Coordinate Transform Notes

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1 Notations

1.1 Nabla

$\nabla\phi$, $\nabla\cdot\phi$, $\nabla^2\phi$ represent the gradient, divergence and Laplacian in Euclidean space without specifying a coordinate system, their values are irrelevant to the coordinate system.

∇_X and ∇_Ξ are differential operators with respect to specified coordinate systems

$$\nabla_X = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \qquad \nabla_\Xi = \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{bmatrix} \tag{1.1}$$

1.2 Vectors

$\boldsymbol{\phi}$ is a vector in Euclidean space without specifying a coordinate system, the vector itself is irrelevant to the coordinate system.

$\boldsymbol{\phi}^X$ and $\boldsymbol{\phi}^\Xi$ represent the coordinates of vector under different coordinate systems.

$\boldsymbol{\phi}$, $\boldsymbol{\phi}^X$, $\boldsymbol{\phi}^\Xi$ are equivalent for representing the same vector, this is indicated by $A \equiv B$ as

$$\boldsymbol{\phi} \equiv \boldsymbol{\phi}^X \equiv \boldsymbol{\phi}^\Xi$$

Having the same value is indicated by $A = B$ as

$$\boldsymbol{\phi}^\Xi = \Xi_X \cdot \boldsymbol{\phi}^X \tag{3.2}$$

For scalars, $A = B$ is the same as $A \equiv B$.

2 Matrices

2.1 Transform Matrices

Two coordinate systems $X = (x, y, z)$ and $\Xi = (\xi, \eta, \zeta)$ for Euclidean space, the transform matrices between them are

$$X_\Xi = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \qquad \Xi_X = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} \tag{2.1}$$

X_Ξ and Ξ_X are mutually inverse

$$X_\Xi \cdot \Xi_X = \Xi_X \cdot X_\Xi = I \tag{2.2}$$

The Jacobian J is the determinant of X_Ξ

$$J = |X_\Xi| \tag{2.3}$$

2.2 Metric Tensors

The metric tensor

$$g = X_\Xi^\top \cdot X_\Xi \tag{2.4}$$

And the inverse metric tensor

$$g^{-1} = \Xi_X \cdot \Xi_X^\top \tag{2.5}$$

They are indeed inverse to each other because

$$\begin{aligned} g \cdot g^{-1} &= (X_\Xi^\top \cdot X_\Xi) \cdot (\Xi_X \cdot \Xi_X^\top) \\ &= X_\Xi^\top \cdot (X_\Xi \cdot \Xi_X) \cdot \Xi_X^\top \\ &= X_\Xi^\top \cdot \Xi_X^\top \\ &= I \end{aligned} \tag{2.6}$$

3 Basic Transforms

3.1 Vectors

A vector and its components in both coordinates systems

$$\boldsymbol{\phi} \equiv \boldsymbol{\phi}^X = [\phi^x, \phi^y, \phi^z]^\top \equiv \boldsymbol{\phi}^\Xi = [\phi^\xi, \phi^\eta, \phi^\zeta]^\top \tag{3.1}$$

The transform between them

$$\boldsymbol{\phi}^\Xi = \Xi_X \cdot \boldsymbol{\phi}^X \qquad \boldsymbol{\phi}^X = X_\Xi \cdot \boldsymbol{\phi}^\Xi \tag{3.2}$$

3.2 Derivatives

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \Xi_X^\top \cdot \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{bmatrix} \tag{3.3}$$

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{bmatrix} = X_\Xi^\top \cdot \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \tag{3.4}$$

Another way to write this can be seen in http://dx.doi.org/10.5772/2542 Chapter 7.3.2

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \frac{1}{J} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{bmatrix} = \frac{1}{J} C \cdot \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{bmatrix} \tag{3.5}$$

With C being the cofactor matrix of X_Ξ. This is equivalent to the transform given before as

$$\Xi_X = X_\Xi^{-1} = \frac{1}{J} C^\top \tag{3.6}$$

and there is

$$c_{ij} = J \, \partial \xi_j / \partial x_i \tag{3.7}$$

3.3 Gradient

The gradient is a vector and is expressed by the Cartesian coordinate system by default

$$\nabla \phi \equiv [\nabla \phi]^X = \nabla_X \phi = \Xi_X^\top \cdot \nabla_\Xi \phi \tag{3.8}$$

This is the same as https://en.wikipedia.org/wiki/Gradient#General_coordinates, as it gives

$$[\nabla \phi]^X = (X_\Xi \cdot (g^{-1})^\top) \cdot \nabla_\Xi \phi \tag{3.9}$$

and

$$X_\Xi \cdot (g^{-1})^\top = X_\Xi \cdot (\Xi_X \cdot \Xi_X^\top)^\top = X_\Xi \cdot (\Xi_X \cdot \Xi_X^\top) = (X_\Xi \cdot \Xi_X) \cdot \Xi_X^\top = \Xi_X^\top \tag{3.10}$$

And as it is a vector, it can be expressed using the other coordinate system

$$\nabla\phi \equiv [\nabla\phi]^\Xi = \Xi_X \cdot [\nabla\phi]^X = \Xi_X \cdot (\Xi_X^\top \cdot \nabla_\Xi \phi) = g^{-1} \cdot \nabla_\Xi \phi \tag{3.11}$$

3.4 Divergence

$$\nabla \cdot \boldsymbol{\phi} = \nabla_X^\top \cdot \boldsymbol{\phi}^X = \frac{1}{J} \nabla_\Xi^\top \cdot J \boldsymbol{\phi}^\Xi \tag{3.12}$$

https://en.wikipedia.org/wiki/Divergence#General_coordinates gives a more general form as

$$\nabla \cdot \boldsymbol{\phi} = \frac{1}{\rho} \nabla_\Xi^\top \cdot \rho \boldsymbol{\phi}^\Xi \tag{3.13}$$

With ρ being the volume coefficient, it can be defined as

$$\rho = \sqrt{|\det g|} \tag{3.14}$$

or directly defined as the Jacobian itself

$$\rho = J \tag{3.15}$$

3.5 Laplacian

The Laplacian of a scalar is the divergence of the gradient vector

$$\nabla^2 \phi = \nabla \cdot \nabla \phi \tag{3.16}$$

so there is

$$\begin{aligned} \nabla^2 \phi &= \frac{1}{J} \nabla_\Xi^\top \cdot J[\nabla\phi]^\Xi \\ &= \frac{1}{J} \nabla_\Xi^\top \cdot (J g^{-1} \cdot \nabla_\Xi \phi) \end{aligned} \tag{3.17}$$

3.6 Volume Integration

The volume element

$$dV = dV_X = J dV_\Xi \tag{3.18}$$

so the volume integration of a scalar ϕ in a control volume is

$$\int_V \phi \; dV = \int_V \phi \; dV_X = \int_V J\phi \; dV_\Xi \tag{3.19}$$

3.7 Flux

The amount of some scalar ϕ flowing across a face dS in dt is

$$Q = \phi \boldsymbol{u} \cdot d\boldsymbol{S} dt \tag{3.20}$$

Where the area normal vector $d\boldsymbol{S} = \boldsymbol{n} dS$ is the unit normal vector multiplied by the face area and $V = \boldsymbol{u} \cdot \boldsymbol{S} dt$ is the volume that goes across the face in dt. The volume is calculated differently under different coordinate systems

$$V^X = (\boldsymbol{u}^X)^\top \cdot d\boldsymbol{S}_X dt = V^\Xi = J(\boldsymbol{u}^\Xi)^\top \cdot d\boldsymbol{S}_\Xi dt \tag{3.21}$$

Here, $d\boldsymbol{S}_X$ and $d\boldsymbol{S}_\Xi$ represent the area normal vectors with respect to different coordinate systems, they are **not** the same vector, while V^X and V^Ξ represent the same volume V calculated in different ways.

So the flux vectors for each coordinate system that $Q = \boldsymbol{f}_X \cdot d\boldsymbol{S}_X dt = \boldsymbol{f}_\Xi \cdot d\boldsymbol{S}_\Xi dt$ should be

$$[\boldsymbol{f}_X]^X = \phi \boldsymbol{u}^X \qquad [\boldsymbol{f}_\Xi]^\Xi = J\phi \boldsymbol{u}^\Xi \tag{3.22}$$

They are **not** the same vector.

And from the Lagrangian viewpoint, the flux is the amount of ϕ in a small parcel of fluid moving by the velocity

$$\boldsymbol{F} \equiv \boldsymbol{u}\phi dV \tag{3.23}$$

So with respect to different coordinate systems, there should be

$$[\boldsymbol{F}]^X = \boldsymbol{u}^X \phi dV_X \;\equiv\; [\boldsymbol{F}]^\Xi = J\boldsymbol{u}^\Xi \phi dV_\Xi \tag{3.24}$$

So the flux vectors for each coordinate system that $\boldsymbol{F} \equiv \boldsymbol{f}_X dV_X \equiv \boldsymbol{f}_\Xi dV_\Xi$ should be

$$[\boldsymbol{f}_X]^X = \phi \boldsymbol{u}^X \qquad [\boldsymbol{f}_\Xi]^\Xi = J\phi \boldsymbol{u}^\Xi \tag{3.22}$$

Again, they are **not** the same vector.

3.8 Gauss's Divergence Theorem

$$\int_V \nabla \cdot \boldsymbol{\phi}\ dV = \oint_S \boldsymbol{\phi} \cdot d\boldsymbol{S} \tag{3.25}$$

In Cartesian coordinate system

$$\int_V \nabla_X^\top \cdot \boldsymbol{\phi}^X\ dV_X = \oint_S (\boldsymbol{\phi}^X)^\top \cdot d\boldsymbol{S}_X \tag{3.26}$$

In Ξ coordinate system

$$\int_V \nabla_\Xi^\top \cdot J\boldsymbol{\phi}^\Xi\ dV_\Xi = \oint_S J(\boldsymbol{\phi}^\Xi)^\top \cdot d\boldsymbol{S}_\Xi \tag{3.27}$$

4 Control Equations

4.1 Continuity Equation

The differential form

$$\nabla \cdot \boldsymbol{u} = 0 \tag{4.1}$$

The integral form

$$\oint_S \boldsymbol{u} \cdot d\boldsymbol{S} = 0 \tag{4.2}$$

So the continuity equation in general coordinate will be

$$\frac{1}{J}\nabla_\Xi^\top \cdot J\boldsymbol{u}^\Xi = 0 \tag{4.3}$$

and

$$\oint_S J(\boldsymbol{u}^\Xi)^\top \cdot d\boldsymbol{S}_\Xi \tag{4.4}$$

4.2 Convection-Diffusion Equation

The differential form

$$\frac{\partial \phi}{\partial t} + \nabla \cdot \phi \boldsymbol{u} - \nabla \cdot \nu \nabla \phi = \psi \tag{4.5}$$

The integral form

$$\int_V \frac{\partial \phi}{\partial t} dV + \oint_S \phi \boldsymbol{u} \cdot d\boldsymbol{S} - \oint_S \nu \nabla \phi \cdot d\boldsymbol{S} = \int_V \psi\, dV \tag{4.6}$$

So in general coordinate, it will be

$$\frac{\partial \phi}{\partial t} + \frac{1}{J}\nabla_\Xi^\top \cdot J\phi\boldsymbol{u}^\Xi - \frac{1}{J}\nabla_\Xi^\top \cdot (J\nu g^{-1} \cdot \nabla_\Xi \phi) = \psi \tag{4.7}$$

and

$$\int_V J\frac{\partial \phi}{\partial t} dV_\Xi + \oint_S (J\phi\boldsymbol{u}^\Xi)^\top \cdot d\boldsymbol{S}_\Xi - \oint_S (J\nu g^{-1} \cdot \nabla_\Xi \phi)^\top \cdot d\boldsymbol{S}_\Xi \tag{4.8}$$