CAS 760
Simple Type Theory
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9 Developments

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Outline

1. Theory developments.
2. Example: Peano Arithmetic.
3. Example: Real Number Mathematics.

1. Theory Developments

How to Build a Mathematics Library

1. Build theories by:
   a. Defining new theories.
   b. Extending existing theories.
2. Build theory develpments by:
   a. Defining new concepts.
   b. Stating and proving new facts.
3. Build morphisms that connect theories and theory developments.
4. Build realms from theories and morphisms.
5. Transport definitions and theorems from one theory development to another using morphisms.

Definition Packages

- A definition package is a tuple $P = (n, \mathbf{c}_\alpha, \mathbf{A}_\alpha, \pi)$ where:
  1. n is the name of the package.
  2. $\mathbf{c}_\alpha$ is a constant.
  3. $\mathbf{A}_\alpha$ is a closed expression.
  4. π is a proof (either a traditional proof or a formal proof).
- P is valid in a theory $T = ((\mathcal{B}, \mathcal{C}), \Gamma)$ if

$$T[P] = ((\mathcal{B}, \mathcal{C} \cup \{\mathbf{c}_\alpha\}), \Gamma \cup \{\mathbf{c}_\alpha = \mathbf{A}_\alpha\})$$

  is a simple definitional extension of T and π is a proof of $\mathbf{A}_\alpha\!\downarrow$ from Γ.

Theorem Packages

- A theorem package is a tuple $P = (n, \mathbf{A}_o, \pi)$ where:
  1. n is the name of the package.
  2. $\mathbf{A}_o$ is a sentence.
  3. π is a proof.
- P is valid in a theory $T = (L, \Gamma)$ if $\mathbf{A}_o$ is a sentence of L and π is a proof of $\mathbf{A}_o$ from Γ.

Developments [1/2]

A theory development (or development for short) of Alonzo is a pair $D = (T, \Xi)$ such that:

1. T is a theory called the bottom theory of D.
2. Ξ is a (possibly empty) list $[P_1, \ldots, P_n]$ of definition and theorem packages where $n \geq 0$.
3. For each i with $1 \leq i \leq n$, $\mathbf{B}_o^i$ is $\mathbf{c}_\alpha = \mathbf{A}_\alpha$ if $P_i = (n, \mathbf{c}_\alpha, \mathbf{A}_\alpha, \pi)$ and is $\mathbf{A}_o$ if $P_i = (n, \mathbf{A}_o, \pi)$.
4. There is a list $[T_0, T_1, \ldots, T_n]$ of theories such that:
    a. $T_0 = T$.
    b. For all i with $0 \leq i \leq n-1$, P_{i+1} is valid in T_i and $T_{i+1} = T[P_{i+1}]$ if P_{i+1} is a definition package and $T_{i+1} = T_i$ if P_{i+1} is a theorem package.
5. T_n is called the top theory of D.

Developments [2/2]

- D is a trivial if $\Xi = [\,]$ and so $T_n = T$.
- We will identify a theory T with its trivial development $(T, [\,])$.
- Proposition. Let $D = (T, \Xi)$ be a nontrivial development that defines the list $[T_0, T_1, \ldots, T_n]$ of theories. Then $T_i \trianglelefteq_{\mathrm{sd}} T_{i+1}$ or $T_i = T_{i+1}$ for all i with $0 \leq i \leq n-1$, and thus $T_0 \trianglelefteq_{\mathrm{d}} T_n$.

Development Definition Module

Development Definition X.Y

Name: Name.
Bottom theory: Name-of-bottom-theory.
Definitions and theorems:

$$P_1 \qquad \text{(description of } P_1\text{).}$$

$$\vdots$$

$$P_n \qquad \text{(description of } P_n\text{).}$$

Development Extensions

- Let $D_i = (T, \Xi_i)$ be a development of T for $i \in \{1, 2\}$.
- D_2 is an extension of D_1 (or D_1 is a subdevelopment of D_2), written $D_1 \leq D_2$, if there is a list Ξ of definition and theorem packages such that $\Xi_2 = \Xi_1 ++ \Xi$.

Development Extension Module

Development Extension X.Y

Name: Name.

Extends Name-of-subdevelopment.

New definitions and theorems:

P_1 (description of P_1).

$\vdots$

P_n (description of P_n).

2. Example: Peano Arithmetic

Example: Development of PA

- We define a development of PA as a series of extensions of an initial development:
  1. Basic definitions and theorems.
  2. Algebraic structure: commutative semiring.
  3. Order structure: weak total order.
  4. Divides lattice structure.
- NAT is the top theory of the development.

3. Example: Real Number Mathematics

Structure of Real Number Arithmetic

- $(\mathbb{R}, 0, 1, +, *, -, \cdot^{-1})$, real number arithmetic, is one of the most important and useful structures in mathematics.
- It contains the following substructures:
  - $(\mathbb{N}, 0, 1, +, *)$, natural number arithmetic.
  - $(\mathbb{Z}, 0, 1, +, *, -)$, integer arithmetic.
  - $(\mathbb{Q}, 0, 1, +, *, -, \cdot^{-1})$, rational number arithmetic.
- It is contained in the following superstructures:
  - $(\mathbb{C}, 0, 1, +, *, -, \cdot^{-1})$, complex number arithmetic.
  - $(\mathbb{H}, 0, 1, +, *, -, \cdot^{-1})$, quaternion arithmetic.
  - $(\mathbb{O}, 0, 1, +, *, -, \cdot^{-1})$, octonion arithmetic.
  - $({}^*\mathbb{R}, 0, 1, +, *, -, \cdot^{-1})$, hyperreal number arithmetic.
  - $(\mathbb{S}, 0, 1, +, *, -, \cdot^{-1})$, surreal number arithmetic.

Theory of Complete Ordered Fields [1/3]

The theory of complete ordered fields is $\mathsf{COF} = ((\mathcal{B}, \mathcal{C}), \Gamma)$ where:

- $\mathcal{B} = \{R\}$.
- $\mathcal{C}$ contains the following constants:
  1. 0_R.
  2. 1_R.
  3. $+_{R \rightarrow R \rightarrow R}$.
  4. $*_{R \rightarrow R \rightarrow R}$.
  5. $-_{R \rightarrow R}$.
  6. ${\cdot^{-1}}_{R \rightarrow R}$.
  7. $\mathsf{pos}_{R \rightarrow o}$.
  8. $\leq_{R \rightarrow R \rightarrow o}$.
  9. $\mathsf{ub}_{R \rightarrow \{R\} \rightarrow o}$.
  10. $\mathsf{lub}_{R \rightarrow \{R\} \rightarrow o}$.

Theory of Complete Ordered Fields [2/3]

- Γ contains the following sentences:
  1. $\forall x, y, z : R \,.\, (x + y) + z = x + (y + z)$.
  2. $\forall x, y : R \,.\, x + y = y + x$.
  3. $\forall x : R \,.\, x + 0 = x$.
  4. $\forall x : R \,.\, x + (-x) = 0$
  5. $\forall x, y, z : R \,.\, (x * y) * z = x * (y * z)$.
  6. $\forall x, y : R \,.\, x * y = y * x$.
  7. $\forall x : R \,.\, x * 1 = x$.
  8. $\forall x : R \,.\, x \neq 0 \Rightarrow x * x^{-1} = 1$.
  9. $0^{-1}\uparrow$.
  10. $0 \neq 1$.
  11. $\forall x, y, z : R \,.\, x * (y + z) = (x * y) + (x * z)$.

Theory of Complete Ordered Fields [3/3]

12. $\forall x : R \,.\, (x = 0 \wedge \neg(\mathrm{pos}\, x) \wedge \neg(\mathrm{pos}\,(-x))) \vee$
    $(x \neq 0 \wedge \mathrm{pos}\, x \wedge \neg(\mathrm{pos}\,(-x))) \vee$
    $(x \neq 0 \wedge \neg(\mathrm{pos}\, x) \wedge \mathrm{pos}\,(-x))$.
13. $\forall x, y : R \,.\, (\mathrm{pos}\, x \wedge \mathrm{pos}\, y) \Rightarrow \mathrm{pos}\,(x + y)$.
14. $\forall x, y : R \,.\, (\mathrm{pos}\, x \wedge \mathrm{pos}\, y) \Rightarrow \mathrm{pos}\,(x * y)$.
15. $\forall x, y : R \,.\, x \leq y \Leftrightarrow (x = y \vee \mathrm{pos}\,(y + (-x)))$.

16. $\forall x : R, s : \{R\} \,.\, \mathrm{ub}\, x\, s \Leftrightarrow \forall y : s \,.\, y \leq x$.
17. $\forall x : R, s : \{R\} \,.\, \mathrm{lub}\, x\, s \Leftrightarrow$
    $(\mathrm{ub}\, x\, s \wedge (\forall y : R \,.\, \mathrm{ub}\, y\, s \Rightarrow x \leq y))$.
18. $\forall s : \{R\} \,.\, (s \neq \emptyset_{\{R\}} \wedge \exists x : R \,.\, \mathrm{ub}\, x\, s) \Rightarrow$
    $\exists x : R \,.\, \mathrm{lub}\, x\, s$.

- Proposition. COF specifies in the standard sense the set of complete ordered fields.
- Theorem. COF is categorical in the standard sense.

Alternative Constructions to COF

1. Construction of a different theory of complete ordered fields in a single step as with COF.
2. Construction of COF (or some other theory of complete ordered fields) as the last theory in a sequence of extensions.
3. Construction of a theory of real number arithmetic as the last theory in a sequence of definitional extensions.
   a. Start with PA.
   b. Define natural number arithmetic.
   c. Define integer arithmetic.
   d. Define rational number arithmetic.
   e. Define real number arithmetic.

Example: Development of COF

- We define a development of COF as a series of extensions of an initial development:
  1. Basic definitions and theorems.
  2. Naturals, integers, and rationals.
  3. Iterated sum and product operators.
  4. Calculus.
  5. Euclidean Space.
- REAL is the top theory of the development.

Notational Definitions for REAL

$(-\mathbf{A}_R)$	stands for	$-_{R \to R}\, \mathbf{A}_R$.
$\mathbf{A}_R{}^{-1}$	stands for	$\cdot^{-1}{}_{R \to R}\, \mathbf{A}_R$.
$\left(\frac{\mathbf{A}_R}{\mathbf{B}_R}\right)$	stands for	$/_{R \to R \to R}\, \mathbf{A}_R\, \mathbf{B}_R$.
$\lvert \mathbf{A}_R \rvert$	stands for	$\lvert \cdot \rvert_{R \to R}\, \mathbf{A}_R$.
$\sqrt{\mathbf{A}_R}$	stands for	$\sqrt{\cdot}_{R \to R}\, \mathbf{A}_R$.
$\lVert \mathbf{A}_{R \to R} \rVert$	stands for	$\lVert \cdot \rVert_{(R \to R) \to R} \mathbf{A}_{R \to R}$.
$\left(\sum_{\mathbf{i} = \mathbf{M}_R}^{\mathbf{N}_R} \mathbf{A}_R\right)$	stands for	$\mathrm{sum}_{R \to R \to (R \to R) \to R}\, \mathbf{M}_R\, \mathbf{N}_R\, (\lambda\, \mathbf{i} : R \,.\, \mathbf{A}_R)$.
$\left(\prod_{\mathbf{i} = \mathbf{M}_R}^{\mathbf{N}_R} \mathbf{A}_R\right)$	stands for	$\mathrm{prod}_{R \to R \to (R \to R) \to R}\, \mathbf{M}_R\, \mathbf{N}_R\, (\lambda\, \mathbf{i} : R \,.\, \mathbf{A}_R)$.
$\left(\lim_{\mathbf{x} \to \mathbf{a}_R} \mathbf{B}_R\right)$	stands for	$\mathrm{lim}_{(R \to R) \to R \to R}\, (\lambda\, \mathbf{x} : R \,.\, \mathbf{B}_R)\, \mathbf{a}_R$.
$\left(\lim_{\mathbf{x} \to \mathbf{a}_R{}^+} \mathbf{B}_R\right)$	stands for	$\text{right-lim}_{(R \to R) \to R \to R}\, (\lambda\, \mathbf{x} : R \,.\, \mathbf{B}_R)\, \mathbf{a}_R$.
$\left(\lim_{\mathbf{x} \to \mathbf{a}_R{}^-} \mathbf{B}_R\right)$	stands for	$\text{left-lim}_{(R \to R) \to R \to R}\, (\lambda\, \mathbf{x} : R \,.\, \mathbf{B}_R)\, \mathbf{a}_R$.
$\left(\lim_{\mathbf{n} \to \infty} \mathbf{B}_R\right)$	stands for	$\text{lim-seq}_{(R \to R) \to R}\, (\lambda\, \mathbf{n} : N_{\{R\}} \,.\, \mathbf{B}_R)$.
$\left(\int_{\mathbf{A}_R}^{\mathbf{B}_R} \mathbf{C}_R\, d\mathbf{x}\right)$	stands for	$\mathrm{integral}_{(R \to R) \to R \to R \to R}\, (\lambda\, \mathbf{x} : R \,.\, \mathbf{C}_R)\, \mathbf{A}_R\, \mathbf{B}_R$.

Skolem's Paradox

- REAL is satisfiable, so REAL has a frugal model M.
  - M is nonstandard.
- Skolem's paradox:
  1. D_R^M is countable since $|D_R^M| \leq \|M\| \leq \|L\| = \omega$.
  2. Thm22 says D_R^M is uncountable.
- Resolution:
  1. Thm22 actually says there is no bijection from $V_\varphi^M(N_{\{R\}})$ to $V_\varphi^M(U_{\{R\}})$ in $D_{R \rightarrow R}^M$.
  2. Thm22 is true in M since $D_{R \rightarrow R}^M$ contains no bijection from $V_\varphi^M(N_{\{R\}})$ to $V_\varphi^M(U_{\{R\}})$.
- Summary:
  1. In a standard model, Thm22 says that D_R^M is uncountable.
  2. In a nonstandard, frugal model, Thm22 says there is no bijection from $V_\varphi^M(N_{\{R\}})$ to $V_\varphi^M(U_{\{R\}})$ in $D_{R \rightarrow R}^M$, which enables D_R^M to be countable.

The End.