10 Morphisms

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Outline

1. Little theories method.
2. Theory morphisms.
3. Development morphisms.
4. Mathematics Libraries.
5. Theory graph combinators.

1. Little Theories Method

A Motivating Example

- The structure $(\mathbb{R}, 0, 1, +, *, -, \cdot^{-1})$ contains two monoids: $(\mathbb{R}, 0, +)$ and $(\mathbb{R}, 1, *)$.

- Making definitions and proving theorems in COF about these two substructures involves duplication of work.

- We should be able to do the work in MON and then transport the results to COF.

- We can define translations from MON to the additive and multiplicative contexts of COF.

- The structure $(\mathbb{R}, 0, 1, +, *, -, \cdot^{-1})$ also contains two groups: $(\mathbb{R}, 0, +, -)$ and $(\mathbb{R}', 1, *', \cdot^{-1'})$ where $\mathbb{R}' = \{r \in \mathbb{R} \mid r \neq 0\}$ and $*'$ and $\cdot^{-1'}$ are the functions $*$ and $\cdot^{-1}$ restricted to domain $\mathbb{R}'$.

- We can define more general translations from GRP to additive and multiplicative contexts of COF.

Methods for Formalizing Mathematical Knowledge

Two principal ways for formalizing mathematical knowledge:

1. Big theory method.
   - ▶ All knowledge resides in one big theory.
   - ▶ Language includes a huge number of defined concepts.
   - ▶ The method is foundational.
   - ▶ Redundancy is a major concern.

2. Little theories method.
   - ▶ A body of knowledge is represented as a theory graph whose nodes are theories and directed edges are theorem-preserving mappings called (theory) morphisms.
   - ▶ Each mathematical topic is developed in its own "little theory" with it own language and axioms.
   - ▶ The definitions and theorems produced in a little theory are transported to other theories via morphisms.
   - ▶ The method is foundational if desired.

Advantages of the Little Theories Method

The little theories method supports:

1. Abstraction: Each mathematical topic can be developed in the little theory that has the most convenient level of abstraction and the most convenient vocabulary. Clarity is maximized!

2. Reuse: Definitions and theorems in one theory can be transported, as needed, to other theories via morphisms.

3. Redundancy elimination: Redundant concepts and facts can are replaced by instances of more general concepts and facts. Redundancy is minimized!

4. Parallel development: Theories can be developed separately but share their results via morphisms.

5. Foundational development: All theories can be mapped to a chosen foundational theory if desired.

2. Theory Morphisms

Overview

- Let T_1 and T_2 be theories.
- A translation from T_1 to T_2 is a mapping from the expressions of T_1 to the expressions of T_2 defined by:
    1. Each base type is mapped to a non-Boolean type or quasitype.
    2. Each constant of type α is mapped to a closed expression of the type to which α is mapped.
- A translation is a morphism if it maps theorems to theorems.

Theory Translations [1/4]

- Let $T_i = (L_i, \Gamma_i)$, where $L_i = (\mathcal{B}_i, \mathcal{C}_i)$, be a theory for $i \in \{1, 2\}$.

- In addition, for $i \in \{1, 2\}$, let:
  - $\mathcal{T}_i = \mathcal{T}(L_i)$.
  - $\mathcal{T}_i^- = \{\alpha \in \mathcal{T}_i \mid \alpha \neq o\}$.
  - $\mathcal{E}_i = \mathcal{E}(L_i)$.
  - $\mathcal{Q}_i$ be the set of closed quasitypes in $\mathcal{E}_i$.
  - $\mathcal{Q}_i^- = \{\mathbf{Q}_{\{\alpha\}} \in \mathcal{Q}_i \mid \alpha \neq o\}$.

- Let $\tau(\alpha) = \alpha$ and $\tau(\mathbf{Q}_{\{\alpha\}}) = \alpha$.

Theory Translations [2/4]

- If $\mu : \mathcal{B}_1 \to \mathcal{T}_2^- \cup \mathcal{Q}_2^-$ is total, then $\overline{\mu} : \mathcal{T}_1 \to \mathcal{T}_2 \cup \mathcal{Q}_2$ is defined by:
  1. $\overline{\mu}(o) \equiv o$.
  2. $\overline{\mu}(\mathbf{a}) \equiv \mu(\mathbf{a})$.
  3. $\overline{\mu}(\alpha \to \beta) \equiv \overline{\mu}(\alpha) \to \overline{\mu}(\beta)$.
  4. $\overline{\mu}(\alpha \times \beta) \equiv \overline{\mu}(\alpha) \times \overline{\mu}(\beta)$.
- Lemma. Let $\mu : \mathcal{B}_1 \to \mathcal{T}_2^- \cup \mathcal{Q}_2^-$ be total.
  1. $\overline{\mu} : \mathcal{T}_1 \to \mathcal{T}_2 \cup \mathcal{Q}_2$ is well-defined and total.
  2. $\tau(\overline{\mu}(\alpha)) = o$ iff $\alpha = o$ for all $\alpha \in \mathcal{T}_1$.
  3. If $\alpha \in \mathcal{T}(L_\emptyset)$, then $\overline{\mu}(\alpha) = \alpha$.

Theory Translations [3/4]

- A (theory) translation from T_1 to T_2 of Alonzo is a pair $\Phi = (\mu, \nu)$, where

  $\mu : \mathcal{B}_1 \to \mathcal{T}_2^- \cup \mathcal{Q}_2^-$ and
  $\nu : \mathcal{C}_1 \to \mathcal{E}_2$

  are total, such that $\nu(\mathbf{c}_\alpha)$ is a closed expression of type $\tau(\overline{\mu}(\alpha))$ for all $\mathbf{c}_\alpha \in \mathcal{C}_1$.

- Let $\overline{\nu} : \mathcal{E}_1 \to \mathcal{E}_2$ be defined by:

  1. $\overline{\nu}((\mathbf{x} : \alpha)) \equiv (\mathbf{x} : \tau(\overline{\mu}(\alpha)))$.
  2. $\overline{\nu}(\mathbf{c}_\alpha) \equiv \nu(\mathbf{c}_\alpha)$.
  3. $\overline{\nu}(\mathbf{A}_\alpha = \mathbf{B}_\alpha) \equiv (\overline{\nu}(\mathbf{A}_\alpha) = \overline{\nu}(\mathbf{B}_\alpha))$.
  4. $\overline{\nu}(\mathbf{F}_{\alpha\to\beta}\,\mathbf{A}_\alpha) \equiv \overline{\nu}(\mathbf{F}_{\alpha\to\beta})\,\overline{\nu}(\mathbf{A}_\alpha)$.
  5. $\overline{\nu}(\lambda\,\mathbf{x} : \alpha\,.\,\mathbf{B}_\beta) \equiv \lambda\,\mathbf{x} : \overline{\mu}(\alpha)\,.\,\overline{\nu}(\mathbf{B}_\beta)$.
  6. $\overline{\nu}(\mathrm{I}\,\mathbf{x} : \alpha\,.\,\mathbf{A}_o) \equiv \mathrm{I}\,\mathbf{x} : \overline{\mu}(\alpha)\,.\,\overline{\nu}(\mathbf{A}_o)$.
  7. $\overline{\nu}((\mathbf{A}_\alpha, \mathbf{B}_\beta)) \equiv (\overline{\nu}(\mathbf{A}_\alpha), \overline{\nu}(\mathbf{B}_\beta))$.

Theory Translations [4/4]

- Φ is:
  - ▶ An identity translation if $\overline{\mu}$ and $\overline{\nu}$ are identity mappings.
  - ▶ An injective translation if $\overline{\mu}$ and $\overline{\nu}$ are injective mappings.
  - ▶ Normal if $\mu : \mathcal{B}_1 \to \mathcal{T}_2^-$.

- Lemma. Let $\Phi = (\mu, \nu)$ be a translation from T_1 to T_2.
  1. $\overline{\nu} : \mathcal{E}_1 \to \mathcal{E}_2$ is well-defined and total.
  2. $\overline{\nu}(\mathbf{A}_\alpha)$ is of type $\tau(\overline{\mu}(\alpha))$ for all $\mathbf{A}_\alpha \in \mathcal{E}_1$.
  3. If $\mathbf{A}_\alpha$ is closed, then $\overline{\nu}(\mathbf{A}_\alpha)$ is closed.
  4. If $\mathbf{A}_\alpha \in \mathcal{E}(L_\emptyset)$, then $\overline{\nu}(\mathbf{A}_\alpha) = \mathbf{A}_\alpha$.
  5. If μ and ν are identity mappings, then Φ is an identity translation.
  6. If $\mu : \mathcal{B}_1 \to \mathcal{B}_2$ and $\nu : \mathcal{C}_1 \to \mathcal{C}_2$ are injective mappings, then Φ is injective.

Theory Translation Definition Module

Theory Translation Definition X.Y

Name: Name.
Source theory: Name-of-source-theory.
Target theory: Name-of-target-theory.
Base type mapping:

$\mathbf{a}_1 \mapsto \zeta_1$ where ζ_1 is a type or closed quasitype.

$\vdots$

$\mathbf{a}_m \mapsto \zeta_m$ where ζ_m is a type or closed quasitype.

Constant mapping:

$\mathbf{c}^1_{\alpha_1} \mapsto \mathbf{B}^1_{\beta_1}$.

$\vdots$

$\mathbf{c}^n_{\alpha_n} \mapsto \mathbf{B}^n_{\beta_n}$.

Monoids to Groups

Theory Translation Definition (MON to GRP)

Name: MON-to-GRP.
Source theory: MON.
Target theory: GRP.
Base type mapping:

1. $S \mapsto S$.

Constant mapping:

1. $\cdot_{S \to S \to S} \mapsto \cdot_{S \to S \to S}$.
2. $e_S \mapsto e_S$.

Peano Arithmetic with 1 to Peano Arithmetic

Theory Translation Definition (PA-with-1 to PA)

Name: PA-with-1-to-PA.
Source theory: PA-with-1.
Target theory: PA.
Base type mapping:

1. $N \mapsto N$.

Constant mapping:

1. $0_N \mapsto 0_N$.
2. $S_{N \to N} \mapsto S_{N \to N}$.
3. $1_N \mapsto S_{N \to N} \, 0_N$.

Proposition. $T_1 \trianglelefteq_d T_2$ implies there is a normal morphism from T_2 to T_1.

Dense Weak Total Orders without Endpoints to COF

Theory Translation Definition (DWTOWE to COF)

Name: DWTOWE-to-COF.

Source theory: DWTOWE.

Target theory: COF.

Base type mapping:

1. $S \mapsto R$.

Constant mapping:

1. $\leq_{S \to S \to o} \mapsto \leq_{R \to R \to o}$.

Robinson Arithmetic to Peano Arithmetic

Theory Translation Definition (RA to PA)

Name: RA-to-PA.

Source theory: RA.

Target theory: PA.

Base type mapping:

1. $N \mapsto N$.

Constant mapping:

1. $0_N \mapsto 0_N$.

2. $S_{N \to N} \mapsto S_{N \to N}$.

3. $+_{N \to N \to N} \mapsto$
   $\mathrm{I}\, f : N \to N \to N \,.\, \forall\, x, y : N \,.\, f\, x\, 0 = x \wedge f\, x\, (S\, y) = S\, (f\, x\, y)$.

4. $*_{N \to N \to N} \mapsto$
   $\mathrm{I}\, g : N \to N \to N \,.$
   $\quad \forall\, x, y : N \,.\, g\, x\, 0 = 0 \wedge g\, x\, (S\, y) = \mathbf{F}_{N \to N \to N}\, (g\, x\, y)\, x$
   where $\mathbf{F}_{N \to N \to N}$ is
   $\mathrm{I}\, f : N \to N \to N \,.\, \forall\, x, y : N \,.\, f\, x\, 0 = x \wedge f\, x\, (S\, y) = S\, (f\, x\, y)$.

Monoids to Additive COF

Theory Translation Definition (MON to COF with +)

Name: MON-to-COF-+.
Source theory: MON.
Target theory: COF.
Base type mapping:

1. $S \mapsto R$.

Constant mapping:

1. $\cdot_{S \to S \to S} \mapsto +_{R \to R \to R}$.
2. $e_S \mapsto 0_R$.

Monoids to Multiplicative COF

Theory Translation Definition (MON to COF with $*$)

Name: MON-to-COF-$*$.
Source theory: MON.
Target theory: COF.
Base type mapping:

1. $S \mapsto R$.

Constant mapping:

1. $\cdot_{S \to S \to S} \mapsto *_{R \to R \to R}$.
2. $e_S \mapsto 1_R$.

Theory Translation Extensions

- Let $\Phi = (\mu, \nu)$ be a translation from T_1 to T_2.
- A translation $\Phi' = (\mu', \nu')$ from T_1' to T_2' is an extension of Φ (or Φ is a subtranslation of Φ'), written $\Phi \leq \Phi'$, if $T_1 \leq T_1'$, $T_2 \leq T_2'$, $\mu \sqsubseteq \mu'$, and $\nu \sqsubseteq \nu'$.

Theory Translation Extension Module

Theory Translation Extension X.Y

Name: Name.

Extends Name-of-subtranslation.

New source theory: Name-of-new-source-theory.

New target theory: Name-of-new-target-theory.

New base type mapping:

$\mathbf{a}_1 \mapsto \zeta_1$ where ζ_1 is a type or closed quasitype.

$\vdots$

$\mathbf{a}_m \mapsto \zeta_m$ where ζ_m is a type or closed quasitype.

New constant mapping:

$\mathbf{c}^1_{\alpha_1} \mapsto \mathbf{B}^1_{\beta_1}$.

$\vdots$

$\mathbf{c}^n_{\alpha_n} \mapsto \mathbf{B}^n_{\beta_n}$.

Groups to Additive COF

Theory Translation Extension (GRP to COF with $+$)

Name: GRP-to-COF-$+$.

Extends: MON-to-COF-$+$.

New source theory: GRP.

New target theory: COF.

New base type mapping:

(none).

New constant mapping:

3. $\cdot^{-1}{}_{S \to S} \mapsto -_{R \to R}$.

Groups to Multiplicative COF

Theory Translation Extension (GRP **to** COF **with** $*$**)**

Name: GRP-to-COF-$*$.

Extends: MON-to-COF-$*$.

New source theory: GRP.

New target theory: COF.

New base type mapping:

(none).

New constant mapping:

3. $\cdot^{-1}{}_{S \to S} \mapsto \left(\cdot^{-1}\right)_{R \to R}$.

Groups to Nonzero Multiplicative COF

Theory Translation Definition (GRP **to Nonzero** COF**)**

Name: GRP-to-nonzero-COF.
Source theory: GRP.
Target theory: COF.
Base type mapping:

1. $S \mapsto \{r : R \mid r \neq 0_R\}$.

Constant mapping:

1. $\cdot_{S \to S \to S} \mapsto$
   $\lambda x : \{r : R \mid r \neq 0_R\} . \lambda y : \{r : R \mid r \neq 0_R\} . *_{R \to R \to R} x\, y$.
2. $e_S \mapsto 1_R$.
3. $\cdot^{-1}{}_{S \to S} \mapsto \lambda x : \{r : R \mid r \neq 0_R\} . \left(\cdot^{-1}\right)_{R \to R} x$.

Theory Morphisms [1/2]

- Φ is a (theory) morphism from T_1 to T_2 if $T_1 \vDash \mathbf{A}_o$ implies $T_2 \vDash \overline{\nu}(\mathbf{A}_o)$ for all sentences $\mathbf{A}_o$ of L_1.
- Examples: All the translations given above.
- Nonexamples: GRP-to-COF-$*$.
- Φ is an inclusion if it is an identity translation that is a morphism.
- Φ is an embedding if it is a normal injective translation that is a morphism.
- Proposition. $T_1 \preceq T_2$ iff there is an inclusion from T_1 to T_2.

Theory Morphisms [2/2]

- Corollary. If $T_1 \leq T_2$, then the identity translation from T_1 to T_2 is an inclusion.

- Remark. A morphism from a theory T is a generalization of an extension of T. The corollary shows that a theory extension can be viewed as an inclusion.

Obligations

- An obligation of Φ is any one of the following sentences of L_2:
    1. $\overline{\nu}(U_{\{\mathbf{a}\}} \neq \emptyset_{\{\mathbf{a}\}})$ where $\mathbf{a} \in \mathcal{B}_1$ and $\mu(\mathbf{a}) \in \mathcal{Q}_2^-$.
    2. $\overline{\nu}(\mathbf{c}_\alpha \downarrow U_{\{\alpha\}})$ where $\mathbf{c}_\alpha \in \mathcal{C}_1$.
    3. $\overline{\nu}(\mathbf{A}_o)$ where $\mathbf{A}_o \in \Gamma_1$.
- Proposition. If $\overline{\nu}(\mathbf{B}_o)$ is an obligation of Φ, then $T_1 \vDash \mathbf{B}_o$.
- Lemma. A normal translation has no obligations of the first kind corresponding to base types mapped to quasitypes.
- Lemma. If $\nu(\mathbf{c}_\alpha) = \mathbf{d}_\beta$, then the obligation of Φ of the second kind for $\mathbf{c}_\alpha$ is valid in T_2.

Morphism Theorem

- Lemma. Let $\Phi = (\mu, \nu)$ be a translation from T_1 to T_2.
    1. If $\alpha \in \mathcal{T}(L_1)$ and $\overline{\mu}(\alpha) \in \mathcal{Q}_2$, then
       $T_2 \vDash \overline{\mu}(\alpha) \downarrow \Rightarrow \overline{\nu}(U_{\{\alpha\}}) = \overline{\mu}(\alpha)$.
    2. If $\alpha \in \mathcal{T}(L_1)$ and $\overline{\mu}(\alpha) \in \mathcal{Q}_2$, then
       $T_2 \vDash \overline{\mu}(\alpha) \uparrow \Rightarrow \overline{\nu}(U_{\{\alpha\}}) = \emptyset_{\{\tau(\overline{\mu}(\alpha))\}}$.
    3. If $\mathbf{a} \in \mathcal{B}_1$ and $\mu(\mathbf{a}) \in \mathcal{Q}_2$, then
       $T_2 \vDash \overline{\nu}(U_{\{\mathbf{a}\}} \neq \emptyset_{\{\mathbf{a}\}}) \Leftrightarrow (\mu(\mathbf{a}) \downarrow \land \mu(\mathbf{a}) \neq \emptyset_{\{\tau(\mu(\mathbf{a}))\}})$.
    4. If $\mathbf{c}_\alpha \in \mathcal{C}_1$ and $\overline{\mu}(\alpha) \in \mathcal{T}_2$, then
       $T_2 \vDash \overline{\nu}(\mathbf{c}_\alpha \downarrow U_{\{\alpha\}}) \Leftrightarrow \nu(\mathbf{c}_\alpha) \downarrow$.
    5. If $\mathbf{c}_\alpha \in \mathcal{C}_1$ and $\overline{\mu}(\alpha) \in \mathcal{Q}_2$, then
       $T_2 \vDash \overline{\mu}(\alpha) \downarrow \Rightarrow \overline{\nu}(\mathbf{c}_\alpha \downarrow U_{\{\alpha\}}) \Leftrightarrow \nu(\mathbf{c}_\alpha) \downarrow \overline{\mu}(\alpha)$.

- Morphism Theorem. If each obligation of Φ is valid in T_2, then Φ is a morphism from T_1 to T_2.

- Relative Satisfiability Theorem. Let Φ be a morphism from T_1 to T_2. If T_2 is satisfiable, then T_1 is satisfiable.

Groups to Dual Groups

Theory Translation Definition (GRP **to Dual** GRP**)**

Name: GRP-to-dual-GRP.

Source theory: GRP.

Target theory: GRP.

Base type mapping:

1. $S \mapsto S$.

Constant mapping:

1. $\cdot_{S \to S \to S} \mapsto \lambda x : S \,.\, \lambda y : S \,.\, \cdot_{S \to S \to S} \, y \, x$.

2. $e_S \mapsto e_S$.

3. $\cdot^{-1}{}_{S \to S} \mapsto \cdot^{-1}{}_{S \to S}$.

Function Composition to COF

Let FUN-COMP be the theory $((\{A, B, C, D\}, \emptyset), \emptyset)$.

Theory Translation Definition (FUN-COMP to COF)
 Name: FUN-COMP-to-COF.
 Source theory: FUN-COMP.
 Target theory: FUN-COMP.
 Base type mapping:

1. $A \mapsto R$.
2. $B \mapsto R$.
1. $C \mapsto R$.
2. $D \mapsto R$.

Constant mapping:

(none).

Faithful Morphisms

- A morphism Φ from T_1 to T_2 is faithful if $T_2 \vDash \overline{\nu}(\mathbf{A}_o)$ implies $T_1 \vDash \mathbf{A}_o$ for all sentences $\mathbf{A}_o$ of L_1.
- Examples:
    1. PA-with-1-to-PA.
    2. GRP-to-dual-GRP.
- Proposition. If $T_1 \unlhd T_2$, then the identity translation from T_1 to T_2 is a faithful inclusion.
- Remark. A faithful morphism from a theory T is a generalization of a conservative extension of T. The proposition shows that a conservative theory extension can be viewed as a faithful inclusion.
- Equisatisfiability Theorem. Let Φ be a faithful morphism from T_1 to T_2. Then T_1 is satisfiable iff T_2 is satisfiable.

Interfaces

- An interface for a theory T is a faithful embedding Φ from some theory T' to T.
  - ▶ T' is called the front of the interface.
  - ▶ T is called the back of the interface.
- An interface is intended to be a convenient means for accessing T.
- Proposition. If $T_1 \trianglelefteq T_2$, then T_1 is the front of an interface for T_2.

An Example Theory Graph
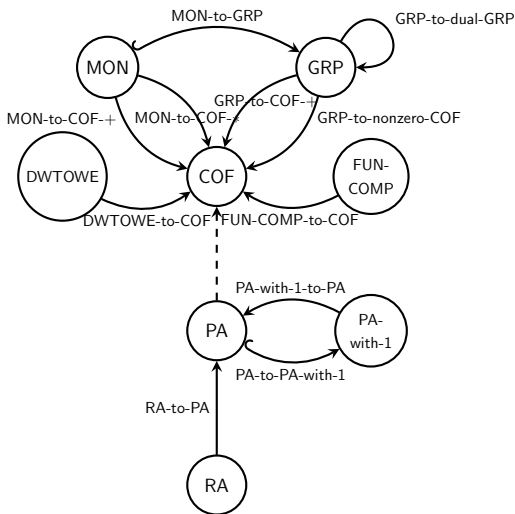
3. Development Morphisms

What should a Development Morphism Be?

- Let $D_i = (T_i, \Xi_i)$ be a development where T_i' is the top theory of D_i for $i \in \{1, 2\}$.
  - $T_1 \trianglelefteq_d T_1'$.
  - L_1' has the form $(\mathcal{B}_1, \mathcal{C}_1 \cup \{\mathbf{c}_{\alpha_1}^1, \ldots, \mathbf{c}_{\alpha_n}^n\})$.
  - Γ_1' has the form $\Gamma_1 \cup \{\mathbf{c}_{\alpha_1}^1 = \mathbf{A}_{\alpha_1}^1, \ldots, \mathbf{c}_{\alpha_n}^n = \mathbf{A}_{\alpha_n}^n\}$.

- Minimum approach: A translation from D_1 to D_2 is a translation Ψ from T_1 to T_2.
  - Each $\mathbf{c}_{\alpha_i}^i$ is mapped to the image of $\mathbf{A}_{\alpha_i}^i$ under Ψ.
  - The image of $\mathbf{c}_{\alpha_i}^i$ can explode if $\mathbf{A}_{\alpha_i}^i$ contains other $\mathbf{c}_{\alpha_j}^j$s.

- Maximum approach: A translation from D_1 to D_2 is a translation Ψ' from T_1' to T_2'.
  - Ψ' must map all the $\mathbf{c}_{\alpha_i}^i$, even those of no concern to us.
  - Ψ' must be extended each time D_1 is extended.

Development Morphisms [1/2]

- A (development) translation from D_1 to D_2 is a tuple $\Phi = (\mu, \nu)$ such that:
  1. $\mu : \mathcal{B}_1 \to \mathcal{T}_2^- \cup \mathcal{Q}_2^-$ is total.
  2. $\nu : \mathcal{C}_1' \to \mathcal{E}_2'$ is partial or total.
  3. $\nu\restriction_{\mathcal{C}_1}$ is total.
  4. $\nu(\mathbf{c}_\alpha)$ is a closed expression of type $\tau(\overline{\mu}(\alpha))$ for all $\mathbf{c}_\alpha \in \mathcal{C}_1'$ for which $\nu(\mathbf{c}_\alpha)$ is defined.

- Proposition.
  1. If $\nu = \nu\restriction_{\mathcal{C}_1}$, then Φ is a translation from T_1 to T_2 (minimum approach).
  2. If ν is total, then Φ is a translation from T_1' to T_2' (maximum approach).

Development Morphisms [2/2]

- Let $\widetilde{\nu} : \mathcal{E}'_1 \to \mathcal{E}'_2$ be the canonical extension of ν defined as follows by recursion and pattern matching:

  1. $\widetilde{\nu}((\mathbf{x} : \alpha)) \equiv (\mathbf{x} : \tau(\overline{\mu}(\alpha)))$.
  2. $\widetilde{\nu}(\mathbf{c}_\alpha) \equiv \nu(\mathbf{c}_\alpha)$ if $\nu(\mathbf{c}_\alpha)$ is defined.
  3. $\widetilde{\nu}(\mathbf{c}^i_{\alpha_i}) \equiv \widetilde{\nu}(\mathbf{A}^i_{\alpha_i})$ if $\nu(\mathbf{c}^i_{\alpha_i})$ is undefined.
  4. $\widetilde{\nu}(\mathbf{A}_\alpha = \mathbf{B}_\alpha) \equiv (\widetilde{\nu}(\mathbf{A}_\alpha) = \widetilde{\nu}(\mathbf{B}_\alpha))$.
  5. $\widetilde{\nu}(\mathbf{F}_{\alpha \to \beta} \, \mathbf{A}_\alpha) \equiv \widetilde{\nu}(\mathbf{F}_{\alpha \to \beta}) \, \widetilde{\nu}(\mathbf{A}_\alpha)$.
  6. $\widetilde{\nu}(\lambda \, \mathbf{x} : \alpha \, . \, \mathbf{B}_\beta) \equiv \lambda \, \mathbf{x} : \overline{\mu}(\alpha) \, . \, \widetilde{\nu}(\mathbf{B}_\beta)$.
  7. $\widetilde{\nu}(\mathrm{I} \, \mathbf{x} : \alpha \, . \, \mathbf{A}_o) \equiv \mathrm{I} \, \mathbf{x} : \overline{\mu}(\alpha) \, . \, \widetilde{\nu}(\mathbf{A}_o)$.
  8. $\widetilde{\nu}((\mathbf{A}_\alpha, \mathbf{B}_\beta)) \equiv (\widetilde{\nu}(\mathbf{A}_\alpha), \widetilde{\nu}(\mathbf{B}_\beta))$.

- Define $\widetilde{\Phi} = (\mu, \widetilde{\nu}\!\restriction_{\mathcal{C}'_1})$.

- Proposition. $\widetilde{\Phi}$ is a translation from T'_1 to T'_2.

Development Translation Definition Module

Development Translation Definition X.Y

Name: Name.

Source development: Name-of-source-development.

Target development: Name-of-target-development.

Base type mapping:

$$\mathbf{a}_1 \mapsto \zeta_1 \text{ where } \zeta_1 \text{ is a type or closed quasitype.}$$

$$\vdots$$

$$\mathbf{a}_m \mapsto \zeta_m \text{ where } \zeta_m \text{ is a type or closed quasitype.}$$

Constant mapping:

$$\mathbf{c}^1_{\alpha_1} \mapsto \mathbf{B}^1_{\beta_1}.$$

$$\vdots$$

$$\mathbf{c}^{n'}_{\alpha'_n} \mapsto \mathbf{B}^{n'}_{\beta_{n'}}.$$

Development Translation Extensions

- Let $\Phi = (\mu, \nu)$ be a translation from D_1 to D_2.
- A translation $\Phi' = (\mu', \nu')$ from D_1' to D_2' is an extension of Φ (or Φ is a subtranslation of Φ'), written $\Phi \leq \Phi'$, if $D_1 \leq D_1'$, $D_2 \leq D_2'$, $\mu = \mu'$, and $\nu \sqsubseteq \nu'$.

Development Translation Extension Module

Development Translation Extension X.Y

Name: Name.

Extends Name-of-subtranslation.

New source development:
  Name-of-new-source-development.

New target development:
  Name-of-new-target-development.

New defined constant mapping:

$$\mathbf{c}^1_{\alpha_1} \mapsto \mathbf{B}^1_{\beta_1}.$$

$$\vdots$$

$$\mathbf{c}^n_{\alpha_n} \mapsto \mathbf{B}^n_{\beta_n}.$$

Example: Peano Arithmetic to COF

Development Translation Definition (PA **to** COF **1)**

Name: PA-to-COF.
Source development: PA-dev-4.
Target development: COF-dev-5.
Base type mapping:

1. $N \mapsto N_{\{R\}}$.

Constant mapping:

1. $0_N \mapsto 0_R$.
2. $S_{N \to N} \mapsto \lambda\, x : N_{\{R\}} \,.\, x + 1$.
3. $1_N \mapsto 1_R$.
4. $P_{N \to N} \mapsto \lambda\, x : N_{\{R\}} \,.\, x \neq 0 \mapsto x - 1 \mid \bot_R$.
5. $+_{N \to N \to N} \mapsto \lambda\, x : N_{\{R\}} \,.\, \lambda\, y : N_{\{R\}} \,.\, x + y$.
6. $*_{N \to N \to N} \mapsto \lambda\, x : N_{\{R\}} \,.\, \lambda\, y : N_{\{R\}} \,.\, x * y$.

Handling Constant Definitions

- Suppose $\mathbf{c}_\alpha = \mathbf{A}_\alpha$ is a definition of a development D_1 but $\mathbf{c}_\alpha$ is not defined by a morphism Φ from D_1 to D_2.
- There are four ways of dealing with this situation:
  1. Leave the morphism as it is.
  2. Extend Φ so that $\mathbf{c}_\alpha$ is mapped to an appropriate constant of D_2 (if there is one).
  3. Extend Φ so that $\mathbf{c}_\alpha$ is mapped to an appropriate nonconstant expression of D_2 (if its size is tolerable).
  4. Transport the definition $\mathbf{c}_\alpha = \mathbf{A}_\alpha$ via Φ.

Definition Transportations

- Let $\Phi = (\mu, \nu)$ be a morphism from D_1 to D_2.
- Assume $\mathbf{c}_\alpha = \mathbf{A}_\alpha$ is a definition of D_1 such that $\mathbf{c}_\alpha \notin \operatorname{dom}(\nu)$.
- A transportation of $\mathbf{c}_\alpha = \mathbf{A}_\alpha$ from D_1 to D_2 via Φ is a pair (D_2', Φ') where $D_2' = (T_2, \Xi_2 \mathbin{++} [P])$ is an extension of D_2, P is a definition package of the form $(n, \mathbf{d}_{\tau(\overline{\mu}(\alpha))}, \widetilde{\nu}(\mathbf{A}_\alpha), \pi)$ such that $\mathbf{d}_{\tau(\overline{\mu}(\alpha))} \notin \mathcal{C}_2$, and $\Phi' = (\mu, \nu')$ is an extension of Φ such that

$$\nu'(x) = \begin{cases} \nu(x) & \text{if } x \in \operatorname{dom}(\nu); \\ \mathbf{d}_{\tau(\overline{\mu}(\alpha))} & \text{if } x = \mathbf{c}_\alpha. \end{cases}$$

- Hence:
    1. The definition $\mathbf{c}_\alpha = \mathbf{A}_\alpha$ of D_1 is translated to a new definition $\mathbf{d}_{\tau(\overline{\mu}(\alpha))} = \widetilde{\nu}(\mathbf{A}_\alpha)$ that is added to the D_2.
    2. Φ is extended to include the mapping $\mathbf{c}_\alpha \mapsto \mathbf{d}_{\tau(\overline{\mu}(\alpha))}$.

Theorem Transportations

- Let $\Phi = (\mu, \nu)$ be a morphism from D_1 to D_2.
- Assume $\mathbf{B}_o$ is a theorem of D_1.
- A transportation of $\mathbf{B}_o$ from D_1 to D_2 via Φ is a development $D_2' = (T_2, \Xi_2 \mathbin{+\!\!+} [P])$ where P is a theorem package of the form $(n, \widetilde{\nu}(\mathbf{B}_o), \pi)$.
- Hence the theorem $\mathbf{B}_o$ of D_1 is translated to a new theorem $\widetilde{\nu}(\mathbf{B}_o)$ that is added to the D_2.

Definition Transportation Modules

Definition Transportation X.Y

Name: Name-1.
Source development: Name-2.
Target development: Name-3.
Development morphism: Name-4.
Definition: P.
Transported definition: P'.
New target development: Name-5.
New development morphism: Name-6.

Example: Transportation of Divides

Definition Transportation (Transport of $|$ **to** COF**)**

Name: Divides-via-PA-to-COF.
Source Development: PA-dev-4.
Target Development: COF-dev-5.
Development morphism: PA-to-COF.
Definition:

Def6: $|_{N \to N \to o} = \lambda x : N . \lambda y : N . \exists z : N . x * z = y$
$$\text{(divides).}$$

Transported Definition:

Def6-via-PA-to-COF: $|_{R \to R \to o} =$
$\lambda x : N_{\{R\}} . \lambda y : N_{\{R\}} . \exists z : N_{\{R\}} . x * z = y$ (divides).

New target development: COF-dev-6.

New development morphism: PA-to-COF-1.

Theorem Transportation Modules

Theorem Transportation X.Y

Name: Name-1.
Source development: Name-2.
Target development: Name-3.
Development morphism: Name-4.
Theorem: P.
Transported theorem: P'.
New target development: Name-5.

Example: Transportation of "0 is Top"

Theorem Transportation ("0 is Top" to COF)

Name: 0-is-Top-via-PA-to-COF-1.
Source Development: PA-dev-4.
Target Development: COF-dev-6.
Development morphism: PA-to-COF-1.
Theorem:

Thm22: $\forall x : N . x \mid_{N \to N \to o} 0$ (0 is top).

Transported theorem:

Thm22-via-PA-to-COF-1:
    $\forall x : N_{\{R\}} . x \mid_{R \to R \to o} 0$ (0 is top).

New target development: COF-dev-7.

Group Transportation Modules

Group Transportation X.Y

Name: Name-1.

Source development: Name-2.

Target development: Name-3.

Development morphism: Name-4.

Definitions and theorems:

P_1 (description of P_1).

$\vdots$

P_n (description of P_n).

Transported definitions and theorems:

P_1' (description of P_1').

$\vdots$

P_n' (description of P_n').

New target development: Name-5.

New development morphism: Name-6.

4. Mathematics Libraries

Theory Graphs

- A theory graph of Alonzo is a directed graph $G = (\mathcal{N}, \mathcal{V})$ where:
  1. $\mathcal{N}$ is a set of theories of Alonzo.
  2. $\mathcal{V}$ is a set of theory morphisms of Alonzo from T_1 to T_2 where $T_1, T_2 \in \mathcal{N}$.
- Two theories T_1 and T_2 (which may have different languages) are equivalent, written $T_1 \equiv T_2$, if there is a morphism from T_1 to T_2 and a morphism from T_2 to T_1.
- An axiomatization of T is any theory T' that is equivalent to T.
- Proposition. $T_1 \trianglelefteq_d T_2$ implies $T_1 \equiv T_2$.

Development Graphs [1/2]

- A development graph of Alonzo is a directed graph $G = (\mathcal{N}, \mathcal{V})$ where:
    1. $\mathcal{N}$ is a set of developments of Alonzo.
    2. $\mathcal{V}$ is a set of development morphisms of Alonzo from D_1 to D_2 where $D_1, D_2 \in \mathcal{N}$.
- We identify each theory T with $(T, [\,])$, its trivial development.
- Thus $\mathcal{N}$ includes theories viewed as developments, and $\mathcal{V}$ includes theory morphisms viewed as development morphisms.

Development Graphs [2/2]

A development graph supports four ways of developing mathematical knowledge:

1. New theories can be added to the graph, and theories already in the graph can be extended.
2. The theories can be developed by defining new concepts and stating and proving new facts.
3. Theories in the graph can be interconnected with theory morphisms, and developments in the graph can be interconnected with development modules.
4. Definitions and theorems can be transported from one development to another.

Realm Graphs [1/2]

- A "realm" is a structure for consolidating knowledge about a mathematical theory.

- A realm of Alonzo is a tuple $R = (F, \mathcal{D}, \mathcal{M}, \mathcal{I})$ where:
  1. F is a theory called the face of R.
  2. $\mathcal{D}$ is a set $\{D_1, D_2, \ldots, D_n\}$ of developments.
  3. $\mathcal{M}$ is a set of theory morphisms that establish that the bottom theories $T_1, T_2, \ldots, T_n$ of $D_1, D_2, \ldots, D_n$, respectively, are pairwise equivalent.
  4. $\mathcal{I}$ is a set $\{I_1, I_2, \ldots, I_n\}$ of interfaces such that F is the front of I_i and the top theory T_i' of D_i is the back of I_i for each i with $1 \leq i \leq n$.

- We identify a development D with bottom theory T and top theory T' with the trivial realm $(T, \{D\}, \{\,\}, \{\Phi\})$ where Φ is the faithful inclusion from T to T'.

Realm Graphs [2/2]

- A realm graph of Alonzo is a directed graph $G = (\mathcal{N}, \mathcal{V})$ where:

    1. $\mathcal{N}$ is a set of realms of Alonzo.
    2. $\mathcal{V}$ is a set of development morphisms of Alonzo that connect the theories and developments in a realm to theories and developments in other realms in $\mathcal{N}$.

5. Theory Graph Combinators

Basic Definitions

- A theory graph combinator is a function that takes a theory graph G plus some optional arguments $A_1, \ldots, A_n$ as input and returns as output a theory graph G' that is an extension of G obtained by adding theories and morphisms to G.

- Examples:
  1. Theory extension.
  2. Theory renaming.
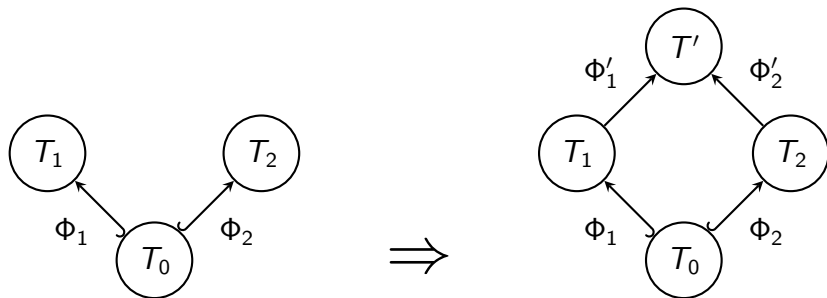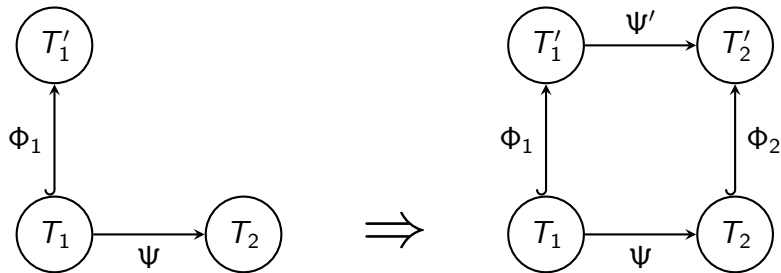 3. Theory combination.
  4. Theory instantiation

Theory Extension

$$T \quad \Longrightarrow \quad T \xrightarrow{\Phi} T'$$
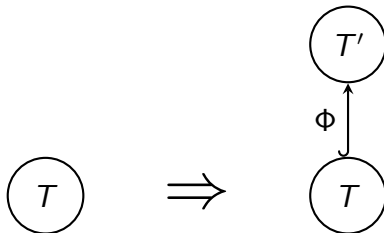
Theory Renaming
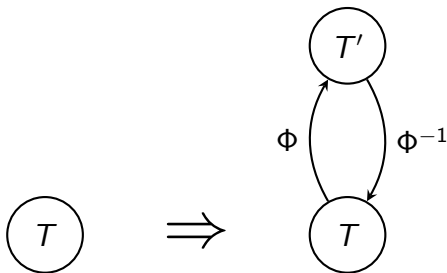
Theory Combination

Theory Instantiation

The End.