CAS 760
Simple Type Theory
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5 Proof Systems

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Outline

1. Background.
2. A proof system for Alonzo.
3. Soundness and completeness.

1. Background

Proofs and Theorems

- A proof system $\mathfrak{P}$ for Alonzo consists of a decidable set of axioms and rules of inference.
- A proof of $\mathbf{A}_o$ in $\mathfrak{P}$ is a finite sequence Π of formulas of Alonzo ending with $\mathbf{A}_o$ such that every formula in Π is:
  1. An axiom of $\mathfrak{P}$ or
  2. Inferred from previous formulas in Π by one of the rules of inference of $\mathfrak{P}$.
- $\vdash_{\mathfrak{P}} \mathbf{A}_o$ asserts that there is a proof of $\mathbf{A}_o$ in $\mathfrak{P}$.
- A theorem of $\mathfrak{P}$ is a formula such that $\vdash_{\mathfrak{P}} \mathbf{A}_o$.

Proofs from Premises

- Let Γ be a set of formulas.
- A proof of $\mathbf{A}_o$ from Γ in $\mathfrak{P}$ is a pair (Π_1, Π_2) of finite sequences of formulas of Alonzo ending with $\mathbf{A}_o$ such that Π_1 is a proof in $\mathfrak{P}$, Π_2 ends with $\mathbf{A}_o$, and every formula in Π_2 is:
  1. A member of Γ,
  2. A member of Π_1 (and thus a theorem of $\mathfrak{P}$), or
  3. Inferred from previous formulas in Π_2 by one of the rules of inference of $\mathfrak{P}$ modified, if necessary, so that the free variables in members of Γ are treated as constants instead of as universally quantified variables as in axioms.
- $\Gamma \vdash_{\mathfrak{P}} \mathbf{A}_o$ asserts that there is a proof of $\mathbf{A}_o$ from Γ in $\mathfrak{P}$.

Soundness and Completeness

- $\mathfrak{P}$ is sound if

  $$\Gamma \vdash_{\mathfrak{P}} \mathbf{A}_o \text{ implies } \Gamma \vDash \mathbf{A}_o,$$

  where Γ is a set of formulas.

- $\mathfrak{P}$ is complete (in the general sense) if

  $$\Gamma \vDash \mathbf{A}_o \text{ implies } \Gamma \vdash_{\mathfrak{P}} \mathbf{A}_o,$$

  where Γ is a set of sentences.

- $\mathfrak{P}$ is complete in the standard sense if

  $$\Gamma \vDash^{\mathsf{s}} \mathbf{A}_o \text{ implies } \Gamma \vdash_{\mathfrak{P}} \mathbf{A}_o,$$

  where Γ is a set of sentences.

Henkin's Soundness and Completeness Theorem

- Theorem. There is no sound proof system for Alonzo that is complete in the standard sense.

  Proof. Theorem 8.2 proved in Chapter 9.

- Theorem (Henkin, 1950). There is a sound proof system for Church's type theory that is complete in the general sense.

- Proposition. If $\mathfrak{P}$ is sound and complete proof system for Alonzo and Γ is a set of sentences that is inconsistent in $\mathfrak{P}$, then $\Gamma \vdash_{\mathfrak{P}} \mathbf{A}_o$ for all formulas $\mathbf{A}_o$.

2. A Proof System for Alonzo

A Proof System for Alonzo [1/4]

$\mathfrak{A}$ (named for Peter Andrews) is a proof system for Alonzo that has the following axioms and rules of inference:

A1 (Truth Values)

$$((g : o \to o)\, T_o \wedge (g : o \to o)\, F_o) \Leftrightarrow \forall\, x : o \,.\, (g : o \to o)\, x.$$

A2 (Leibniz's Law)

$$(x : \alpha) = (y : \alpha) \Rightarrow$$
$$((h : \alpha \to o)\, (x : \alpha) \Leftrightarrow (h : \alpha \to o)\, (y : \alpha)).$$

A3 (Extensionality)

$$(f : \alpha \to \beta) = (g : \alpha \to \beta) \Leftrightarrow$$
$$\forall\, x : \alpha \,.\, (f : \alpha \to \beta)\, x \simeq (g : \alpha \to \beta)\, x.$$

A4 (Beta-Reduction)

$$\mathbf{A}_\alpha\!\downarrow\, \Rightarrow (\lambda\, \mathbf{x} : \alpha \,.\, \mathbf{B}_\beta)\, \mathbf{A}_\alpha \simeq \mathbf{B}_\beta[(\mathbf{x} : \alpha) \mapsto \mathbf{A}_\alpha]$$

provided $\mathbf{A}_\alpha$ is free for $(\mathbf{x} : \alpha)$ in $\mathbf{B}_\beta$.

A Proof System for Alonzo [2/4]

A5 (Definedness)

1. $(\mathbf{x} : \alpha)\downarrow$.
2. $\mathbf{c}_\alpha\downarrow$.
3. $(\mathbf{A}_\alpha = \mathbf{B}_\alpha)\downarrow$.
4. $(\mathbf{A}_\alpha = \mathbf{B}_\alpha) \Rightarrow \mathbf{A}_\alpha\downarrow$.
5. $(\mathbf{A}_\alpha = \mathbf{B}_\alpha) \Rightarrow \mathbf{B}_\alpha\downarrow$.
6. $(\mathbf{F}_{\alpha \rightarrow o}\, \mathbf{A}_\alpha)\downarrow$.
7. $\mathbf{F}_{\alpha \rightarrow o}\, \mathbf{A}_\alpha \Rightarrow \mathbf{F}_{\alpha \rightarrow o}\downarrow$.
8. $\mathbf{F}_{\alpha \rightarrow o}\, \mathbf{A}_\alpha \Rightarrow \mathbf{A}_\alpha\downarrow$.
9. $(\mathbf{F}_{\alpha \rightarrow \beta}\, \mathbf{A}_\alpha)\downarrow \Rightarrow \mathbf{F}_{\alpha \rightarrow \beta}\downarrow$ where $\beta \neq o$.
10. $(\mathbf{F}_{\alpha \rightarrow \beta}\, \mathbf{A}_\alpha)\downarrow \Rightarrow \mathbf{A}_\alpha\downarrow$ where $\beta \neq o$.
11. $(\lambda\, \mathbf{x} : \alpha \,.\, \mathbf{B}_\beta)\downarrow$.
12. $\mathbf{A}_\alpha\downarrow \Rightarrow (\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha) \simeq (\mathbf{A}_\alpha = \mathbf{B}_\alpha)$.
13. $\mathbf{B}_\alpha\downarrow \Rightarrow (\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha) \simeq (\mathbf{A}_\alpha = \mathbf{B}_\alpha)$.

A Proof System for Alonzo [3/4]

A6 (Definite Description)

1. $(\exists!\, x : \alpha \,.\, \mathbf{A}_o) \Rightarrow (\mathrm{I}\, x : \alpha \,.\, \mathbf{A}_o) \in \{x : \alpha \mid \mathbf{A}_o\}$.
2. $\neg(\exists!\, x : \alpha \,.\, \mathbf{A}_o) \Rightarrow (\mathrm{I}\, x : \alpha \,.\, \mathbf{A}_o)\uparrow$.

A7 (Pairs)

1. $\forall\, x : \alpha,\ y : \beta \,.\, (x, y)\downarrow$.
2. $(\mathbf{A}_\alpha, \mathbf{B}_\beta)\downarrow \Rightarrow \mathbf{A}_\alpha\downarrow$.
3. $(\mathbf{A}_\alpha, \mathbf{B}_\beta)\downarrow \Rightarrow \mathbf{B}_\alpha\downarrow$.
4. $\forall\, p : \alpha \times \beta \,.\, p = (\mathsf{fst}_{(\alpha\times\beta)\to\alpha}\, p, \mathsf{snd}_{(\alpha\times\beta)\to\beta}\, p)$.
5. $\forall\, x, x' : \alpha,\ y, y' : \beta \,.\, (x, y) = (x', y') \Rightarrow (x = x' \wedge y = y')$.

A Proof System for Alonzo [4/4]

R1 (Modus Ponens) From $\mathbf{A}_o$ and $\mathbf{A}_o \Rightarrow \mathbf{B}_o$ infer $\mathbf{B}_o$.

R2 (Quasi-Equality Substitution) From $\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$ and $\mathbf{C}_o$ infer the result of replacing one occurrence of $\mathbf{A}_\alpha$ in $\mathbf{C}_o$ by an occurrence of $\mathbf{B}_\alpha$, provided that the occurrence of $\mathbf{A}_\alpha$ in $\mathbf{C}_o$ is not the first argument of a function abstraction or a definite description.

Proofs from Premises in $\mathfrak{A}$

- Let $\mathbf{A}_o$ be a formula and Γ be a set of formulas of Alonzo.
- A proof of $\mathbf{A}_o$ from Γ in $\mathfrak{A}$ is a pair (Π_1, Π_2) of finite sequences of formulas of Alonzo such that Π_1 is a proof in $\mathfrak{A}$, Π_2 ends with $\mathbf{A}_o$, and every formula $\mathbf{D}_o$ in Π_2 satisfies one of the following conditions:
  1. $\mathbf{D}_o$ is a member of Γ.
  2. $\mathbf{D}_o$ is in Π_1 (and thus a theorem of $\mathfrak{A}$).
  3. $\mathbf{D}_o$ is inferred from two previous formulas in Π_2 by R1.
  4. $\mathbf{D}_o$ is obtain from previous formulas $\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$ and $\mathbf{C}_o$ in Π_2 by replacing one occurrence of $\mathbf{A}_\alpha$ in $\mathbf{C}_o$ by an occurrence of $\mathbf{B}_\alpha$, provided that the occurrence of $\mathbf{A}_\alpha$ in $\mathbf{C}_o$ is not the first argument of a function abstraction or a definite description and not in the second argument of a function abstraction $\lambda\, \mathbf{x} : \beta \,.\, \mathbf{E}_\gamma$ or a definite description $\mathrm{I}\, \mathbf{x} : \beta \,.\, \mathbf{E}_o$ where $(\mathbf{x} : \beta)$ is free in a member of Γ and free in $\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$.

3. Soundness and Completeness

Soundness and Consistency Theorems

- Theorem (Soundness Theorem). $\mathfrak{A}$ is sound.

  Proof. Theorem B.11 in Appendix B.

- Corollary (Consistency Theorem). If a set of formulas is satisfiable, then it is consistent in $\mathfrak{A}$.

Frugal Models

- A general model M of L is frugal if $\|M\| \leq \|L\|$.
- Theorem. Every frugal infinite general model of a language of power ω is a nonstandard model.
- Theorem (Henkin's Theorem). Every set of sentences in Alonzo that is consistent in $\mathfrak{A}$ has a frugal general model.

  Proof. Theorem C.3 in Appendix C.

Completeness Theorems

- Theorem (Provability equals Validity). $\mathfrak{A}$ is sound and complete.

  Proof. Follows from Soundness Theorem, Henkin's Theorem, and metatheorems proved in Appendix A.

- Corollary (Consistency equals Satisfiability). Let Γ be a set of sentences. Then Γ is consistent in $\mathfrak{A}$ iff Γ is satisfiable.
- Corollary (Compactness Theorem). Let Γ be a set of sentences. Then Γ is satisfiable iff every finite subset of Γ is satisfiable.
- Theorem (Nonstandard Models Exist). Let Γ be a set of sentences that has infinite models. Then Γ has nonstandard models.

The End.