Homework 3 Edited

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Problem 1

Consider a room that is paved with $n \times n$ square tiles which are labeled from 1 to n^2 in some order. A frog performs a random walk by hopping from one tile to a randomly chosen adjacent tile in each time step. All adjacent tiles are chosen with the same probability. The frog can never hop into a wall of the room.

True or not true: The transition matrix for this random walk is symmetric, that is, it satisfies $P(X_{i+1} = k|X_i = j) = P(X_{i+1} = j|X_i = k)$ for all i and all possible states $1 \leq j,\ k \leq n^2$. Explain your answer.

Problem 2

Let $X \sim B(80, .2)$ and $Y \sim B(100, .7)$ be independent binomial random variables. Let $Z = X + Y$. Find the following conditional quantities, using R simulations:

a) $P(X < 12|X < 18)$ and $E(X|X < 18)$
b) the cumulative distribution function of $X|(12 \leq X \leq 20)$ (plot of the ecdf)
c) the cumulative distribution function of $X|Z = 90$ (plot of the ecdf)
d) $E(Z|X = k)$ for $k = 10, 15, 20$.
e) $E(X|Z = k)$ for $k = 80, 90, 100$.

Problem 3

Suppose X has an exponential distribution with parameter $\lambda = 1$ and $Y|X = x$ has a Poisson distribution with parameter x.

a) Generate at least 1000 random samples from the marginal distribution of X and make a probability histogram.

b) Generate at least 1000 random samples from the conditional distribution of $Y|X = 1.5$ and make a probability histogram.

c) Generate at least 1000 random samples from the marginal distribution of Y and make a probability histogram.

d) Generate at least 1000 random samples from the conditional distribution of $X|Y = 2$ and make a probability histogram.

Problem 4

Suppose X and Y have independent standard normal distributions. Make at least 1,000 random samples from Z, defined as $Z = Y|(X + Y \geq 1)$. Do you think that Z has a normal distribution? What are its approximate mean and standard deviation?

Problem 5 (Submit only part 1 and part 4; *practice and discuss the parts 2 and 3 with classmates*)

Mixtures. Let Y_1 and Y_2 be two random variables which have the same range R, and let w_1, w_2 probabilities with $w_1 + w_2 = 1$. Then the mixture Y of Y_1 and Y_2 is defined as follows:

- Select $X \in \{1, 2\}$ at random, with $P(X = 1) = w_1$, $P(X = 2) = w_2$.
- If $X = 1$, draw a sample Y_1 and set $Y = Y_1$. Otherwise, draw a sample Y_2 and set $Y = Y_2$.

1. Suppose $E(Y_1) = \mu_1$ and $E(Y_2) = \mu_2$. What is $E(Y|X = 1)$? What is $E(Y|X = 2)$? Use this to show that $E(Y) = w_1\mu_1 + w_2\mu_2$.
2. Suppose $var(Y_1) = \sigma_1^2$ and $var(Y_2) = \sigma_2^2$. Explain why $E(Y^2|X = 1) = \sigma_1^2 + \mu_1^2$ and $E(Y^2|X = 2) = \sigma_2^2 + \mu_2^2$. Use this to find a formula for $E(Y^2)$.
3. Use the results of a) and b) to find a formula for $var(Y)$.
4. Generate a sample of size 10,000 from $Y_1 \sim N(-2, 1)$, $Y_2 \sim N(2, 2)$, $w_1 = \frac{1}{3}$, $w_2 = \frac{2}{3}$ and make a probability histogram. Clearly this is not a normal distribution, and a mixture is not a sum!

BONUS

Bob's preferred bet in American roulette consists in betting \$1 on black numbers and simultaneously \$2 on even numbers (see the roulette board in the course slides). Find all possible outcomes of a single game and their probabilities, that is, find the probability distribution of the outcome of a single bet. Then compute its expected value.

(Hint: Since the question is asking about a single bet, you can calculate this by hand)