Coordinate Transformation between Cartesian and Orthogonal Non-uniform Grids

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1 Orthogonal Non-uniform Grids

We have two coordinate systems here, the original Cartesian system X and a new coordinate system Ξ, and we have the relationship between them as

$$\begin{cases} \xi = \xi(x) \\ \eta = \eta(y) \\ \zeta = \zeta(z) \end{cases} \tag{1}$$

and vice versa

$$\begin{cases} x = x(\xi) \\ y = y(\eta) \\ z = z(\zeta) \end{cases} \tag{2}$$

from which we have the transform matrices

$$\Xi_X = \begin{bmatrix} \xi_x & 0 & 0 \\ 0 & \eta_y & 0 \\ 0 & 0 & \zeta_z \end{bmatrix} \tag{3}$$

$$X_\Xi = \begin{bmatrix} x_\xi & 0 & 0 \\ 0 & y_\eta & 0 \\ 0 & 0 & z_\zeta \end{bmatrix} \tag{4}$$

and also the Jacobian $J = |X_\Xi| = x_\xi y_\eta z_\zeta$.

2 Differential Operators

2.1 Divergence

Suppose we have a vector $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$. Then we have its divergence in the Cartesian coordinate as

$$\nabla \cdot \boldsymbol{\phi} = \nabla_X \cdot \boldsymbol{\phi} = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} + \frac{\partial \phi_3}{\partial z} \tag{5}$$

To write the same vector in the Ξ coordinates, we have $\boldsymbol{\Phi} = (\Phi_1, \Phi_2, \Phi_3) = \Xi_x \cdot \boldsymbol{\phi} = (\xi_x \phi_1, \eta_y \phi_2, \zeta_z \phi_3)$, then we have the divergence of the same vector in Ξ coordinates as

$$\nabla \cdot \boldsymbol{\phi} = \frac{1}{J} \nabla_\Xi \cdot J\boldsymbol{\Phi} = \frac{1}{J} \left(\frac{\partial J \xi_x \phi_1}{\partial \xi} + \frac{\partial J \eta_y \phi_2}{\partial \eta} + \frac{\partial J \zeta_z \phi_3}{\partial \zeta} \right) \tag{6}$$

We can check it out by expanding the derivatives

$$\begin{aligned} \frac{\partial J \xi_x \phi_1}{\partial \xi} &= \frac{\partial J \xi_x}{\partial \xi} \phi_1 + J \xi_x \frac{\partial \phi_1}{\partial \xi} = \frac{\partial y_\eta z_\zeta}{\partial \xi} \phi_1 + J \frac{\partial \phi_1}{\partial x} = J \frac{\partial \phi_1}{\partial x} \\ \frac{\partial J \eta_y \phi_2}{\partial \eta} &= \frac{\partial J \eta_y}{\partial \eta} \phi_2 + J \eta_y \frac{\partial \phi_2}{\partial \eta} = \frac{\partial x_\xi z_\zeta}{\partial \eta} \phi_2 + J \frac{\partial \phi_2}{\partial y} = J \frac{\partial \phi_2}{\partial y} \\ \frac{\partial J \zeta_z \phi_3}{\partial \eta} &= \frac{\partial J \zeta_z}{\partial \zeta} \phi_3 + J \zeta_z \frac{\partial \phi_3}{\partial \zeta} = \frac{\partial x_\xi y_\eta}{\partial \zeta} \phi_3 + J \frac{\partial \phi_3}{\partial z} = J \frac{\partial \phi_3}{\partial z} \end{aligned}$$

So when we add them up, formula (6) will be exactly the same as (5).
When we have a tensor, the formula is still applicable as

$$\nabla \cdot \boldsymbol{\tau} = \nabla_X \cdot \boldsymbol{\tau} = \frac{1}{J}\nabla_\Xi \cdot J\boldsymbol{T} \tag{7}$$

where $\boldsymbol{T} = \Xi_X \cdot \boldsymbol{\tau}$

3 Navier-Stokes Equation

3.1 Advection

The advection of a scalar ϕ by the influence of flow velocity $\boldsymbol{u}$ is expressed by the advection term as

$$Adv = \nabla \cdot \boldsymbol{u}\phi \tag{8}$$

Convert this into our Ξ coordinate, we will have

$$Adv = \frac{1}{J}\nabla_\Xi \cdot J\boldsymbol{U}\phi \tag{9}$$

Where $\boldsymbol{U} = \Xi_X \cdot \boldsymbol{u}$

3.2 Diffusion

The diffusion of a scalar ϕ for incompressible Newton fluid with constant kinetic viscosity is

$$Vis = \nabla \cdot (\nu \nabla\phi) = \nu\nabla \cdot (\nabla\phi) \tag{10}$$

Convert this into our Ξ coordinate, we will have

$$Vis = \frac{\nu}{J}\nabla_\Xi \cdot (J\Xi_X \cdot \nabla\phi) \tag{11}$$

Expand it to look at the details

$$\begin{aligned}
Vis &= \frac{\nu}{J}\nabla_\Xi \cdot (J\Xi_X \cdot \nabla\phi) \\
&= \frac{\nu}{J}\left(\frac{\partial}{\partial\xi}\left(J\xi_x\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial\eta}\left(J\eta_y\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial\zeta}\left(J\zeta_z\frac{\partial\phi}{\partial z}\right)\right) \\
&= \frac{\nu}{J}\left(\frac{\partial}{\partial\xi}\left(J\xi_x^2\frac{\partial\phi}{\partial \xi}\right) + \frac{\partial}{\partial\eta}\left(J\eta_y^2\frac{\partial\phi}{\partial \eta}\right) + \frac{\partial}{\partial\zeta}\left(J\zeta_z^2\frac{\partial\phi}{\partial \zeta}\right)\right) \\
&= \frac{\nu}{J}\left(\frac{\partial}{\partial\xi}\left(\gamma^{11}\frac{\partial\phi}{\partial \xi}\right) + \frac{\partial}{\partial\eta}\left(\gamma^{22}\frac{\partial\phi}{\partial \eta}\right) + \frac{\partial}{\partial\zeta}\left(\gamma^{33}\frac{\partial\phi}{\partial \zeta}\right)\right)
\end{aligned}$$

where

$$\gamma^{ij} = J\frac{\partial\xi_i}{\partial x_k}\frac{\partial\xi_j}{\partial x_k} \tag{12}$$

If we further expand this formula, we will have

$$\begin{aligned}
\frac{1}{J}\frac{\partial}{\partial\xi}\left(J\xi_x^2\frac{\partial u}{\partial\xi}\right) &= \xi_x^2\frac{\partial^2\phi}{\partial\xi^2} + \frac{1}{J}\frac{\partial J\xi_x^2}{\partial\xi}\frac{\partial\phi}{\partial\xi} \\
&= \xi_x^2\frac{\partial^2\phi}{\partial\xi^2} + \frac{1}{x_\xi y_\eta z_\zeta}\frac{\partial \xi_x y_\eta z_\zeta}{\partial\xi}\frac{\partial\phi}{\partial\xi} \\
&= \xi_x^2\frac{\partial^2\phi}{\partial\xi^2} + \frac{1}{x_\xi}\frac{\partial \xi_x}{\partial\xi}\frac{\partial\phi}{\partial\xi} \\
&= \xi_x^2\frac{\partial^2\phi}{\partial\xi^2} - \frac{x_{\xi\xi}}{x_\xi^3}\frac{\partial\phi}{\partial\xi}
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{J}\frac{\partial}{\partial\eta}\left(J\eta_y\frac{\partial\phi}{\partial\eta}\right) &= \eta_y^2\frac{\partial^2\phi}{\partial\eta^2} + \frac{1}{J}\frac{\partial J\eta_y^2}{\partial eta}\frac{\partial\phi}{\partial\eta} \\
&= \eta_y^2\frac{\partial^2\phi}{\partial\eta^2} + \frac{1}{x_\xi y_\eta z_\zeta}\frac{\partial x_\xi \eta_y z_\zeta}{\partial\eta}\frac{\partial\phi}{\partial\eta} \\
&= \eta_y^2\frac{\partial^2\phi}{\partial\eta^2} + \frac{1}{y_\eta}\frac{\partial\eta_y}{\partial\eta}\frac{\partial\phi}{\partial\eta} \\
&= \eta_y^2\frac{\partial^2\phi}{\partial\eta^2} - \frac{y_{\eta\eta}}{y_\eta^3}\frac{\partial\phi}{\partial\eta}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{J}\frac{\partial}{\partial\zeta}\left(J\zeta_z\frac{\partial\phi}{\partial\zeta}\right) &= \zeta_z^2\frac{\partial^2\phi}{\partial\zeta^2} + \frac{1}{J}\frac{\partial J\zeta_z^2}{\partial\zeta}\frac{\partial\phi}{\partial\zeta} \\
&= \zeta_z^2\frac{\partial^2\phi}{\partial\zeta^2} + \frac{1}{x_\xi y_\eta z_\zeta}\frac{\partial x_\xi y_\eta \zeta_z}{\partial\zeta}\frac{\partial\phi}{\partial\zeta} \\
&= \zeta_z^2\frac{\partial^2\phi}{\partial\zeta^2} + \frac{1}{z_\zeta}\frac{\partial\zeta_z}{\partial\zeta}\frac{\partial\phi}{\partial\zeta} \\
&= \zeta_z^2\frac{\partial^2\phi}{\partial\zeta^2} - \frac{z_{\zeta\zeta}}{z_\zeta^3}\frac{\partial\phi}{\partial\zeta}
\end{aligned}
$$

This matches with our conversion of the ∇^2 operator