CAS 760
Simple Type Theory
Winter 2026

3 Preliminary Concepts

William M. Farmer

Department of Computing and Software
McMaster University

December 29, 2025

Outline

1. What is mathematics?.
2. Mathematical values.
3. Binders.
4. Undefinedness.
5. Mathematical structures.

1. What is Mathematics?

What is Mathematics?

- Mathematics is a process for understanding the mathematical aspects of the world involving such things as time, space, measure, pattern, and logical consequence.
- The process is based on mathematical models consisting of objects, concepts, and facts.
  1. A model is created to describe a particular mathematical phenomenon exhibited in the world.
  2. The model is explored in various ways to discover new objects, concepts, and facts related to the model.
  3. The resulting enriched model then provides a deeper understanding of the mathematical phenomenon.

Tetrapod Model

- Mathematical models are explored by:
  1. Inference — e.g,., proving a conjecture.
  2. Computation — e.g., simplifying a mathematical expression.
  3. Concretization — e.g., collecting examples.
  4. Narration — e.g., presenting a linear development of a model in natural language.
  5. Organization — e.g., identifying common structure in a collection of models.
- These are the five aspects of the tetrapod model of mathematical knowledge.

2. Mathematical Values

Mathematical Values

- A mathematical value (value for short) is any concrete or abstract object used in mathematics.
- There are various kinds of values:
  - Sets.
  - Sequences.
    - Finite (tuples and lists).
    - Infinite.
  - Relations.
  - Functions.
    - Partial.
    - Total.
  - Boolean values.
  - Predicates.
    - Boolean operators.

Equivalence Relations

- A binary relation on a set is an equivalence relation if it is reflexive, symmetric, and transitive.
- Example. An identity relation is an equivalence relation.
- Given an equivalence relation R on a set S, the equivalence class of $a \in S$ is the set

  $$\{b \in S \mid a \; R \; b\}.$$

- Theorem.
  1. The equivalence classes of an equivalence relation on a set S form a partition of S.
  2. Given a partition of a set S, there is an equivalence relation on S whose equivalence classes are the members of the partition.

(Unary) Functions

- Definition 1: A function is a rule $f : I \to O$ that associates members of I (inputs) with members of O (outputs).
  - Every input is associated with at most one output.
  - Some inputs may not be associated with an output.
    Example: $f : \mathbb{Z} \to \mathbb{Q}$ where $x \mapsto 1/x$.
- Definition 2: A function is a relation $f \subseteq I \times O$ such that if $(x, y), (x, y') \in f$, then $y = y'$.
- Each function f has a domain $D \subseteq I$ and a range $R \subseteq O$.
  - f is total if $D = I$ and partial if $D \subset I$.
- A set or relation can be represented as a special kind of function (e.g., as a predicate, a characteristic function, or an indicator).

Lambda Notation

- Lambda notation is a precise, convenient way to describe functions.
- If B is an expression of type β,

$$\lambda\, x : \alpha \,.\, B$$

  denotes a function $f : \alpha \to \beta$ such that $f(a) = B[x \mapsto a]$.
- Example: Let $f = \lambda\, x : \mathbb{R} \,.\, x * x$.
  - $f(2) = (\lambda\, x : \mathbb{R} \,.\, x * x)(2) = 2 * 2$.
  - f denotes the squaring function.
- Lambda notation is used in many languages to express ideas about functions.
- Examples:
  - Lambda Calculus (a model of computability).
  - Simple Type Theory (a higher-order predicate logic).
  - Lisp (a functional programming language).

n-Ary Functions

- Definition 1: For $n \geq 0$, an n-ary function is a rule $f : I_1, \ldots, I_n \to O$ that associates members of $I_1, \ldots, I_n$ (inputs) with members of O (outputs).
  - Every sequence of inputs is associated with at most one output.
  - Some sequence of inputs may not be associated with an output.
- Definition 2: For $n \geq 0$, an n-ary function is a relation $f \subseteq I_1 \times \cdots \times I_n \times O$ such that if $(x_1, \ldots, x_n, y), (x_1, \ldots, x_n, y') \in f$, then $y = y'$.
- Each function f has a domain $D \subseteq I_1 \times \cdots \times I_n$ and a range $R \subseteq O$.

Representing n-Ary Functions as Unary Functions

For $n \geq 2$, there are two ways of representing a n-ary function as a unary function:

1. As a function on tuples: $f : I_1, \ldots, I_n \rightarrow O$ is represented as

$$f' : I_1 \times \cdots \times I_n \rightarrow O$$

where

$$f(x_1, \ldots, x_n) = f'((x_1, \ldots, x_n)).$$

2. As a Curryed function: $f : I_1, \ldots, I_n \rightarrow O$ is represented as

$$f'' : I_1 \rightarrow (I_2 \rightarrow (\cdots (I_n \rightarrow O) \cdots))$$

where

$$f(x_1, \ldots, x_n) = f''(x_1) \cdots (x_n).$$

Example: Binary Functions as Unary Functions

- Let $f : \mathbb{R}, \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(a, b) = a^2 + b^2$.
- $f' = \lambda\, p : \mathbb{R} \times \mathbb{R} \,.\, [\mathsf{fst}(p)]^2 + [\mathsf{snd}(p)]^2$.

  $$\begin{aligned} f'((a, b)) &= (\lambda\, p : \mathbb{R} \times \mathbb{R} \,.\, [\mathsf{fst}(p)]^2 + [\mathsf{snd}(p)]^2)((a, b)) \\ &= [\mathsf{fst}((a, b))]^2 + [\mathsf{snd}((a, b))]^2 \\ &= a^2 + b^2. \end{aligned}$$

- $f'' = \lambda\, x : \mathbb{R} \,.\, \lambda\, y : \mathbb{R} \,.\, x^2 + y^2$.

  $$\begin{aligned} f''(a)(b) &= (\lambda\, x : \mathbb{R} \,.\, \lambda\, y : \mathbb{R} \,.\, x^2 + y^2)(a)(b) \\ &= (\lambda\, y : \mathbb{R} \,.\, a^2 + y^2)(b) \\ &= a^2 + b^2. \end{aligned}$$

3. Binders

Binders

- A binder B is a special operator that is applied to a variable x, a set S of values, and an expression E.
- An application of a binder has the form $B\,x \in S \,.\, E$.
- The value of $B\,x \in S \,.\, E$ is derived from the set $\{(a, \mathrm{val}(E, x, a)) \mid a \in S\}$ by the definition of B.
- Examples:
  - Set abstraction: $Ⓢ\,x \in S \,.\, E = \{x \in S \mid E\}$.
  - Function abstraction: $\lambda\,x \in S \,.\, E$.
  - Universal quantification: $\forall\,x \in S \,.\, E$.
  - Existential quantification: $\exists\,x \in S \,.\, E$.
  - Definite description: $\mathrm{I}\,x \in S \,.\, E$.
  - Indefinite description: $\epsilon\,x \in S \,.\, E$.

  Note that in each example, except function abstraction, E is a boolean expression.

4. Undefinedness

The Undefinedness Problem

- A mathematical expression is undefined if it has no prescribed meaning or if it denotes a nonexistent value.
- Undefined terms are commonplace and unavoidable in mathematics and computing.
- Sources of undefinedness:
  1. Improper function applications:
     $\frac{17}{0}$, $\sqrt{-4}$, $\tan(\frac{\pi}{2})$, $\lim_{x \to 0} \sin(\frac{1}{x})$, $\mathsf{top}(\mathsf{empty\_stack})$
  2. Improper definite descriptions:
     "the x such that $x^2 = 4$"
  3. Improper indefinite descriptions:
     "some x such $x^2 = -4$"

Some Approaches for Handling Undefinedness

1. A partial function f is represented as a predicate that denotes the graph of f.
2. A partial function is represented as a total function plus a domain predicate.
3. A partial function is a total function on the type of its domain.
4. An expression is undefined if its value is completely unspecified.
5. The value of an undefined expression of type α is some error value in α.
6. The value of an undefined expression of type α is some error value added to α to produce a new type.
7. The value of an undefined expression is determined by the traditional approach to undefinedness.

Traditional Approach to Undefinedness [1/2]

The traditional approach to undefinedness, which is widely practiced in mathematics, is based on three principles:

1. Atomic expressions (i.e., variables and constants) are always defined.
2. Compound expressions may be undefined.
   - A function application $f(a)$ is undefined if f is undefined, a is undefined, or $a \not\in \mathrm{dom}(f)$.
   - A definite description $\mathrm{I}\, x \in S \;.\; E$ is undefined if there is not exactly one $a \in S$ such that $\mathrm{val}(E, x, a) = \mathrm{T}$.
   - An indefinite description $\epsilon\, x \in S \;.\; E$ is undefined if there is no $a \in S$ such that $\mathrm{val}(E, x, a) = \mathrm{T}$.
3. Formulas are always true or false and hence defined.
   - So, by convention, a predicate application $p(a) = \mathrm{F}$ if p is undefined, a is undefined, or $a \not\in \mathrm{dom}(p)$.

Traditional Approach to Undefinedness [2/2]

There are two kinds of equality:

1. Equality: $a = b$ if a and b are defined and equal.
2. Quasi-equality: $a \simeq b$ if $a = b$ or a and b are undefined.

Benefits of the Traditional Approach

- Meaningful statements can involve undefined expressions.

  $\forall\, x \in \mathbb{R} \,.\, 0 \leq x \Rightarrow (\sqrt{x})^2 = x.$
  $0 \leq -2 \Rightarrow (\sqrt{-2})^2 = -2.$

- Function domains can be implicit.

  $k(x) \simeq \frac{1}{x} + \frac{1}{x-1}.$
  $\left(\frac{f}{g}\right)(x) \simeq \frac{f(x)}{g(x)}.$

- Definedness assumptions can be implicit, and as a result, expressions involving undefinedness can be very concise.

  $\forall\, x, y, z \in \mathbb{R} \,.\, \frac{x}{y} = z \Rightarrow x = y * z.$

- Values can be defined implicitly using definite or indefinite description.

  $\sqrt{x} \simeq \mathrm{I}\, y \in \mathbb{R} \,.\, 0 \leq y \wedge y^2 = x.$

5. Mathematical Structures

Mathematical Structures

- A mathematical structure (structure for short) is a pair $S = (\mathcal{D}, \mathcal{A})$ where:
  1. $\mathcal{D}$ is a nonempty finite set of base domains that are nonempty sets of values.
  2. $\mathcal{A}$ is a set of distinguished values that are members of the domains in $\{\mathbb{B}\} \cup \mathcal{D}$ or domains constructed from these domains by the function space, power set, Cartesian product, and Kleene star operations.
- When $\mathcal{D} = \{D_1, D_2, \ldots, D_m\}$ and $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$, S can be conveniently written as the tuple

$$(D_1, D_2, \ldots, D_m, a_1, a_2, \ldots, a_n).$$

Domain Building Operations

- Function space: $A \to B$.
  - The set of partial and total functions from A to B.
- Power set: $\mathcal{P}(A)$.
  - The set of subsets of A.
  - $|A| < |\mathcal{P}(A)|$.
- Cartesian product: $A_1 \times \cdots \times A_n$ for $n \geq 1$.
  - The set of tuples $(a_1, \ldots, a_n)$ where $a_i \in A_i$.
  - $A^0 = \{(\,)\}$.
  - $A^n = A \times \cdots \times A$ (n times) for $n \geq 1$.
- Kleene star: A^*.
  - The set of tuples $(a_1, \ldots, a_n)$ where $n \geq 0, a_i \in A$.
  - $A^* = A^0 \cup A^1 \cup A^2 \cup \cdots$.
  - A^* is a domain of lists over A.

The End.