Coordinate Transformation between Cartesian and Orthogonal Non-uniform Grids

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1 Orthogonal Non-uniform Grids

We have two coordinate systems here, the original Cartesian system X and a new coordinate system Ξ, and we have the relationship between them as

$$\begin{cases} \xi = \xi(x) \\ \eta = \eta(y) \\ \zeta = \zeta(z) \end{cases} \tag{1}$$

and vice versa

$$\begin{cases} x = x(\xi) \\ y = y(\eta) \\ z = z(\zeta) \end{cases} \tag{2}$$

from which we have the transform matrices

$$\Xi_X = \begin{bmatrix} \xi_x & 0 & 0 \\ 0 & \eta_y & 0 \\ 0 & 0 & \zeta_z \end{bmatrix} \tag{3}$$

$$X_\Xi = \begin{bmatrix} x_\xi & 0 & 0 \\ 0 & y_\eta & 0 \\ 0 & 0 & z_\zeta \end{bmatrix} \tag{4}$$

and also the Jacobian $J = |X_\Xi| = x_\xi y_\eta z_\zeta$.

2 Vectors

Let $\boldsymbol{\phi}$ be a vector in Cartesian coordinates, the contravariant form of $\boldsymbol{\phi}$ (i.e. the same vector in Ξ coordinates) is

$$\boldsymbol{\Phi} = \Xi_X \cdot \boldsymbol{\phi} \tag{5}$$

3 Differential Operators

3.1 Derivative

The relation between the derivatives of a scalar ϕ in X coordinates and in Ξ coordinates is as

$$\begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix} = \Xi_X \cdot \begin{bmatrix} \frac{\partial \phi}{\partial \xi} \\ \frac{\partial \phi}{\partial \eta} \\ \frac{\partial \phi}{\partial \zeta} \end{bmatrix} \tag{6}$$

In our orthogonal non-uniform Ξ, this means

$$\frac{\partial\phi}{\partial x} = \xi_x \frac{\partial\phi}{\partial\xi}$$
$$\frac{\partial\phi}{\partial y} = \eta_y \frac{\partial\phi}{\partial\eta}$$
$$\frac{\partial\phi}{\partial z} = \zeta_z \frac{\partial\phi}{\partial\zeta}$$

3.2 Gradient

The gradient vector of a scalar ϕ is

$$\nabla\phi = \nabla_X\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \tag{7}$$

And according to 3.1, we have

$$\nabla\phi = \nabla_X\phi = \Xi_X \cdot \nabla_\Xi\phi = \Xi_X \cdot \left(\frac{\partial\phi}{\partial\xi}, \frac{\partial\phi}{\partial\eta}, \frac{\partial\phi}{\partial\zeta}\right) \tag{8}$$

3.3 Divergence

The divergence of a vector $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ is

$$\nabla \cdot \boldsymbol{\phi} \tag{9}$$

Under the Cartesian coordinate, it can be written as

$$\nabla \cdot \boldsymbol{\phi} = \nabla_X \cdot \boldsymbol{\phi} = \frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_2}{\partial y} + \frac{\partial\phi_3}{\partial z} \tag{10}$$

Let $\boldsymbol{\Phi}$ be the contravariant form of $\boldsymbol{\phi}$, i.e. $\boldsymbol{\Phi} = \Xi_x \cdot \boldsymbol{\phi}$, then we have

$$\nabla \cdot \boldsymbol{\phi} = \frac{1}{J}\nabla_\Xi \cdot J\boldsymbol{\Phi} \tag{11}$$

We can check it out by expanding the derivatives

$$\frac{1}{J}\nabla_\Xi \cdot J\boldsymbol{\Phi} = \frac{1}{J}\left(\frac{\partial J\xi_x\phi_1}{\partial\xi} + \frac{\partial J\eta_y\phi_2}{\partial\eta} + \frac{\partial J\zeta_z\phi_3}{\partial\zeta}\right)$$
$$\frac{\partial J\xi_x\phi_1}{\partial\xi} = \frac{\partial J\xi_x}{\partial\xi}\phi_1 + J\xi_x\frac{\partial\phi_1}{\partial\xi} = \frac{\partial y_\eta z_\zeta}{\partial\xi}\phi_1 + J\frac{\partial\phi_1}{\partial x} = J\frac{\partial\phi_1}{\partial x}$$
$$\frac{\partial J\eta_y\phi_2}{\partial\eta} = \frac{\partial J\eta_y}{\partial\eta}\phi_2 + J\eta_y\frac{\partial\phi_2}{\partial\eta} = \frac{\partial x_\xi z_\zeta}{\partial\eta}\phi_2 + J\frac{\partial\phi_2}{\partial y} = J\frac{\partial\phi_2}{\partial y}$$
$$\frac{\partial J\zeta_z\phi_3}{\partial\eta} = \frac{\partial J\zeta_z}{\partial\zeta}\phi_3 + J\zeta_z\frac{\partial\phi_3}{\partial\zeta} = \frac{\partial x_\xi y_\eta}{\partial\zeta}\phi_3 + J\frac{\partial\phi_3}{\partial z} = J\frac{\partial\phi_3}{\partial z}$$

So when we add them up, formula (11) will be exactly the same as (10).
When we have a tensor, the formula is still applicable as

$$\nabla \cdot \boldsymbol{\tau} = \nabla_X \cdot \boldsymbol{\tau} = \frac{1}{J}\nabla_\Xi \cdot J\boldsymbol{T} \tag{12}$$

where $\boldsymbol{T} = \Xi_X \cdot \boldsymbol{\tau}$

3.4 Laplacian

The laplacian of a scalar ϕ is

$$\nabla^2\phi = \nabla\cdot(\nabla\phi) \tag{13}$$

Under the Cartesian coordinate, it is written as

$$\nabla^2\phi = \nabla_X\cdot(\nabla\phi) = \nabla_X\cdot(\nabla_X\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{14}$$

We can convert this using what we have got in 3.2 and 3.3 as

$$\nabla^2\phi = \frac{1}{J}\nabla_\Xi\cdot(J\boldsymbol{Grad}\,\phi) = \frac{1}{J}\nabla_\Xi\cdot(J\Xi_X\cdot(\Xi_X\cdot\nabla_\Xi\phi)) = \frac{1}{J}\nabla_\Xi\cdot(J\Xi_X^2\cdot\nabla_\Xi\phi) \tag{15}$$

Where $\boldsymbol{Grad}\,\phi = \Xi_x\cdot\nabla\phi$ is the contravariant form of the gradient vector. If we expand (15), we will have

$$\begin{aligned}\nabla^2\phi &= \frac{1}{J}\nabla_\Xi\cdot(J\Xi_X^2\cdot\nabla_\Xi\phi)\\ &= \frac{1}{J}\left(\frac{\partial}{\partial\xi}\left(J\xi_x^2\frac{\partial\phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(J\eta_y^2\frac{\partial\phi}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(J\zeta_z^2\frac{\partial\phi}{\partial\zeta}\right)\right)\end{aligned} \tag{16}$$

If we further expand it, we will have

$$\begin{aligned}\frac{1}{J}\frac{\partial}{\partial\xi}\left(J\xi_x^2\frac{\partial\phi}{\partial\xi}\right) &= \xi_x^2\frac{\partial^2\phi}{\partial\xi^2} + \frac{1}{J}\frac{\partial J\xi_x^2}{\partial\xi}\frac{\partial\phi}{\partial\xi}\\ &= \xi_x^2\frac{\partial^2\phi}{\partial\xi^2} + \frac{1}{x_\xi y_\eta z_\zeta}\frac{\partial\xi_x y_\eta z_\zeta}{\partial\xi}\frac{\partial\phi}{\partial\xi}\\ &= \xi_x^2\frac{\partial^2\phi}{\partial\xi^2} + \frac{1}{x_\xi}\frac{\partial\xi_x}{\partial\xi}\frac{\partial\phi}{\partial\xi}\\ &= \xi_x^2\frac{\partial^2\phi}{\partial\xi^2} - \frac{x_{\xi\xi}}{x_\xi^3}\frac{\partial\phi}{\partial\xi}\end{aligned}$$

$$\begin{aligned}\frac{1}{J}\frac{\partial}{\partial\eta}\left(J\eta_y\frac{\partial\phi}{\partial\eta}\right) &= \eta_y^2\frac{\partial^2\phi}{\partial\eta^2} + \frac{1}{J}\frac{\partial J\eta_y^2}{\partial eta}\frac{\partial\phi}{\partial\eta}\\ &= \eta_y^2\frac{\partial^2\phi}{\partial\eta^2} + \frac{1}{x_\xi y_\eta z_\zeta}\frac{\partial x_\xi\eta_y z_\zeta}{\partial\eta}\frac{\partial\phi}{\partial\eta}\\ &= \eta_y^2\frac{\partial^2\phi}{\partial\eta^2} + \frac{1}{y_\eta}\frac{\partial\eta_y}{\partial\eta}\frac{\partial\phi}{\partial\eta}\\ &= \eta_y^2\frac{\partial^2\phi}{\partial\eta^2} - \frac{y_{\eta\eta}}{y_\eta^3}\frac{\partial\phi}{\partial\eta}\end{aligned}$$

$$\begin{aligned}\frac{1}{J}\frac{\partial}{\partial\zeta}\left(J\zeta_z\frac{\partial\phi}{\partial\zeta}\right) &= \zeta_z^2\frac{\partial^2\phi}{\partial\zeta^2} + \frac{1}{J}\frac{\partial J\zeta_z^2}{\partial\zeta}\frac{\partial\phi}{\partial\zeta}\\ &= \zeta_z^2\frac{\partial^2\phi}{\partial\zeta^2} + \frac{1}{x_\xi y_\eta z_\zeta}\frac{\partial x_\xi y_\eta\zeta_z}{\partial\zeta}\frac{\partial\phi}{\partial\zeta}\\ &= \zeta_z^2\frac{\partial^2\phi}{\partial\zeta^2} + \frac{1}{z_\zeta}\frac{\partial\zeta_z}{\partial\zeta}\frac{\partial\phi}{\partial\zeta}\\ &= \zeta_z^2\frac{\partial^2\phi}{\partial\zeta^2} - \frac{z_{\zeta\zeta}}{z_\zeta^3}\frac{\partial\phi}{\partial\zeta}\end{aligned}$$

And

$$
\begin{aligned}
\nabla^2\phi &= \frac{1}{J}\nabla_\Xi \cdot (J\Xi_X^2 \cdot \nabla_\Xi \phi) \\
&= \xi_x^2 \frac{\partial^2\phi}{\partial\xi^2} - \frac{x_{\xi\xi}}{x_\xi^3}\frac{\partial\phi}{\partial\xi} + \eta_y^2\frac{\partial^2\phi}{\partial\eta^2} - \frac{y_{\eta\eta}}{y_\eta^3}\frac{\partial\phi}{\partial\eta} + \zeta_z^2\frac{\partial^2\phi}{\partial\zeta^2} - \frac{z_{\zeta\zeta}}{z_\zeta^3}\frac{\partial\phi}{\partial\zeta}
\end{aligned} \tag{17}
$$

It is the same as what we will have if we directly convert the second derivatives in (14).

4 Navier-Stokes Equation

4.1 Advection

The advection of a scalar ϕ by the influence of flow velocity $\boldsymbol{u}$ is expressed by the advection term as

$$
Adv = \nabla \cdot \boldsymbol{u}\phi \tag{18}
$$

According to 3.3 have

$$
Adv = \frac{1}{J}\nabla_\Xi \cdot J\boldsymbol{U}\phi \tag{19}
$$

Where $\boldsymbol{U} = \Xi_X \cdot \boldsymbol{u}$ is the contravariant velocity.

4.2 Diffusion

The diffusion of a scalar ϕ for incompressible Newton fluid with constant kinetic viscosity is

$$
Vis = \nabla \cdot (\nu\nabla\phi) = \nu\nabla \cdot (\nabla\phi) = \nu\nabla^2\phi \tag{20}
$$

According to 3.4, have

$$
Vis = \frac{\nu}{J}\nabla_\Xi \cdot (J\Xi_X^2 \cdot \nabla_\Xi\phi) \tag{21}
$$

Expand it to look at the details as in (16)

$$
\begin{aligned}
Vis &= \frac{\nu}{J}\nabla_\Xi \cdot (J\Xi_X^2 \cdot \nabla_\Xi\phi) \\
&= \frac{\nu}{J}\left(\frac{\partial}{\partial\xi}\left(J\xi_x^2\frac{\partial\phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(J\eta_y^2\frac{\partial\phi}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(J\zeta_z^2\frac{\partial\phi}{\partial\zeta}\right)\right) \\
&= \frac{\nu}{J}\left(\frac{\partial}{\partial\xi}\left(\gamma^{11}\frac{\partial\phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\gamma^{22}\frac{\partial\phi}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\gamma^{33}\frac{\partial\phi}{\partial\zeta}\right)\right)
\end{aligned}
$$

where

$$
\gamma^{ij} = J\frac{\partial\xi_i}{\partial x_k}\frac{\partial\xi_j}{\partial x_k} \tag{22}
$$

We can also write it as (17)

$$
\begin{aligned}
Vis &= \frac{\nu}{J}\nabla_\Xi \cdot (J\Xi_X^2 \cdot \nabla_\Xi\phi) \\
&= \nu\left(\xi_x^2 \frac{\partial^2\phi}{\partial\xi^2} - \frac{x_{\xi\xi}}{x_\xi^3}\frac{\partial\phi}{\partial\xi} + \eta_y^2\frac{\partial^2\phi}{\partial\eta^2} - \frac{y_{\eta\eta}}{y_\eta^3}\frac{\partial\phi}{\partial\eta} + \zeta_z^2\frac{\partial^2\phi}{\partial\zeta^2} - \frac{z_{\zeta\zeta}}{z_\zeta^3}\frac{\partial\phi}{\partial\zeta}\right)
\end{aligned}
$$