CAS 760

Simple Type Theory

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7 Inductive Sets and Types

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Outline

1. Inductive sets.
2. Inductive types.
3. Inductive type theory extensions.

1. Inductive Sets

Inductive Sets [1/5]

- Let A be a nonempty set of values.
- A generator for A is a value g that satisfies one of the following four conditions:
  1. $g \in A$.
  2. $g : B \rightarrow A$ is a total, injective function such that B is a nonempty set of values.
  3. $g : (A \times \cdots \times A) \rightarrow A$ is a total, injective function such that the length of the Cartesian product is $m \geq 1$.
  4. $g : (B \times A \times \cdots \times A) \rightarrow A$ is a total, injective function such that B is a nonempty set of values and the length of the Cartesian product is $m + 1 \geq 2$.
- g is nullary if g is the first kind of generator.
- g is pure if g is the first or third kind of generator.
- g is recursive if g is the third or fourth kind of generator.

Inductive Sets [2/5]

- Let X_g be the formula, in which $p : A \to \mathbb{B}$ is a predicate, defined as follows:
  1. If g is the first kind of generator, then X_g is
     $$p(g).$$
  2. If g is the second of generator, then X_g is
     $$\forall\, y \in B \,.\, p(g(y))\ .$$
  3. If g is the third kind of generator, then X_g is
     $$\begin{array}{l}\forall\, x_1 \in A, \ldots, x_m \in A \,. \\ \quad (p(x_1) \wedge \cdots \wedge p(x_m)) \Rightarrow p(g(x_1, \ldots, x_m)).\end{array}$$
  4. If g is the fourth kind of generator, then X_g is
     $$\begin{array}{l}\forall\, x_1 \in A, \ldots, x_m \in A \,. \\ \quad (p(x_1) \wedge \cdots \wedge p(x_m)) \Rightarrow \forall\, y \in B \,.\, p(g(y, x_1, \ldots, x_m)).\end{array}$$

Inductive Sets [3/5]

- Let G be a finite set of generators for A.
- The set of values generated by G, written $\mathrm{gen}(G)$, is the set $A' \subseteq A$ defined inductively by the following membership rules:
  1. For each $g \in G$ of the first kind, $g \in A'$.
  2. For each $g \in G$ of the second kind with $g : B \to A$, if $y \in B$, then $g(y) \in A'$.
  3. For each $g \in G$ of the third kind with $g : (A \times \cdots \times A) \to A$ such that the length of the Cartesian product is $m \geq 1$, if $x_1, \ldots, x_m \in A'$, then $g(x_1, \ldots, x_m) \in A'$.
  4. For each $g \in G$ of the fourth kind with $g : (B \times A \times \cdots \times A) \to A$ such that the length of the Cartesian product is $m + 1 \geq 2$, if $x_1, \ldots, x_m \in A'$ and $y \in B$, then $g(y, x_1, \ldots, x_m) \in A'$.

Inductive Sets [4/5]

- A is freely generated by G if:
  1. The sentences

     $$g \neq g',$$
     $$g \not\in \mathrm{ran}(h), \text{ and}$$
     $$\mathrm{ran}(h) \cap \mathrm{ran}(h') = \emptyset$$

     are true for all distinct nullary $g, g' \in G$ and distinct nonnullary $h, h' \in G$. These sentences, together with the injectivity of the nonnullary generators, ensure that the generators in G produce distinct members of A.
  2. The sentence

     $$\forall\, p : A \to \mathbb{B} \,.\, (X_{g_1} \wedge \cdots \wedge X_{g_n}) \Rightarrow \forall\, x \in A \,.\, p(x)$$

     is true where $G = \{g_1, \ldots, g_n\}$. This sentence is called the structural induction principle for A with respect to $g_1, \ldots, g_n$. It ensures $A = \mathrm{gen}(G)$.

Inductive Sets [5/5]

- An inductive set is a pair (A, G) where A is a nonempty set and G is a finite set of generators for A such that A is freely generated by G.
- Theorem (No Junk). Let (A, G) be an inductive set. Then $A = \mathrm{gen}(G)$.
- Theorem (No Confusion). Let (A, G) be an inductive set. Then each member of A is generated by G in exactly one way.

Example: Inductive Set of Boolean Values

- Let $\mathbb{B} = \{\mathrm{F}, \mathrm{T}\}$, the set of Boolean values.
- Then $(\mathbb{B}, \{\mathrm{F}, \mathrm{T}\})$ is an inductive set.
- The structural induction principle for $\mathbb{B}$ with respect to F, T is

$$\forall\, p : \mathbb{B} \rightarrow \mathbb{B} \,.\, (p(\mathrm{F})) \wedge p(\mathrm{T})) \Rightarrow \forall\, x \in \mathbb{B} \,.\, p(x).$$

- $(\mathbb{B}, \{\mathrm{F}, \mathrm{T}\})$ is an example of an inductive set with an enumerated structure.

Example: Inductive Set of Natural Numbers

- Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, the set of natural numbers, and let $S : \mathbb{N} \rightarrow \mathbb{N}$ be the successor function.
- Then $(\mathbb{N}, \{0, S\})$ is an inductive set.
- The structural induction principle for $\mathbb{N}$ with respect to $0, S$ is

$$\begin{array}{l} \forall\, p : \mathbb{N} \rightarrow \mathbb{B} \,. \\ \quad (p(0) \wedge \forall\, x_1 \in \mathbb{N} \,.\, p(x_1) \Rightarrow p(S(x_1)) \Rightarrow \forall\, x \in \mathbb{N} \,.\, p(x). \end{array}$$

- $(\mathbb{N}, \{0, S\})$ is the standard example of an inductive set that has a hierarchical structure.

2. Inductive Types

Inductive Types [1/5]

- Let $\mathbf{a}$ be a base type.
- A constructor for $\mathbf{a}$ is one of the following:
  1. A constant $\mathbf{c}_{\mathbf{a}}$.
  2. A pair $(\mathbf{c}_{\beta \rightarrow \mathbf{a}}, \mathbf{B}_{\{\beta\}})$ where β does not contain $\mathbf{a}$ and $\mathbf{B}_{\{\beta\}}$ is closed.
  3. A constant $\mathbf{c}_{\alpha}$ where $\alpha = (\mathbf{a} \times \cdots \times \mathbf{a}) \rightarrow \mathbf{a}$ and the length in the extended product type is $m \geq 1$.
  4. A pair $(\mathbf{c}_{\alpha}, \mathbf{B}_{\{\beta\}})$ where $\alpha = (\beta \times \mathbf{a} \times \cdots \times \mathbf{a}) \rightarrow \mathbf{a}$ and the length of the extended product type is $m + 1 \geq 2$, β does not contain $\mathbf{a}$, and $\mathbf{B}_{\{\beta\}}$ is closed.
- Let c be a constructor for $\mathbf{a}$.
- Define

$$\overline{c} = \begin{cases} \mathbf{c}_{\alpha} & \text{if } c = \mathbf{c}_{\alpha}; \\ \mathbf{c}_{\alpha} & \text{if } c = (\mathbf{c}_{\alpha}, \mathbf{B}_{\{\beta\}}). \end{cases}$$

- c is nullary if c is the first kind of constructor.

Inductive Types [2/5]

- c is pure if c is the first or third kind of constructor.
- c is recursive if c is the third or fourth kind of constructor.
- Let $\mathbf{X}_o^c$ be the formula of Alonzo defined as follows:
  1. If $c = \mathbf{c_a}$ is the first kind of constructor, $\mathbf{X}_o^c$ is
  $$(p : \mathbf{a} \rightarrow o)\, \mathbf{c_a},$$
  2. If $c = (\mathbf{c}_{\beta \rightarrow \mathbf{a}}, \mathbf{B}_{\{\beta\}})$ is the second kind of constructor, then $\mathbf{X}_o^c$ is
  $$\forall\, y : \mathbf{B}_{\{\beta\}} \,.\, (p : \mathbf{a} \rightarrow o)\, (\mathbf{c}_{\beta \rightarrow \mathbf{a}}\, y).$$
  3. If $c = \mathbf{c}_\alpha$ is the third kind of constructor, $\mathbf{X}_o^c$ is
  $$\begin{array}{l} \forall\, x_1, \ldots, x_m : \mathbf{a} \,. \\ \quad ((p : \mathbf{a} \rightarrow o)\, x_1 \wedge \cdots \wedge (p : \mathbf{a} \rightarrow o)\, x_m) \Rightarrow \\ \quad (p : \mathbf{a} \rightarrow o)\, (\mathbf{c}_\alpha\, (x_1, \ldots, x_m)). \end{array}$$
  4. If $c = (\mathbf{c}_\alpha, \mathbf{B}_{\{\beta\}})$ is the fourth kind of constructor, then $\mathbf{X}_o^c$ is
  $$\begin{array}{l} \forall\, x_1, \ldots, x_m : \mathbf{a} \,. \\ \quad ((p : \mathbf{a} \rightarrow o)\, x_1 \wedge \cdots \wedge (p : \mathbf{a} \rightarrow o)\, x_m) \Rightarrow \\ \quad \forall\, y : \mathbf{B}_{\{\beta\}} \,.\, (p : \mathbf{a} \rightarrow o)\, (\mathbf{c}_\alpha\, (y, x_1, \ldots, x_m)). \end{array}$$

Inductive Types [3/5]

- Let C be a finite set of constructors for $\mathbf{a}$.
- The set of obligations for C, written $\mathrm{obl}(C)$, is the set of sentences defined by:
  1. For each $(\mathbf{c}_{\beta \rightarrow \mathbf{a}}, \mathbf{B}_{\{\beta\}}) \in C$, the obligations of C are:
     a. $\mathbf{B}_{\{\beta\}} \neq \emptyset_{\{\beta\}}$.
     b. $\mathbf{c}_{\beta \rightarrow \mathbf{a}} \downarrow (\mathbf{B}_{\{\beta\}} \rightarrow U_{\{\mathbf{a}\}})$.
     c. $\mathrm{TOTAL\text{-}ON}(\mathbf{c}_{\beta \rightarrow \mathbf{a}}, \mathbf{B}_{\{\beta\}}, U_{\{\mathbf{a}\}})$.
     d. $\mathrm{INJ}(\mathbf{c}_{\beta \rightarrow \mathbf{a}})$.
  2. For each $\mathbf{c}_\alpha \in C$ where $\alpha = (\mathbf{a} \times \cdots \times \mathbf{a}) \rightarrow \mathbf{a}$, the obligations of C are:
     a. $\mathrm{TOTAL}(\mathbf{c}_\alpha)$.
     b. $\mathrm{INJ}(\mathbf{c}_\alpha)$.

Inductive Types [4/5]

3. For each $(\mathbf{c}_\alpha, \mathbf{B}_{\{\beta\}}) \in C$ where $\alpha = (\beta \times \mathbf{a} \times \cdots \times \mathbf{a}) \rightarrow \mathbf{a}$, the obligations of C are:
   a. $\mathbf{B}_{\{\beta\}} \neq \emptyset_{\{\beta\}}$.
   b. $\mathbf{c}_\alpha \downarrow ((\mathbf{B}_{\{\beta\}} \times U_{\{\mathbf{a}\}} \times \cdots \times U_{\{\mathbf{a}\}}) \rightarrow U_{\{\mathbf{a}\}})$.
   c. $\mathsf{TOTAL\text{-}ON}(\mathbf{c}_\alpha, \mathbf{B}_{\{\beta\}} \times U_{\{\mathbf{a}\}} \times \cdots \times U_{\{\mathbf{a}\}}, U_{\{\mathbf{a}\}})$.
   d. $\mathsf{INJ}(\mathbf{c}_\alpha)$.
4. For all distinct nullary $c, c' \in C$ and distinct nonnullary $d, d' \in C$ where $\overline{d}$ has type $\alpha \rightarrow \mathbf{a}$ and $\overline{d'}$ has type $\alpha' \rightarrow \mathbf{a}$, the obligations of C are:
   a. $c \neq c'$,
   b. $c \not\in (\mathsf{ran}_{(\alpha \rightarrow \mathbf{a}) \rightarrow \{\mathbf{a}\}}\, \overline{d})$.
   c. $(\mathsf{ran}_{(\alpha \rightarrow \mathbf{a}) \rightarrow \{\mathbf{a}\}}\, \overline{d}) \cap (\mathsf{ran}_{(\alpha' \rightarrow \mathbf{a}) \rightarrow \{\mathbf{a}\}}\, \overline{d'}) = \emptyset_{\{\mathbf{a}\}}$.
5. The sentence
   $$\forall\, p : \mathbf{a} \rightarrow o \,.\, (\mathbf{X}_o^{c_1} \wedge \cdots \wedge \mathbf{X}_o^{c_n}) \Rightarrow \forall\, x : \mathbf{a} \,.\, p\, x,$$
   where $C = \{c_1, \ldots, c_n\}$, is an obligation of C. This sentence is called the structural induction principle for $\mathbf{a}$ with respect to $c_1, \ldots, c_n$.

Inductive Types [5/5]

- Let $T = (L, \Gamma)$ be a theory where $L = (\mathcal{B}, \mathcal{C})$, $\mathbf{a}$ is a base type, and C is a finite set of constructors for $\mathbf{a}$.
- $(\mathbf{a}, C)$ belongs to T if $\mathbf{a} \in \mathcal{B}$, $\{\overline{c} \mid c \in C\} \subseteq \mathcal{C}$, and $\mathbf{B}_\beta \in \mathcal{E}(L)$ for all $(\mathbf{c}_\alpha, \mathbf{B}_\beta) \in C$.
- $\mathbf{a}$ is freely constructed by C in T if:
  1. $(\mathbf{a}, C)$ belongs to T.
  2. $T \models \mathbf{A}_o$ for all $\mathbf{A}_o \in \mathrm{obl}(C)$.
- An inductive type in T is a pair $(\mathbf{a}, C)$ where $\mathbf{a}$ is a base type and C is a finite set of constructors for $\mathbf{a}$ such that $\mathbf{a}$ is freely constructed by C in T.
- Theorem (Inductive Types Theorem). Let T be a theory and M be a standard model of T. If $(\mathbf{a}, C)$ is an inductive type in T, then $(D^M_{\mathbf{a}}, G)$ is an inductive set where

$$\begin{aligned} G = {} & \{I^M(c) \mid c \in C \text{ is pure}\} \cup \\ & \{\mathrm{gts}(I^M(\overline{c})) \mid c \in C \text{ is nonpure}\}. \end{aligned}$$

Example: Theory of Boolean Values

Theory Definition (Boolean Values)

Name: BOOLE.
Base types: B.
Constants: f_B, t_B.
Axioms:

1. $\mathsf{f}_B \neq \mathsf{t}_B$ (Boolean values are distinct).
2. $\forall\, b : B \,.\, b = \mathsf{f}_B \vee b = \mathsf{t}_B$. (all truth values are Boolean).

Example: Theory of Natural Numbers

Theory Definition (Peano Arithmetic)

Name: PA.
Base types: N.
Constants: $0_N,\ S_{N \to N}$.
Axioms:

1. $\mathsf{TOTAL}(S)$ (S is total).
2. $\forall\, x : N \,.\, 0 \neq S\,x$ (0 has no predecessor).
3. $\forall\, x, y : N \,.\, S\,x = S\,y \Rightarrow x = y$. ($S$ is injective).
4. $\forall\, p : N \to o \,.$
   $(p\,0 \wedge \forall\, x : N \,.\, (p\,x \Rightarrow p\,(S\,x))) \Rightarrow \forall\, x : N \,.\, p\,x$
   (induction principle).

3. Inductive Type Theory Extensions

Inductive Type Theory Extensions [1/2]

- Let $L_i = (\mathcal{B}_i, \mathcal{C}_i)$ be a language and $T_i = (L_i, \Gamma_i)$ be a theory for $i \in \{1, 2\}$.
- Let $\mathbf{a}$ be a base type and C be a finite set of constructors for $\mathbf{a}$.
- T_2 is an inductive type theory extension of T_1 by $(\mathbf{a}, C)$ if:
  1. $\mathcal{B}_2 = \mathcal{B}_1 \cup \{\mathbf{a}\}$ with $\mathbf{a} \not\in \mathcal{B}_1$.
  2. $\mathcal{C}_2 = \mathcal{C}_1 \cup \{\overline{c} \mid c \in C\}$ with $\{\overline{c} \mid c \in C\} \cap \mathcal{C}_1 = \emptyset$.
  3. $\mathbf{B}_\beta \in \mathcal{E}(L_1)$ for all $(\mathbf{c}_\alpha, \mathbf{B}_\beta) \in C$.
  4. $\Gamma_2 = \Gamma_1 \cup \mathrm{obl}(C)$.
- Proposition. Let T_2 be an inductive type theory extension of T_1 by $(\mathbf{a}, C)$. Then $(\mathbf{a}, C)$ is an inductive type in T_2.

Inductive Type Theory Extensions [2/2]

- Theorem. Let T_2 be an inductive type theory extension of T_1 by $(\mathbf{a}, C)$. Then the following statements hold:
  1. If each $c \in C$ is pure, then $T_1 \trianglelefteq_{\mathrm{m}} T_2$.
  2. If $T_1 \models \mathbf{B}_{\{\beta\}} \neq \emptyset_{\{\beta\}}$ for all $(\mathbf{c}_\alpha, \mathbf{B}_{\{\beta\}}) \in C$, then $T_1 \trianglelefteq_{\mathrm{m}} T_2$.
- Remark. The proof of this theorem shows that, if $(\mathbf{a}, C)$ is an inductive type, then $\mathbf{a}$ can be interpreted, roughly speaking, as the set of expressions constructed from the members of C.

Inductive Type Theory Extension Module

Inductive Type Theory Extension X.Y

Name: Name.
Extends Name-of-subtheory.
New base type: $\mathbf{a}$.
Constructors:

1. c_1 with the form $\mathbf{c}_\alpha$ or $(\mathbf{c}_\alpha, \mathbf{B}_{\{\beta\}})$.

   $\vdots$

p. c_p with the form $\mathbf{c}_\alpha$ or $(\mathbf{c}_\alpha, \mathbf{B}_{\{\beta\}})$.

Example: Ind. Type Thy. Ext. of Boolean Values

Inductive Type Theory Extension (Boolean Values)

Name: BOOLE-ALT.
Extends: MIN.
New Base Type: B.
Constructors:

1. f_B.
2. t_B.

Example: Ind. Type Thy. Ext. of Nat. Numbers

Inductive Type Theory Extension (Natural Numbers)

Name: PA-ALT.
Extends: MIN.
New Base Type: N.
Constructors:

1. 0_N.
2. $S_{N \to N}$.

The End.