

CAS 760
Simple Type Theory
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8 Sequences

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Outline

1. Background.
2. Systems of natural numbers.
3. Notational definitions for sequences.

1. Background

Representation of Domain Constructors in Alonzo

1. Function space: $\alpha \rightarrow \beta$.
2. Power set: $\{\alpha\} = \alpha \rightarrow o$.
3. Cartesian product: $\alpha \times \beta$.
4. Kleene star: $[\alpha]$. What is this?

Benefits of having Undefined Expressions

The traditional approach to undefinedness facilitates the formalization of the following values:

1. Partial functions.
2. Definite descriptions.
3. Indefinite descriptions.
4. Function subtypes.
5. Finite sequences.
6. Recursively defined functions.

Basic Definitions

- An **infinite sequence** of values from a set A can be represented as a total function $s : \mathbb{N} \rightarrow A$.
- A **finite sequence** can be represented as a partial function $s : \mathbb{N} \rightarrow A$ such that, for some $n \in \mathbb{N}$, $s(m)$ is defined iff $m < n$.

2. Systems of Natural Numbers

Systems of Natural Numbers [1/2]

Let $T = (L, \Gamma)$ be a theory. A set

$$\mathcal{E}_{\text{nat}} = \{\mathbf{C}_{\{\alpha\}}^N, \mathbf{C}_\alpha^0, \mathbf{C}_{\alpha \rightarrow \alpha}^S, \mathbf{C}_{\alpha \rightarrow \alpha}^P, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^*, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow o}^\leq\}$$

of closed expressions is a **system of natural numbers** in T if $\mathcal{E}_{\text{nat}} \subseteq \mathcal{E}(L)$ and the following sentences are theorems of T :

1. $\mathbf{C}_\alpha^0 \in \mathbf{C}_{\{\alpha\}}^N$.
2. $\mathbf{C}_{\alpha \rightarrow \alpha}^S \in \mathbf{C}_{\{\alpha\}}^N \rightarrow \mathbf{C}_{\{\alpha\}}^N$.
3. TOTAL-ON($\mathbf{C}_{\alpha \rightarrow \alpha}^S, \mathbf{C}_{\{\alpha\}}^N, \mathbf{C}_{\{\alpha\}}^N$).
4. $\forall x : \mathbf{C}_{\{\alpha\}}^N \cdot \mathbf{C}_\alpha^0 \neq \mathbf{C}_{\alpha \rightarrow \alpha}^S x$.
5. $\forall x, y : \mathbf{C}_{\{\alpha\}}^N \cdot \mathbf{C}_{\alpha \rightarrow \alpha}^S x = \mathbf{C}_{\alpha \rightarrow \alpha}^S y \Rightarrow x = y$.
6. $\forall p : \mathbf{C}_{\{\alpha\}}^N \rightarrow o$.
$$(p \mathbf{C}_\alpha^0 \wedge \forall x : \mathbf{C}_{\{\alpha\}}^N \cdot (p x \Rightarrow p(\mathbf{C}_{\alpha \rightarrow \alpha}^S x))) \Rightarrow \forall x : \mathbf{C}_{\{\alpha\}}^N \cdot p x.$$
7. $\mathbf{C}_{\alpha \rightarrow \alpha}^P = \lambda x : \mathbf{C}_{\{\alpha\}}^N \cdot \lambda y : \mathbf{C}_{\{\alpha\}}^N \cdot \mathbf{C}_{\alpha \rightarrow \alpha}^S y = x$.
8. $\mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+ = \lambda f : \mathbf{C}_{\{\alpha\}}^N \rightarrow \mathbf{C}_{\{\alpha\}}^N \rightarrow \mathbf{C}_{\{\alpha\}}^N \cdot \forall x, y : \mathbf{C}_{\{\alpha\}}^N \cdot$
$$f x 0 = x \wedge f x (\mathbf{C}_{\alpha \rightarrow \alpha}^S y) = \mathbf{C}_{\alpha \rightarrow \alpha}^S (f x y)$$
.
9. $\mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^* = \lambda f : \mathbf{C}_{\{\alpha\}}^N \rightarrow \mathbf{C}_{\{\alpha\}}^N \rightarrow \mathbf{C}_{\{\alpha\}}^N \cdot \forall x, y : \mathbf{C}_{\{\alpha\}}^N \cdot$
$$f x 0 = 0 \wedge f x (\mathbf{C}_{\alpha \rightarrow \alpha}^S y) = \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+ (f x y) x$$
.
10. $\mathbf{C}_{\alpha \rightarrow \alpha \rightarrow o}^\leq = \lambda x : \mathbf{C}_{\{\alpha\}}^N \cdot \lambda y : \mathbf{C}_{\{\alpha\}}^N \cdot \exists z : \mathbf{C}_{\{\alpha\}}^N \cdot \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+ x z = y$.

Systems of Natural Numbers [2/2]

- The set
$$\{N_{\{N\}}, 0_N, S_{N \rightarrow N}, P_{N \rightarrow N}, +_{N \rightarrow N \rightarrow N}, *_{N \rightarrow N \rightarrow N}, \leq_{N \rightarrow N \rightarrow o}\}$$
is a system of natural numbers in PA' .
- Thus:
 - ▶ Any extension of PA' contains a system of natural numbers.
 - ▶ Any theory T that contains a system of natural numbers contains a copy of PA' .
- Let T_{nat} be a theory in which
$$\{\mathbf{C}_{\{\alpha\}}^N, \mathbf{C}_\alpha^0, \mathbf{C}_{\alpha \rightarrow \alpha}^S, \mathbf{C}_{\alpha \rightarrow \alpha}^P, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^*, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow o}^{\leq}\}$$
is a system of natural numbers.
- We next introduce notation for sequences in T_{nat} .

3. Notation Definitions for Sequences

Notational Definitions for Sequences [1/2]

| | | |
|---|------------|---|
| sequences $_{\{\alpha \rightarrow \beta\}}$ | stands for | $\mathbf{C}_{\{\alpha\}}^N \rightarrow \beta$. |
| $\langle\langle \beta \rangle\rangle$ | stands for | sequences $_{\{\alpha \rightarrow \beta\}}$. |
| streams $_{\{\alpha \rightarrow \beta\}}$ | stands for | $\{s : \langle\langle \beta \rangle\rangle \mid \text{TOTAL}(s)\}$. |
| $\langle \beta \rangle$ | stands for | streams $_{\{\alpha \rightarrow \beta\}}$. |
| lists $_{\{\alpha \rightarrow \beta\}}$ | stands for | $\{s : \langle\langle \beta \rangle\rangle \mid \exists n : \mathbf{C}_{\{\alpha\}}^N . \forall m : \mathbf{C}_{\{\alpha\}}^N .$ $(s m) \downarrow \Leftrightarrow \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow o}^{\leq} m (\mathbf{C}_{\alpha \rightarrow \alpha}^P n)\}$. |
| $[\beta]$ | stands for | lists $_{\{\alpha \rightarrow \beta\}}$. |
| cons $_{\beta \rightarrow (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)}$ | stands for | $\lambda x : \beta . \lambda s : \langle\langle \beta \rangle\rangle . \lambda n : \mathbf{C}_{\{\alpha\}}^N .$ $n = \mathbf{C}_{\alpha}^0 \mapsto x \mid s (\mathbf{C}_{\alpha \rightarrow \alpha}^P n)$. |
| $(\mathbf{A}_\beta :: \mathbf{B}_{\alpha \rightarrow \beta})$ | stands for | cons $_{\beta \rightarrow (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)} \mathbf{A}_\beta \mathbf{B}_{\alpha \rightarrow \beta}$. |
| hd $_{(\alpha \rightarrow \beta) \rightarrow \beta}$ | stands for | $\lambda s : \langle\langle \beta \rangle\rangle . \text{I } x : \beta . \exists t : \langle\langle \beta \rangle\rangle .$ $s = (x :: t)$. |
| tl $_{(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)}$ | stands for | $\lambda s : \langle\langle \beta \rangle\rangle . \text{I } t : \langle\langle \beta \rangle\rangle . \exists x : \beta .$ $s = (x :: t)$. |

Notational Definitions for Sequences [2/2]

| | | |
|---|------------|---|
| $\text{nil}_{\alpha \rightarrow \beta}$ | stands for | $\Delta_{\alpha \rightarrow \beta}.$ |
| $[]_{\alpha \rightarrow \beta}$ | stands for | $\text{nil}_{\alpha \rightarrow \beta}.$ |
| $[\mathbf{A}_\beta]$ | stands for | $(\mathbf{A}_\beta :: []_{\alpha \rightarrow \beta}).$ |
| $[\mathbf{A}_\beta^1, \dots, \mathbf{A}_\beta^n]$ | stands for | $(\mathbf{A}_\beta^1 :: [\mathbf{A}_\beta^2, \dots, \mathbf{A}_\beta^n])$ where $n \geq 2.$ |
| $\text{len}_{(\alpha \rightarrow \beta) \rightarrow \alpha}$ | stands for | $\text{If } f : [\beta] \rightarrow \mathbf{C}_{\{\alpha\}}^N . f []_{\alpha \rightarrow \beta} = \mathbf{C}_\alpha^0 \wedge \forall x : \beta, s : [\beta] . f(x :: s) = \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+ (f s) (\mathbf{C}_{\alpha \rightarrow \alpha}^S \mathbf{C}_\alpha^0).$ |
| $ \mathbf{A}_{\alpha \rightarrow \beta} $ | stands for | $\text{len}_{(\alpha \rightarrow \beta) \rightarrow \alpha} \mathbf{A}_{\alpha \rightarrow \beta}.$ |
| $\text{nlists}_{\alpha \rightarrow \{\alpha \rightarrow \beta\}}$ | stands for | $\lambda n : \mathbf{C}_{\{\alpha\}}^N . \{s : [\beta] \mid s = n\}.$ |
| $[\beta]^{\mathbf{N}_\alpha}$ | stands for | $\text{nlists}_{\alpha \rightarrow \{\alpha \rightarrow \beta\}} \mathbf{N}_\alpha.$ |

Facts about Sequences

- **Proposition.** The following sentences are valid in T_{nat} for every type β of T_{nat} :

1. $\forall x : \beta, s : \langle \beta \rangle . (x :: s) \in \langle \beta \rangle$.
2. $\forall x : \beta, s : [\beta] . (x :: s) \in [\beta]$.
3. $\forall s : \langle \beta \rangle . (\text{tl}_{(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)} s) \in \langle \beta \rangle$.
4. $\forall s : [\beta] . s \neq []_{\alpha \rightarrow \beta} \Rightarrow (\text{tl}_{(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)} s) \in [\beta]$.
5. $(\text{hd}_{(\alpha \rightarrow \beta) \rightarrow \beta} []_{\alpha \rightarrow \beta}) \uparrow$.
6. $(\text{tl}_{(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)} []_{\alpha \rightarrow \beta}) \uparrow$.
7. $\forall s : \langle \beta \rangle . (\text{len}_{(\alpha \rightarrow \beta) \rightarrow \alpha} s) \uparrow$.
8. $\forall s : [\beta] . (\text{len}_{(\alpha \rightarrow \beta) \rightarrow \alpha} s) \downarrow$.

- If $[\beta]$ denotes D_β^* , the set of lists over D_β , then $[\beta]^n$ denotes D_β^n , the set of lists over D_β of length n .

- ▶ $D_\beta^n \subseteq D_\beta^* \subseteq \text{"N" } \rightarrow D_\beta \subseteq D_\alpha \rightarrow D_\beta$.
- ▶ $[\beta]^n$ is a dependent quasitype.
- ▶ $[\beta]^n$ is a subquasitype of $[\beta]$, $\langle \beta \rangle$, and $\langle \langle \beta \rangle \rangle$.
- ▶ $[\beta]^n$ is a subtype of $\alpha \rightarrow \beta$.

The End.