Cross-section and polarization of neutrino-produced τ's made simple

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Abstract

Practical formulae are derived for the cross-section and polarization vector of the τ lepton produced in deep inelastic neutrino-nucleon scattering in the frame of the simple quark-parton model.

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Introduction.

The increasing amount of evidence for $\nu_\mu \leftrightarrow \nu_\tau$ oscillations entails that some experiments being prepared should actually 'see' the τ produced by C.C. ν_τ interaction. Accordingly, papers have appeared of late with the purpose of giving the experimental groups the necessary tools to prepare their Monte-Carlo and the analysis of their future data. One of the themes of these papers is of course the calculation of the τ production cross-section, both differential and integrated, and that of the τ polarization which is necessary to predict the angular distribution of the decay products into various channels, see e.g. [1][2][3]
However, not all these papers give simple recipes which can easily be implemented in simulation programs. We therefore thought it useful to write this note which describes the way we calculated the cross-section and polarisation and implemented them in the simulation which served many years ago to prepare the proposal for the NOMAD experiment [8]. The following is essentially an improvement over an unpublished internal note [9]

We shall only treat the deep inelastic scattering (DIS) case in the quark-parton model. This makes things particularly simple since quarks are then described by Dirac spinors. However, the procedure can be straightforwardly extended to quasi-elastic or resonance production, through the introduction of form factors. These complicate the expression for the hadronic current, but the main result, viz the extraction of the τ polarization 4-vector can be done by mimicking what follows.

Formulae for C.C. neutrino interactions without neglect of the lepton mass were first published in [5] (see also [6]). However, they were written in terms of the scaling functions $F_i's$ and no connection was given with the basic quark/anti-quark densities given by the available structure functions library packages. Moreover, [5] gives a 'polarized' part of the cross-section which is not very useful for Monte-Carlo, whereas the polarization vector to be used in decay matrix elements is not explicitly written. It could be extracted from the given formulae but the result would only be an average with respect to the various elementary nuclear constituents. In using Monte-Carlo of the Lund type, where proper simulation of the hadronic final state requires knowledge of the type of quark/anti-quark which has been hit, it is more appropriate to derive a polarization for each case.

In the spirit of the impulse approximation, we shall therefore start from the elementary interactions and derive what should represent the observable cross-section and polarization by summing and/or averaging. We also make explicit the derivation of the kinematical limits in x and y which are only stated in the above mentionned papers and can be used as training for the students.

Notations and basic formulae are listed in section 1; in section 2 we derive the kinematical bounds for neutrino-nucleon C.C. scattering with a massive lepton in the final state. Going over to the quark-parton model, the elementary subprocesses cross-sections and polarization 4-vectors for the lepton are calculated in section 3. Section 4 sums up these results with a neutrino-nucleon cross-section formula and an average lepton polarization.

1 Notations

The reaction to be described is $\nu_\tau \; N \to \tau \; X$.
Four momenta : $k = p_\nu, p = p_N, p_\tau, q = p_\nu - p_\tau$
Masses : m_τ for τ , M for target nucleon, W for X.
Kinematical variables:
C.M.S. energy squared: $s = M^2 + 2ME$ with E the neutrino energy in the target rest frame.
Four-momentum transfer squared: $Q^2 = -q^2$
Squared mass of final hadronic system: $W^2 = (q+p)^2$
Bjorken's variables x: $x = Q^2/2p.q \quad y = p.q/p.k$
C.M.S. scattering angle: θ
Useful relations:

$$Q^2 = xy(s - M^2) = -m_\tau^2 + 2E_\nu^{c.m.}(E_\tau^{c.m.} - p_\tau^{c.m.} cos\theta) \tag{1}$$

$$W^2 = s + m_\tau^2 - 2E_\tau^{c.m.}\sqrt{s} \tag{2}$$

$$x = \frac{Q^2}{W^2 - M^2 + Q^2} \tag{3}$$

$$y = \frac{W^2 - M^2 + Q^2}{s - M^2} = \frac{W^2 - M^2}{(1-x)(s-M^2)} \tag{4}$$

2 $\nu - N \to l - X$ kinematics with non zero lepton mass.

From (3) and $W^2 \geq M^2$ one finds $x \leq 1$ as usual. Replacing W^2 taken from (2) into (4) leads to the following expression for $E_\tau^{c.m.}$:

$$E_\tau^{c.m.} = \frac{m_\tau^2 + (s - M^2)(1 - y + xy)}{2\sqrt{s}} \tag{5}$$

On the other hand, the second of relations (1) and $|cos\theta| \leq 1$ yield:

$$|xy(s - M^2) + m_\tau^2 - 2E_\tau^{c.m.}E_\nu^{c.m.}| \leq 2p_\tau^{c.m.}E_\nu^{c.m.}$$

Now define $\delta_\tau = \frac{m_\tau^2}{s-M^2}$, $\delta_N = \frac{M^2}{s-M^2}$, and use $E_\nu^{c.m.} = \frac{s-M^2}{2\sqrt{s}}$ to rewrite this:

$$|xy + \delta_\tau - \frac{E_\tau^{c.m.}}{\sqrt{s}}| \leq \frac{p_\tau^{c.m.}}{\sqrt{s}}$$

or letting $h \stackrel{def}{=} xy + \delta_\tau$ and squaring:

$$h^2 - 2h\frac{E_\tau^{c.m.}}{\sqrt{s}} + \frac{m_\tau^2}{s} \leq 0$$

Using (5) for $E_\tau^{c.m.}$ and the definitions of δ_τ, δ_N, this can be transformed to:

$$h^2\delta_N - h(1-y) + \delta_\tau \leq 0 \tag{6}$$

or, by re-expressing y as function of h:

$$(1 + x\delta_N)h^2 - (x + \delta_\tau)h + x\delta_\tau \leq 0 \tag{7}$$

Both inequalities (6) and (7) lead to the kinematical limits in terms of x and y, albeit in slightly different forms. Solving the first one for h will give limits on x as a function of y and conversely for the second. This later inequality leads to the limits quoted in [5] as follows: (7) is possible only if: $\Delta = (x - \delta_\tau)^2 - 4x^2\delta_\tau\delta_N \geq 0$, i.e.:

$$|x - \delta_\tau| \geq 2x\sqrt{\delta_\tau\delta_N} \tag{8}$$

And the limits on y for given x are then found from the roots of the trinomial in (7):

$$\frac{x - \delta_\tau - 2x\delta_\tau\delta_N - \sqrt{\Delta}}{2(1 + x\delta_N)x} \leq y \leq \frac{x - \delta_\tau - 2x\delta_\tau\delta_N + \sqrt{\Delta}}{2(1 + x\delta_N)x} \tag{9}$$

Clearly, one must have $x > \delta_\tau$ so that (8) is to be understood as :

$$x \geq x^{min} = \frac{\delta_\tau}{1 - 2\sqrt{\delta_\tau\delta_N}} = \frac{\delta_\tau}{1 - 2\frac{m_\tau M}{s - M^2}} = \frac{\delta_\tau}{1 - \frac{m_\tau}{E}} \tag{10}$$

Note that the upper limit for x derived from (6) is irrelevant (above 1) but that as soon as one demands a minimum value for W above M (DIS should mean $W > M_\Delta$ at least), then relation (4) entails an upper limit on x as a function of y

3 Elementary interactions.

To derive formulae for the differential cross-section and polarization, we use the simplest quark-parton model and then sum over partonic contributions.

3.1 Matrix elements.

The kinematics is $p_\nu + p_q = p_\tau + p_{q'}$ [1] and use shall be made of the τ polarization four-vector S

The reaction amplitude is:

$$\frac{G_F}{\sqrt{2}}\bar{u}(\tau)\gamma^\alpha(1-\gamma^5)u(\nu)\bar{u}(q')\gamma_\alpha(1-\gamma^5)u(q)B.W.$$

[1] **we will use this notation even for anti-quarks and anti-neutrinos except in the final x-section formulae**

where

$$B.W. = \frac{M_W^2}{M_W^2 - q^2 - i\Gamma_W M_W}$$

is a W's propagator correction to the pure Fermi amplitude, where terms arising from the longitudinal ($\frac{q^\alpha q^\beta}{M_W^2}$) part have been neglected.
Squaring and introducing the density matrices one gets:

$$\frac{G_F^2}{2} Tr(\rho_\tau \gamma^\alpha (1-\gamma^5)\rho_\nu \gamma^\beta (1-\gamma^5)) Tr(\rho_{q'} \gamma_\alpha (1-\gamma^5)\rho_q \gamma_\beta (1-\gamma^5))$$

Here $\rho_\tau = \frac{1}{2}(\gamma \cdot p_\tau + m_\tau)(1+\gamma^5 \gamma \cdot S)$ but the other density matrices are summed (averaged) over polarizations and $|B.W.|^2$ is understood here and in what follows.
The first trace is the so-called leptonic tensor, which, with $\rho_\nu = \gamma \cdot k$ ($p_\nu = k$ here, for ease of notation) reads explicitly:

$$L_\nu^{\alpha,\beta} = 4(L^\alpha k^\beta + L^\beta k^\alpha - g^{\alpha\beta} L \cdot k - i\epsilon^{\mu\alpha\nu\beta} L_\mu k_\nu)$$

where $L \stackrel{def}{=} k - m_\tau S$
For anti-neutrinos, L should read $k + m_\tau S$ and the sign of the antisymmetric part should be reversed.

Taking the trace of the quark tensor and contracting, one finds:

$$|T|^2 = 32 G_F^2 p_\nu \cdot p_q (p_\tau - m_\tau S) \cdot p_{q'} \tag{11}$$

We shall use the S dependence of this result later. For the moment, we sum it over polarizations to get the transition probability:

$$64 G_F^2 p_\nu \cdot p_q p_\tau \cdot p_{q'}$$

For an anti-quark target, one gets instead of (11):

$$32 G_F^2 p_\nu \cdot p_{\overline{q'}} (p_\tau - m_\tau S) \cdot p_{\overline{q}} \tag{12}$$

so that the roles of the initial and final partons are permuted. Finally for an anti-neutrino going over into τ^+ one finds:

$$32 G_F^2 p_{\overline{\nu}} \cdot p_{q'} (p_\tau + m_\tau S) \cdot p_q \tag{13}$$

and

$$32 G_F^2 p_{\overline{\nu}} \cdot p_{\overline{q}} (p_\tau + m_\tau S) \cdot p_{\overline{q'}} \tag{14}$$

for a quark and anti-quark target respectively.

3.2 Cross-sections.

The differential cross-section for each of the elementary processes considered above is:

$$d\sigma = \frac{1}{F}|T|^2(2\pi)^{-2}\delta^4(p_{q'} + p_\tau - p_q - p_\nu)\frac{d^3p_{q'}}{2p^0_{q'}}\frac{d^3p_\tau}{2p^0_\tau}$$

with F the Möller flux factor and $|T|$ is the matrix element computed above. Standard manipulations and integration with respect to the τ azimuthal angle reduce this expression to

$$d\sigma = \frac{1}{8\pi F}|T|^2 dy$$

with y defined in section 1 as the leptonic fractional energy loss in the target nucleon rest-frame. The flux factor is simply $F = 4p_\nu \cdot p_q$.

3.3 Polarization

Here shall be found the main simplification of our presentation w.r.t. others. Setting up a polarization basis is useful for testing symmetries or conservation laws in the production process, but is of no use for simulations which only require the τ polarization four-vector in any specified frame, to be fed, for example, in a decay matrix element.
The calculation is most simply done by following the reasoning of ([4]): the squared amplitudes written above are proportional to the probabilities for finding the final $\tau^\pm$ in a given state of polarization described by the four-vector S. If the true polarization from the production process is S_f, then those probabilities are equally found by projecting the true density matrix $1 + \gamma^5\gamma \cdot S_f$ on $1 + \gamma^5\gamma \cdot S$ which represents the given state for which we want the probability. In other words, $|T|^2 \propto Tr(1 + \gamma^5\gamma \cdot S)(1 + \gamma^5\gamma \cdot S_f) \propto 1 - S \cdot S_f$
Hence we find for the $\nu - q$ case:

$$S \cdot S_f = \frac{S \cdot p_{q'} m_\tau}{p_\tau \cdot p_{q'}}$$

Since $S_f \cdot p_\tau = 0$ we cannot simply invoke the arbitrariness of S to cross it away but we must allow for a term proportional to p_τ:

$$S_f = \frac{m_\tau p_{q'}}{p_\tau \cdot p_{q'}} + \lambda p_\tau$$

λ is now determined by projecting this equality onto p_τ and found to be $-\frac{1}{m_\tau}$ so that the final result reads:

$$S_f = \frac{m_\tau p_{q'}}{p_\tau \cdot p_{q'}} - \frac{p_\tau}{m_\tau} \quad (15)$$

In the τ rest-frame we see that the time component $S^0_f = 0$ as it should be and that the space part (the polarization in its usual sense) is:

$$\vec{\mathcal{P}}_f = \frac{\vec{p_{q'}}}{E_{q'}}$$

For the three other cases one finds:

$\nu - \bar{q}$	$\bar{\nu} - q$	$\bar{\nu} - \bar{q}$
$\vec{\mathcal{P}}_f = \frac{\vec{p_{\overline{q}}}}{E_{\overline{q}}}$	$\vec{\mathcal{P}}_f = -\frac{\vec{p_q}}{E_q}$	$\vec{\mathcal{P}}_f = -\frac{\vec{p_{\overline{q'}}}}{E_{\overline{q'}}}$

It is important to note that what has been done here with the simple quark tensor can be adapted to more complicated forms of the hadronic current and of the tensor built from it. For example, if τ is produced through a quasi-elastic C.C. interaction, the matrix element of the hadronic current between the neutron and the proton states is described by six form factors (two of which are zero in this case) but exactly the same procedure can be used to identify S_f in terms of 4-momenta, kinematical invariants and these form factors.

4 Interactions with Nucleons

4.1 ν_τ - Nucleon cross-section

The elementary cross-sections written in 3.2 read explicitly

$$d\sigma(\nu - q) = \frac{2G_F^2}{\pi p_\nu \cdot p_q}(p_\nu \cdot p_q \; p_\tau \cdot p_{q'})dy$$

$$d\sigma(\nu - \overline{q}) = \frac{2G_F^2}{\pi p_\nu \cdot p_{\overline{q}}}(p_\nu \cdot p_{\overline{q'}} \; p_\tau \cdot p_{\overline{q}})dy$$

$$d\sigma(\overline{\nu} - q) = \frac{2G_F^2}{\pi p_{\overline{\nu}} \cdot p_q}(p_{\overline{\nu}} \cdot p_{q'} \; p_{\tau^+} \cdot p_q)dy$$

$$d\sigma(\overline{\nu} - \overline{q}) = \frac{2G_F^2}{\pi p_{\overline{\nu}} \cdot p_{\overline{q}}}(p_{\overline{\nu}} \cdot p_{\overline{q}} \; p_{\tau^+} \cdot p_{\overline{q'}})dy$$

Let ξ be the nucleon momentum fraction carried by the struck (anti-) quark: $p_q = \xi p$. Energy-momentum conservation says that:

$$p_\nu + \xi p = p_\tau + p_{q'} \tag{16}$$

$$\text{or} \qquad q + \xi p = p_{q'} \tag{17}$$

by squaring (17) we get:

$$2(\xi - x)p \cdot q + \xi^2 M^2 = m_{q'}^2 \tag{18}$$

In Bjorken's limit, $q^2 \to \infty, p \cdot q \to \infty$ (18) shows that $\xi = x$
Squaring now (16), combining with (18) and using $\xi = x$ yields:

$$2p_\tau \cdot p_{q'} = x(s - M^2) - m_\tau^2$$

Therefore :

$$d\sigma(\nu - q) = d\sigma(\overline{\nu} - \overline{q}) = \frac{2G_F^2}{\pi} MEx(1 - \frac{\delta_\tau}{x})dy \tag{19}$$

Taking the scalar product of (16) by p_ν, using again $\xi = x$ and the definition of q yields:

$$2p_{q'} \cdot p_\nu = x(s - M^2)(1 - \frac{\delta_\tau}{x} - y)$$

On the other hand:

$$2p_q \cdot p_\tau = x(1-y)(s - M^2)$$

and

$$F = 4p_\nu \cdot p_q = 2x(s - M^2)$$

Hence:

$$d\sigma(\nu - \overline{q}) = d\sigma(\overline{\nu} - q) = \frac{2G_F^2}{\pi} MEx(1 - \frac{\delta_\tau}{x} - y)(1-y)dy \tag{20}$$

Formulae (19) and (20) solve the cross-section question. Multiplying them by the appropriate quark/anti-quark distribution functions, summing over flavors and re-introducing the W boson propagator factor, one finds for a neutrino beam:

$$\frac{d\sigma}{dx\, dy} = \frac{2G_F^2 EMx}{\pi} \left[(1 - \frac{\delta_\tau}{x})\mathcal{Q}(x, Q^2) + (1 - \frac{\delta_\tau}{x} - y)(1-y)\overline{\mathcal{Q}}(x, Q^2)\right] |B.W.(Q^2)|^2 \tag{21}$$

where $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ are the appropriate mixtures of (Q.C.D.- evolved) quark distribution functions for the nucleon which is hit.
For an anti-neutrino beam, the coefficients of the quark and anti-quark distribution functions must evidently be exchanged and the distributions themselves adequately modified for the given nucleon, in order to take into account charge conservation at the constituent level.

Expressions (19)(20) and/or (21) are directly usable with standard quark distribution function libraries; from that point of vue, they are more practical than expressions found in the quoted articles where nucleons are described by structure functions or their scaling limits. In this same limit, the connection between the two descriptions is easily made along the lines of what can be found, for example, in ([7]) for the electromagnetic interactions: one merely has to identify the general hadronic tensor with the sum of quark and antiquark tensors decomposed on the same basis; the various relations $F_2 = 2xF_1$, $F_4 = 0$, $F_5 = F_1$ follow and after simplifications, formula (21) is retrieved.

4.2 τ polarization in ν-nucleon scattering

The two terms in (21) are to be interpreted as the relative probabilities for scattering from a quark and an anti-quark in the target nucleon. The τ polarization vector is therefore the weighted average of the two relevant $\mathcal{P}_f$ of section 3.3.

Let us call $P_Q = (1 - \frac{\delta_\tau}{x})\mathcal{Q}$ and $P_{\bar{Q}} = (1-y)(1-y-\frac{\delta_\tau}{x})\overline{\mathcal{Q}}$
The coefficients of P_Q and $P_{\bar{Q}}$ were found in section 3.3 to be: $\frac{\vec{p_{q'}}}{E_{q'}}$ and $\frac{\vec{p_{\overline{q}}}}{E_{\overline{q}}}$.
Rewriting the denominators in Lorenz-invariant form as:
$E_{q'} = \frac{1}{m_\tau} p_\tau \cdot (q + xp) = \frac{s-M^2}{2m_\tau} x(1 - \frac{\delta_\tau}{x})$ and $E_{\overline{q}} = \frac{p_\tau \cdot p_{\overline{q}}}{m_\tau} = \frac{s-M^2}{2m_\tau} x(1-y)$

we find the average τ polarization in neutrino scattering:

$$\vec{\mathcal{P}}_\tau = \frac{2m_\tau}{s-M^2}\left[(\vec{p} + \frac{\vec{q}}{x})\mathcal{Q}(x,Q^2) + \vec{p}(1 - \frac{\delta_\tau}{x} - y)\overline{\mathcal{Q}}(x,Q^2)\right](P_Q + P_{\bar{Q}})^{-1}$$

and in anti-neutrino scattering:

$$\vec{\mathcal{P}}_{\tau^+} = -\frac{2m_\tau}{s-M^2}\left[(\vec{p} + \frac{\vec{q}}{x})\overline{\mathcal{Q}}(x,Q^2) + \vec{p}(1 - \frac{\delta_\tau}{x} - y)\mathcal{Q}(x,Q^2)\right](P_Q + P_{\bar{Q}})^{-1}$$

with the same proviso as in the preceding subsection as to the contents of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$

In these expressions, all 3-vectors are expressed in the τ rest-frame. Should one need S_f in another frame, one could either boost the four vector $(0, \vec{\mathcal{P}})$ or perform the same averaging directly with the S_f vectors extracted from the transition probabilities as done in section (3.3). It is clear that the relative probabilities are made of Lorentz invariants only and therefore valid in any frame, as it must be physically.
Let us remark, however, that for a Lund-type Monte-Carlo where the struck quark is identified, no averaging takes place at the event level and one uses directly the results of (3.3).

5 Conclusion

We have given lowest order but simple formulae for cross-section and polarization of τ 's produced by ν_τ 's charged current interactions valid in the DIS regime. We leave it to the interested reader to use her/his favorite quark distribution functions package to calculate and study cross-sections and polarizations with the help of these formulae through Monte-Carlo or numerical integration.

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