

**CAS 760**  
**Simple Type Theory**  
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# 8 Sequences

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# Outline

1. Background.
2. Systems of natural numbers.
3. Notational definitions for sequences.

# 1. Background

# Representation of Domain Constructions in Alonzo

1. Function space:  $\alpha \rightarrow \beta$ .
2. Power set:  $\{\alpha\} = \alpha \rightarrow \mathbf{o}$ .
3. Cartesian product:  $\alpha \times \beta$ .
4. Kleene star:  $[\alpha]$ . What is this?

# Benefits of having Undefined Expressions

The **traditional approach to undefinedness** facilitates the formalization of the following values:

1. Partial functions.
2. Definite descriptions.
3. Indefinite descriptions.
4. Function subtypes.
5. **Finite sequences.**
6. Recursively defined functions.

# Basic Definitions

- An **infinite sequence** of values from a set  $A$  can be represented as a total function  $s : \mathbb{N} \rightarrow A$ .
- A **finite sequence** can be represented as a partial function  $s : \mathbb{N} \rightarrow A$  such that, for some  $n \in \mathbb{N}$ ,  $s(m)$  is defined iff  $m < n$ .

## 2. Systems of Natural Numbers

# Systems of Natural Numbers [1/2]

Let  $T = (L, \Gamma)$  be a theory. A set

$$\mathcal{E}_{\text{nat}} = \{\mathbf{C}_{\{\alpha\}}^N, \mathbf{C}_{\alpha}^0, \mathbf{C}_{\alpha \rightarrow \alpha}^S, \mathbf{C}_{\alpha \rightarrow \alpha}^P, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^*, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow o}^{\leq}\}$$

of closed expressions is a **system of natural numbers in  $T$**  if

$\mathcal{E}_{\text{nat}} \subseteq \mathcal{E}(L)$  and the following sentences are theorems of  $T$ :

1.  $\mathbf{C}_{\alpha}^0 \in \mathbf{C}_{\{\alpha\}}^N$ .
2.  $\mathbf{C}_{\alpha \rightarrow \alpha}^S \in \mathbf{C}_{\{\alpha\}}^N \rightarrow \mathbf{C}_{\{\alpha\}}^N$ .
3. TOTAL-ON( $\mathbf{C}_{\alpha \rightarrow \alpha}^S, \mathbf{C}_{\{\alpha\}}^N, \mathbf{C}_{\{\alpha\}}^N$ ).
4.  $\forall x : \mathbf{C}_{\{\alpha\}}^N . \mathbf{C}_{\alpha}^0 \neq \mathbf{C}_{\alpha \rightarrow \alpha}^S x$ .
5.  $\forall x, y : \mathbf{C}_{\{\alpha\}}^N . \mathbf{C}_{\alpha \rightarrow \alpha}^S x = \mathbf{C}_{\alpha \rightarrow \alpha}^S y \Rightarrow x = y$ .
6.  $\forall p : \mathbf{C}_{\{\alpha\}}^N \rightarrow o .$   
 $(p \mathbf{C}_{\alpha}^0 \wedge \forall x : \mathbf{C}_{\{\alpha\}}^N . (p x \Rightarrow p(\mathbf{C}_{\alpha \rightarrow \alpha}^S x))) \Rightarrow \forall x : \mathbf{C}_{\{\alpha\}}^N . p x$ .
7.  $\mathbf{C}_{\alpha \rightarrow \alpha}^P = \lambda x : \mathbf{C}_{\{\alpha\}}^N . \text{I } y : \mathbf{C}_{\{\alpha\}}^N . \mathbf{C}_{\alpha \rightarrow \alpha}^S y = x$ .
8.  $\mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+ = \text{I } f : \mathbf{C}_{\{\alpha\}}^N \rightarrow \mathbf{C}_{\{\alpha\}}^N \rightarrow \mathbf{C}_{\{\alpha\}}^N . \forall x, y : \mathbf{C}_{\{\alpha\}}^N .$   
 $f \times 0 = x \wedge f \times (\mathbf{C}_{\alpha \rightarrow \alpha}^S y) = \mathbf{C}_{\alpha \rightarrow \alpha}^S (f \times y)$ .
9.  $\mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^* = \text{I } f : \mathbf{C}_{\{\alpha\}}^N \rightarrow \mathbf{C}_{\{\alpha\}}^N \rightarrow \mathbf{C}_{\{\alpha\}}^N . \forall x, y : \mathbf{C}_{\{\alpha\}}^N .$   
 $f \times 0 = 0 \wedge f \times (\mathbf{C}_{\alpha \rightarrow \alpha}^S y) = \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+ (f \times y) x$ .
10.  $\mathbf{C}_{\alpha \rightarrow \alpha \rightarrow o}^{\leq} = \lambda x : \mathbf{C}_{\{\alpha\}}^N . \lambda y : \mathbf{C}_{\{\alpha\}}^N . \exists z : \mathbf{C}_{\{\alpha\}}^N . \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+ x z = y$ .

# Systems of Natural Numbers [2/2]

- The set

$$\{N_{\{N\}}, 0_N, S_{N \rightarrow N}, P_{N \rightarrow N}, +_{N \rightarrow N \rightarrow N}, *_{N \rightarrow N \rightarrow N}, \leq_{N \rightarrow N \rightarrow o}\}$$

is a system of natural numbers in  $PA'$ .

- Thus:

- ▶ Any extension of  $PA'$  contains a system of natural numbers.
- ▶ Any theory  $T$  that contains a system of natural numbers contains a copy of  $PA'$ .

- Let  $T_{\text{nat}}$  be a theory in which

$$\{\mathbf{C}_{\{\alpha\}}^N, \mathbf{C}_{\alpha}^0, \mathbf{C}_{\alpha \rightarrow \alpha}^S, \mathbf{C}_{\alpha \rightarrow \alpha}^P, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^*, \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow o}^{\leq}\}$$

is a system of natural numbers.

- We next introduce notation for sequences in  $T_{\text{nat}}$ .

### 3. Notation Definitions for Sequences

# Notational Definitions for Sequences [1/2]

sequences $_{\{\alpha \rightarrow \beta\}}$	stands for	$\mathbf{C}_{\{\alpha\}}^N \rightarrow \beta$ .
$\langle\langle\beta\rangle\rangle$	stands for	sequences $_{\{\alpha \rightarrow \beta\}}$ .
streams $_{\{\alpha \rightarrow \beta\}}$	stands for	$\{s : \langle\langle\beta\rangle\rangle \mid \text{TOTAL}(s)\}$ .
$\langle\beta\rangle$	stands for	streams $_{\{\alpha \rightarrow \beta\}}$ .
lists $_{\{\alpha \rightarrow \beta\}}$	stands for	$\{s : \langle\langle\beta\rangle\rangle \mid \exists n : \mathbf{C}_{\{\alpha\}}^N . \forall m : \mathbf{C}_{\{\alpha\}}^N .$ $(s\ m) \downarrow \Leftrightarrow \mathbf{C}_{\alpha \rightarrow \alpha \rightarrow o}^{\leq} m(\mathbf{C}_{\alpha \rightarrow \alpha}^P n)\}$ .
$[\beta]$	stands for	lists $_{\{\alpha \rightarrow \beta\}}$ .
cons $_{\beta \rightarrow (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)}$	stands for	$\lambda x : \beta . \lambda s : \langle\langle\beta\rangle\rangle . \lambda n : \mathbf{C}_{\{\alpha\}}^N .$ $n = \mathbf{C}_{\alpha}^0 \mapsto x \mid s(\mathbf{C}_{\alpha \rightarrow \alpha}^P n)$ .
$(\mathbf{A}_{\beta} :: \mathbf{B}_{\alpha \rightarrow \beta})$	stands for	cons $_{\beta \rightarrow (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)} \mathbf{A}_{\beta} \mathbf{B}_{\alpha \rightarrow \beta}$ .
hd $_{(\alpha \rightarrow \beta) \rightarrow \beta}$	stands for	$\lambda s : \langle\langle\beta\rangle\rangle . \text{I } x : \beta . \exists t : \langle\langle\beta\rangle\rangle .$ $s = (x :: t)$ .
tl $_{(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)}$	stands for	$\lambda s : \langle\langle\beta\rangle\rangle . \text{I } t : \langle\langle\beta\rangle\rangle . \exists x : \beta .$ $s = (x :: t)$ .

# Notational Definitions for Sequences [2/2]

$\text{nil}_{\alpha \rightarrow \beta}$	stands for	$\Delta_{\alpha \rightarrow \beta}$ .
$[ ]_{\alpha \rightarrow \beta}$	stands for	$\text{nil}_{\alpha \rightarrow \beta}$ .
$[ \mathbf{A}_{\beta} ]$	stands for	$(\mathbf{A}_{\beta} :: [ ]_{\alpha \rightarrow \beta})$ .
$[ \mathbf{A}_{\beta}^1, \dots, \mathbf{A}_{\beta}^n ]$	stands for	$(\mathbf{A}_{\beta}^1 :: [ \mathbf{A}_{\beta}^2, \dots, \mathbf{A}_{\beta}^n ])$ where $n \geq 2$ .
$\text{len}_{(\alpha \rightarrow \beta) \rightarrow \alpha}$	stands for	$\text{I } f : [\beta] \rightarrow \mathbf{C}_{\{\alpha\}}^N . f [ ]_{\alpha \rightarrow \beta} = \mathbf{C}_{\alpha}^0 \wedge$ $\forall x : \beta, s : [\beta] . f (x :: s) =$ $\mathbf{C}_{\alpha \rightarrow \alpha \rightarrow \alpha}^+ (f s) (\mathbf{C}_{\alpha \rightarrow \alpha}^S \mathbf{C}_{\alpha}^0)$ .
$  \mathbf{A}_{\alpha \rightarrow \beta}  $	stands for	$\text{len}_{(\alpha \rightarrow \beta) \rightarrow \alpha} \mathbf{A}_{\alpha \rightarrow \beta}$ .
$\text{nlists}_{\alpha \rightarrow \{\alpha \rightarrow \beta\}}$	stands for	$\lambda n : \mathbf{C}_{\{\alpha\}}^N . \{ s : [\beta] \mid  s  = n \}$ .
$[\beta]^{\mathbf{N}_{\alpha}}$	stands for	$\text{nlists}_{\alpha \rightarrow \{\alpha \rightarrow \beta\}} \mathbf{N}_{\alpha}$ .

# Facts about Sequences

- **Proposition.** The following sentences are valid in  $T_{\text{nat}}$  for every type  $\beta$  of  $T_{\text{nat}}$ :

1.  $\forall x : \beta, s : \langle \beta \rangle . (x :: s) \in \langle \beta \rangle .$
2.  $\forall x : \beta, s : [\beta] . (x :: s) \in [\beta] .$
3.  $\forall s : \langle \beta \rangle . (\text{tl}_{(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)} s) \in \langle \beta \rangle .$
4.  $\forall s : [\beta] . s \neq []_{\alpha \rightarrow \beta} \Rightarrow (\text{tl}_{(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)} s) \in [\beta] .$
5.  $(\text{hd}_{(\alpha \rightarrow \beta) \rightarrow \beta} []_{\alpha \rightarrow \beta}) \uparrow .$
6.  $(\text{tl}_{(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)} []_{\alpha \rightarrow \beta}) \uparrow .$
7.  $\forall s : \langle \beta \rangle . (\text{len}_{(\alpha \rightarrow \beta) \rightarrow \alpha} s) \uparrow .$
8.  $\forall s : [\beta] . (\text{len}_{(\alpha \rightarrow \beta) \rightarrow \alpha} s) \downarrow .$

- If  $[\beta]$  denotes  $D_\beta^*$ , the set of lists over  $D_\beta$ , then  $[\beta]^n$  denotes  $D_\beta^n$ , the set of lists over  $D_\beta$  of length  $n$ .

- ▶  $D_\beta^n \subseteq D_\beta^* \subseteq \text{"N"} \rightarrow D_\beta \subseteq D_\alpha \rightarrow D_\beta .$
- ▶  $[\beta]^n$  is a dependent quasitype.
- ▶  $[\beta]^n$  is a subquasitype of  $[\beta]$ ,  $\langle \beta \rangle$ , and  $\langle\langle \beta \rangle\rangle$ .
- ▶  $[\beta]^n$  is a subtype of  $\alpha \rightarrow \beta$ .

The End.