CAS 760
Simple Type Theory
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6 Theories

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Outline

1. Axiomatic theories.
2. Theory extensions.
3. Conservative theory extensions.
4. Categorical theories.
5. Complete theories.
6. Fundamental Form of a Mathematical Problem.

1. Theories

Theories [1/3]

- An axiomatic theory (theory for short) of Alonzo is a pair $T = (L, \Gamma)$ where:
  1. L is a language of Alonzo.
  2. Γ is a set of sentences of L called the axioms of T.

  The axioms serve as the assumptions of T.
- A general model M of L is a model of T, written $M \vDash T$, if $M \vDash \Gamma$.
- A formula $\mathbf{A}_o$ of L is a valid in T or a theorem of T, written $T \vDash \mathbf{A}_o$, if $\Gamma \vDash \mathbf{A}_o$.
- T is satisfiable if Γ is satisfiable.
- The minimum theory is the theory $T_{\text{min}} = (L_{\text{min}}, \emptyset)$.

Theories [2/3]

- The closure of a theory T, written $\overline{T}$, is the set of theorems of T.
- Two theories T_1 and T_2 having the same language are equivalent, written $T_1 \equiv T_2$, if $\overline{T_1} = \overline{T_2}$.
- A theory T_1 is included in a theory T_2, written $T_1 \preceq T_2$, if $\overline{T_1} \subseteq \overline{T_2}$.
- An axiomatization of T is any theory $T' = (L, \Gamma')$ that is equivalent to T.
  - T is finitely axiomatizable if there is an axiomatization $T' = (L, \Gamma')$ of T such that Γ' is finite.
  - T is recursively axiomatizable if there is an axiomatization $T' = (L, \Gamma')$ of T such that Γ' is decidable.

Theories [3/3]

- Two flavors of theories:
  1. Specifies a single structure.
  2. Specifies a collection of structures.
- Examples of computing-based theories:
  1. Theory of labeled binary trees.
     - Labeled binary trees is an inductive set.
     - Modeled on Peano arithmetic.
  2. Theory of the IP protocol.
     - Internet as a bipartite graph.
     - IP addresses.
     - IP datagrams.
     - Routing mechanism.

Mathematical Knowledge Modules

- Mathematical knowledge module (modules for short) define the following Alonzo objects:
  - Theories and theory extensions.
  - Developments and development extensions.
  - Theory translations and theory translation extensions.
  - Development translations and development translation extensions.
  - Definition, theorem, and group transportations.
- Modules are used to construct libraries of mathematical knowledge.

Theory Definition Module

Theory Definition X.Y

Name: Name.

Base types: $\mathbf{a}_1, \ldots, \mathbf{a}_m$.

Constants: $\mathbf{c}^1_{\alpha_1}, \ldots, \mathbf{c}^n_{\alpha_n}$.

Axioms:

1. $\mathbf{A}^1_o$ (description of $\mathbf{A}^1_o$).

$\vdots$

p. $\mathbf{A}^p_o$ (description of $\mathbf{A}^p_o$).

Theory of Monoids

Theory Definition (Monoids)

Name: MON.

Base types: S.

Constants: $\cdot_{S \to S \to S}, e_S$.

Axioms:

1. $\forall x, y, z : S \,.\, x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity).
2. $\forall x : S \,.\, e \cdot x = x$ (left identity).
3. $\forall x : S \,.\, x \cdot e = x$ (right identity).

Theory of Weak Partial Orders

Theory Definition (Weak Partial Orders)

Name: WPO.
Base types: S.
Constants: $\leq_{S \rightarrow S \rightarrow o}$.
Axioms:

1. $\forall\, x : S \,.\, x \leq x$ (reflexivity).
2. $\forall\, x, y : S \,.\, (x \leq y \wedge y \leq x) \Rightarrow x = y$ (antisymmetry).
3. $\forall\, x, y, z : S \,.\, (x \leq y \wedge y \leq z) \Rightarrow x \leq z$ (transitivity).

Peano Arithmetic [1/2]

Theory Definition (Peano Arithmetic)

Name: PA.
Base types: N.
Constants: $0_N,\ S_{N \rightarrow N}$.
Axioms:

1. $\mathsf{TOTAL}(S)$ (S is total).
2. $\forall\, x : N \,.\, 0 \neq S\, x$ (0 has no predecessor).
3. $\forall\, x, y : N \,.\, S\, x = S\, y \Rightarrow x = y$. ($S$ is injective).
4. $\forall\, p : N \rightarrow o \,.$
   $(p\, 0 \wedge \forall\, x : N \,.\, (p\, x \Rightarrow p\,(S\, x))) \Rightarrow \forall\, x : N \,.\, p\, x$
   (induction principle).

Peano Arithmetic [2/2]

- Proposition. There is a standard model of PA that defines the structure $(\mathbb{N}, 0, S)$ where $\mathbb{N}$ is the set of natural numbers, $0 \in \mathbb{N}$ is the natural number 0, and $S : \mathbb{N} \to \mathbb{N}$ is the successor function on $\mathbb{N}$.
- Let M be a model of PA. A member of D_N^M is a finite element of M if it is a member of

$$\{V_\varphi^M(S^m\,0) \mid m \in \mathbb{N} \text{ and } \varphi \in \mathsf{assign}(M)\},$$

  where $S^m\,0$ is the expression formed by applying $S_{N \to N}$ to 0_N $m \geq 0$ times.
- Theorem. A standard model of PA has no infinite elements.

Robinson Arithmetic

Theory Definition (Robinson Arithmetic)

Name: RA.

Base types: N.

Constants: 0_N, $S_{N \rightarrow N}$, $+_{N \rightarrow N \rightarrow N}$, $*_{N \rightarrow N \rightarrow N}$.

Axioms:

1. $\mathrm{TOTAL}(S)$ (S is total).
2. $\mathrm{TOTAL2}(+)$ ($+$ is total).
3. $\mathrm{TOTAL2}(*)$ ($*$ is total).
4. $\forall x : N \,.\, 0 \neq S\,x$ (0 has no predecessor).
5. $\forall x, y : N \,.\, S\,x = S\,y \Rightarrow x = y$. ($S$ is injective).
6. $\forall x : N \,.\, (x = 0 \vee (\exists y : N \,.\, S\,y = x))$ (every element is 0 or a successor).
7. $\forall x : N \,.\, x + 0 = x$ (definition of addition).
8. $\forall x, y : N \,.\, x + S\,y = S\,(x + y)$ (definition of addition).
9. $\forall x : N \,.\, x * 0 = 0$ (definition of multiplication).
10. $\forall x, y : N \,.\, x * S\,y = (x * y) + x$ (definition of multiplication).

2. Theory Extensions

Theory Extensions

- Let $L_i = (\mathcal{B}_i, \mathcal{C}_i)$ be a language and $T_i = (L_i, \Gamma_i)$ be a theory for $i \in \{1, 2\}$.
- T_2 is an extension of T_1 (or T_1 is a subtheory of T_2), written $T_1 \le T_2$, if $L_1 \le L_2$ and $\Gamma_1 \subseteq \Gamma_2$.
- Proposition. $T_{\mathrm{min}} \le T$ for every theory T.
- Proposition. $T_1 \le T_2$ implies $T_1 \preceq T_2$.
- Proposition. Let $T_1 \le T_2$. T_2 is satisfiable implies T_1 is satisfiable.

Theory Extension Module

Theory Extension X.Y

Name: Name.

Extends Name-of-subtheory.

New base types: $\mathbf{a}_1, \ldots, \mathbf{a}_m$.

New constants: $\mathbf{c}^1_{\alpha_1}, \ldots, \mathbf{c}^n_{\alpha_n}$.

New axioms:

p. $\mathbf{A}^p_o$ (description of $\mathbf{A}^p_o$).

$\vdots$

q. $\mathbf{A}^q_o$ (description of $\mathbf{A}^q_o$).

Theory of Groups

Theory Extension (Groups)

Name: GRP.
Extends: MON.
New base types: (none).
New constants: ${\cdot^{-1}}_{S \rightarrow S}$.
New axioms:

4. $\forall x : S \,.\, x^{-1} \cdot x = e$ (left inverse).
5. $\forall x : S \,.\, x \cdot x^{-1} = e$ (right inverse).

Theory of Weak Total Orders

Theory Extension (Weak Total Orders)

Name: WTO.
Extends: WPO.
New base types: (none).
New constants: (none).
New axioms:

4. $\forall x, y : S \,.\, x \leq y \vee y \leq x$ (totality).

Dense Weak Total Orders without Endpoints

Theory Extension (Dense WTOs without Endpoints)

Name: DWTOWE.
Extends: WTO.
New base types: (none).
New constants: (none).
New axioms:

5. $\exists\, x, y : S \,.\, x \neq y$ (more than one element).
6. $\forall\, x, y : S \,.\, x < y \Rightarrow (\exists\, z : S \,.\, x < z < y)$ (density).
7. $\forall\, x : S \,.\, \exists\, y : S \,.\, y < x$ (no minimum).
8. $\forall\, x : S \,.\, \exists\, y : S \,.\, x < y$ (no maximum).

Semantics and Theory Extensions

- Let $T_1 \leq T_2$.
- Then the theorems of T_1 are theorems of T_2.
- T_2 can compromise the semantics of T_1 in two ways:
  1. Some nontheorems of T_1 become theorems of T_2.
  2. Some models of T_1 cannot be expanded to models of T_2.
- Extreme case: T_1 is satisfiable, but T_2 is not.
- T_2 will not compromise the semantics of T_1 if it is a conservative extension of T_1.

3. Conservative Theory Extensions

Basic Definitions and Results

- Let $T_1 \leq T_2$.
- T_2 is a conservative extension of T_1, written $T_1 \trianglelefteq T_2$, if, for all formulas $\mathbf{A}_o$ of L_1, $T_2 \vDash \mathbf{A}_o$ implies $T_1 \vDash \mathbf{A}_o$.
- T_2 is a model conservative extension of T_1, written $T_1 \trianglelefteq_{\mathrm{m}} T_2$, if every model of T_1 expands to a model of T_2.
- Proposition. Let $T_1 \trianglelefteq T_2$. Then T_1 is satisfiable iff T_2 is satisfiable.
- Proposition. $T_1 \trianglelefteq_{\mathrm{m}} T_2$ implies $T_1 \trianglelefteq T_2$.
- We will define an extension PA^+ of PA that is conservative, but not model conservative.

Definitional Extensions

- T_2 is a simple definitional extension of T_1, written $T_1 \trianglelefteq_{\mathrm{sd}} T_2$, if:
  1. $\mathbf{c}_\alpha \not\in \mathcal{C}_1$.
  2. $L_2 = (\mathcal{B}_1, \mathcal{C}_1 \cup \{\mathbf{c}_\alpha\})$.
  3. $\mathbf{A}_\alpha$ is a closed expression of L_1.
  4. $T_1 \models \mathbf{A}_\alpha\!\downarrow$.
  5. $\Gamma_2 = \Gamma_1 \cup \{\mathbf{c}_\alpha = \mathbf{A}_\alpha\}$.
- T_2 is a definitional extension of T_1, written $T_1 \trianglelefteq_{\mathrm{d}} T_2$, if there is a sequence of theories $U_1, \ldots, U_n$ with $n \geq 2$ such that $T_1 = U_1 \trianglelefteq_{\mathrm{sd}} \cdots \trianglelefteq_{\mathrm{sd}} U_n = T_2$.

Example: Simple Definitional Extension

Theory Extension (Peano Arithmetic with 1)

Name: PA-with-1.
Extends: PA.
New base types: (none).
New constants: 1_N.
New axioms:

5. $1 = S\,0$.

Specificational Extensions

- T_2 is a simple specificational extension of T_1, written $T_1 \trianglelefteq_{\mathsf{ss}} T_2$, if:
  1. $\mathbf{c}_\alpha \not\in \mathcal{C}_1$.
  2. $L_2 = (\mathcal{B}_1, \mathcal{C}_1 \cup \{\mathbf{c}_\alpha\})$.
  3. $\exists\, \mathbf{x} : \alpha \,.\, \mathbf{B}_o$ is a sentence of L_1.
  4. $T_1 \models \exists\, \mathbf{x} : \alpha \,.\, \mathbf{B}_o$.
  5. $\Gamma_2 = \Gamma_1 \cup \{\mathbf{B}_o[(\mathbf{x} : \alpha) \mapsto \mathbf{c}_\alpha]\}$.
- T_2 is a specificational extension of T_1, written $T_1 \trianglelefteq_{\mathsf{s}} T_2$, if there is a sequence of theories $U_1, \ldots, U_n$ with $n \geq 2$ such that $T_1 = U_1 \trianglelefteq_{\mathsf{ss}} \cdots \trianglelefteq_{\mathsf{ss}} U_n = T_2$.

Example: Simple Specificational Extension

Theory Extension (Peano Arithmetic with a Successor)

Name: PA-with-a-successor.
Extends: PA.
New base types: (none).
New constants: $\textsf{a-successor}_N$.
New axioms:

5. $0 \neq \textsf{a-successor}$.

Definition and Specification Extensions

Lemma.

1. $T_1 \trianglelefteq_{\mathrm{d}} T_2$ implies that, for all formulas $\mathbf{A}_o$ of L_2, there is formula $\mathbf{B}_o$ of L_1 such that $T_2 \vDash \mathbf{A}_o \Leftrightarrow \mathbf{B}_o$.
2. $T_1 \trianglelefteq_{\mathrm{d}} T_2$ implies $T_1 \trianglelefteq_{\mathrm{s}} T_2$.
3. $T_1 \trianglelefteq_{\mathrm{d}} T_2$ implies that every model of T_1 can be expanded to a unique model of T_2.
4. $T_1 \trianglelefteq_{\mathrm{s}} T_2$ implies that every model of T_1 can be expanded to a model of T_2.
5. $T_1 \trianglelefteq_{\mathrm{d}} T_2$ implies $T_1 \trianglelefteq_{\mathrm{m}} T_2$.
6. $T_1 \trianglelefteq_{\mathrm{s}} T_2$ implies $T_1 \trianglelefteq_{\mathrm{m}} T_2$.

Recursive Definitions

There are three ways of defining a function (e.g., $+_{N \to N \to N}$) recursively in a theory (e.g., PA):

1. Theory extension.
   - Add $+_{N \to N \to N}$ to the constants of PA.
   - Add the following two sentences to the axioms of PA:

     $\forall x : N \,.\, x + 0 = x.$

     $\forall x, y : N \,.\, x + S\,y = S\,(x + y).$
2. Simple specificational extension.
   - Add $+_{N \to N \to N}$ to the constants of PA.
   - Add the following sentence to the axioms of PA:

     $\forall x, y : N \,.\, x + 0 = x \wedge x + S\,y = S\,(x + y).$
3. Simple definitional extension.
   - Add $+_{N \to N \to N}$ to the constants of PA.
   - Add the following sentence to the axioms of PA:

     $$+ = \mathrm{I}\, f : N \to N \to N \,. \\ \forall x, y : N \,.\, f\,x\,0 = x \wedge f\,x\,(S\,y) = S\,(f\,x\,y).$$

Recursive Definition Schemes

Schemes for verifying that a recursive definition of a function is well-defined:

1. Use the rules of primitive recursion to define the function.
   - Restricted to functions on natural numbers.
   - Guarantees that the function is total and computable.
2. Use the rules of general recursion to define a function.
   - Restricted to functions on natural numbers.
   - Guarantees that the function is computable.
3. Use well-founded recursion to defined the function.
   - Most general form of recursion.
   - Special cases: structural recursion and ordinal recursion.
4. Use a monotone functional to define the function.
   - The monotone functional operates on partial functions.
   - Every monotone functional has a least fixed point.
   - The function is defined as this least fixed point.

Peano Arithmetic with $+$ and $*$ [1/2]

Theory Extension (Peano Arithmetic with $+$ and $*$)

Name: PA′.

Extends: PA.

New base types: (none).

New constants: $P_{N \rightarrow N}$, $+_{N \rightarrow N \rightarrow N}$, $*_{N \rightarrow N \rightarrow N}$, $\leq_{N \rightarrow N \rightarrow o}$, $N_{\{N\}}$.

New axioms:

5. $P = \lambda\, x : N \,.\, \mathrm{I}\, y : N \,.\, S\, y = x$ (definition of predecessor function).
6. $+ = \mathrm{I}\, f : N \rightarrow N \rightarrow N \,.$
   $\forall\, x, y : N \,.\, f\, x\, 0 = x \wedge f\, x\, (S\, y) = S\, (f\, x\, y)$
   (definition of addition function).
7. $* = \mathrm{I}\, f : N \rightarrow N \rightarrow N \,.$
   $\forall\, x, y : N \,.\, f\, x\, 0 = 0 \wedge f\, x\, (S\, y) = (f\, x\, y) + x$
   (definition of multiplication function).
8. $\leq = \lambda\, x : N \,.\, \lambda\, y : N \,.\, \exists\, z : N \,.\, x + z = y$
   (definition of weak total order).
9. $N_{\{N\}} = U_{\{N\}}$ (definition of quasitype for the type N).

Peano Arithmetic with $+$ and $*$ [2/2]

- Proposition. $\mathsf{PA} \trianglelefteq_{\mathsf{d}} \mathsf{PA}'$.
- Proposition. $\mathsf{RA} \preceq \mathsf{PA}'$.

Peano Arithmetic with an Infinite Element

Theory Extension (Peano Arithmetic with ∞ Element)

Name: PA^{+}.
Extends: PA.
New base types: (none).
New constants: C_N.
New axioms:

1. $C \neq 0$.
2. $C \neq S\,0$.
3. $C \neq S\,(S\,0)$.

$\vdots$

Properties of PA^+

- Theorem.
  1. PA^+ is satisfiable.
  2. Every model of PA^+ has an infinite element.
  3. Every model of PA^+ is a nonstandard model.
  4. $\mathsf{PA} \trianglelefteq \mathsf{PA}^+$.
  5. $\mathsf{PA} \ntrianglelefteq_{\mathsf{m}} \mathsf{PA}^+$.
- Corollary. PA has a nonstandard model that has an infinite element.
- Corollary. There is a conservative extension that is not a model conservative extension.

4. Categorical Theories

Maximal Theories

- A theory T can be used to specify:
  1. A collection of structures (i.e., the structures defined by the models of T).
  2. A collection of sentences that are valid in the structures (i.e., the theorems of T).
- A categorical theory is a theory that is a maximal specification of the first kind.
- A complete theory is a theory that is a maximal specification of the second kind.

Basic Definitions and Results [1/2]

- A theory is categorical (in the general sense) if it has a unique model (up to isomorphism).[1]
- A theory is categorical in the standard sense if it has a unique standard model (up to isomorphism).
- Proposition. Let $T_1 \trianglelefteq_{\mathrm{d}} T_2$. Then:
  1. T_1 is categorical iff T_2 is categorical.
  2. T_1 is categorical in the standard sense iff T_2 is categorical in the standard sense.
- Theorem (Löwenheim-Skolem Theorem). Let T be a theory. If T has an infinite general model, then T has a general model of size and power κ for every cardinal $\kappa \geq \|L\|$.

[1] This concept was introduced by Oswald Veblen in 1904.

Basic Definitions and Results [2/2]

- Corollary. Every theory that has an infinite general model is noncategorical.
- Corollary. Let T be a theory. Then the following are equivalent:
  1. T is categorical.
  2. T has a unique finite general model (up to isomorphism) and no infinite general models.
  3. T is categorical in the standard sense and has no infinite general models.

Categoricity of PA

- Theorem. PA is categorical in the standard sense.
- Corollary. PA$'$ is categorical in the standard sense.
- Corollary. There is a theory categorical in the standard sense that uniquely specifies $(\mathbb{N}, 0, 1, +, *)$, the structure of natural number arithmetic.

Rational Numbers Order [1/2]

Theory Extension (Rational Numbers Order)

Name: RAT.

Extends: DWTOWE.

New base types: (none).

New constants: (none).

New axioms:

9. $\mathsf{COUNT}(U_{\{S\}})$. ($S$ is countable.)

Rational Numbers Order [2/2]

- Proposition. RAT is a specification in the standard sense of all countable dense weak total orders without endpoints.
- Theorem. RAT is categorical in the standard sense.
- The proof uses:

  Theorem (Cantor). Every two countable dwtowes are order isomorphic.

5. Complete Theories

Basic Definitions and Results

- Let $T = (L, \Gamma)$ be a theory.
- T is semantically complete if either $T \models \mathbf{A}_o$ or $T \models \neg\mathbf{A}_o$ holds for all sentences $\mathbf{A}_o$ of L.
- T is syntactically complete in $\mathfrak{A}$, if either $T \vdash_{\mathfrak{A}} \mathbf{A}_o$ or $T \vdash_{\mathfrak{A}} \neg\mathbf{A}_o$ holds for all sentences $\mathbf{A}_o$ of L.
- Theorem. T is semantically complete iff T is syntactically complete in $\mathfrak{A}$.
- Theorem. Every complete theory is satisfiable.
- Proposition. Let $T_1 \trianglelefteq_{\mathrm{d}} T_2$. Then T_1 is complete iff T_2 is complete.
- Examples of complete theories:
  - The complete theory of any general model.
  - Any categorical theory.

Gödel's First Incompleteness Thm. for Alonzo

- A theory T is sufficiently strong if $\mathrm{RA} \preceq T$.
- A theory T is essentially incomplete if T is incomplete and every satisfiable, recursively axiomatizable extension of T is incomplete.
- Theorem (Gödel's FIT for Alonzo). Let T be a theory of Alonzo that is (1) satisfiable, (2) recursively axiomatizable, and (3) sufficiently strong. Then T is essentially incomplete.
- Corollary. RA is essentially incomplete.
- Corollary. PA is essentially incomplete.
- Corollary. There is no sound proof system for Alonzo that is complete in the standard sense.

Recursively Axiomatizable Theories

- There are complete, satisfiable, recursively axiomatizable theories that have only finite models.
  - Any categorical theory is an example.
- Is there a complete, satisfiable, recursively axiomatization theory that has infinite general models?
- Theorem. Let T be a satisfiable, recursively axiomatizable theory. If T has only infinite general models, then T is essentially incomplete.
- Corollary. Let T be a satisfiable, recursively axiomatizable theory. Then T is complete iff it is categorical.
- So the answer is no!

Complete First-Order Theories

- There are several well-known first-order theories that can be shown to be complete in first-order logic using the method of quantifier elimination:
  - Presburger arithmetic.
  - Theory of real closed fields.
  - Theory of algebraically closed fields.
  - Theory of dense total orders without endpoints.
- The analogs of these theories expressed in Alonzo are usually incomplete.
- Theorem. The theory DWTOWE is incomplete.

6. Fundamental Form of a Mathematical Problem

Fundamental Form of a Mathematical Problem

- The fundamental form of a mathematical problem (FFMP) in Alonzo is

  $$T \models^{\mathrm{s}} \mathbf{A}_o$$

  where $T = (L, \Gamma)$ is a theory of Alonzo and $\mathbf{A}_o$ is a sentence of L.
- The problem is positively solved by proving that this statement holds and is negatively solved by proving that this statement does not hold.
- There are three principal approaches for solving a problem of this form:
  1. Model-theoretic approach.
  2. Proof-theoretic approach.
  3. Decision procedure approach.
- FFMP cannot generally be expressed in first-order logic.

The End.