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BOOK REVIEW

The Fairest Cut of All

Steven J. Brams and Alan D. Taylor
Fair Division: From Cake–Cutting to Dispute Resolution. Cambridge: Cambridge Univ. Press, 1996. Pp. ix + 272. \$59.95 (hb); \$18.95 (pb).

Reviewed by Peter Fishburn

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Fair Division is about the theory and application of procedures for allocating resources, broadly construed, to two or more agents such as individuals, political parties, or nations, in ways perceived as fair. Because situations that involve issues of fairness, equity, or distributive justice are ubiquitous at all levels of society, and because unfair or unjust allocations can have dire consequences for interpersonal relationships on up to political stability and international amity, the need for thoughtful advocacy of fair allocation procedures is enormous. Brams and Taylor's book and other writings do much to fill this need while pointing the way toward further work that will enrich both the theoretical foundations and the applicability of fair allocation procedures.

Two basic facts are evident throughout the book. First, different allocation problems can have different structures and constraints that call for different approaches. Fortunately, there is a manageable number of distinct abstract structures that encompass many situations. These play a key role in the book's topical organization.

The second basic fact is that no one conception of fairness is suitable for all situations, and several notions of fairness can apply within one situation. The most salient and universally applicable may be the notion of an envy-free allocation, wherein no agent prefers another agent's share to its own share. Envy-freeness, which has been discussed by economists for some time, is given center stage in *Fair Division*. Other fairness notions include proportional equity, equitability, and Pareto efficiency, a mainstay of economic analysis for many decades.

By the authors' own account (p. 1), their presentation of the books' topics is guided by three factors:

(1) setting forth explicit *criteria* that characterize different notions of fairness;

(2) providing step-by-step *constructive procedures* for achieving a fair division;

(3) illustrating these procedures with *applications* to real-life situations.

These organizing principles reflect their three aims for *Fair Division* (p. 3):

(i) to present the latest constructive results on fair-division procedures that have rejuvenated theoretical study in the field;

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(ii) to illustrate the application of these procedures to important practical problems; and

(iii) to trace out some of the history of fair division, including who discovered what.

The last of these will help readers appreciate the authors' own innovative contributions, which are primarily responsible for (i), within a historical context.

The book integrates old and new contributions to fair division in an easy-going style. As suggested above, it strongly favors constructive results in contrast to nonconstructive existence theorems, and illustrates an array of allocation procedures with numerous examples and sketches of applications. Mathematical symbolism is minimized, and only elementary mathematics is used in proofs, which are often placed in appendices. More involved mathematical arguments that depend on measure theory or combinatorics are available in references.

The preceding features are intended to make the work accessible to a wide audience that includes social scientists, business people, lawyers, and others interested in fairness and justice issues.

CUTTING QUESTIONS

The main problem addressed in *Fair Division* is how n people can cut a cake into n portions, one for each person, so that each receives a fair share when they value different parts of the cake differently. Although the following question has a slightly different emphasis, it provides access to key concerns of the book. We take $n=3$ but consider only a two-part division.

QUESTION 1. *Is there an algorithm or constructive procedure for dividing a cake into two portions so that each of* three *people regards the portions as equal*?

This is posed as an open problem (pp. 27, 125, 128). Before we can address it in the way intended by the authors, other questions must be answered:

What is a cake?

What is a portion of the cake?

What does it mean to say that a person regards two portions as equal?

What is an algorithm or constructive procedure for dividing the cake?

If the cake consists entirely of three indivisible crumbs, if a portion is a subset of crumbs, and if a person regards two portions as equal precisely when they have the same number of crumbs, then the whole cake can not be divided into two portions that anyone regards as equal. This is obviously not the scenario envisioned for Question 1.

For that question and many others, a cake C is viewed as a two-dimensional rectangle, and a portion is the union of a finite number of subareas of C bounded by line segments. A more general formulation might define C as a bounded, connected, and separable subset of a finite-dimensional Euclidean space, with a portion being any subset of C that is a union of open or closed sets in the usual topology. As noted earlier, the authors assiduously avoid this type of generalization.

Later in the book, a cake takes rather different forms than just described. For example, in Chapter 10, the "cake" is a legislature whose "crumbs" are seats, and the "persons" are political parties or interest groups. For present discussion, I maintain the view of C as a two-dimensional rectangle.

The question of equal portions for a person refers to the person's values or preferences for portions. It is assumed that values are additive over disjoint portions and can be represented by a nonnegative and additive real-valued function μ defined on a family of subsets of C that includes all portions. This is a very strong assumption and it often fails in practice, but because additivity is assumed in the book as well as in most of the literature, I adopt it here. When μ is normalized with $\mu(C)=1$, it can be regarded as a probability measure. It can also be thought of as an additive utility function, with portion A preferred to B if and only if $\mu(A)>\mu(B)$: see, for example, Fishburn (1992) and Barbanel and Taylor (1995). To avoid crumb-like problems, μ may be presumed to be spikeless and smooth or, more formally, to be nonatomic and perhaps countably additive and/or continuous. By definition, a person with additive utility function μ regards portions A and B as equal if and only if $\mu(A)=\mu(B)$.

The preceding definition of μ assumes that a person values every portion positively, or at least nonnegatively. Thus C is a "good." When the opposite case obtains and each person wishes to avoid as much of the cake as possible, we can view $\mu \geqslant 0$ as a disutility function with A preferred to B if and only if $\mu(A)<\mu(B)$. To deal with the mixed case in which people desire some portions and wish to avoid others, μ can be generalized as a signed measure with positive and negative values. Additivity is maintained in these variations, and we continue to say that A and B are equal with respect to μ if and only if $\mu(A)=\mu(B)$. In what follows, I use the initial "good" definition with $\mu(C)=1$.

We return to Question 1. We designate the three people as 1, 2, and 3 and denote their additive utility functions on subsets of C by μ_1, μ_2, and μ_3 respectively. Then each person regards A and its complementary portion $C\backslash A$ as equal in a two-part division of the cake precisely when

$$\mu_i(A) = \mu_i(C\backslash A) = \tfrac{1}{2} \quad \text{for } i = 1, 2, 3.$$

Under preceding assumptions, it is known (Neyman, 1946) that there *exists* a division of C into two portions that each person regards as equal. However, this does not settle Question 1 positively because it says nothing about a procedure that the people might actually use to effect such a division.

The book's glossary (pp. 237–247) defines an algorithm as "a constructive procedure with finitely many steps." This is not mathematically rigorous, but it suffices for the authors' purposes. Most of us have a good intuitive understanding of what it means to use a step-by-step constructive procedure to solve a problem in a finite number of steps. In the authors' terminology, all algorithms are procedures, but not all procedures are algorithms. I will follow their lead and use "procedure" when "algorithm" seems unsuitable. Further discussion of these and related terms is available in Brams and Taylor (1995) and Brams, Taylor, and Zwicker (1997).

The simplest algorithm in the book is divide-and-choose for two people. Person 1, the *cutter*, divides C into two portions. Person 2, the *chooser*, selects one portion, and the cutter gets the other. The cutter ensures a fair share with any partition $\{A, C\backslash A\}$ for which $\mu_1(A) = \mu_1(C\backslash A) = 1/2$. The chooser then does as well as possible by selecting A if $\mu_2(A) \geqslant \mu_2(C\backslash A)$, and selecting $C\backslash A$ otherwise. If $\mu_2(A) \neq \mu_2(C\backslash A)$, person 2 gets more than a fair share in terms of μ_2 but not in terms of μ_1. With the indicated strategies, it follows that divide-and-choose can be envy-free, and its aura of fairness can be further enhanced by flipping a coin to determine who cuts.

Because person 2 gets more than a fair share from 2's perspective if $\mu_2(A) \neq \mu_2(C\backslash A)$, divide-and-choose does not provide an affirmative answer to the following simplification of Question 1.

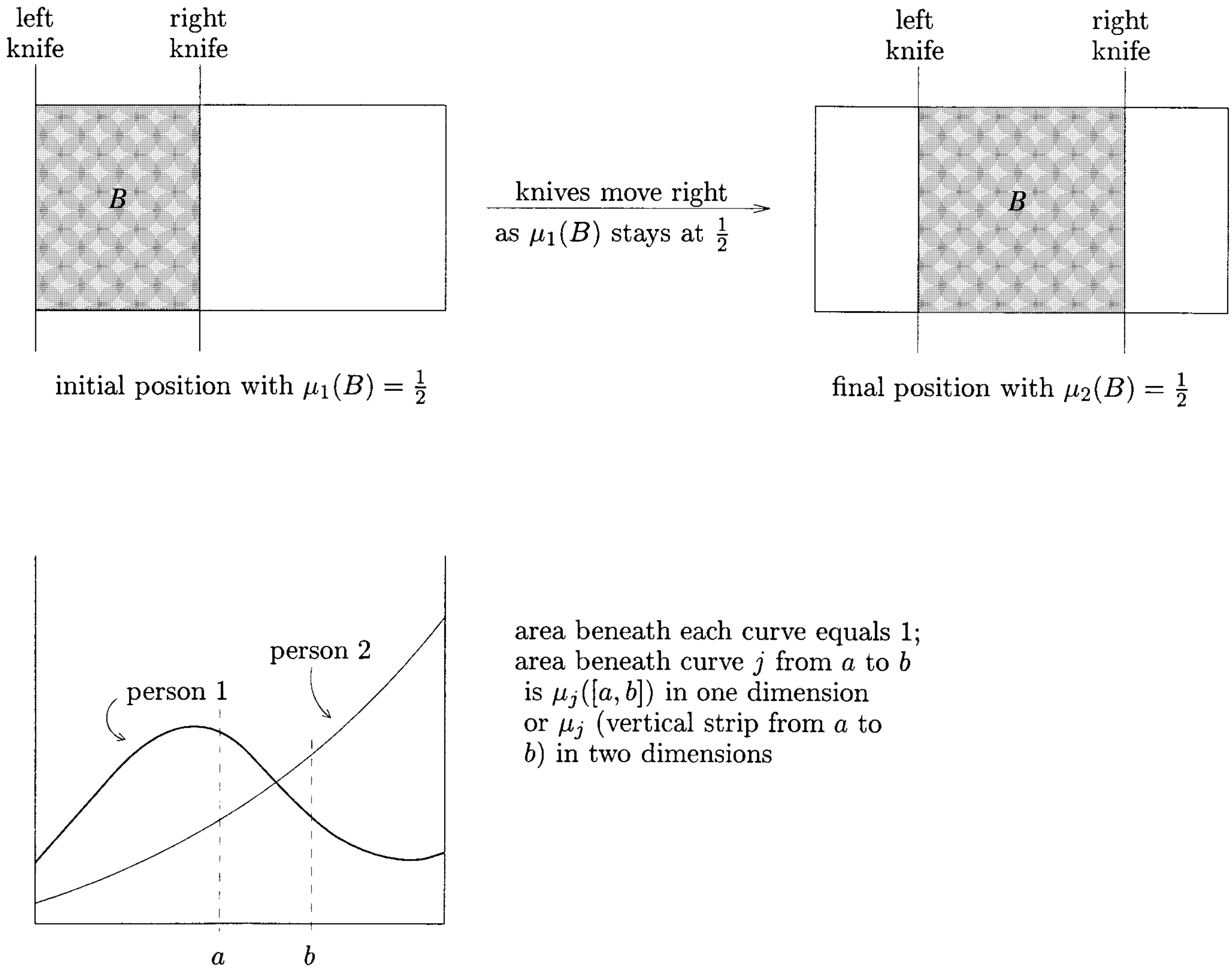

FIGURE 1

QUESTION 2. *Is there a procedure for dividing a cake into two portions so that each of* two *people regards the portions as equal*?

A moving-knife procedure (pp. 25 and 26) due to Austin (1982) gives a positive answer so long as each person assigns positive value to each vertical strip of a rectangular cake. In one variation, person 1 places two knives perpendicular to the bottom edge, one at the left edge and the other through the interior so that the portions left and right of the interior knife are equal for that person. Let B denote the portion between the knives. They are then moved simultaneously to the right at rates that maintain $\mu_1(B) = \frac{1}{2}$. The rates of travel of the two knives will often differ unless μ_1 is a uniform or rectangular distribution. When $\mu_2(B)$ first equals $\frac{1}{2}$, person 2 stops the process: the final B and its complement define a two-part division whose portions are equal for both people. The existence of such a B is assured by additivity, continuity, and the foregoing positivity assumption. Note in particular that if the right knife arrives at the right edge then the left knife will duplicate the initial position of the right knife, so $\mu_2(B)$ cannot remain strictly less than $\frac{1}{2}$ or strictly more than $\frac{1}{2}$ throughout. Austin's procedure is not referred to as an algorithm because of its continuous movement feature.

Figure 1 illustrates Austin's procedure. The lower diagram indicates μ values by areas beneath curves and shows that the procedure works as well in one dimension (C a line segment) as in two. In one dimension, the value of μ_j on the interval $[a, b]$ along the line is the area between curve j and the abscissa from a to b. For one dimension, the knives shrink to points that are moved left-to-right along the line.

It is not presently known if an extension of Austin's ideas yields an affirmative answer to Question 1, even when "constructive procedure" allows continuous knife movements in more than one dimension. For example, there are various ways that persons 1 and 2 can divide a circular cake into two equal portions for both. If it were possible to do this by a continuous series of divisions that continuously transform the equal parts in a certain way, then the continuity argument used in Austin's $n = 2$ procedure would yield an equal-parts division for person 3 as well as 1 and 2.

NOTIONS OF FAIRNESS

Two main assumptions underlie most of the fairness notions in the book. The first is *individual autonomy*, which means that what is fair for an agent depends solely on that agent's preferences. This may involve shares allocated to others, as in envy-freeness, but considered from the viewing agent's perspective rather than from the preferences of others. Ancillary to this is a presumption that agents act in their own interest. The challenge for fair allocation, as in discussions of incentive-compatible mechanisms in the economics' literature, is to design procedures that encourage equitable distribution in the face of self-interested behavior on the part of agents.

The second main assumption is *additivity* of individual preferences or values, as described earlier. Additivity is vulnerable to preference interdependencies arising from substitutability and complementarity effects, as when the value of an item received depends on what else is received. It should be treated with caution in practice and never be taken lightly. The authors observe that little is known about their types of procedures when utility is not additive, and they cite the need for new research on the subject. Because additivity is often inapplicable, I strongly second their call for research on fair division with nonadditive utilities.

To define fairness notions used in the book, assume that C is to be divided among n people. Let μ_i denote a non-negative and additive utility function for person i with $\mu_i(C) = 1$. Also let $P = (P_1, P_2, ..., P_n)$ and $Q = (Q_1, Q_2, ..., Q_n)$ denote ordered partitions of C in which i receives portion P_i or Q_i. An allocation P is

proportional if $\mu_i(P_i) \geqslant 1/n$ for $i = 1, ..., n$;

envy-free if $\mu_i(P_i) \geqslant \mu_i(P_j)$ for all i and j;

Pareto efficient if no allocation Q has $\mu_i(Q_i) \geqslant \mu_i(P_i)$ for all i, and $\mu_i(Q_i) > \mu_i(P_i)$ for some i.

It follows from additivity that every envy-free allocation is proportional, but not conversely except when $n = 2$. On the other hand, there are no implications between Pareto efficiency and the other two criteria. Indeed, Pareto efficiency is not an obvious fairness criterion since an efficient allocation can leave some people in unenviable positions. Its role in fair division might be most appropriate as a secondary criterion, to improve allocations that are proportional or envy-free, or which satisfy the Rawlsian proportionality criterion of maximizing the minimum $\mu_i(P_i)$.

None of the preceding criteria depends in any way on a notion of interpersonal comparison of utility. If we drop the normalization convention of $\mu_i(C) = 1$ and let $\mu_i(C) = c_i > 0$, then the effects of the definitions are unchanged so long as proportionality is replaced by $\mu_i(P_i) \geqslant c_i/n$ and the Rawlsian criterion is to maximize the minimum $\mu_i(P_i)/c_i$.

On the other hand, the authors' notion of *equitability*, which is defined only for $n = 2$, presumes commensurate scales for μ_1 and μ_2 since P is said to be equitable if and only if $\mu_1(P_1) = \mu_2(P_2)$. Because judgments of equality in this sense assume knowledge of the other person's utilities, and there is no guarantee that their disclosures will be sincere or truthful, Brams and Taylor use *announced valuations*, say μ_1^* and μ_2^* with $\mu_i^*(C) = 1$, as the basis of equitability, which they characterize by $\mu_1^*(P_1) = \mu_2^*(P_2)$.

The issue of truthful disclosure, or more generally of strategies that agents use in allocation processes, is prominent throughout the book. Invulnerability of allocation procedures to strategic manipulation, or encouragement of truthful revelation of preferences, is listed with other fairness criteria. The authors give many examples of ways in which agents could try to gain advantage by falsifying their true values, and they are careful to point out the associated risks. It is also argued that the information about others' values or likely behaviors needed to gain advantage is sufficiently daunting to discourage falsification in many instances. In any event, because most procedures attain fairness only through truthful revelation or action, there is a premium on procedures that encourage and reward such behavior.

Another issue of concern is the treatment of entitlements. An example is a will with two beneficiaries who are bequeathed 2/3 and 1/3 of an estate. Differential entitlements can sometimes be handled with clones, an idea due to Steinhaus (1948). Suppose persons 1, 2, and 3 are entitled to 2/9, 3/9, and 4/9 of C, respectively. One could then replace person i by $i+1$ " people" with i's preferences, apply an allocation procedure to nine equally entitled people, and aggregate the resulting portions in the obvious way.

OVERVIEW

Fair Division has 10 chapters in addition to an introduction and an excellent concluding chapter. I divide the 10 into four parts for summary purposes.

Part 1 (Chapters 1–3) deals with proportional equity. It includes divide-and-choose and Austin's moving-knife procedure for two people, then looks at extensions of these and a variety of others that assure proportionality with a continuously divisible cake and three or more people. Chapter 3 describes two proportional schemes for a finite list of indivisible goods. One of these, proposed by B. Knaster in the 1940s, involves sealed bids by each player for items in the list along with side payments to compensate for discrepancies caused by indivisibility.

None of the proportional procedures of Part 1 is envy-free when $n \geqslant 3$, and an argument on p. 48 shows that no algorithmic procedure can be both proportional and Pareto efficient when $n \geqslant 2$ and C is divisible. However, Knaster's procedure with side payments is efficient.

Part 2 (Chapters 4 and 5) begins a series of chapters that emphasize envy-free allocations. Part 2 focuses on two algorithms for $n=2$ in which each person distributes 100 points over the goods in a finite set of goods, each of which is divisible. The points a person assigns to a good is that person's *announced valuation* of the good. The first algorithm begins with a tentative assignment of whole goods according to larger announced valuations and then makes divisible adjustments. It is envy-free, Pareto efficient, and equitable in terms of the announced valuations, but can be vulnerable to strategic manipulation. The second algorithm, which divides each good on a point-proportional basis, is relatively nonmanipulable and is envy-free and equitable, but not Pareto efficient in terms of announced valuations. For $n \geqslant 3$, no point-distribution algorithm is generally envy-free, efficient, and equitable, but algorithms have been devised to guarantee any two of these three criteria.

Part 3 (Chapters 6 and 7) is devoted to envy-free allocations of a continuously divisible cake for $n \geqslant 3$. The authors cite Steinhaus' proof (1949) for the existence of such allocations and note that it was not until about 1960 that John L. Selfridge and John H. Conway independently discovered a constructive algorithm for an envy-free allocation among three people.

The Selfridge–Conway algorithm has two stages and requires at most five cuts. In stage 1, person 1 cuts C into three μ_1-equal portions, person 2 trims at most one of the three to create two largest μ_2-equal portions, then persons 3, 2, and 1 choose portions (minus the trimmings) in that order with the proviso that 2 gets the trimmed portion if it wasn't chosen by 3. Stage 2 allocates the trimmings T, if any, cut off by person 2. Supposing that 3 chose the trimmed portion at the end of stage 1, person 2 now cuts T into three μ_2-equal portions, and 3, 1, and 2 choose portions in that order. The entire C is now allocated, and it is fairly straightforward to prove that the allocation is envy-free. If person 2 or 3 cuts first, a different envy-free allocation can occur.

Chapter 6 continues with moving-knife envy-free procedures for $n \in \{3, 4\}$. One $n=4$ procedure, due to Brams, Taylor, and Zwicker (1997), features two applications of a simple extension of Austin's procedure. The extension shows how two people can cut C into k equal parts. The $n=4$ procedure has persons 1 and 2 divide C into four μ_1-equal and μ_2-equal portions. Person 3 trims at most one of these to create two largest μ_3-equal portions, and the trimmings T are set aside. Portions (minus T) are chosen by 4, 3, 2, and 1 in that order, with the proviso that 3 gets the trimmed portion if 4 didn't choose it. Let x denote the trimmed-portion holder. Persons 2 and x now divide T into four μ_2-equal and μ_x-equal portions, which are chosen by the one of 3 and 4 who is not x, 1, 2, and x in that order. It is easily checked that there is no envy in either stage, so additivity implies that the complete allocation of C is envy-free.

Chapter 7 describes approximate and exact envy-free allocations for all n, with subsequent commentaries on indivisible goods, Pareto efficiency, and entitlements. The chapter highlights recent contributions by Brams and Taylor, including the first finite-stages algorithm that produces envy-free allocations for all n and uses only discrete actions by the players as opposed, for example, to the continuous actions of moving-knife procedures. One technical advantage of their procedures is that the μ_i only need to be finitely and not countably additive in discrete-move schemes. The innovative idea that makes their procedure work is to have the first cutter cut C into *more* than n equal portions.

Consider the following stage-1 algorithm for $n = 4$;

person 1 cuts C into *five* μ_1-equal portions;

person 2 trims one or two of the five to create three largest μ_2-equal portions and sets the trimmings aside;

person 3 trims at most one of the resulting five portions to create two largest μ_3-equal portions and sets the trimmings aside;

4, 3, 2, and 1 each choose one of the five portions in that order with the provisos that 3 gets the portion he trimmed if 4 didn't choose it, and 2 must choose a portion she trimmed if one is still available.

Let C_1 be the union of the four chosen portions. Assuming self-interested behavior, which is presumed throughout, the allocation of C_1 is envy-free. The same algorithm can be used on $C \backslash C_1$ in stage 2 to give an envy-free allocation of part of $C \backslash C_1$. Additional iterations leave a smaller and smaller remainder, but an infinite or transfinite number of iterations are needed to reduce the remainder to zero.

What is amazing is that Brams and Taylor found a way to modify the foregoing iterated algorithm so that the remainder vanishes after a finite number of cuts. Descriptions of their finite-stages algorithm for $n = 4$ are presented in the book (pp. 139–143) and in their *American Mathematical Monthly* article (Brams and Taylor, 1995). The *Monthly* article is an informative exposition of divisible-cake allocation and a valuable supplement to the book.

Part 4 (Chapters 8–10) discusses fairness issues in three other topics of long-term interest to the authors. Chapter 8 presents a new approach to the divide-the-dollar game in which two players bid for portions of a 100-unit homogeneous good that they value equally. The division of the good depends on the bids. Their approach uses new rules for the division that attempt to induce the egalitarian solution in which each player gets 50 units.

Chapter 9 analyzes a new and unorthodox two-stage auction scheme that is intended to be fairer to bidders than Vickrey second-price auctions in which the highest bidder wins but pays only the amount bid by the second-highest bidder. Stage 1 uses sealed bids that are made public without naming who bid each amount. In stage 2, each person submits a new bid that is identical to some first-stage bid. The high bidder in stage 2 wins the item.

Chapter 10 discusses voting systems for allocating seats in a representational legislature in a way that proportionally approximates the sizes of different factions or interest groups thus represented. Procedures intended to achieve proportional representation and ensure fair seat shares for minority interests include the single-transferable-vote quota system, a Borda point-assignment method, cumulative voting, additional-member systems, and a constrained form of approval voting. A few of these are widely used in practice, and all have an array of virtues and vices described in the chapter.

Note. *Fair Division* succeeds handsomely in presenting and illustrating the latest constructive results on fair-division procedures in abstract and applied settings. Its informal and historically grounded presentation adds considerably to one's reading enjoyment. Readers who wish to probe the mathematical foundations in greater depth will be well served by the book's bibliography.

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