = \operatorname{Nm}_{K_1}(\operatorname{Disc}_p(L_1/K_1))$. Thus, $\operatorname{C}_p(L_1) = \operatorname{Disc}_p(K_1) \cdot \operatorname{Disc}_p(K_2)$. However, when $I_p = \langle \sigma^2 \rangle$, Table 1 implies that $\operatorname{Disc}_p(K_1) = \operatorname{Disc}_p(K_2) = 1$, but $\operatorname{Nm}_{K_1}(\operatorname{Disc}_p(L_1/K_1)) = \operatorname{Nm}_{K_2}(\operatorname{Disc}_p(L_2/K_2)) = p^2$. Thus, we have that $\operatorname{C}_p(L_1) = p^2 \cdot \operatorname{Disc}_p(K_1) \cdot \operatorname{Disc}_p(K_2)$.

We are now ready to define the two fundamental invariants of a D_4 -field.

Definition 2.7. If L_1 is a D_4 -field with quadratic subfield K_1 , define the fundamental invariants of L_1 as

$$q(L_1) := \frac{\operatorname{Disc}(L_1)}{\operatorname{Disc}(K_1)^2}$$
 and $d(L_1) := \operatorname{Disc}(K_1)$.

Remark 2.8. For a D_4 -field L_1 , there is a global restriction on the integers $q(L_1)$ and $d(L_1)$, namely that they are each congruent to 0 or 1 mod 4.

Proposition 2.4 can be reformulated as $\operatorname{Cond}(\rho_M) = |\operatorname{q}(L_1) \cdot \operatorname{d}(L_1)| = |\operatorname{q}(L_2) \cdot \operatorname{d}(L_2)|$ for a D_4 -field L_1 as in (5). Define

$$J(L_1) := \frac{\mathrm{C}(L_1)}{\mathrm{Disc}(K_1) \cdot \mathrm{Disc}(\phi(K_1))} = \left| \frac{\mathrm{q}(L_1)}{\mathrm{d}(\phi(L_1))} \right|,\tag{7}$$

and for a prime p, let $J_p(L_1)$ denote the p-part of $J(L_1)$. Proposition 2.6 determines that for an odd prime p, $J_p(L_1)$ is equal to p^2 if and only if the inertia group I_p at p is equal to $\langle \sigma^2 \rangle \subset D_4$. Note that it is always true that $J(L_1) = J(\phi(L_1))$. Furthermore, it will not be necessary to compute J_2 ; it will be enough that J_2 is absolutely bounded.

3 Heuristics for counting D_4 -fields by conductor

In [3], Bhargava developed heuristics for the asymptotics of the number of S_n -fields of degree n ordered by discriminant. The framework used to formulate these heuristics was expanded by Kedlaya [18] for families of Galois representations ordered by their Artin conductor. Additionally, Wood [26] predicted asymptotics (including mass formulae for the constants) for fixed-degree families of number fields whose normal closures have a fixed Galois group when such fields are ordered by invariants including the conductor. In this section, we adapt their heuristics to the family of D_4 -fields ordered by our two fundamental invariants, q and d (see Definition 2.7). We recover the predictions in [26] for the number of D_4 -fields ordered either by conductor or discriminant, and we additionally verify that the conjectured mass formula when ordering by discriminant is not equal to the constant c determined by Cohen-Diaz y Diaz-Olivier in [8].

3.1 The expected number of D_4 -fields for a given pair of fundamental invariants

Let v be a place of \mathbb{Q} , and let $K_v \subset L_v$ be étale algebras of \mathbb{Q}_v of degrees 2 and 4, respectively. When v corresponds to a finite prime p (resp. infinity), we say that such a pair (L_v, K_v) is compatible with a pair of integers (q, d) if the p-parts (resp. signs) of $\operatorname{Nm}_{K_v}(\operatorname{Disc}(L_v/K_v))$, the norm in \mathbb{Q}_v of the relative discriminant of L_v over K_v , and $\operatorname{Disc}(K_v)$, the discriminant of K_v , agree with the p-parts (resp. signs) of q and q0, respectively. Note that when q > 0, q1 can be positive or negative; however, when q < 0, q2 must be positive, otherwise no such compatible pairs (L_∞, K_∞) exist.

Given a place v of \mathbb{Q} and integers q and d, let $\Sigma_v(q, d)$ denote the set of pairs of \mathbb{Q}_v -algebras (L_v, K_v) that are compatible with (q, d). Let the weighted local mass $E_v(q, d)$ be defined by

$$E_{v}(\mathbf{q}, \mathbf{d}) := \sum_{(L_{v}, K_{v}) \in \Sigma_{v}(\mathbf{q}, \mathbf{d})} \frac{1}{\# \operatorname{Aut}(L_{v}, K_{v})},$$

where $\operatorname{Aut}(L_v, K_v)$ is the group of automorphisms of L_v that restrict to endomorphisms of the subalgebra K_v . The following result evaluates $E_v(\mathbf{q}, \mathbf{d})$ for all places v.

Proposition 3.1. We have:

- (1) If q and d are nonzero, then $E_{\infty}(q, d) = 1/4$ when at least one of q or d is positive.
- (2) If v corresponds to an odd prime p, then $E_p(q, d)$ is nonempty if and only if the p-parts of (q, d) are one of (1, 1), (p, 1), $(p^2, 1)$, (1, p), or (p, p). In each case, we have $E_p(q, d) = 1$.
- (3) The values of $E_2(q, d)$ are given below.

q	d = 1	$d = 2^2$	$d = 2^3$
1	1	1	2
2^{2}	1	1	2
2^{3}	2	_	_
$ \begin{array}{ccc} 2^{2} \\ 2^{3} \\ 2^{4} \\ 2^{5} \\ 2^{6} \end{array} $	2	2	4
2^{5}	2	4	8
2^{6}	4	_	

Table 2: The value of $E_2(q, d)$

Proof. The proposition follows from a direct computation using a database of local fields away from p = 2. The code for the case p = 2 using [19, 17] can be found at http://github.com/khwilson/D4Counting.

The framework in [3, 26] depends on the basic heuristic assumption that for a family of number fields of fixed degree and fixed associated Galois group, the completions at different places behave independently of one another. This implies that the expected number of such number fields having given invariants is equal to the infinite product over all places v of \mathbb{Q} of the weighted number of local extensions of \mathbb{Q}_v that are compatible with those invariants. More precisely:

Assumption 3.2. If E(q, d) denotes the expected number of isomorphism classes of D_4 -fields with fundamental invariants equal to q and d, we assume

$$E(\mathbf{q}, \mathbf{d}) = \frac{1}{2} \cdot E_{\infty}(\mathbf{q}, \mathbf{d}) \cdot \prod_{p} E_{p}(\mathbf{q}, \mathbf{d}). \tag{8}$$

The extra factor of $\frac{1}{2}$ above arises from two issues: (1) there is a global restriction on the invariants q and d (see Remark 2.8), which occurs $\frac{1}{4}$ of the time, and is not taken into account by the local masses, and (2) the product of the local masses $E_v(q, d)$ determines the expected weighted number of D_4 -fields with invariants q and d, where a D_4 -field L is weighted by $\# \operatorname{Aut}(L)^{-1} = \frac{1}{2}$.

3.2 Predicting the global distribution of D_4 -fields using double Dirichlet series

To determine the asymptotics of $\sum E(q, d)$, we study the behavior of the double Dirichlet series

$$\xi(s,t) := \sum_{\mathbf{d}} \sum_{\mathbf{q}} \frac{E(\mathbf{q}, \mathbf{d})}{|\mathbf{q}|^s |\mathbf{d}|^t},$$

which converges absolutely for s, t > 1. Since there are three possible sign configurations for the pair of integers (q, d), the archimedean contribution to $\xi(s, t)$ is exactly 3/4. Additionally, $2 \cdot E(q, d)$ is multiplicative with respect to both q and d, and so it follows from Proposition 3.1 that $\xi(s, t)$ can be expressed as

$$\xi(s,t) = \frac{3}{8} \cdot \prod_{p} \xi_p(s,t),$$

where

$$\xi_p(s,t) = \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^t} \left(1 + \frac{1}{p^s}\right)\right)$$

when p is odd, and

$$\xi_2(s,t) = \left(1 + \frac{1}{2^{2s}} + \frac{2}{2^{3s}} + \frac{2}{2^{4s}} + \frac{2}{2^{5s}} + \frac{4}{2^{6s}} + \frac{1}{2^{2t}} \left(1 + \frac{1}{2^{2s}} + \frac{2}{2^{4s}} + \frac{4}{2^{5s}}\right) + \frac{2}{2^{3t}} \left(1 + \frac{1}{2^{2s}} + \frac{2}{2^{4s}} + \frac{4}{2^{5s}}\right)\right).$$

Define the correction factor at 2 to be

$$\widetilde{\xi}_2(s,t) := \xi_2(s,t) / \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^t} \left(1 + \frac{1}{2^s}\right)\right).$$

We can rewrite $\xi(s,t)$ as

$$\xi(s,t) = \frac{3}{8} \cdot \widetilde{\xi}_2(s,t) \cdot \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^t} \left(1 + \frac{1}{p^s} \right) \right)$$

$$= \frac{3}{8} \cdot \widetilde{\xi}_2(s,t) \cdot \zeta(s) \cdot \zeta(t) \cdot \prod_p (1 - p^{-2t} - p^{-t-2s} - p^{-3s} + p^{-2t-2s} + p^{-t-3s}).$$

Therefore, the function $\xi(s,t)$ is holomorphic in the region $t > \frac{1}{2}$, $s > \frac{1}{3}$ aside from poles at the lines s = 1 and t = 1.

3.3 Heuristics

We now consider families of D_4 -fields under different orderings. If X and Y be positive real numbers going to infinity, let $E_{q,d}(X,Y)$ denote the expected number of isomorphism classes of D_4 -fields L such that |q(L)| < X and |d(L)| < Y, i.e.

$$E_{\mathbf{q},\mathbf{d}}(X,Y) := \sum_{\substack{|\mathbf{q}| < X \\ |\mathbf{d}| < Y}} E(\mathbf{q},\mathbf{d}).$$

Then, by computing the residue of $\xi(s,t)$ at (1,1), we obtain the heuristic

$$E_{q,d}(X,Y) \sim \frac{3}{8} \cdot \prod_{p} \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4} \right) \cdot X \cdot Y.$$
 (9)

Here, the correction factor $\tilde{\xi}_2(1,1) = 1$. Note that this heuristic only relies on the Assumption 3.2. If we were to take (8) as a definition, then we have completely verified the main term (9), and we can additionally obtain a power-saving.

Heuristics for the family of D_4 -fields ordered by conductor

Next, we consider the family of D_4 -fields ordered by conductor. Let $E_{\rm C}(X)$ denote the expected number of isomorphism classes of D_4 -fields L such that $|{\rm C}(L)| < X$. If we let $E({\rm C})$ denote the expected number of D_4 -fields with conductor C, then we have

$$\sum_{C} \frac{E(C)}{|C|^s} = \xi(s, s)$$

since C(L) = q(L)d(L). The function $\xi(s,s)$ has a double pole at 1 and, by computing its residue, we obtain

$$E_{\rm C}(X) \sim \frac{3}{8} \cdot \prod_{p} \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4} \right) \cdot X \log(X).$$
 (10)

The correction factor at 2 is again $\tilde{\xi}_2(1,1) = 1$.

Heuristics for the family of D_4 -fields ordered by discriminant

Finally, we consider the family of D_4 -fields ordered by discriminant. Let $E_{\text{Disc}}(X)$ denote the expected number of isomorphism classes of D_4 -fields L such that $|\operatorname{Disc}(L)| < X$. If we let $E(\operatorname{Disc})$ denote the expected number of D_4 -fields L with discriminant equal to Disc, then we have

$$\sum_{\text{Disc}} \frac{E(\text{Disc})}{|\text{Disc}|^s} = \xi(s, 2s)$$

since $\operatorname{Disc}(L) = \operatorname{q}(L)\operatorname{d}(L)^2$. The function $\xi(s,2s)$ has a simple pole at 1 and, by computing the residue, we obtain

$$E_{\text{Disc}}(X) \sim \frac{3 \cdot 11^2}{2^6 \cdot 17} \cdot \prod_{p} \left(1 + \frac{1}{p^2} - \frac{1}{p^3} - \frac{1}{p^4} \right) \cdot X.$$
 (11)

In this case, the correction factor at 2 is $\widetilde{\xi}_2(1,2) = 11^2/(17 \cdot 2^3)$.

Cohen-Diaz y Diaz-Olivier showed in Proposition 6.2 of [8] that the number of D_4 -fields having discriminant bounded by X is $\sim cX$ where $c \approx 0.052$, whereas the constant on the right hand side of (11) is ≈ 0.406 . This implies that when ordering D_4 -fields by discriminant, the completions of such fields at different primes do *not* behave independently of one another in the sense of [3], and so Assumption 3.2 is false.

4 Counting D_4 -fields using analytic methods

In this section, we obtain asymptotics for the number of D_4 -fields, ordered by conductor, whose quadratic subfield has small discriminant, following the methods of Cohen-Diaz y Diaz-Olivier [8] where similar asymptotics for the number of such D_4 -fields ordered by discriminant are determined. By refining their arguments, we are able to count D_4 -fields of bounded conductor whose quadratic subfields have small discriminant and satisfy a prescribed set of splitting conditions at a finite number of primes. We begin with a few definitions before giving the precise statement of the main theorem of the section.

Definition 4.1. If K is a quadratic field and L is a quadratic extension of K, define the *conductor* of the pair (L, K) as

$$C(L,K) := \frac{\operatorname{Disc}(L)}{\operatorname{Disc}(K)}.$$
(12)

If L is a D_4 -field and K denotes its (unique) quadratic subfield, then C(L, K) = C(L).

We refine the notion of a collection of local specifications described in the introduction. Let $\Sigma_v^{\text{full}} = \{(\varsigma_v(L), \varsigma_v(K))\}$ be the set of all pairs consisting of a possible splitting type for a place v in a D_4 -field L and a consistent splitting type at v for its quadratic subfield K. We refer to a collection $\Sigma = (\Sigma_v)_v$ as a set of local specifications if for each v, $\Sigma_v \subseteq \Sigma_v^{\text{full}}$.

Definition 4.2. A set of local specifications $\Sigma = (\Sigma_v)_v$ is *stable* if for every prime p and every quadratic splitting type ς'_p (equal to either (11), (2), or (1²)), the set Σ_p either contains all possible pairs $(*, \varsigma'_p)$ or none of them.

Additionally, we denote by $\mathcal{L}(\Sigma)$ the set of D_4 -fields L with quadratic subfield K such that $(\varsigma_v(L), \varsigma_v(K)) \in \Sigma_v$ for all v. Similarly, let $\mathcal{K}(\Sigma)$ denote the set of quadratic subfields of $\mathcal{L}(\Sigma)$. Note that when Σ is stable, the set $\mathcal{L}(\Sigma)$ consists of all D_4 -fields that are quadratic extensions of all the fields in $\mathcal{K}(\Sigma)$.

For a set of local specifications Σ , let $N_{\rm C}(\Sigma;X,Y)$ be the number of isomorphism classes of D_4 -fields $L \in \mathcal{L}(\Sigma)$ such that $|{\rm C}(L)| < X$ and $|{\rm d}(L)| < Y$. Additionally, set $N_{\rm C}(X,Y) := N_{\rm C}(\Sigma^{\rm full};X,Y)$. In this section, we compute asymptotics for $N_{\rm C}(\Sigma;X,X^\beta)$ when Σ is stable and $\beta < 2/3$. More precisely, our goal is to prove the following theorem:

Theorem 4.3. Let Σ be a stable set of local specifications. Then, for every $\beta < 2/3$, we have

$$N_{\mathbf{C}}(\Sigma; X, X^{\beta}) = \frac{1}{2\zeta(2)} \cdot \left(\sum_{\substack{K \in \mathcal{K}(\Sigma) \\ |\operatorname{Disc}(K)| < X^{\beta}}} \frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} \cdot \frac{2^{-r_2(K)}}{|\operatorname{Disc}(K)|} \right) \cdot X + o_{\beta}(X),$$

We do so by first demonstrating that the number of quadratic extensions of quadratic number fields that are not D_4 -quartic fields is negligible, thus we can compute $N_{\rm C}(\Sigma;X,X^\beta)$ in terms of these towers of quadratic extensions. In [8], the authors define a Dirichlet series for each quadratic field K whose residue at s=1 is shown to be equal to the number of quadratic extensions of K. We then carry out a smooth count for the quartic fields in $\mathcal{L}(\Sigma)$ that are quadratic extensions of K and subsequently obtain the theorem by summing over all $K \in \mathcal{K}(\Sigma)$.

4.1 Quadratic extensions of quadratic number fields

If L is a quadratic extension of a quadratic field K, then L is either a D_4 -field or it is Galois with $Gal(L/\mathbb{Q}) = C_4$ or V_4 . In the following lemma, we prove a bound for the number of pairs (L, K) having bounded conductor, where L is a Galois quartic field and K is a quadratic subfield of L having small discriminant.

Lemma 4.4. Let $\beta < 1$ be fixed. The number of pairs of (L, K), where L is a Galois quartic field, K is a quadratic subfield of L, the conductor C(L, K) < X, and $|\operatorname{Disc}(K)| < X^{\beta}$ is bounded by $O_{\epsilon}(X^{(1+\beta)/2+\epsilon})$.

Proof. Let (L, K) be a pair satisfying the conditions of the lemma. By the relative discriminant formula, we have

$$|\operatorname{Disc}(L)| = |\operatorname{Disc}(K) \cdot \operatorname{C}(L, K)| < X^{1+\beta}.$$

It is known from §2.4 and §2.5 of [9] that the number of Galois quartic fields whose discriminant have absolute value less than X is bounded by $O_{\epsilon}(X^{1/2+\epsilon})$. The lemma follows immediately.

For stable Σ , we can thus prove Theorem 4.3 by counting the number of quadratic extensions over quadratic fields in $\mathcal{K}(\Sigma)$ whose relative discriminants have bounded norm. To this end, we consider the Dirichlet series $\Phi_{K,2}(s) = \Phi_{K,2}(C_2, s)$ defined in [8] for any number field K as

$$\Phi_{K,2}(s) := \sum_{[L:K]=2} \frac{1}{\operatorname{Nm}_K(\operatorname{Disc}(L/K))^s}.$$

It is proved in Theorem 1.1 of [8] that

$$\Phi_{K,2}(s) = -1 + \frac{2^{-r_2(K)}}{\zeta_K(2s)} \cdot \sum_{\mathfrak{c} \mid 2} \frac{\operatorname{Nm}_K(2/\mathfrak{c})}{\operatorname{Nm}_K(2/\mathfrak{c})^{2s}} \cdot \sum_{\chi \in \operatorname{Cl}(K,\mathfrak{c}^2)^{\vee}} L_K(s,\chi),$$

where $r_2(K)$ denotes the number of pairs of complex embeddings of K, \mathfrak{c} runs over all integral ideals of K dividing 2, χ runs over all quadratic characters of the ray class group modulo \mathfrak{c}^2 , and $L_K(s,\chi)$ is the L-function of K for χ . It is also proven in Corollary 1.2 of [8] that the rightmost pole of $\Phi_{K,2}(s)$ is at s=1 with residue given by

$$\operatorname{Res}_{s=1} \Phi_{K,2}(s) = \frac{2^{-r_2(K)}}{\zeta(2)} \cdot \frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})}.$$
 (13)

We can then obtain "smooth counts" of the number of quadratic extensions of quadratic fields K:

Lemma 4.5. Let φ be a smooth compactly supported function $\varphi: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. If K is quadratic, then

$$\begin{split} \sum_{[L:K]=2} \varphi \Big(\frac{|\operatorname{Disc}(K) \cdot \operatorname{Nm}_K(\operatorname{Disc}(L/K)|}{X} \Big) &= \operatorname{Vol}(\varphi) \cdot \operatorname{Res}_{s=1} \Phi_{K,2}(s) \cdot \frac{X}{|\operatorname{Disc}(K)|} \\ &+ O_{\epsilon,\varphi}(2^{\omega(\operatorname{Disc}(K))} |\operatorname{Disc}(K)|^{-1/4+\epsilon} X^{1/2+\epsilon}), \end{split}$$

where $\omega(d)$ denotes the number of prime divisors of d, and $\operatorname{Vol}(\varphi)$ denotes $\int \varphi(t)dt$.

Proof. Let $\tilde{\varphi}$ denote the Mellin transform of φ . By Mellin inversion, we see that the left hand side of the above equation is equal to

$$\frac{1}{2\pi i} \int_{\operatorname{Re}(s)=2} \tilde{\varphi}(s) \cdot \frac{X^s}{|\operatorname{Disc}(K)|^s} \cdot \Phi_{K,2}(s) \ ds.$$

Shifting the line of integration to $\operatorname{Re}(s) = 1/2 + \epsilon$, we pick up the main term from the pole at 1 since $\tilde{\varphi}(1) = \operatorname{Vol}(\varphi)$. The error term follows by using the convexity bound of $O_{\epsilon}(|\operatorname{Disc}(K)|^{1/4+\epsilon})$ near s = 1/2 at each of the $O\left(2^{\omega(\operatorname{Disc}(K))}\right)$ L-functions used to define $\Phi_{K,2}(s)$.

The next lemma, whose proof is very similar to that of Lemma 4.5, gives a bound when imposing ramification on the quadratic extensions of a fixed quadratic field K and will only be used in Section 8:

Lemma 4.6. If K is a quadratic field and \mathfrak{f} is a squarefree product of prime ideals in K, then the number of quadratic extensions L over K such that $\operatorname{Nm}_K(\operatorname{Disc}(L/K)) < X$ and every prime dividing \mathfrak{f} ramifies in L is bounded by

$$O_{\epsilon}\left(\frac{L(1, K/\mathbb{Q})}{\operatorname{Nm}(\mathfrak{f})} \cdot X + \frac{|\operatorname{Disc}(K)|^{-1/4}}{\operatorname{Nm}(\mathfrak{f})^{1/2}} \cdot X^{1/2+\epsilon}\right).$$

Proof. We can (and do) assume that f is odd. Since an upper bound for the number of quadratic extensions of K can be obtained with a smooth sum, we proceed as in the proof of Lemma 4.5. The only difference is that we use, instead of $\Phi_{K,2}(s)$, the Dirichlet series $\Phi_{K,2,f}(s)$ corresponding to extensions L of K that are ramified at every prime dividing \mathfrak{f} :

$$\begin{split} \Phi_{K,2,\mathfrak{f}}(s) & := \sum_{\substack{[L:K]=2\\L \text{ ramified at }\mathfrak{f}}} \frac{1}{\mathrm{Nm}_K(\mathrm{Disc}(L/K))^s} \\ & = -1 + \frac{2^{-r_2(K)}}{\zeta_K(2s)} \cdot \sum_{\mathfrak{c}\mid 2} \mathrm{Nm}_K(\mathfrak{c})^{2s-1} \cdot \sum_{\chi \in \mathrm{Cl}(K,\mathfrak{c}^2)^\vee} \bigg(\sum_{\substack{\mathfrak{a} \text{ squarefree}\\ (\mathfrak{a},\mathfrak{c})=1, \ \mathfrak{f}\mid \mathfrak{a}}} \frac{\chi(\mathfrak{a})}{\mathrm{Nm}_K(\mathfrak{a})^s} \bigg), \end{split}$$

where the notation is as in the definition of $\Phi_{K,2}(s)$. Since the residue of $\Phi_{K,2,f}(s)$ at 1, its rightmost pole, is $\ll L(1, K/\mathbb{Q})/Nm(\mathfrak{f})$, the lemma follows from an argument identical to the proof of Lemma 4.5.

4.2 Proof of Theorem 4.3

We are now ready to prove the main result of this section. From Lemma 4.4, it follows that we may estimate $N_{\rm C}(\Sigma; X, X^{\beta})$ by counting quadratic extensions L of quadratic fields K. Let $\chi_{[0,1]}$ denote the characteristic function of [0,1]. Then

$$N_{\mathcal{C}}(\Sigma; X, X^{\beta}) = \frac{1}{2} \sum_{\substack{K \in \mathcal{K}(\Sigma) \\ |\operatorname{Disc}(K)| < X^{\beta}}} \sum_{[L:K]=2} \chi_{[0,1]} \left(\frac{|\operatorname{Disc}(K) \cdot \operatorname{Nm}_{K}(\operatorname{Disc}(L/K))|}{X} \right). \tag{14}$$

The factor of 1/2 in the right hand side of (14) is to account for the fact that a D_4 -fields L and its conjugate

L' both contribute to the inner sum, while the left hand side of (14) counts D_4 -fields up to conjugacy. For $\epsilon > 0$, choose φ^{\pm} to be smooth compactly supported functions such that $\varphi^{\pm} - \chi_{[0,1]}$ takes values in \mathbb{R}^{\pm} and such that $\operatorname{Vol}(\varphi^{\pm}) = 1 \pm \epsilon$. Lemma 4.5 together with (13) implies that

$$\sum_{\substack{K \in \mathcal{K}(\Sigma) \\ |\operatorname{Disc}(K)| < X^{\beta}}} \sum_{\substack{[L:K]=2}} \varphi^{\pm} \left(\frac{|\operatorname{Disc}(K) \cdot \operatorname{Nm}_{K}(\operatorname{Disc}(L/K))|}{X} \right) = \sum_{\substack{K \in \mathcal{K}(\Sigma) \\ |\operatorname{Disc}(K)| < X^{\beta}}} \frac{1 \pm \epsilon}{\zeta(2)} \cdot \frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} \cdot \frac{2^{-r_{2}(K)}}{|\operatorname{Disc}(K)|} \cdot X + O_{\epsilon} \left(\sum_{\substack{K \in \mathcal{K}(\Sigma) \\ |\operatorname{Disc}(K)| < X^{\beta}}} |\operatorname{Disc}(K)|^{-\frac{1}{4} + \epsilon} X^{\frac{1}{2} + \epsilon} \right).$$

In the above equation, the left hand side corresponding to φ^+ (resp. φ^-) is an upper bound (resp. lower bound) for $N_{\rm C}(\Sigma; X, X^{\beta})$. Meanwhile the error term on the right hand side is bounded by $O_{\epsilon}(X^{\frac{1}{2} + \frac{3\beta}{4} + \epsilon})$ which, when $\beta < 2/3$, is bounded by o(X). Therefore, Theorem 4.3 follows by letting ϵ tend to 0.

Remark 4.7. We note that standard analytic methods (namely, Perron's formula in conjunction with hybrid (sub)convexity bounds on the growth of Hecke L-functions in the critical strip) yield a power saving in the error bound in Theorem 4.3. However, we do not include the arguments since they will not be necessary for the results of this paper.

Mass formulae for families of D_4 -fields 5

We now turn to the proof of Theorem 2. In the previous section, the constant in the asymptotic number of D_4 -fields with bounded conductor whose quadratic subfield has small discriminant was determined as a sum of L-values. In $\S5.1$ and $\S5.2$, we prove an identity relating the constant in Theorem 4.3 to an Euler product matching the predicted mass formula described in §3.3 by proving that the main contribution of the sum in the right hand side of Theorem 4.3 comes from certain diagonal terms. Finally, in §5.3, we study the family of D_4 -fields ordered by discriminant, and we prove an interesting identity between the analogous diagonal terms and the heuristic predicted by (11).

5.1 Isolating the diagonal terms in Theorem 4.3

We first prove a lemma that will be used in bounding the non-diagonal terms when we calculate the sum of L-values that appear in Theorem 4.3 in terms of a weighted Möbius sum.

Lemma 5.1. For any $\epsilon > 0$,

$$\sum_{\substack{0 < D < X \\ D \text{ square free}}} \frac{1}{D} \cdot \left(\sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ mn \neq \square}}^{D^{\frac{1}{2}+\epsilon}} \frac{\mu(m)}{m^2 n} \left(\frac{D}{mn} \right) \right) = O_{\epsilon}(1),$$

where (÷) denotes the Legendre symbol.

Proof. The m-sum is absolutely convergent, so we will focus on the n and the D-sums. Interchanging the n and D-sums yields

$$\sum_{\substack{0 < D < X \\ \text{D squarefree}}} \frac{1}{D} \cdot \sum_{\substack{n < D^{\frac{1}{2} + \epsilon} \\ mn \neq \square}} \frac{1}{n} \left(\frac{D}{mn}\right) = \sum_{\substack{n < X^{\frac{1}{2} + \epsilon} \\ mn \neq \square}} \frac{1}{n} \cdot \sum_{\substack{\frac{2}{n^{\frac{1}{1 + 2\epsilon}} < D < X} \\ D \text{ squarefree}}} \frac{1}{D} \left(\frac{D}{mn}\right). \tag{15}$$

We will now apply a simple squarefree sieve to complete the D-sum and then use the Pólya-Vinogradov inequality to finish the estimate. In particular, we can rewrite (15) as

$$\sum_{\substack{n < X^{\frac{1}{2} + \epsilon} \\ mn \neq \square}} \frac{1}{n} \cdot \left(\sum_{\alpha < n^{\frac{1}{1 + 2\epsilon}}} \left(\frac{\mu(\alpha)}{\alpha^2} \cdot \sum_{n^{\frac{2}{1 + 2\epsilon}} \leq \alpha^2 d < X} \frac{1}{d} \cdot \left(\frac{\alpha^2 d}{mn} \right) \right) + \sum_{n^{\frac{1}{1 + 2\epsilon}} \leq \alpha < X^{\frac{1}{2}}} \left(\frac{\mu(\alpha)}{\alpha^2} \cdot \sum_{n^{\frac{2}{1 + 2\epsilon}} \leq \alpha^2 d < X} \frac{1}{d} \cdot \left(\frac{\alpha^2 d}{mn} \right) \right) \right).$$

Thus, (15) is bounded by

$$\begin{split} & \ll \sum_{\substack{n < X^{\frac{1}{2} + \epsilon} \\ mn \neq \square}} \frac{1}{n} \cdot \left(\sum_{\alpha < n^{\frac{1}{1 + 2\epsilon}}} \frac{1}{\alpha^2} \cdot \Big| \sum_{\alpha^{-2} n^{\frac{2}{1 + 2\epsilon}} \leq d < \alpha^{-2} X} \frac{1}{d} \cdot \left(\frac{\alpha^2 d}{mn} \right) \Big| \right. \\ & + \sum_{\substack{n = 1 \\ n \neq \square}} \frac{1}{\alpha^2} \cdot \left| \sum_{\alpha < X^{\frac{1}{2}}} \frac{1}{d} \cdot \left(\frac{\alpha^2 d}{mn} \right) \Big| \right) \\ & \ll \sum_{\substack{n < X^{\frac{1}{2} + \epsilon} \\ mn \neq \square}} \frac{m^{\frac{1}{2}} \log(n)}{n^{\frac{1}{2} + \frac{1}{1 + 2\epsilon}}} \\ & = O_{\epsilon}(m^{\frac{1}{2}}). \end{split}$$

The lemma then follows from the absolute convergence of $\sum m^{-\frac{3}{2}}$.

The next result is the key input in obtaining the mass formula. Using Lemma 5.1, we rewrite the sum of L-values appearing in Theorem 4.3 in terms of a weighted Möbius sum that we will later show is equal to an Euler product. When ordering D_4 -fields by discriminant, there is no known analogue to Proposition 5.2.

Proposition 5.2. We have:

$$\sum_{\substack{[K:\mathbb{Q}]=2\\0<\operatorname{Disc}(K)< X}} \frac{L(1,K/\mathbb{Q})}{L(2,K/\mathbb{Q})} \cdot \frac{1}{|\operatorname{Disc}(K)|} = \sum_{\substack{[K:\mathbb{Q}]=2\\0<\operatorname{Disc}(K)< X}} \frac{1}{|\operatorname{Disc}(K)|} \cdot \sum_{\substack{0

$$\sum_{\substack{[K:\mathbb{Q}]=2\\-X<\operatorname{Disc}(K)<0}} \frac{L(1,K/\mathbb{Q})}{L(2,K/\mathbb{Q})} \cdot \frac{1}{|\operatorname{Disc}(K)|} = \sum_{\substack{[K:\mathbb{Q}]=2\\-X<\operatorname{Disc}(K)<0}} \frac{1}{|\operatorname{Disc}(K)|} \cdot \sum_{\substack{0
(16)$$$$

Proof. Let χ_K denote the quadratic character associated with K by class field theory so that we have $L(1, K/\mathbb{Q}) = \sum_{n=1}^{\infty} \frac{\chi_K(n)}{n}$. From the absolutely convergent Euler product, it is straightforward to see that

 $L(2, K/\mathbb{Q}) > (\zeta(4)/\zeta(2))^2 > 0$, and hence $1/L(2, K/\mathbb{Q})$ is uniformly bounded independent of K. Using partial summation and the Pólya-Vinogradov inequality, for any $\epsilon > 0$ we get

$$\frac{1}{L(2, K/\mathbb{Q})} \cdot \sum_{n > |\operatorname{Disc}(K)|^{\frac{1}{2} + \epsilon}} \frac{\chi_K(n)}{n} = O_{\epsilon} \left(\frac{\log(|\operatorname{Disc}(K)|)}{|\operatorname{Disc}(K)|^{\epsilon}} \right).$$

Thus, we can conclude that

$$\frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} = \frac{1}{L(2, K/\mathbb{Q})} \cdot \sum_{n=1}^{|\operatorname{Disc}(K)|^{\frac{1}{2}+\epsilon}} \frac{\chi_K(n)}{n} + O_{\epsilon} \left(\frac{\log(|\operatorname{Disc}(K)|)}{|\operatorname{Disc}(K)|^{\epsilon}} \right). \tag{17}$$

Using (17), the left hand sides of (16) are equal to

$$\sum_{\substack{[K:\mathbb{Q}]=2\\0<\mathrm{Disc}(K)< X}} \frac{1}{|\mathrm{Disc}(K)|} \cdot \left(\frac{1}{L(2,K/\mathbb{Q})} \cdot \sum_{n=1}^{\mathrm{Disc}(K)^{\frac{1}{2}+\epsilon}} \frac{\chi_K(n)}{n} + O_{\epsilon}\left(\frac{\log(\mathrm{Disc}(K))}{\mathrm{Disc}(K)^{\epsilon}}\right)\right);$$

$$\sum_{\substack{[K:\mathbb{Q}]=2\\0<\mathrm{Disc}(K)<0}} \frac{1}{|\mathrm{Disc}(K)|} \cdot \left(\frac{1}{L(2,K/\mathbb{Q})} \cdot \sum_{n=1}^{|\mathrm{Disc}(K)|^{\frac{1}{2}+\epsilon}} \frac{\chi_K(n)}{n} + O_{\epsilon}\left(\frac{\log(|\mathrm{Disc}(K)|)}{|\mathrm{Disc}(K)|^{\epsilon}}\right)\right). \tag{18}$$

In either case, the sum of the O_{ϵ} terms is itself $O_{\epsilon}(1)$, and so we focus on the remaining term. Using the absolute convergence of the Euler product of $L(2, K/\mathbb{Q})^{-1}$, we have

$$\frac{1}{L(2, K/\mathbb{Q})} \cdot \left(\sum_{n=1}^{|\operatorname{Disc}(K)|^{\frac{1}{2}+\epsilon}} \frac{\chi_K(n)}{n} \right) = \left(\sum_{m=1}^{\infty} \frac{\mu(m)\chi_K(m)}{m^2} \right) \cdot \left(\sum_{n=1}^{|\operatorname{Disc}(K)|^{\frac{1}{2}+\epsilon}} \frac{\chi_K(n)}{n} \right). \tag{19}$$

The key observation we make is that the main contribution to the right hand side of (19) comes from the "diagonal" terms, i.e., when mn is a square. By pulling out these terms, we may rewrite (19) as

$$\sum_{\substack{0 < a, b < \infty \\ (\mathrm{Disc}(K), ab) = 1}} \frac{\mu(a)}{a^3 b^2} + \sum_{n=1}^{|\mathrm{Disc}(K)|^{\frac{1}{2} + \epsilon}} \frac{\chi_K(n)}{n} \cdot \sum_{\substack{m=1 \\ mn \neq \square}}^{\infty} \frac{\mu(m)\chi_K(m)}{m^2}.$$
 (20)

Substituting (20) back into (18) implies that the left hand sides of (16) are equal to

$$\sum_{\substack{[K:\mathbb{Q}]=2\\0<\mathrm{Disc}(K)< X}}\frac{1}{\mathrm{Disc}(K)}\cdot\left(\sum_{\substack{0< a,b<\infty\\(\mathrm{Disc}(K),ab)=1}}\frac{\mu(a)}{a^3b^2}+\sum_{n=1}^{\mathrm{Disc}(K)^{\frac{1}{2}+\epsilon}}\sum_{m=1}^{\infty}\frac{\mu(m)\chi_K(mn)}{m^2n}\right)+O_{\epsilon}(1);$$

$$\sum_{\substack{[K:\mathbb{Q}]=2\\ Y \neq \operatorname{Disc}(K) \neq 0}} \frac{1}{|\operatorname{Disc}(K)|} \left(\sum_{\substack{0 < a,b < \infty\\ (\operatorname{Disc}(K) | ab) = 1}} \frac{\mu(a)}{a^3 b^2} + \sum_{n=1}^{|\operatorname{Disc}(K)|^{\frac{1}{2} + \epsilon}} \sum_{\substack{m=1\\ mn \neq \square}}^{\infty} \frac{\mu(m) \chi_K(mn)}{m^2 n} \right) + O_{\epsilon}(1).$$

By Lemma 5.1,

$$\sum_{\substack{[K:\mathbb{Q}]=2\\|\operatorname{Disc}(K)|\leq X}} \frac{1}{|\operatorname{Disc}(K)|} \cdot \left(\sum_{n=1}^{|\operatorname{Disc}(K)|^{\frac{1}{2}+\epsilon}} \sum_{\substack{m=1\\mn\neq\square}}^{\infty} \frac{\mu(m)\chi_K(mn)}{m^2n}\right) = O_{\epsilon}(1).$$

Noting that the remaining term does not depend on ϵ , we obtain the proposition.

5.2 Proof of Theorem 2

We now turn to the proof of Theorem 2. From the identity

$$\sum_{\substack{0 < a, b < \infty \\ (\text{Disc}(K), ab) = 1}} \frac{\mu(a)}{a^3 b^2} = \frac{\zeta(2)}{\zeta(3)} \cdot \prod_{\substack{p \mid \text{Disc}(K)}} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}},$$
(21)

we immediately obtain:

$$\sum_{\substack{[K:\mathbb{Q}]=2\\0<\mathrm{Disc}(K)< X}} \frac{1}{\mathrm{Disc}(K)} \cdot \sum_{\substack{0< a,b < \infty\\(\mathrm{Disc}(K),ab)=1}} \frac{\mu(a)}{a^3b^2} = \frac{\zeta(2)}{\zeta(3)} \cdot \left(\sum_{\substack{[K:\mathbb{Q}]=2\\0<\mathrm{Disc}(K)< X}} \frac{1}{\mathrm{Disc}(K)} \cdot \prod_{\substack{p \mid \mathrm{Disc}(K)}} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}}\right);$$

$$\sum_{\substack{[K:\mathbb{Q}]=2\\-X<\mathrm{Disc}(K)<0}} \frac{1}{|\mathrm{Disc}(K)|} \cdot \sum_{\substack{0< a,b < \infty\\(\mathrm{Disc}(K),ab)=1}} \frac{\mu(a)}{a^3b^2} = \frac{\zeta(2)}{\zeta(3)} \cdot \left(\sum_{\substack{[K:\mathbb{Q}]=2\\-X<\mathrm{Disc}(K)<0}} \frac{1}{|\mathrm{Disc}(K)|} \cdot \prod_{\substack{p \mid \mathrm{Disc}(K)}} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}}\right). \tag{22}$$

Decomposing the right hand sides of (22) into sums over squarefree integers in a fixed congruence class mod 4, we obtain that the left hand sides of (22) are equal to

$$\frac{\zeta(2)}{\zeta(3)} \cdot \left(\sum_{\substack{1 < D < X \\ D \equiv 1 \bmod 4 \\ D \text{ squarefree}}} \frac{1}{D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{1 < D < X \\ D \equiv 3 \bmod 4 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{1 < D < X \\ D \equiv 3 \bmod 4 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{1 < D < X \\ D \equiv 1 \bmod 2 \\ D \equiv 1 \bmod 4}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{-X < D < -1 \\ D \equiv 3 \bmod 4 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{-X < D < -1 \\ D \equiv 3 \bmod 4 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{-X < D < -1 \\ D \equiv 3 \bmod 4 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{-X < D < -1 \\ D \equiv 3 \bmod 4 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3}{14D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} + \sum_{\substack{D \le 1 \bmod 2 \\ D \text{ squarefree}}} \frac{3$$

Consider the limit

$$\lim_{X \to \infty} \frac{\zeta(2)}{\zeta(3) \log(X)} \cdot \sum_{\substack{1 < D < X \\ D \text{ squarefree}}} \frac{1}{D} \cdot \prod_{p|D} \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}} = \lim_{X \to \infty} \frac{\zeta(2)}{\zeta(3) \log(X)} \cdot \sum_{\substack{1 < D < X \\ D \text{ squarefree}}} \prod_{p|D} \frac{1}{p} \cdot \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}}$$

$$= \frac{\zeta(2)}{\zeta(3)} \cdot \prod_{p} \left(1 + \frac{1}{p} \cdot \frac{1 - \frac{1}{p^2}}{1 - \frac{1}{p^3}}\right) \left(1 - \frac{1}{p}\right)$$

$$= \zeta(2) \cdot \prod_{p} \left(1 + \frac{1}{p} - \frac{2}{p^3}\right) \left(1 - \frac{1}{p}\right)$$

$$= \zeta(2) \cdot \prod_{p} \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4}\right).$$

Carrying out the analogous computation for each term in both equations of (23) yields Theorem 2.

Theorems 2 and 4.3 immediately imply the following result.

Theorem 5.3. Let $\beta < 2/3$ be fixed. For the family of all D_4 -fields,

$$N_{\rm C}(X, X^{\beta}) = \frac{3\beta}{8} \cdot \prod_{p} \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4} \right) \cdot X \log(X) + O(X).$$

It follows from the heuristics of §3.3 that the family of D_4 -fields L satisfying $|d(L)| \leq |C(L)|^{2/3}$ satisfies the mass formula (10) when such fields are ordered by their conductors. In Sections 6-8, we prove a refinement of Theorem 5.3 by adapting the arguments in [2] in conjunction with the analytic techniques used in Section A

5.3 Diagonal terms for the family of D_4 -fields ordered by discriminant

We would like to conclude this section by considering asymptotics for the analogous "diagonal terms" that arise when counting the number of D_4 -fields having bounded discriminants. Let $N_{\text{Disc}}(X)$ denote the number of isomorphism classes of D_4 -fields L with |Disc(L)| < X. By Corollary 1.4 of [8] (or following the proof of Theorem 4.3 with discriminant in place of conductor), we have

$$N_{\text{Disc}}(X) = \frac{X}{2\zeta(2)} \cdot \sum_{\substack{[K:\mathbb{Q}]=2\\|\text{Disc}(K)| < X}} \frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} \cdot \frac{2^{-r_2(K)}}{|\text{Disc}(K)|^2} + o(X).$$
 (24)

Applying the reasoning in the proof of Proposition 5.2, we obtain that combining Equation (19) and (24) implies

$$N_{\text{Disc}}(X) = \frac{X}{2\zeta(2)} \cdot \sum_{\substack{[K:\mathbb{Q}]=2\\|\text{Disc}(K)| < X}} \left(\sum_{m=1}^{\infty} \frac{\mu(m)\chi_K(m)}{m^2} \right) \cdot \left(\sum_{n=1}^{\infty} \frac{\chi_K(n)}{n} \right) \cdot \frac{2^{-r_2(K)}}{|\text{Disc}(K)|^2} + o(X).$$

We replace the product of sums

$$\left(\sum_{m=1}^{\infty} \frac{\mu(m)\chi_K(m)}{m^2}\right) \cdot \left(\sum_{n=1}^{\infty} \frac{\chi_K(n)}{n}\right),\,$$

with simply the "diagonal" terms (i.e., the terms where mn is a square).

Theorem 5.4. We have the following:

$$\frac{1}{2\zeta(2)} \cdot \sum_{\substack{[K:\mathbb{Q}]=2\\ |\operatorname{Disc}(K)| < X}} \frac{2^{-r(K)}}{|\operatorname{Disc}(K)|^2} \cdot \left(\sum_{\substack{0 < a,b < \infty \\ (\operatorname{Disc}(K),ab) = 1}} \frac{\mu(a)}{a^3b^2} \right) \cdot X \quad \sim \quad \frac{3 \cdot 11^2}{2^6 \cdot 17} \cdot \prod_{p} \left(1 + \frac{1}{p^2} - \frac{1}{p^3} - \frac{1}{p^4} \right) \cdot X. \quad (25)$$

Proof. This follows from an argument analogous to the proof of Theorem 2.

This term agrees exactly with the heuristic in (11)! The non-diagonal terms, as in §5.2, again give an error term of O(X). In the case when D_4 -fields were ordered by conductor, this error term was negligible compared to the main term of $\approx X \log X$. This time, however, the main term of $\approx X$ does not automatically dominate the error term. In fact, the comparison of the constant $c \approx 0.0523$ from [8] and the constant ≈ 0.406 on the right hand side of (25) implies that the non-diagonal terms do make a non-negligible contribution.

6 Parameterizing D_4 -fields via pairs of ternary quadratic forms

We next give a proof using geometry-of-numbers techniques in conjunction with arithmetic invariant theory methods for determining asymptotics on the number of D_4 -fields with $|\mathbf{q}| < X$ and $|\mathbf{d}| \ll X$. We obtain worse error estimates in this second proof, but we are able to prove more refined statements for a wider class of collections of local specifications. We begin with a parametrization of D_4 -fields via certain pairs of ternary quadratic forms, following Bhargava [1] and Wood [27]. In §6.1, we describe the arithmetic invariant theory for orbits of such pairs of ternary quadratic forms and compare it to the invariants defined in Definitions 2.7 and 2.5 for the corresponding D_4 -fields. We additionally define splitting types, and we compute the p-adic densities for pairs of ternary quadratic forms corresponding to D_4 -fields with fixed splitting type at p. These results allow us to employ geometry-of-numbers methods carried out in Section 7 to count the relevant orbits parametrizing D_4 -fields with $|\mathbf{q}| < X$ and $|\mathbf{d}| \ll X$.

In [1], Bhargava proved that isomorphism classes of pairs (Q, C), where Q is a quartic ring and C is a cubic resolvent ring of Q are in bijection with $GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$ -orbits on $\mathbb{Z}^2 \otimes Sym^2(\mathbb{Z}^3)$, the space of pairs of integral ternary quadratic forms. If Q is a maximal quartic ring, then it has a unique cubic resolvent ring, so this bijection (when restricted to maximal rings) can be viewed as a parametrization of quartic fields.

We write a pair of ternary quadratic forms as a pair of symmetric 3×3 matrices (A, B) whose diagonal entries are integers and nondiagonal entries are half-integers. The group $GL_2 \times SL_3$ acts on pairs of ternary quadratic forms as follows:

$$(g_2, g_3) \cdot (A, B) = (g_3 A g_3^t, g_3 B g_3^t) \cdot g_2^t.$$

For quartic rings Q containing a quadratic subring, Wood [27] gives a more specialized bijection: For any ring R, let $V'(R) \subset R^2 \otimes \operatorname{Sym}^2(R^3)$ denote the space of pairs of ternary quadratic forms (A, B) satisfying

$$(A,B) = \begin{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{22} & \frac{a_{23}}{2} \\ 0 & \frac{a_{23}}{2} & a_{33} \end{bmatrix}, \begin{bmatrix} b_{11} & \frac{b_{12}}{2} & \frac{b_{13}}{2} \\ \frac{b_{12}}{2} & b_{22} & \frac{b_{23}}{2} \\ \frac{b_{13}}{2} & \frac{b_{23}}{2} & b_{33} \end{bmatrix} \end{pmatrix},$$

where a_{22} , a_{23} , a_{33} , b_{11} , b_{12} , b_{22} , b_{23} , and b_{33} are elements of R with $b_{11} \neq 0$. The subgroup G'(R) of $GL_2(R) \times SL_3(R)$ consisting of elements (g_2, g_3) such that

$$g_2 = \begin{bmatrix} \pm 1 & 0 \\ * & \pm 1 \end{bmatrix}$$
, and $g_3 = \begin{bmatrix} \pm 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$,

acts on V'(R). Then the $G'(\mathbb{Z})$ -orbits on $V'(\mathbb{Z})$ are in bijection with triples (Q, C, T) consisting of a quartic ring Q, a cubic resolvent C of Q, and a quadratic subring $T \subset Q$. More precisely:

Theorem 6.1 (Thm. 7.3.5 of [27]). For any principal ideal domain R of characteristic different from 2, there is a bijection between G'(R)-equivalence classes of elements of V'(R) with isomorphism classes of triples (Q, C, T) where

- Q is a quartic ring over R,
- C is a cubic resolvent of Q with map $\varphi: Q \to C$, and
- $T \subset Q$ is a primitive quadratic subalgebra such that $\varphi(T) \neq 0$.

In order to obtain a parametrization of maximal orders in D_4 -fields, we first make a few definitions.

Definition 6.2. An element of $v \in V'(\mathbb{Z})$ is *generic* if the quartic ring corresponding to v under Theorem 6.1 is a D_4 -order, i.e., an order in a D_4 -field. Additionally, an element $V'(\mathbb{Z})$ is said to be maximal if it corresponds to a maximal quartic ring.

Let (A, B) be an element of $V'(\mathbb{Z})$ and let Q be the quartic ring corresponding to it. It follows from Lemma 22 of [1], that Q is nonmaximal at every prime dividing b_{11} . Hence Q is maximal only when $b_{11} = \pm 1$. Furthermore, by replacing (A, B) with $(-\operatorname{Id}, \operatorname{Id}) \cdot (A, B) = (-A, -B)$, if necessary, we may assume that $b_{11} = 1$. We define $V(\mathbb{Z}) \subset V'(\mathbb{Z})$ to be the subspace of pairs

$$(A,B) = \begin{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{22} & \frac{a_{23}}{2} \\ 0 & \frac{a_{23}}{2} & a_{33} \end{bmatrix}, \begin{bmatrix} 1 & \frac{b_{12}}{2} & \frac{b_{13}}{2} \\ \frac{b_{12}}{2} & b_{22} & \frac{b_{23}}{2} \\ \frac{b_{13}}{2} & \frac{b_{23}}{2} & b_{33} \end{bmatrix} \end{pmatrix}, \tag{26}$$

and we define the subgroup $G(\mathbb{Z}) \subset G'(\mathbb{Z})$ to be the set of pairs $(g_2, g_3) \in G'(\mathbb{Z})$ such that

$$g_2 = \begin{bmatrix} \pm 1 & 0 \\ * & 1 \end{bmatrix}$$
, and $g_3 = \begin{bmatrix} \pm 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$.

Moreover, for any ring R, we analogously define the space V(R) and the group G(R). We have the following proposition:

Proposition 6.3. There is a bijection between (isomorphism classes of) D_4 -fields and maximal generic $G(\mathbb{Z})$ -orbits on $V(\mathbb{Z})$.

Proof. Proposition 6.3 follows immediately from Theorem 6.1 and the above discussion, in conjunction with the fact that a maximal D_4 -order has a unique resolvent ring and primitive quadratic subalgebra.

6.1 Invariant theory

We next discuss the invariant theory for the action of G on V. The action of $G(\mathbb{C}) \cap SL_3(\mathbb{C})$ on $V(\mathbb{C})$ turns out to have ring of invariants freely generated by 3 elements. We can describe these invariants in terms of the *cubic resolvent* of (A, B), i.e., the binary cubic form Det(Ax + By). It is straightforward to check that if (A, B) is as in (26), then the coefficient of x^3 in Det(Ax + By) is equal to zero. For a ring R, let U(R) denote the subspace of space of binary cubic forms consisting of elements

$$f(x,y) = bx^2y + cxy^2 + dy^3,$$

where b, c, and d are elements of R. Define N_1 to be the group of lower triangular 2×2 -matrices with top left entry ± 1 and bottom right entry 1. Then N_1 acts on U via the action

$$g \cdot f(x, y) = f((x, y) \cdot g^t).$$

Additionally, if $\pi: V(R) \to U(R)$ denotes the resolvent map $(A, B) \mapsto 4 \operatorname{Det}(Ax + By)$, then for $(g_2, g_3) \in G(R)$,

$$\pi((g_2, g_3) \cdot (A, B)) = g_2 \cdot \pi(A, B).$$

The coefficients b, c, and d of the binary cubic form $\pi(A, B)$ are the invariants for the action of $G(\mathbb{C}) \cap SL_3(\mathbb{C})$ on $V(\mathbb{C})$. Therefore, the ring of invariants for the action of all of $G(\mathbb{C})$ on $V(\mathbb{C})$ is the same as the ring of invariants for the action of $N_1(\mathbb{C})$ on $U(\mathbb{C})$. The latter ring is freely generated by two elements d and q, which can be associated to an element of V as follows:

Definition 6.4. If $(A, B) \in V(R)$ has resolvent $f(x, y) = bx^2y + cxy^2 + dy^3$, then

$$\begin{array}{rcl} \operatorname{d}(A,B) & \coloneqq & \operatorname{d}(f) & \coloneqq & -b \\ \operatorname{q}(A,B) & \coloneqq & \operatorname{q}(f) & \coloneqq & c^2 - 4bd. \end{array}$$

Additionally, we set

$$C(A, B) := d(A, B) \cdot q(A, B)$$
 and $Disc(A, B) := d(A, B)^2 \cdot q(A, B)$.

We now relate the invariants d and q of an element $(A, B) \in V(\mathbb{Z})$ with the invariants of the quartic ring corresponding to the $G(\mathbb{Z})$ -orbit of (A, B) in Theorem 6.1.

Proposition 6.5. Let $(A, B) \in V(\mathbb{Z})$ be an element with nonzero invariants d(A, B) and q(A, B). Let Q denote the quartic ring corresponding to (A, B) and let T be its quadratic subalgebra. Then we have $\operatorname{Disc}(T) = d(A, B)$ and $\operatorname{Nm}_T(\operatorname{Disc}(Q/T)) = |q(A, B)|$.

Proof. Let (A, B) be as in (26) with resolvent $f(x, y) = bx^2y + cxy^2 + dy^3$, and assume that d(A, B) and q(A, B) are nonzero. As described in Section 3 of [1], one can describe the multiplicative structure on a (normal) \mathbb{Z} -basis of Q using the matrix coefficients of (A, B). Indeed, if $\langle 1, \alpha_1, \alpha_2, \alpha_3 \rangle$ is a \mathbb{Z} -basis for Q, we can write its multiplication table as

$$\alpha_i \alpha_j = c_{ij}^0 + c_{ij}^1 \alpha_1 + c_{ij}^2 \alpha_2 + c_{ij}^3 \alpha_3,$$

where $c_{ij}^k \in \mathbb{Z}$ for $1 \le i, j \le 3$ and $k \in \{0, 1, 2, 3\}$ are determined completely by a_{ij} and b_{ij} . Equations 20, 21, and 22 of [1] with $a_{11} = a_{12} = a_{13} = 0$ and $b_{11} = 1$ imply that $c_{11}^2 = c_{11}^3 = 0$. Additionally, we obtain $c_{11}^0 = -a_{33}a_{22}$ and $c_{11}^1 = a_{23}$, i.e.,

$$\alpha_1^2 = -a_{33}a_{22} + a_{23}\alpha_1,$$

and so $\mathbb{Z}[\alpha_1] = \langle 1, \alpha_1 \rangle$ is a quadratic subalgebra of Q. By the description of T given in the proof of Corollary 7.2.2 of [27], we have that $T = \mathbb{Z}[\alpha_1]$, and

$$Disc(T) = Disc(\alpha_1) = a_{23}^2 - 4a_{33}a_{22}.$$

Using the resolvent map $\pi(A, B) = 4 \operatorname{Det}(Ax + By) = bx^2y + cxy^2 + dy^3$, we deduce that

$$b = 4a_{22}a_{33} - a_{23}^2.$$

Thus, $\operatorname{Disc}(T) = \operatorname{d}(A, B)$, as necessary.

Furthermore, by Proposition 10 of [1] and since $\operatorname{Disc}(A, B) = \operatorname{Disc}(\pi(A, B))$, we have

$$Disc(Q) = Disc(A, B) = Disc(bx^{2}y + cxy^{2} + dy^{3}) = b^{2}c^{2} - 4b^{3}d.$$

The relative discriminant formula implies that $|\operatorname{Disc}(Q)| = |\operatorname{Disc}(T)^2 \cdot \operatorname{Nm}_T(\operatorname{Disc}(Q/T))|$, and so we conclude that

$$\operatorname{Nm}_T(\operatorname{Disc}(Q/T)) = \left|\frac{\operatorname{Disc}(Q)}{\operatorname{Disc}(T)^2}\right| = |c^2 - 4bd|.$$

The proposition follows.

We slightly generalize the notion of conductor given in the introduction for étale quartic algebras over R. For a pair (Q, T), where Q is a étale quartic algebra over R and T is a primitive quadratic subalgebra of Q, we set C(Q, T) := Disc(Q)/Disc(T). The following lemma will be useful in obtaining a bound on the number of $G(\mathbb{Z})$ -orbits that correspond to non-maximal D_4 -orders in Section 8.

Lemma 6.6. If $(A, B) \in V(\mathbb{Z}_p)$ corresponds to a non-maximal quartic order Q contained in a degree 4 étale extension L_p of \mathbb{Q}_p and a quartic subalgebra T contained in a quadratic subextension K_p of L_p , then

$$p^2 \mid \frac{\mathrm{C}(A,B)}{\mathrm{C}(L_p,K_p)}.$$

Proof. Since the index of Q in the maximal order of L is a multiple of p, it follows that the discriminant of (A, B) differs from the discriminant of L by a factor of at least p^2 . Since C(A, B) = Disc(A, B)/Disc(T) and $C(L_p, K_p) = Disc(L_p)/Disc(K_p)$, the lemma follows unless $Disc(T)/Disc(K_p)$ is divisible by p, or equivalently, unless T is not maximal in the ring of integers of K_p .

Assume that T is not maximal in the ring of integers of K_p , and thus has discriminant divisible by p^2 . We know from Proposition 6.5 that $-\operatorname{Disc}(T)$ is the discriminant of the quadratic form $a_{22}y^2 + a_{23}yz + a_{33}z^2$ corresponding to A. Hence, either A is a multiple of p or, after a change of variables, we may assume that $p^2 \mid a_{22}$ and $p \mid a_{23}$. In the first case, the pair of rings (Q_1, T_1) corresponding to $(A/p, B) \in V(\mathbb{Z})$ are over-orders of Q and T such that $p^4 \mid C(Q, T)/C(Q_1, T_1)$. In the second case, the argument is similar: we use the pair (A_2, B_2) obtained by multiplying the third row and column of A and B by p, and dividing A by p^2 , which yield a pair (Q_2, T_2) of over-orders of (Q, T) such that $p^2 \mid C(Q, T)/C(Q_1, T_1)$. The lemma follows.

6.2 Splitting types of pairs of ternary quadratic forms

Let p be a fixed prime. We say that a pair $(A, B) \in V(\mathbb{F}_p)$ is nondegenerate if the zero sets in $\mathbb{P}^2(\overline{\mathbb{F}}_p)$ of the two ternary quadratic forms corresponding to A and B intersect at four points counted with multiplicity. For nondegenerate elements $(A, B) \in V(\mathbb{F}_p)$ such that A is nonzero, we define the quartic splitting type at p to be

$$\varsigma_p(A, B) = (f_1^{e_1} f_2^{e_2} \cdots),$$

where the f_i 's are the degrees over \mathbb{F}_p of the field of definition of these points, and the e_i 's are their multiplicities. Furthermore, recall that the top row and column of A is 0, and so in the notation of (26), A corresponds to a quadratic form $g(y,z) = a_{22}y^2 + a_{23}yz + a_{33}z^2$. We define the quadratic splitting type $\varsigma_p'(A,B)$ of (A,B) to be (11) if g(x,y) has two distinct roots in $\mathbb{P}^1(\mathbb{F}_p)$, to be (2) if g(x,y) has a pair of conjugate roots defined over a quadratic extension of \mathbb{F}_p , and (1²) if g(x,y) has a double root. We then say that the pair $(\varsigma_p(A,B),\varsigma_p'(A,B))$ is the splitting type of (A,B) at p.

If (A, B) is an element in $V(\mathbb{Z})$ or $V(\mathbb{Z}_p)$, we define the splitting type of (A, B) at p to be the splitting type of the reduction modulo p of (A, B), assuming it is nondegenerate. Let Q be the quartic ring corresponding to (A, B), and let T denote the quadratic subring of Q arising from Theorem 6.1. It follows from §4.1 of [1], that the quartic splitting type of (A, B) is equal to the splitting type of Q. We have seen that the quadratic subring T of Q corresponding to the pair (A, B) is the quadratic ring whose discriminant is the same as that of the binary quadratic form corresponding to A. Hence, the quadratic splitting type of (A, B) is the same as the splitting type of T.

Given a pair (L_p, K_p) of extensions of \mathbb{Q}_p , whose rings of integers correspond to a pair $(A, B) \in V(\mathbb{Z}_p)$, we define $\varsigma_p(L_p, K_p)$ to equal the splitting type of (A, B). Additionally, there are four possible splitting types $\varsigma = (\varsigma_{\infty}, \varsigma_{\infty}')$ at ∞ for an element in $V(\mathbb{R})$ having nonzero discriminant. The invariants (d, q) of an element $v \in V(\mathbb{R})^{(\varsigma)}$ are constrained in the following way:

$$\begin{split} \varsigma &= ((1111), (11)) & \Rightarrow \quad q > 0, \ d > 0; \\ \varsigma &= ((112), (11)) & \Rightarrow \quad q < 0, \ d > 0; \\ \varsigma &= ((22), (11)) & \Rightarrow \quad q > 0, \ d > 0; \\ \varsigma &= ((22), (2)) & \Rightarrow \quad q > 0, \ d < 0. \end{split}$$

We denote the set of elements in $V(\mathbb{R})$ having splitting type ς by $V(\mathbb{R})^{(\varsigma)}$ and set $V(\mathbb{Z})^{(\varsigma)} = V(\mathbb{Z}) \cap V(\mathbb{R})^{(\varsigma)}$.

6.3 The density of maximal elements

For a prime p and splitting type $(\varsigma_p, \varsigma_p')$, let $T_p(\varsigma_p, \varsigma_p')$ denote the set of elements $(A, B) \in V(\mathbb{Z}_p)$ whose splitting type at p is $(\varsigma_p, \varsigma_p')$ and let $M_p(\varsigma_p, \varsigma_p')$ denote the set of elements $(A, B) \in T_p(\varsigma_p, \varsigma_p')$ that correspond to quartic rings under Theorem 6.1 that are maximal at p. Identifying $V(\mathbb{Z}_p) \cong \mathbb{Z}_p^8$ by regarding the non-fixed entries of (26) as a vector, let μ denote the Haar measure normalized so that $V(\mathbb{Z}_p)$ have volume 1. We have the following result which computes the volumes of the sets $M_p(\varsigma_p, \varsigma_p')$.

Proposition 6.7. We have

$$\mu(M_p((1111),(11))) = \frac{1}{8}(p-1)^3(p+1)/p^4$$

$$\mu(M_p((22),(11))) = \frac{1}{8}(p-1)^3(p+1)/p^4$$

$$\mu(M_p((22),(2))) = \frac{1}{4}(p-1)^3(p+1)/p^4$$

$$\mu(M_p((112),(11))) = \frac{1}{4}(p-1)^3(p+1)/p^4$$

$$\mu(M_p((4),(2))) = \frac{1}{4}(p-1)^3(p+1)/p^4$$

$$\mu(M_p((1^211),(11))) = \frac{1}{2}(p-1)^3(p+1)/p^5$$

$$\mu(M_p((1^22),(11))) = \frac{1}{2}(p-1)^3(p+1)/p^5$$

$$\mu(M_p((1^21^2),(1^2))) = \frac{1}{2}(p-1)^3(p+1)/p^5$$

$$\mu(M_p((2^2),(1^2))) = \frac{1}{2}(p-1)^3(p+1)/p^5$$

$$\mu(M_p((1^4)(1^2))) = (p-1)^3(p+1)/p^6$$

$$\mu(M_p((1^21^2),(11))) = \frac{1}{2}(p-1)^3(p+1)/p^6$$

$$\mu(M_p((2^2),(2))) = \frac{1}{2}(p-1)^3(p+1)/p^6$$

Proof. First note that the splitting type of $v \in V(\mathbb{Z}_p)$ depends only on the reduction of v modulo p. It follows that the densities of the sets $T_p(\varsigma_p, \varsigma_p')$ can be computed by counting elements in $V(\mathbb{F}_p)$. The conditions that ensure the maximality of v are listed in §4.2 of [1]. We prove Proposition 6.7 by computing the densities of $T_p(\varsigma_p, \varsigma_p')$ and then, for each ς_p , determining the probability that $v \in T_p(\varsigma_p, \varsigma_p')$ is maximal.

Let (A, B) be an element of $V(\mathbb{Z}_p)$ having quadratic splitting type (11). It follows that the quadratic form corresponding to A has two distinct roots in $\mathbb{P}^1(\mathbb{F}_p)$. The number of possibilities for \bar{A} , the reduction of A modulo p, is thus equal to (p+1)p(p-1)/2 giving a density of $(p+1)p(p-1)/(2p^3)$ for the possibilities of A. By a change of variables, we may assume that (A, B) is of the form

$$(\bar{A}, \bar{B}) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{bmatrix} \right), \tag{27}$$

when p is odd and

$$(\bar{A}, \bar{B}) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \alpha/2 & \beta/2 \\ \alpha/2 & s & 0 \\ \beta/2 & 0 & t \end{bmatrix} \right), \tag{28}$$

when p = 2. The quartic splitting type of (A, B) then has six options: (1111), (112), (22), (1²11), (1²2), and (1²1²). When p is odd, this splitting type depends on whether s and t are residues modulo p, nonresidues modulo p, or 0. Their relative densities are easy to compute. To obtain the splitting type (1111), both s and t must be residues modulo p which occurs with relative density $(p-1)^2/4p^2$. It is easy to check that for p=2, the relative density of elements with quartic splitting type (1111) is again $1/16 = (p-1)^2/4p^2$. Multiplying with the density $(p^2-1)/2p^2$ of split quadratic forms arising from A yields the density of $T_p((1111),(11))$. Since every element with unramified splitting type is automatically maximal, it follows that $T_p((1111),(11)) = M_p((1111),(11))$, yielding the first part of the proposition.

We now compute the density of $M_p((1^21^2), (11))$. Again, we may assume that (\bar{A}, \bar{B}) is of the form (27) when p is odd and of the form (28) when p = 2. When p is odd, such a pair (A, B) has quartic splitting type (1^21^2) when s = t = 0, and when p = 2 such a form has splitting type (1^21^2) when $\alpha = \beta = 0$. The relative density of such (A, B) is $1/p^2$, and we therefore see that the density of $T_p((1^21^2), (11))$ is $\frac{1}{2}(p-1)^2(p+1)/p^4$. We now compute the probability that an element $(A, B) \in T_p((1^21^2), (11))$ is maximal at p. Let (A, B) in $T_p((1^21^2), (11))$ be fixed. By a change of variables, we may assume (for both p odd and p = 2) that the reduction of (A, B) modulo p is equal to

$$(\bar{A}, \bar{B}) = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right). \tag{29}$$

From Lemma 23 of [1], it follows that (A, B) is maximal if and only if both b_{22} and b_{23} are not divisible by p^2 . Hence the relative density of $M_p((1^21^2), (11))$ in $T_p((1^21^2), (11))$ is $(1 - 1/p)^2$. In conjunction with the previously computed density of $T_p((1^21^2), (11))$, it follows that the density of $M_p((1^21^2), (11))$ is as stated in the proposition. The computations of the densities of the $M_p(\varsigma_p, \varsigma_p')$ for other splitting types $(\varsigma_p, \varsigma_p')$ are very similar to the above two computations and so we omit them.

Proposition 6.7 computes the p-adic splitting densities in V. However, it is possible to extract densities that are more refined. From the splitting types with central inertia in Table 1, it is evident that the splitting type of a D_4 -field L at a prime p is not enough to determine the decomposition groups of L at p. However, if $v \in V(\mathbb{Z})$ corresponds to \mathcal{O}_L , then the $G(\mathbb{Z}_p)$ -orbit of v in $V(\mathbb{Z}_p)$ determines $\mathcal{O}_L \otimes \mathbb{Z}_p$, and hence determines the decomposition groups of L at p. In the following lemma, we compute some of these more refined densities.

If p is an odd prime with splitting type $((1^21^2), (11))$ (resp. $((2^2), (2))$) in a D_4 -field L with quadratic subfield K, then there are two possibilities for the splitting type $(\varsigma_p(\phi(L)), \varsigma_p(\phi(K)))$ of $\phi(L)$ at p (see Definition 2.1), namely ((1111), (11)) (resp. ((22), (11))) or ((22), (2)). Let $M_p^{(11)}(\varsigma_p, \varsigma_p')$ (resp. $M_p^{(2)}(\varsigma_p, \varsigma_p')$) denote the subset of elements $(A, B) \in M_p(\varsigma_p, \varsigma_p')$ corresponding to (Q, T) under Theorem 6.1 such that $\phi(\operatorname{Frac}(T))$ has splitting type (11) (resp. (2)) at p.

Lemma 6.8. Let p be an odd prime and let $(\varsigma_p, \varsigma_p')$ be one of $((1^21^2), (11))$ or $((2^2), (2))$. Then with the notation of the previous paragraph,

$$\mu(M_p^{(2)}(\varsigma_p, \varsigma_p')) = \mu(M_p^{(11)}(\varsigma_p, \varsigma_p')).$$

Proof. We prove the lemma only for the splitting type $((1^21^2), (11))$, since the proof is very similar for $((2^2), (2))$. Let $(A, B) \in V(\mathbb{Z}_p)$ be a maximal element, corresponding to a pair degree 4 étale \mathbb{Z}_p -algebra Q and a quadratic subalgebra T, and let L (respectively, K) denote the fraction field of Q (respectively, T). Assume that the splitting type of (A, B) is $((1^21^2), (11))$, and let $f(x, y) = 4 \det(Ax + By)$ denote the cubic resolvent polynomial of (A, B). Then the x^3 -coefficient of f is 0, and dividing f by g yields a binary quadratic form. Propositions 2.6 and 6.5 imply that we have

$$\operatorname{Disc}_p(f(x,y)/y)/p^2 = \operatorname{Disc}_p(\phi(K)),$$

where Disc_p means the p-part of the discriminant and hence, sign issues don't arise. Since the two possible decomposition groups for the splitting type $((1^21^2),(11))$ are determined by the splitting behaviour of p at $\phi(K)$ (see Table 1), it follows that the relative densities of these decomposition groups can be computed by computing the relative densities of the different possible splitting behaviours of p in the quadratic order whose discriminant is $\operatorname{Disc}_p(f(x,y)/y)/p^2$.

Since p is odd, from the discussion surrounding (29), it follows that we may assume (A, B) satisfies

$$a_{11} = a_{12} = a_{13} = b_{12} = b_{13} = 0, \ b_{11} = 1, \ a_{23} \equiv 1 \pmod{p}, \ a_{22} \equiv b_{22} \equiv b_{23} \equiv b_{33} \equiv 0 \pmod{p}.$$

Consider the pair (A, B_1) , where the B_1 is obtained from B by dividing the b_{22} , b_{23} , and b_{33} by p. Let f_1 denote the cubic resolvent form of (A, B_1) . Then the discriminant of $f_1(x, y)/y$ is exactly the same as the discriminant of $\phi(K)$. Hence the decomposition group of L is determined by the splitting of p in $f_1(x, y)/y$. Since (A, B) was assumed to be maximal, it follows that p does not divide the discriminant of $f_1(x, y)/y$. It is easy to check that the density of elements (A, B_1) such that $f_1(x, y)/y$ has splitting type (11) (resp. (2)) is exactly 1/2, yielding the lemma.

Let \mathcal{M}_p denote the set of elements $(A, B) \in V(\mathbb{Z}_p)$ that are maximal at p, and let \mathcal{U}_p denote the set of elements (A, B) in \mathcal{M}_p that do not have central inertia, i.e., the splitting type of any $(A, B) \in \mathcal{U}_p$ at p is not equal to $((1^4), (1^2)), ((1^21^2), (11))$, or $((2^2), (2))$. Summing the values obtained in Proposition 6.7 we can compute the density of \mathcal{M}_p . To determine the density of \mathcal{U}_p , we add up the values of the first 9 rows.

Theorem 6.9. We have

$$\mu(\mathcal{M}_p) = \left(1 - \frac{1}{p^2}\right) \left(1 - \frac{1}{p^2} - \frac{2}{p^3} + \frac{2}{p^4}\right);$$

$$\mu(\mathcal{U}_p) = \left(1 - \frac{1}{p^2}\right) \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right).$$

7 Counting D_4 -fields using geometry-of-numbers methods

In the previous section, we defined an injective map from D_4 -fields to $G(\mathbb{Z})$ -orbits on $V(\mathbb{Z})$ and determined generators d and q for the ring of invariants for the action of G on V. In this section, our goal is to count generic $G(\mathbb{Z})$ -orbits on $V(\mathbb{Z})$ having bounded invariants.

Recall that an element $v \in V(\mathbb{Z})$ is said to be generic if v corresponds to an order in a D_4 -field, and the subset of elements in $V(\mathbb{Z})$ with infinite splitting type ς is denoted by by $V(\mathbb{Z})^{(\varsigma)}$. For any $G(\mathbb{Z})$ -invariant set $\mathcal{L} \subset V(\mathbb{Z})$ and for any $\delta > 0$, let $\mathcal{N}_q^{(\delta)}(\mathcal{L}; X, Y)$ denote the number of generic $G(\mathbb{Z})$ -orbits v on L such that $X < |q(v)| \le (1 + \delta)X$ and $Y < |d(v)| \le (1 + \delta)Y$. Our goal in this section is to prove the following theorem:

Theorem 7.1. Let X and Y be positive real numbers going to infinity such that $(Y \log Y)^2 = o(X)$. Then we have

$$\mathcal{N}_{\mathbf{q}}^{(\delta)}(V(\mathbb{Z})^{(\varsigma)}; X, Y) = \frac{\zeta(2)}{2\tau_{\varsigma}} \delta^{2} X Y + o_{\delta}(XY),$$

where $\tau_{\varsigma} = 8$ when $\varsigma = ((1111), (11))$ or $\varsigma = ((22), (11))$ and $\tau_{\varsigma} = 4$ otherwise.

To do so, we study the fundamental domain for the action of the non-reductive group $G(\mathbb{Z})$ on $V(\mathbb{R})$. We then compute the volume of a cover of this fundamental set after cutting of the cusps in terms of an Euler product of local densities.

7.1 Construction of fundamental domains

In this section, our goal is to construct a finite cover for a fundamental domain for the action of $G(\mathbb{Z})$ on $V(\mathbb{R})$. As a first step, we describe the $G(\mathbb{R})$ -orbits on $V(\mathbb{R})$, and the sizes of the stabilizers in $G(\mathbb{R})$ of elements in each orbit. Before we do so, it will be convenient to introduce the following group and space: Let $V_{\text{red}} \subset V$ consist of all pairs (A, B) of the form

$$(A,B) = \begin{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_{22} & \frac{a_{23}}{2} \\ 0 & \frac{a_{23}}{2} & a_{33} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & b_{22} & \frac{b_{23}}{2} \\ 0 & \frac{b_{23}}{2} & b_{33} \end{bmatrix} \end{pmatrix}.$$
(30)

The subgroup G_{red} of G acts on V_{red} , where G_{red} consists of elements $(g_2, g_3) \in GL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$ such that

$$g_2 = \begin{bmatrix} \pm 1 & 0 \\ * & 1 \end{bmatrix}, \quad \text{and} \quad g_3 = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$
 (31)

where the lower 2×2 submatrix of g_3 is an element of SL_2^{\pm} . We have the following result.

Proposition 7.2. The orbits for the action of $G(\mathbb{R})$ on the set of elements in $V(\mathbb{R})$ having nonzero invariants q and d correspond to a pair of étale algebras (L_{∞}, K_{∞}) with splitting types and invariants as follows:

- (1) When q > 0 and d > 0, there are two orbits, one with splitting type ((1111), (11)) and one with splitting type ((22), (11));
- (2) When q < 0 and d > 0, there is one orbit with splitting type ((112), (11));
- (3) When g > 0 and d < 0, there is one orbit has splitting type ((22), (2)).

The respective sizes of the stabilizers in $G(\mathbb{R})$ of elements in these orbits are 8 in the first case, and 4 in the second and third cases.

Proof. We start with a few observations. First note that $G(\mathbb{R})$ -orbits on $V(\mathbb{R})$ having fixed invariants d and q are in bijection with $G_{\text{red}}(\mathbb{R})$ -orbits on $V_{\text{red}}(\mathbb{R})$ having invariants q and d. This is because $(A,B) \in V(\mathbb{R})$ is clearly $G(\mathbb{R})$ -equivalent to some $(A_{\text{red}}, B_{\text{red}}) \in V_{\text{red}}(\mathbb{R})$. Futhermore, if two elements in $V_{\text{red}}(\mathbb{R})$ are $G(\mathbb{R})$ -equivalent via some $g \in G(\mathbb{R})$, then g must in fact belong to $G_{\text{red}}(\mathbb{R})$. This latter fact also implies that the stabilizer in $G(\mathbb{R})$ of any element in $V_{\text{red}}(\mathbb{R})$ is contained in $G_{\text{red}}(\mathbb{R})$. Also note that $G_{\text{red}}(\mathbb{R})$ -orbits on $V_{\text{red}}(\mathbb{R})$ having nonzero invariants d and q are in bijection with $G_{\text{red}}(\mathbb{R})$ -orbits on $V_{\text{red}}(\mathbb{R})$ having invariants q/|q| and q/|q| and q/|q| have q/|q| and q/|q| and dividing the lower q/|q| are two elements are clearly the same. Therefore, it suffices to prove the proposition in the case when q and q/|q| are q/|q| and q/|q| are q/|q| and q/|q| are q/|q| are q/|q| and q/|q| and q/|q| are q/|q| and q/|q| and q/|q| are q/|q| and q/|q| and q/|q| and q/|q| are q/|q| and q/|q| and q/|q| are q/|q| and q/|q| are q/|q| and q/|q| and q/|q| and q/|q| are q/|q| and q/|q| and q/|q| are q/|q| and q/|q| and q/|q| are q/|q| are q/|q| and q/|q| and q/|q| and q/|q| and q/|q| are q/|q| and q/|q| and q/|q| and q/|q| are q/|q| and q/|q| are q/|q| are q/|q| are q/|q| and q/|q| are q/|q| are q/|q| and q/|q| are q/|q| and q/|q| are q/|q| are q/|q| and q/|q| are q/|q| are q/|q| are q/|q| and q/|q| are q/|q| are q/|

Consider the case q = d = 1. Let $(A, B) \in V_{red}(\mathbb{R})$ have such invariants. By replacing (A, B) with a $G_{red}(\mathbb{R})$ -translate, we transform it as follows: first, we ensure that $a_{22} = a_{33} = 0$; next, we subtract an appropriate multiple of A from B to ensure that its off-diagonal entries are 0; finally, we use an element of $SL_2(\mathbb{R}) \subset G_{red}(\mathbb{R})$ to ensure that $|b_{22}| = |b_{33}|$. From the fact that q = d = 1, it follows that we have transformed (A, B) into the form

$$\left(\begin{array}{ccc} 0 & & & \\ & & \frac{1}{2} & \\ & \frac{1}{2} & \end{array}\right), \left(\begin{array}{ccc} 1 & & & \\ & \frac{\pm 1}{4} & & \\ & & \frac{\pm 1}{4} & \end{array}\right),$$

where b_{22} and b_{33} are either both positive or both negative. It is easy to check that (A, B) has splitting type ((22), (11)) in the former case and ((1111), (11)) in the latter case. Furthermore, the stabilizer in $G_{\text{red}}(\mathbb{R})$ of (A, B) in either case is seen to consist of the following eight elements.

$$\begin{pmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \end{pmatrix}, \begin{pmatrix} \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \end{pmatrix}, \begin{pmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \end{pmatrix}, \begin{pmatrix} \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \end{pmatrix}, \begin{pmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & 1$$

This concludes the proof of the first item in Proposition 7.2. We omit the proofs of the other two items since they are very similar. \Box

Recall that the set of elements in $V(\mathbb{R})$ with infinite splitting type ς is denoted by $V(\mathbb{R})^{(\varsigma)}$. Given a splitting type ς , let $(A_{\varsigma}, B_{\varsigma}) \in V(\mathbb{R})^{(\varsigma)} \cap V_{\text{red}}(\mathbb{R})$ be an element whose invariants have absolute value 1. By multiplying A_{ς} by $\sqrt{|\mathbf{d}|}$ and multiplying the bottom 2×2 submatrix of B by $\sqrt{|\mathbf{q}|/|\mathbf{d}|}$, we obtain an element with invariants \mathbf{q} and \mathbf{d} , for any pair $(\mathbf{q}, \mathbf{d}) \in \mathbb{R}^2$ having the appropriate signs. We thus obtain the following result which follows immediately from Proposition 7.2.

Proposition 7.3. Fix an infinite splitting type ς . There exists a fundamental set $\mathcal{R}^{(\varsigma)}$ for the action of $G(\mathbb{R})$ on $V(\mathbb{R})^{(\varsigma)}$ such that $\mathcal{R}^{(\varsigma)}$ contains one element (A,B) having invariants q and d for any $(q,d) \in \mathbb{R}^2$ having the appropriate signs. Moreover, $\mathcal{R}^{(\varsigma)}$ may be constructed so that the element $(A,B) \in \mathcal{R}^{(\varsigma)}$ having invariants q and d is such that the coefficients of A are bounded by $O_{\delta}(|d|^{1/2})$ and the coefficients of B are bounded by $O_{\delta}(|d|^{1/2}|d|^{-1/2})$.

Let \mathcal{F} be a fundamental domain for the action of $G(\mathbb{Z})$ on $G(\mathbb{R})$. We may assume that \mathcal{F} is contained in the Siegel domain $\mathcal{S} = \mathcal{S}_1 \mathcal{S}_2$, where

$$\mathcal{S}_{1} = \left\{ \left(\begin{bmatrix} 1 \\ n \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ m_{3} \end{bmatrix} \right] \begin{bmatrix} 1 \\ t^{-1} \\ t \end{bmatrix} \begin{bmatrix} 1 \\ \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \right) : n, m_{3} \in [0, 1), \ t > \frac{1}{2} \right\}, \\
\mathcal{S}_{2} = \left\{ \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ m_{1} & 1 \\ m_{2} & 1 \end{bmatrix} \right) : m_{1}, m_{2} \in [0, 1) \right\}.$$
(32)

We have $\mathcal{F} = \mathcal{F}_2 \mathcal{F}_1$, where $\mathcal{F}_1 \subset \mathcal{S}_1$ and $\mathcal{F}_2 = \mathcal{S}_2$. From an argument identical to that in [5, §2.1], it follows that $\mathcal{F} \cdot \mathcal{R}^{(\varsigma)}$ is a cover of a fundamental domain for the action of $G(\mathbb{Z})$ on $V(\mathbb{R})^{(\varsigma)}$, where the $G(\mathbb{Z})$ -orbit of v is represented m(v) times. Here m(v) is given by

$$m(v) = \# \operatorname{Stab}_{G(\mathbb{R})}(v) / \# \operatorname{Stab}_{G(\mathbb{Z})}(v).$$

Every element in $V(\mathbb{R})$ is fixed by the element $(\mathrm{Id}, g_3) \in G(\mathbb{Z})$, where g_3 is the diagonal 3×3 matrix whose diagonal entries are 1, -1, and -1. Since the set of elements in $V(\mathbb{R})$, that have a stabilizer in $G(\mathbb{Z})$ of size greater than 2, has measure 0, we obtain the following theorem.

Theorem 7.4. The multiset $\mathcal{F} \cdot \mathcal{R}^{(\varsigma)}$ is an $(\tau_{\varsigma})/2$ -fold cover of a fundamental domain for the action of $G(\mathbb{Z})$ on $V(\mathbb{R})^{(\varsigma)}$, where $\tau_{\varsigma} = 8$ for $\varsigma = ((1111), (11))$ or ((22), (11)) and $\tau_{\varsigma} = 4$ for $\varsigma = ((112), (11))$ or ((22), (2)).

7.2 Averaging and cutting off the cusp

Let $\mathcal{L} \subset V(\mathbb{R})^{(\varsigma)}$ be a $G(\mathbb{Z})$ -invariant lattice, and denote the set of generic elements in \mathcal{L} by \mathcal{L}^{gen} . Given a subset W of $V(\mathbb{R})$ and a constant $\delta > 0$, we denote the set of elements $w \in W$ with $X \leq |\mathbf{q}(w)| < (1+\delta)X$ and $Y \leq |\mathbf{d}(w)| < (1+\delta)Y$ by W_{XY} . Since the stabilizer in $G(\mathbb{Z})$ of a generic element in $V(\mathbb{Z})$ has size 2, Theorem 7.4 implies that we have

$$\mathcal{N}_{\mathbf{q}}^{(\delta)}(\mathcal{L}; X, Y) = \frac{2}{\tau_c} \# \{ \mathcal{F} \cdot \mathcal{R}_{XY}^{(\varsigma)} \cap \mathcal{L}^{\mathrm{gen}} \}. \tag{33}$$

We now pick the following bounded open nonempty subset G_0 of $G_{\text{red}}(\mathbb{R})$:

$$G_0 := \left\{ \left(\begin{bmatrix} 1 & \\ n & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ & a & b \\ & c & d \end{bmatrix} \right) : n \in (0, \frac{\sqrt{Y}}{X}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_1 \subset \operatorname{SL}_2(\mathbb{R}) \right\},$$

where G_1 is a bounded open nonempty $SO_2(\mathbb{R})$ -invariant subset of $SL_2(\mathbb{R})$. The reason for the choice of the range of n will become apparent in what follows. Replacing the fundamental domain \mathcal{F} in (33) by $\mathcal{F}_2\mathcal{F}_1g$ and averaging over $g \in G_0$, and following the argument in the proof of [5, Theorem 2.5], we obtain

$$\mathcal{N}_{\mathbf{q}}^{(\delta)}(\mathcal{L}; X, Y) = \frac{2}{\tau_{\varsigma} \operatorname{Vol}(G_{0})} \int_{g \in G_{0}} \# \{ \mathcal{F}_{2} \mathcal{F}_{1} g \cdot \mathcal{R}_{XY}^{(\varsigma)} \cap \mathcal{L}^{\operatorname{gen}} \} dg$$

$$= \frac{2XY^{-1/2}}{\tau_{\varsigma} \operatorname{Vol}(G_{1})} \int_{g \in \mathcal{F}_{1}} \# \{ (\mathcal{F}_{2} g G_{0} \cdot \mathcal{R}_{XY}^{(\varsigma)} \cap \mathcal{L}^{\operatorname{gen}} \} dg,$$

since $Vol(G_0) = X^{1/2}Y^{-1}Vol(G_1)$.

Using coordinates from (32), we write an element in \mathcal{F}_1 as (n, m_3, t, θ) . In the next lemma we show that $\mathcal{F}_2 gG_0 \cdot \mathcal{R}_{XY}^{(\varsigma)}$ has no integral generic points if t is too large.

Lemma 7.5. Suppose $\mathcal{F}_2gG_0 \cdot \mathcal{R}_{XY}^{(\varsigma)} \cap V(\mathbb{Z})^{\text{gen}}$ is nonempty for $g = (n, m_3, t, \theta)$. Then $t \ll Y^{1/4}$.

Proof. Let (A, B) be an element of $\mathcal{R}^{(\varsigma)}$. Then we have $|a_{22}| \ll Y^{1/2}$. Therefore, there exists a constant C such that if $t > CY^{1/4}$, then $|a_{22}| < 1$ for every $(A, B) \in t\theta G_0\mathcal{R}^{(\varsigma)}$. The action of m_3 , n, and \mathcal{F}_2 does not change the value of a_{22} , and it follows that we have $a_{22} = 0$ for every $(A, B) \in \mathcal{F}_2 g G_0 \cdot \mathcal{R}^{(\varsigma)}_{XY} \cap V(\mathbb{Z})$. The lemma now follows since any such (A, B) is not generic.

We let $\mathcal{F}' = \mathcal{F}_2 \mathcal{F}'_1 \subset \mathcal{F}$ consist of all elements $g_2 g_1$ with $g_2 \in \mathcal{F}_2$ and $g_1 = (n, m_3, t, \theta)$, where $t \leq CY^{1/4}$ for the C in the proof of the above lemma. For any lattice \mathcal{L} of $V(\mathbb{Z})$, define

$$\mathcal{N}_{\mathbf{q}}^*(\mathcal{L}; X, Y) := \frac{2X^{-1/2}Y}{\tau_{\varsigma} \operatorname{Vol}(G_1)} \int_{g \in \mathcal{F}'_{\varsigma}} \#\{(\mathcal{F}_2 g G_0 \cdot \mathcal{R}_{XY}^{(\varsigma)} \cap \mathcal{L}\} dg. \tag{34}$$

We use the following result of Davenport [12] to estimate $\mathcal{N}_{q}^{*}(\mathcal{L}; X, Y)$:

Proposition 7.6. Let \mathcal{R} be a bounded, semi-algebraic multiset in \mathbb{R}^n having maximum multiplicity m, and that is defined by at most k polynomial inequalities each having degree at most ℓ . Then the number of integral lattice points (counted with multiplicity) contained in the region \mathcal{R} is

$$Vol(\mathcal{R}) + O(\max\{Vol(\bar{\mathcal{R}}), 1\}),$$

where $\operatorname{Vol}(\bar{\mathcal{R}})$ denotes the greatest d-dimensional volume of any projection of \mathcal{R} onto a coordinate subspace obtained by equating n-d coordinates to zero, where d takes all values from 1 to n-1. The implied constant in the second summand depends only on n, m, k, and ℓ .

In fact, the proof of the above proposition implies that we may replace $Vol(\bar{R})$ by the maximum of the d-dimensional volumes of the projections of any unipotent translate of R.

Now, for $g \in \mathcal{F}_1$, the set $gG_0 \cdot R_{XY}^{(\varsigma)}$ is a bounded set contained in $V_{\text{red}}(\mathbb{R})$. Hence, the b_{12} - and b_{13} -coefficients of elements in $\mathcal{F}_2 gG_0 \cdot R_{XY}^{(\varsigma)}$ must lie in [0,2). They can only be integral when they are 0 or 1. Therefore, every integral point in $\mathcal{F}_2 gG_0 \cdot R_{XY}^{(\varsigma)}$ lies on one of four hyperplanes in $V(\mathbb{R})$: the hyperplanes corresponding to $(b_{12}, b_{13}) = (0,0)$, (0,1), (1,0), and (1,1). Moreover, these hyperplanes are unipotent translates of each other, in fact, by the elements in \mathcal{F}_2 with $m_1, m_2 \in \{0, 1/2\}$. It follows that the four hyperplane sections have the same volume, and Proposition 7.6 applied to them yields the same error estimates. Therefore, we have

$$\mathcal{N}_{\mathbf{q}}^{*}(\mathcal{L}; X, Y) = \frac{8X^{-1/2}Y}{\tau_{\varsigma} \operatorname{Vol}(G_{1})} \int_{g \in \mathcal{F}_{1}'} \operatorname{Vol}_{\mathcal{L}}(gG_{0} \cdot \mathcal{R}_{XY}^{(\varsigma)}) dg + O(\mathcal{E}(X, Y)), \tag{35}$$

where $\operatorname{Vol}_{\mathcal{L}}$ is computed with Euclidean measure normalized so that \mathcal{L} has covolume 1, and

$$\mathcal{E}(X,Y) = X^{-1/2}Y \int_{g=(0,0,t,0)\in\mathcal{F}_1'} MP(gG_0 \cdot \mathcal{R}_{XY}^{(\varsigma)}) t^{-2} d^{\times} t.$$

The quantity MP(g) denotes the maximal volume of the projections of $gG_0 \cdot \mathcal{R}_{XY}^{(\varsigma)}$ onto its coordinate-hyperplanes. Every element (A,B) in $\mathcal{R}^{(\varsigma)}$ is such that the coefficients of A are bounded by $Y^{1/2}$ and the coefficients of B are bounded by $X^{1/2}/Y^{1/2}$. By construction of G_0 , the same is true for every element in $G_0 \cdot \mathcal{R}^{(\varsigma)}$. Then the error integral is easily bounded: as long as $Y \ll X$, the maximum projection is on to the coordinate subspace obtained by setting b_{22} to 0 since the ranges of the other coordinates are clearly $\gg 1$ for every value of $g \in \mathcal{F}'_1$. We therefore have

$$\mathcal{E}(X,Y) \ll YX^{-1/2} \int_{t=1}^{Y^{1/4}} Y^{1/2} Cd^{\times} t \ll Y^{3/2} X^{1/2} \log Y.$$
 (36)

We next have the following bound on the number of non-generic $G(\mathbb{Z})$ -orbits on $V(\mathbb{Z})$.

Proposition 7.7. We have

$$\frac{X^{-1/2}Y}{\operatorname{Vol}(G_1)} \int_{g \in \mathcal{F}_1'} \#\{gG_0 \cdot \mathcal{R}_{XY}^{(\varsigma)} \cap V(\mathbb{Z}) \setminus V(\mathbb{Z})^{\operatorname{gen}}\} dg = o(XY).$$

Proof. If $v \in V(\mathbb{Z})$ is not generic, then there exists unramified splitting types $\varsigma'' = (\varsigma_p'', \varsigma_p''')_p$ for all primes p such that $(\varsigma_p(v), \varsigma_p'(v)) \neq (\varsigma_p'', \varsigma_p''')$ for all primes p. Given any unramified splitting type, there exists a constant $c(\varsigma_p'', \varsigma_p''') < 1$ such that the density of elements in $V(\mathbb{Z}_p)$ that do not have splitting type $(\varsigma_p'', \varsigma_p''')$ is bounded above by $c(\varsigma_p'', \varsigma_p''')$. From (37), we therefore obtain for any fixed integer M:

$$\frac{X^{-1/2}Y}{\operatorname{Vol}(G_1)} \int_{g \in \mathcal{F}_1'} \#\{gG_0 \cdot \mathcal{R}_{XY}^{(\varsigma)} \cap V(\mathbb{Z}) \setminus V(\mathbb{Z})^{\operatorname{gen}}\} dg \ll \sum_{\varsigma''} XY \cdot \prod_{p < M} c(\varsigma_p'', \varsigma_p''').$$

Letting M tend to infinity, we obtain the result.

From (35), (36) and Proposition 7.7, we see that if X and Y go to infinity such that $Y(\log Y)^2 = o(X)$, then

$$\mathcal{N}_{\mathbf{q}}^{(\delta)}(\mathcal{L}; X, Y) = \frac{8X^{-1/2}Y}{\tau_{\varsigma} \operatorname{Vol}(G_{1})} \operatorname{Vol}(\mathcal{F}_{1}) \operatorname{Vol}_{\mathcal{L}}(G_{0} \cdot \mathcal{R}_{XY}^{(\varsigma)}) + o(XY)$$

$$= \frac{8}{\tau_{\varsigma} \operatorname{Vol}(G_{0})} \operatorname{Vol}(\mathcal{F}_{1}) \operatorname{Vol}_{\mathcal{L}}(G_{0} \cdot \mathcal{R}_{XY}^{(\varsigma)}) + o(XY). \tag{37}$$

To compute the volumes of $G_0 \cdot \mathcal{R}_{XY}^{(\varsigma)}$, we have the following result, which follows immediately from a Jacobian change of variables computation.

Proposition 7.8. Let dv_i be the standard Euclidean measures on $V_{\text{red}}(\mathbb{R})$, let dh denote the Haar-measure on $G_{\text{red}}(\mathbb{R})$ obtained from the $\bar{N}AN$ decomposition of $SL_2(\mathbb{R})$, and let dd dq be a measure on $\mathcal{R}^{(\varsigma)}$. We have a natural map $G_{\text{red}}(\mathbb{R}) \times \mathcal{R}^{(\varsigma)} \to V_{\text{red}}(\mathbb{R})$. Then the Jacobian change of variables is 1/16, i.e., for any measurable function φ on $V_{\text{red}}(\mathbb{R})$, we have

$$\int_{v \in G_{\text{red}}(\mathbb{R}) \cdot \mathcal{R}^{(\varsigma)}} \varphi(v) dv = \frac{1}{16} \int_{r \in \mathcal{R}^{(\varsigma)}} \int_{h \in G_{\text{red}}(\mathbb{R})} \varphi(g \cdot r) dh \, d\mathbf{q}(r) \, d\mathbf{q}(r). \tag{38}$$

Therefore, we obtain the following theorem from which Theorem 7.1 follows immediately.

Theorem 7.9. Let \mathcal{L} denote a finite union of $G(\mathbb{Z})$ -invariant lattice in $V(\mathbb{R})^{(\varsigma)}$. Then, for positive real numbers X, Y going to infinity such that $Y(\log Y)^2 = o(X)$, we have

$$\mathcal{N}_{\mathbf{q}}^{(\delta)}(\mathcal{L}; X, Y) = \frac{\zeta(2)}{2\tau_{\varsigma}} \delta^{2} X Y \prod_{p} \operatorname{Vol}(\mathcal{L}_{p}) + o_{\delta}(XY),$$

where \mathcal{L}_p denotes the closure of \mathcal{L} in $V(\mathbb{Z}_p)$, the volumes of sets in $V(\mathbb{Z}_p)$ are taken with respect to the usual Euclidean measure, and τ_s are as in Theorem 7.1.

Proof. The theorem follows from (37) and Proposition 7.8 since the volume of \mathcal{F}_1 under the measure dh is $\zeta(2)$, the volume of $\mathcal{R}_{XY}^{(\varsigma)}$ is XY and $\operatorname{Vol}_{\mathcal{L}}$ differs from normal Euclidean measure by a factor of $\prod_p \operatorname{Vol}(\mathcal{L}_p)$.

8 Uniformity estimates and sieving to maximal D_4 -orders

In order to use our results from §4, §5, and §6, §7 to prove our main theorems, we will employ simple sieves. In this section, we start by collecting the requisite tail estimates. First, we need a bound on the number of D_4 -fields having central inertia at some large prime, which is established in §8.1. On the other hand, we obtain an estimate on the number of $G(\mathbb{Z})$ -orbits on $V(\mathbb{Z})$ that are non-maximal at some large prime in §8.2. It is interesting to note that the results in §7 are not strong enough for these estimates, and so we employ techniques from §4.

8.1 Bounding the number of D_4 -fields with large central inertia

We start with a preliminary lemma bounding the number of D_4 -fields with fixed conductor.

Lemma 8.1. For any positive integer N, the number of D_4 -fields with conductor N is bounded by $O_{\epsilon}(N^{\epsilon})$.

Proof. Let L be a D_4 -field with conductor N, and let K be the quadratic subfield of L. Then the discriminant of K divides N. Hence the number of choices for K is bounded by twice the number of divisors of N. Given a fixed quadratic field K whose discriminant D divides N, the number of quartic D_4 -fields of conductor N whose quadratic subfield is K is bounded by $4 \cdot \# \operatorname{Cl}_2(K)$ times the number of squarefree ideals dividing 4N (see §3 of [8]). But $4 \cdot \# \operatorname{Cl}_2(K) \ll_{\epsilon} D^{\epsilon}$ and the number of divisors of 4N is $\ll_{\epsilon} N^{\epsilon}$. Combining these estimates yields the lemma.

Next, we prove the required estimate on D_4 -fields having central inertia at large primes p by combining the previous lemma with Lemma 4.6.

Proposition 8.2. Fix $\epsilon > 0$. Let X and Y be integers such that $X \geq Y$. Then the number of D_4 -fields L such that $X \leq q(L) < 2X$, $Y \leq d(L) < 2Y$, and L has central inertia at a prime p is bounded by $O_{\epsilon}(XY/p^{2-\epsilon})$.

Proof. We consider two ranges of p. We fix a large positive real number M (any M > 16 will suffice). When $p \geq X^{1/M}$, we have $p \geq (XY)^{1/2M}$. The number of possible conductors for a D_4 -field L satisfying the conditions of the proposition is bounded by $O(XY/p^2)$, since the conductor is bounded by 4XY and is divisible by p^2 . Hence, from Lemma 8.1, it follows that the number of such fields L is bounded by $O_{\epsilon}((XY)^{1+\epsilon}/n) = O_{\epsilon}(XY/n^{1-\epsilon})$.

For $p \leq X^{1/M}$, we see from Lemma 4.6 that the number of D_4 -fields satisfying the conditions of the proposition and having splitting type $((1^21^2), (11))$ or $((2^2), (2))$ is bounded by

$$O\left(\frac{X}{p^{2-\epsilon}} \cdot \sum_{\substack{[K:\mathbb{Q}]=2\\ |\operatorname{Disc}(K)| < Y}} L(1,K/\mathbb{Q})\right)$$

which is bounded by $O(XY/n^{1-\epsilon})$ by §3 of [25]. Similarly, the number of D_4 -fields satisfying the conditions of the proposition and having splitting type $((1^4), (1^2))$ is bounded by

$$O\left(\frac{X}{p^{1-\epsilon}} \cdot \sum_{\substack{[K:\mathbb{Q}]=2\\p|\operatorname{Disc}(K)\\|\operatorname{Disc}(K)|< Y}} L(1,K/\mathbb{Q})\right).$$

When Y grows faster than a large power of p, say $Y \gg p^M$, the sum is bounded by O(Y/p) by arguments identical to those in §3 of [25]. When $p \gg Y^{1/M}$, we instead use the classical bound of $O_{\epsilon}(\operatorname{Disc}(K)^{\epsilon})$ on $L(1, K/\mathbb{Q})$ to obtain the proposition.

Corollary 8.3. Let X and Y be integers such that $X \geq Y$. Then the number of D_4 -fields L such that $X \leq q(L) < 2X$, $Y \leq d(L) < 2Y$, and J(L) = n is bounded by $O_{\epsilon}(XY/n^{1-\epsilon})$.

Proof. If $p \mid J(L)$ for some odd prime p, then by Proposition 2.6, the p-part of J(L) is p^2 and the splitting type of L at p is $((1^21^2), (11))$ or $((2^2), (22))$. The corollary therefore follows from a proof identical to that of Proposition 8.2.

8.2 Bounding the number of non-maximal $G(\mathbb{Z})$ -orbits on $V(\mathbb{Z})$

For a fixed prime p, let \mathcal{W}_p denote the set of generic elements in $V(\mathbb{Z})$ that correspond to nonmaximal orders in D_4 -fields. Our next goal is to prove a uniform tail estimate for the number of $G(\mathbb{Z})$ -orbits on \mathcal{W}_p having bounded invariants. We start with the following lemmas.

Lemma 8.4. The number of D_4 -fields L with |q(L)| < X and |d(L)| < Y is bounded by O(XY).

Proof. From Lemma 4.5, we see that for $X \geq Y$ (in fact for $Y \ll_{\epsilon} X^{3-\epsilon}$), the number of D_4 -fields with invariants q and d less than X and Y, respectively, is bounded by

$$O\left(X \cdot \sum_{|\operatorname{Disc}(K)| < Y} L(1, K/\mathbb{Q})\right),$$

which is O(XY) by the results in §3 of [25].

When X < Y, we bound the number of D_4 -fields L by instead bounding the number of fields $\phi(L)$ (see Definition 2.1). If L satisfies the conditions of the lemma, then it follows that $X/n^2 \le |\mathrm{d}(\phi(L))| < 2X/n^2$, $Yn^2 \le |\mathrm{d}(\phi(L))| < 2Yn^2$, where $n = J(L) = J(\phi(L))$ is a square at every odd prime factor. From Corollary 8.3, it follows that the number of such $\phi(L)$ is bounded by

$$O\left(\sum_{n} XY/n^{1-\epsilon}\right),$$

where n runs over integers that are exactly divisible by p^2 for every odd prime factor. Since the sum converges, the lemma follows.

We now prove the following uniform bound on the number of $G(\mathbb{Z})$ -orbits on \mathcal{W}_p , the set of generic elements in $V(\mathbb{Z})$ that are not maximal at p.

Proposition 8.5. The number of $G(\mathbb{Z})$ -orbits v on \mathcal{W}_p with $X \leq q(v) < 2X$ and $Y \leq d(v) < 2Y$ is bounded by $O_{\epsilon}(XY/p^{2-\epsilon})$.

Proof. An element $v \in \mathcal{W}_p$ gives a quartic ring Q whose field of fractions L is a D_4 -field. Let i(v) denote C(v)/C(L), the ratio of the conductors of v and L. From Lemma 6.6, it follows that i(v) is divisible by p^2 . From Lemma 8.4, it follows that the number of possible fields L that occur this way is bounded by $O(XY/i(v)^{1-\epsilon})$. Next, note that the index of Q in the ring of integers of L divides i(v). The methods of [22] imply that the number of suborders of index $k = \prod p_i^{e_i}$ of a maximal quartic ring is bounded by

$$j(k) \coloneqq \prod p_i^{(2+\epsilon)\lfloor \frac{e_i}{4} \rfloor}.$$

Once the order Q has been determined, there are O(1) choices for the quadratic subring T of Q corresponding to v under Theorem 6.1. Finally, Corollary 4 of [1] asserts that the number of cubic resolvents of ring Q is d(c), the sum of the divisors of the content c of Q. Furthermore, the content c of the quartic ring corresponding to v = (A, B) is equal to be the gcd of the coefficients of A (see §3.6 of [1]), which implies that i(v) is a multiple of c^4 .

Therefore, it follows that the number of $G(\mathbb{Z})$ -orbits on \mathcal{W}_p satisfying the conditions of the proposition is bounded by

$$\sum_{p^2|m} \sum_{c^4|m} \sum_{k|m} j(k)d(c)(X/m^{1-\epsilon}), \tag{39}$$

where m runs over all integers divisible by p^2 . Using the multiplicativity of j and d, the expression (39) is easily seen to be $\ll_{\epsilon} X/p^{2-\epsilon}$ and the proposition follows.

9 Proof of the main theorems

In this section, we conclude the proof of the generalization of Theorem 5.3 that allows for imposing certain local specifications (see Theorem 9.1). As a byproduct of the two asymptotics obtained for $N_{\rm C}(\Sigma; X, X^{1/2})$ by Theorem 4.3 and in §9.1, we prove Theorem 5.

Theorems 1 and 3 are proved in §9.2. First, we obtain asymptotics for $N_{\mathbb{C}}(\Sigma; X, Y)$ when $Y > X^{1/2}$ from counting D_4 -fields $\phi(L)$ with conductor bounded by X and small quadratic discriminant, where $L \in \mathcal{L}(\Sigma)$. We then prove Theorem 3 after employing a bound that follows from the analytic methods in Section 4 and 5. Finally, in §9.3 we prove a refinement of Theorem 4, which follows from Theorem 3 in conjunction with the p-adic volumes determined in Proposition 6.7.

9.1A refinement of Theorem 5.3 and the proof of Theorem 5

Recall that for a collection of local specifications Σ , $\mathcal{L}(\Sigma)$ is the set of D_4 -fields L such that the pair consisting of the splitting type of L and the splitting type of K, its quadratic subfield at a prime p (respectively, at ∞) is contained in Σ_p for all p (respectively, in Σ_{∞}). A set Σ of local specifications (and the corresponding family $\mathcal{L}(\Sigma)$) is said to be acceptable if for all but finitely many primes p, the set Σ_p contains all unramified splitting types and tamely ramified splitting types without central inertia. (In the notation of (5), Σ_p contains exactly the pairs $(\varsigma_p(L_1), \varsigma_p(K_1))$ contained in the first two groups in Table 1.)

Recall that for a prime p and a splitting type $(\varsigma_p, \varsigma_p')$, we computed the density $\mu(M_p(\varsigma_p, \varsigma_p'))$ in Proposition 6.7. We define the density $\mu(\Sigma_p)$ to be the sum of the values of $\mu(M_p(\varsigma_p, \varsigma_p'))$ over $(\varsigma_p, \varsigma_p') \in \Sigma_p$, and define the density of $\mu(\Sigma_p)$ to be the sum of $1/\tau_{\varsigma_p}$ over $\varsigma_p \in \Sigma_p$ (see Theorem 7.1 for the definition of τ_{ς_p} .)

For positive real numbers X and Y, let $\mathcal{N}_q^{(\delta)}(\Sigma; X, Y)$ be as in Section 7. We have the following theorem,

giving another proof that the heuristics of (9) holds for certain ranges of X and Y.

Theorem 9.1. Let Σ be an acceptable set of local specifications. For positive real numbers X and Y such that $Y(\log Y)^2 = o(X)$, we have

$$\mathcal{N}_{\mathbf{q}}^{(\delta)}(\Sigma; X, Y) = \frac{\zeta(2)}{2} \cdot \delta^2 \cdot \mu(\Sigma_{\infty}) \cdot \prod_{p} \mu(\Sigma_{p}) \cdot XY + o_{\delta}(XY).$$

Proof. By Proposition 6.3, it suffices to obtain asymptotics for the number of generic $G(\mathbb{Z})$ -orbits (A,B)on $V(\mathbb{Z})$ such that (A,B) is maximal and the splitting type of (A,B) at each place belongs to Σ . The number of generic $G(\mathbb{Z})$ -orbits on $V(\mathbb{Z})$ satisfying any finite set of congruence conditions has been estimated in Theorem 7.9. Theorem 9.1 then follows from Theorem 7.9 and the uniformity estimates in Propositions 8.2 and 8.5 by means of a simple sieve. We omit the details since they are very similar to those in the proof of Theorem 9.2.

As a consequence of the main terms obtained in Theorems 4.3 and 9.1, we may now prove Theorem 5. It is interesting to note that the proof of Theorem 5 is much more involved than that of Theorem 2. In particular, it is not clear that the arguments in Section 5 can be refined to directly allow for imposing acceptable sets of local specifications.

Proof of Theorem 5. Let K denote the set of quadratic fields with prescribed splitting types ς_p' given at a finite set S of odd primes p. Let $(\Sigma_p)_p$ denote sets, where for each $p \notin S$, $\Sigma_p = \Sigma_p^{\text{full}}$ contains all possible splitting types, and for $p \in S$, $\Sigma_p = \{(*,\varsigma_p')\}$ consists of all possible splitting types compatible with ς_p' . If we let $\Sigma_{\infty}^{(a)} = \{((1111), (11)), ((112), (11)), ((22), (11))\}, \text{ and } \Sigma_{\infty}^{(b)} = \{((22), (2))\}, \text{ we can define } \Sigma^{(*)} \text{ to be the } \Sigma^{(*)} \text{ to be } \Sigma^{(*)} \text{ to be the } \Sigma^{(*)} \text{ to be } \Sigma^{(*)} \text{$ collection $(\Sigma_p)_p$ and $\Sigma_{\infty}^{(*)}$ for *= a or b, which are both acceptable collections.

Recall that $N_{\mathcal{C}}(\Sigma^{(*)}; X, X^{\beta})$ counts the number of isomorphism classes of D_4 -fields $L \in \mathcal{L}(\Sigma^{(*)})$ such that |C(L)| < X and $|d(L)| < X^{\beta}$. As before, let $r_2(K)$ denote the number of pairs of complex embeddings of K. From Theorem 4.3, we have for * = a or b and $\beta < 2/3$,

$$N_{\mathcal{C}}(\Sigma^{(*)}; X, X^{\beta}) = \frac{X}{2\zeta(2)} \cdot \sum_{\substack{K \in \mathcal{K}(\Sigma^{(*)}) \\ |\operatorname{Disc}(K)| < X^{\beta}}} \frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} \cdot \frac{2^{-r_2(K)}}{|\operatorname{Disc}(K)|} + o_{\beta}(X). \tag{40}$$

On the other hand, we can also estimate $N_C(\Sigma^{(*)}; X, X^{\beta})$ using Theorem 9.1 as follows. Consider the region

$$R_{X,\beta} := \{ (d,q) \in \mathbb{R}^2 : |d \cdot q| < X, |d| < X^{\beta} \}.$$

There exist regions $R_{X,\beta}^{(\pm)}$, that are disjoint unions of δ -adic rectangles, such that

$$R_{X,\beta}^{(-)} \subset R_{X,\beta} \subset R_{X,\beta}^{(+)}$$

and such that

$$|\operatorname{Vol}(R_{X,\beta}) - \operatorname{Vol}(R_{X,\beta}^{(\pm)})| \ll \delta \cdot X \log(X^{\beta}).$$

The volume of $R_{X,\beta}$ is $X \log X^{\beta}$. Therefore, from Theorem 9.1, we see that for $\beta < 1/2$,

$$N_{\mathcal{C}}(\Sigma^{(*)}; X, X^{\beta}) = \frac{\zeta(2)}{2} \cdot X \log(X^{\beta}) \cdot \mu(\Sigma_{\infty}^{(*)}) \cdot \prod_{p} \mu(\Sigma_{p}) + o_{\delta}(X \log(X^{\beta})) + O(\delta X \log(X^{\beta})). \tag{41}$$

Equating the right hand sides of (40) and (41), dividing both sides by $X \log(X^{\beta})$, first letting X^{β} tend to infinity, and then finally letting δ tend to 0, we obtain:

$$\frac{1}{2\zeta(2)} \cdot \sum_{\substack{K \in \mathcal{K}(\Sigma^{(*)}) \\ |\operatorname{Disc}(K)| < X}} \frac{L(1, K/\mathbb{Q})}{L(2, K/\mathbb{Q})} \cdot \frac{2^{-r_2(K)}}{|\operatorname{Disc}(K)|} \sim \frac{\zeta(2)}{2} \cdot \mu(\Sigma_{\infty}^{(*)}) \cdot \prod_{p} \mu(\Sigma_p) \cdot \log(X)$$
(42)

It is easy to see that $2^{r_2(K)} \cdot \mu(\Sigma_{\infty}^{(*)})$ is always 1/2, independent of $K \in \mathcal{K}(\Sigma^{(*)})$. Furthermore, the values of $\mu(\Sigma_p) = \sum_{(\varsigma_p,\varsigma_p') \in \Sigma_p} \mu(M_p(\varsigma_p,\varsigma_p'))$ can be computed from Proposition 6.7, and it then follows that the right hand sides of Theorem 5(a) and (b) are asymptotically equal to

$$\frac{\zeta(2)^2}{2} \cdot \prod_p \mu(\Sigma_p) \cdot \log(X).$$

This concludes the proof of Theorem 5.

9.2 The proofs of Theorems 1 and 3

We next obtain asymptotics for $\mathcal{N}_{\mathbf{q}}^{(\delta)}(\Sigma; X, Y)$ when $Y \gg X$. Recall that the outer automorphism ϕ of D_4 provides a non-isomorphic D_4 -field $\phi(L)$ for each D_4 -field L, and the fields L and $\phi(L)$ have the same conductor but (possibly) different invariants. Proposition 2.6 can be used to compute the invariants of $\phi(L)$ in terms of the invariants of L. Note that if d(L) > q(L), then $d(\phi(L)) < q(\phi(L))$. Hence, for a collection of local specifications Σ , we may relate counts of D_4 -fields with d > q to counts of D_4 -fields with d < q.

Given an acceptable collection Σ , let $\phi(\mathcal{L}(\Sigma))$ denote the family defined by

$$\phi(\mathcal{L}(\Sigma)) := \{ \phi(L) : L \in \mathcal{L}(\Sigma) \}$$

It is clear that there exists another acceptable collection $\phi(\Sigma)$ of local specifications such that $\phi(\mathcal{L}(\Sigma)) = \mathcal{L}(\phi(\Sigma))$. Furthermore, for every odd prime p, Table 1 in conjunction with Proposition 6.7 and Lemma 6.8 for an acceptable collection Σ , we have $\mu(\Sigma_p) = \mu(\phi(\Sigma_p))$. An acceptable family Σ is said to be very stable at 2 if the set Σ_2 either contains all splitting types with central inertia (pairs $(\varsigma_2(L_1), \varsigma_2(K_1))$) in the latter two groups of Table 1) or it contains none of them. Our next result computes the number of D_4 -fields in $\mathcal{L}(\Sigma)$ satisfying $X \leq |q(L)| \leq (1+\delta)X$ and $Y \leq |d(L)| < (1+\delta)Y$ when Y is much larger than X.

Theorem 9.2. Let Σ be an acceptable collection of local specifications that is very stable at 2. Let X and Y be positive real numbers such that $X(\log X)^2 = o(Y)$. Then we have

$$\mathcal{N}_{\mathbf{q}}^{(\delta)}(\Sigma; X, Y) = \frac{\zeta(2)}{2} \cdot \delta^2 \cdot \mu(\Sigma_{\infty}) \cdot \prod_{p} \mu(\Sigma_{p}) \cdot XY + o(XY).$$

Proof. For a prime p and an integer $a \ge 1$, let $\mathcal{V}(p,a)$ denote the set of D_4 -fields L such that $J_p(L) = p^a$. (Recall that J(L) is defined in (7), and an odd prime $p \mid J(L)$ if only if L has splitting type $((2^2), (2))$, or $((1^21^2), (11))$.) Let \mathcal{V}_p denote the union of $\mathcal{V}(p,a)$ over all $a \ge 1$, and for any integer $n \ge 1$, let $\mathcal{L}(\Sigma)^{(n)}$ denote the set of fields L in $\mathcal{L}(\Sigma)$ such that J(L) = n. One can check that $\mathcal{L}(\Sigma)^{(n)}$ is defined by an acceptable collection $\Sigma^{(n)}$ of local specifications that is very stable at 2, i.e., $\mathcal{L}(\Sigma^{(n)}) = \mathcal{L}(\Sigma)^{(n)}$. We have

$$\begin{split} \mathcal{N}_{\mathbf{q}}^{(\delta)}(\Sigma;X,Y) &= \sum_{n\geq 1} \mathcal{N}_{\mathbf{q}}^{(\delta)}(\Sigma^{(n)};X,Y) \\ &= \sum_{n\geq 1} \mathcal{N}_{\mathbf{q}}^{(\delta)}(\phi(\Sigma^{(n)});Yn,X/n). \end{split}$$

For a fixed integer M, we use Theorem 9.1 to evaluate $\mathcal{N}_{\mathbf{q}}^{(\delta)}(\phi(\Sigma)^{(n)}; Yn, X/n)$ for $n \leq M$ and Proposition 8.2 to bound $\mathcal{N}_{\mathbf{q}}^{(\delta)}(\phi(\Sigma)^{(n)}; Yn, X/n)$ for n > M. Altogether, we obtain

$$\mathcal{N}_{\mathbf{q}}^{(\delta)}(\Sigma; X, Y) \sim \frac{\zeta(2)}{2} \cdot \delta^{2} \cdot \mu(\phi(\Sigma)_{\infty}) \cdot \left(\sum_{n=1}^{M} \prod_{p^{\alpha} \mid | n} \mu(\phi(\Sigma)_{p} \cap \mathcal{V}(p, a)) \cdot \prod_{p \nmid n} \mu(\phi(\Sigma)_{p} \setminus \mathcal{V}_{p}) \right) \cdot XY$$

up to an error of $o_{\delta,M}(XY) + O_{\epsilon,\delta}(XY/M^{1-\epsilon})$, where we assume that $a \geq 1$. Dividing by XY, letting X and Y tend to infinity, and then letting M tend to infinity, we obtain

$$\lim_{M \to \infty} \lim_{X,Y \to \infty} \frac{\mathcal{N}_{\mathbf{q}}^{(\delta)}(\Sigma; X, Y)}{\delta^{2}XY} = \frac{\zeta(2)}{2} \cdot \mu(\phi(\Sigma)_{\infty}) \cdot \sum_{n \geq 1} \left(\prod_{p^{a} || n} \mu(\phi(\Sigma)_{p} \cap \mathcal{V}(p, a)) \cdot \prod_{p \nmid n} \mu(\phi(\Sigma)_{p} \setminus \mathcal{V}_{p}) \right)$$

$$= \frac{\zeta(2)}{2} \cdot \mu(\phi(\Sigma)_{\infty}) \cdot \sum_{n \geq 1} \left(\prod_{p} \mu(\phi(\Sigma)_{p} \setminus \mathcal{V}_{p}) \cdot \prod_{p^{a} || n} \frac{\mu(\phi(\Sigma)_{p} \cap \mathcal{V}(p, a))}{\mu(\phi(\Sigma)_{p} \setminus \mathcal{V}_{p})} \right)$$

$$= \frac{\zeta(2)}{2} \cdot \mu(\phi(\Sigma)_{\infty}) \cdot \prod_{p} \mu(\phi(\Sigma)_{p} \setminus \mathcal{V}_{p}) \cdot \prod_{p} \left(1 + \sum_{a \geq 1} \frac{\mu(\phi(\Sigma)_{p} \cap \mathcal{V}(p, a))}{\mu(\phi(\Sigma)_{p} \setminus \mathcal{V}_{p})} \right)$$

$$= \frac{\zeta(2)}{2} \cdot \mu(\phi(\Sigma)_{\infty}) \cdot \prod_{p} \mu(\phi(\Sigma)_{p}).$$

Since $\mu(\Sigma_p) = \mu(\phi(\Sigma)_p)$ for all primes p and $\mu(\Sigma_\infty) = \mu(\phi(\Sigma_\infty))$, we obtain the result.

We now have theorems computing $\mathcal{N}_{\mathbf{q}}^{(\delta)}(\Sigma;X,Y)$ when $X(\log X)^2=o(Y)$ (Theorem 9.1) and when $Y(\log Y)^2=o(X)$ (Theorem 9.2 with identical right hand sides. Our last task in proving Theorem 3 is to show that the region that neither Theorem 9.1 nor 9.2 covers contributes negligibly to $\mathcal{N}_{\mathbf{q}}^{(\delta)}(\Sigma;X,Y)$. For that we need the following lemma.

Lemma 9.3. The number of D_4 -fields L such that $|C(L)| \leq X$ and $X(\log X)^{-3} \leq |d(L)| \leq X(\log X)^3$ is bounded by $O(X \log \log X)$.

Proof. The number of D_4 -fields satisfying the conditions of the lemma can be estimated as a sum of ratios of L-values from Theorem 4.3. This sum can be bounded, using Proposition 5.2, by

$$X \cdot \sum_{D=X(\log X)^{-3}}^{X(\log X)^3} \frac{1}{D},$$

yielding the lemma.

Proof of Theorem 3. The invariants d and q of a D_4 field with absolute conductor bounded by X satisfy $|d \cdot q| < X$. Consider the region $R_X := \{(d, q) \in \mathbb{R}^2 : |d \cdot q| < X\}$. We bound the number of D_4 -fields L with $X(\log X)^{-3} \le d(L) \le X(\log X)^3$ using Lemma 9.3, and estimate the rest of the D_4 -fields using Theorems 9.1 and 9.2 with an argument identical to the proof of Theorem 5, obtaining

$$N_{\mathrm{D}_4}(\Sigma, X) \sim \frac{\zeta(2)}{2} \cdot \mu(\Sigma_{\infty}) \cdot \prod_n \mu(\Sigma_p) \cdot X \log(X).$$

Let $\varsigma_p(L_p, K_p)$ denote the splitting type of a pair (L_p, K_p) of local extensions of \mathbb{Q}_p . From Propositions 3.1 and 6.7, we obtain for each prime p,

$$\frac{1}{\# \operatorname{Aut}(L_p, K_p)} \cdot \frac{1}{\operatorname{C}_p(L_p)} \cdot \left(1 - \frac{1}{p}\right)^2 = \frac{\mu(M_p(\varsigma_p(L_p, K_p)))}{1 - p^{-2}},$$

and at ∞ , $\# \operatorname{Aut}(L_{\infty}, K_{\infty}) = \tau_{\varsigma(L_{\infty}),\varsigma(K_{\infty})}$ (see Theorem 7.1 for the definition of τ_{ς}). This concludes the proof of Theorem 3.

Theorem 1 follows directly from Theorem 3 in conjunction with the density computations in Theorem 6.9.

9.3 The proof of Theorem 4

We end this article with the proof and a discussion of Theorem 4.

Let $K \in \mathcal{K}$ be a quadratic field, and for each $p \in S$, let ς denote the prescribed splitting type at p for \mathcal{K} . Since finite abelian groups are isomorphic to their duals, we see that $\#\operatorname{Cl}(K)[4] - \#\operatorname{Cl}(K)[2]$, the number of elements in $\operatorname{Cl}(K)$ having exact order 4, is equal to twice the number of index-4 subgroups of $\operatorname{Cl}(K)$ whose quotients are cyclic. By class field theory, such index-4 subgroups of $\operatorname{Cl}(K)$ are in bijection with isomorphism classes of unramified extensions M of K with $\operatorname{Gal}(M/K) = C_4$. Such an extension M is Galois over $\mathbb Q$ with Galois group D_4 . Conversely, if M is an octic D_4 -field whose splitting type at every prime p lies in the first two quadrants of Table 1, then M is unramified over K, its quadratic subfield fixed by $C_4 \subset D_4$. Furthermore, it is easily checked from Table 1 that M and K have the same conductor.

Now we define three collections of local specifications $\Sigma^{(i)}$ corresponding to the three cases in Theorem 4. First, let $\mathcal{K}^{(*)}$ be the subset of $K \in \mathcal{K}$ with $\varsigma_{\infty}(K) = (11)$ when * = a or c, and $\varsigma_{\infty}(K) = (2)$ when * = b, and define

$$\Sigma_{\infty}^{(*)} := \begin{cases} \{((1111), (11))\} & \text{if } * = a, \\ \{((112), (11)), ((22), (2))\} & \text{if } * = b, \\ \{((1111), (11)), ((22), (11))\} & \text{if } * = c. \end{cases}$$

Next we define Σ_p for all finite primes p. If $p \notin S$, define

$$\Sigma_p = \{(\varsigma_p, \varsigma_p') : (\varsigma_p, \varsigma_p') \text{ lacks central inertia} \}.$$

For $p \in S$, define

$$\Sigma_p = \begin{cases} \{((1111), (11)), ((22), (11)), ((4), (2))\} & \text{if } \varsigma = (11), \\ \{((112), (11)), ((22), (2))\} & \text{if } \varsigma = (2), \\ \{((1^211), (11)), ((1^22), (11)), ((1^21^2), (1^2)), ((2^2), (1^2))\} & \text{if } \varsigma = (1^2); \end{cases}$$

Let $\Sigma^{(*)}$ then denote collection of local specifications consisting of $(\Sigma_p)_p$ along with $\Sigma_{\infty}^{(*)}$ for *= a, b, or c, which is acceptable and very stable at 2. It is easily checked from Table 1 that the Galois closures of fields in $\mathcal{L}(\Sigma^{(*)})$ correspond to cyclic quartic unramified extensions of fields in $\mathcal{K}^{(*)}$. When *= a or b, $\mathcal{L}(\Sigma^{(*)})$ corresponds to order 4 elements in the class groups of $\mathcal{K}^{(*)}$. On the other hand, $\mathcal{L}(\Sigma^{(c)})$ corresponds to cyclic quartic extensions of $\mathcal{K}^{(c)}$ that are unramified at every finite place, but possibly ramified at infinity; thus, they correspond to order 4 elements in the narrow class groups of real quadratic fields in \mathcal{K} . Furthermore, from (5), it follows that exactly two distinct isomorphism classes of D_4 -fields yield the same Galois closure, but additionally every index-4 subgroup of the class group corresponds to two order 4 ideal classes. Thus, we conclude that the left hand sides of Theorem 4(*) are equal to $N_{D_4}(\Sigma^{(*)}, X)$, for *= a, b, and c. The theorem follows from Theorem 3 along with density computations following from Proposition 6.7.

Remark 9.4. Fourry-Klüners [14] prove that the average size of $2^{\text{rk}_4(\text{Cl}^+(K))}$ over imaginary (respectively, real) quadratic fields K ordered by discriminant is equal to 2 (respectively, $\frac{3}{2}$), where $\text{rk}_4(\text{Cl}^+(K)) = \dim_{\mathbb{F}_2}(\text{Cl}^+(K)^4/\text{Cl}^+(K)^2)$. Thus, we have

$$\#\operatorname{Cl}_{4}^{+}(K) - \#\operatorname{Cl}_{2}^{+}(K) = (2^{\operatorname{rk}_{4}(\operatorname{Cl}^{+}(K))} - 1) \cdot (\#\operatorname{Cl}_{2}^{+}(K)).$$

It is well-known from genus theory that for any quadratic field K, $\#\operatorname{Cl}_2^+(K) = 2^{\omega(D)-1}$, where $\omega(D)$ is equal to the number of prime factors of D. Additionally, the asymptotic in Theorem 4(b) (respectively, Theorem 4(c)) is also true if one replaces the summand on the left hand side with simply $\#\operatorname{Cl}_2(K)$ (respectively, $\#\operatorname{Cl}_2^+(K)/2$). Thus, the average value of the product $(2^{\operatorname{rk}_4(\operatorname{Cl}^+(K))} - 1) \cdot (\#\operatorname{Cl}_2^+(K))$ is equal to the product of the average value of $(2^{\operatorname{rk}_4(\operatorname{Cl}^+(K))} - 1)$ and the average size of $\operatorname{Cl}_2^+(K)$. Similar analysis holds for class groups of real quadratic fields.

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