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Underactuated Manipulator Robot Control via H2, H∞, H2/H∞, and μ-Synthesis Approaches: a Comparative Study

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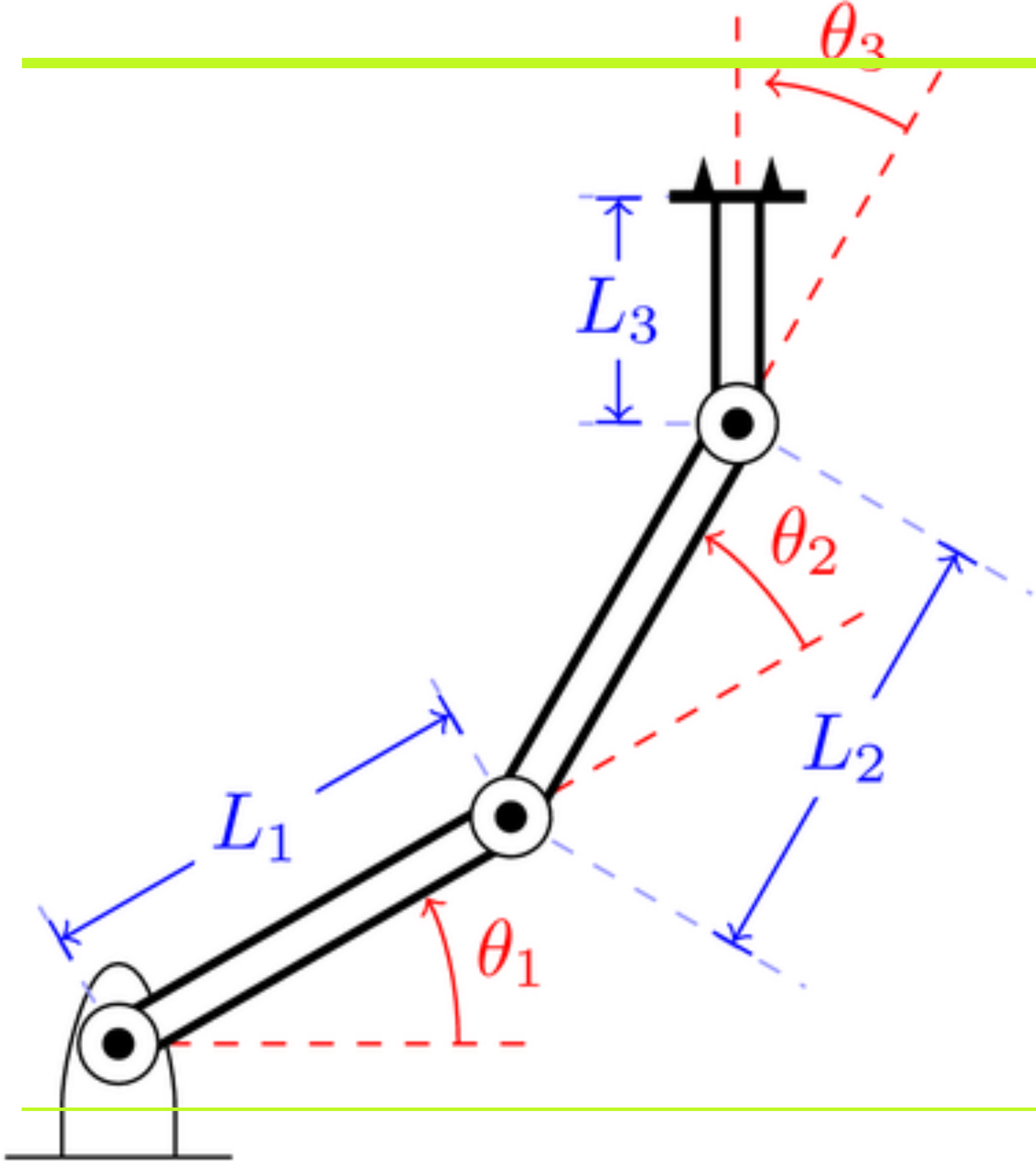
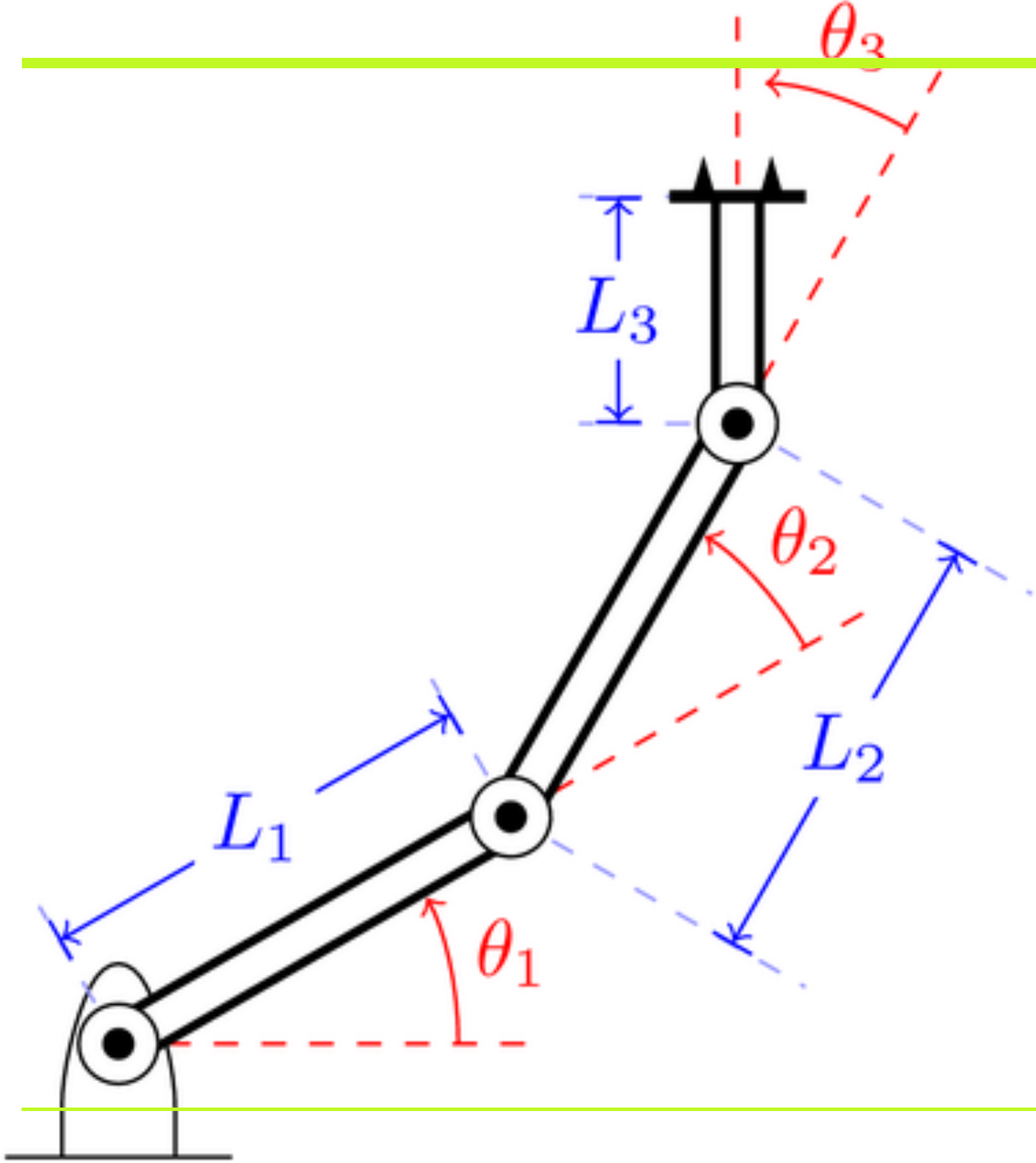


Type of Robot
and its main properties

- Manipulator (UArm II)
- Planar (2D)
- Contains 3 revelatory joints (DC Motors)
- Has 3-DOF (Degrees of Freedom)
- 2 Degrees of freedom for translation (position of the end-effector)
- 1 Degree of freedom for roll (orientation of the end-effector)



Type of Robot
and its main properties

- Fixed Base
- Joint variable output parameters are θ1, θ2, θ3 (q position vector)
- Joint variables are controlled by their corresponding velocities ($\dot{q}$ vector)
- Each joint has an actuator and a brake (active or passive)
- Underactuated (# of Actuators less than # of DOF)

Types of Control

- Torque Control (by Craig (1986) book)
- Robust Control
- Uncertainty with different disturbances not modelled properly:
- 1) joint friction
- 2) torque variation on the actuators
- 3) perturbations due to possible loads carried by the manipulator

Types of Control

- Comparing different types of robust control methods:
- 1) H∞
- 2) H2
- 3) H2/H∞
- 4) μ-Synthesis
- In Design the torque controller is improved - Due to performance reduction for modelling imperfections and external disturbances (Zhou and Doyle (1998))
- All the complex part are omitted as uncertainties - This updated system is linear

Manipulator Dynamics

- Lagrangian Dynamics Method $\longrightarrow$ $\tau = M(q)\ddot{q} + b(q, \dot{q})$ (1)
- q : joint positions $(n \times 1)$
- $\dot{q}$: joint velocities $(n \times 1)$

$\ddot{q}$: joint accelerations $(n \times 1)$

- τ : total torque $(n \times 1)$
- $M(q)$: PD inertia matrix $(n \times n)$
- $b(q, \dot{q})$: Coriolis, centrifugal, gravitational and frictional terms $(n \times 1)$
- $b(q, \dot{q}) = C(q, \dot{q}) + F(\dot{q}) + G(q)$

Manipulator Dynamics

considering number of actuated and passive joints :

$$\begin{bmatrix} \tau_a \\ 0 \end{bmatrix} = \begin{bmatrix} M_{aa}(q) & M_{au}(q) \\ M_{ua}(q) & M_{uu}(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_a \\ \ddot{q}_u \end{bmatrix} + \begin{bmatrix} b_a(q, \dot{q}) \\ b_u(q, \dot{q}) \end{bmatrix} \quad (2)$$

a: actuated u: under-actuated

two control phases (joint 2 is a brake and $n_a > n_u$):

1) $q_a = q_1$ $q_u = q_2$ $q_L = q_3$ L: locked

2) $q_a = [q_1 \;\; q_3]^T$ $q_u = []$ $q_L = q_2$

Manipulator Dynamics

likewise :

$$\tau_a = \bar{M}\ddot{q}_u + \bar{b} \tag{3}$$

- $\bar{M} = M_{au} - M_{aa}M_{ua}^{-1}M_{uu}$
- $\bar{b} = b_a - M_{aa}M_{ua}^{-1}b_u$

Computed Torque Method

To handle the nonlinear dynamics equation "Computed Torque Method" is used :

$$\tau_a = \bar{M}_{est}(q)\tau'_a + \bar{b}_{est}(q, \dot{q}) \tag{4}$$

$\bar{M}_{est}(q)$: robot inertial elements estimated model

$\bar{b}_{est}(q, \dot{q})$: robot non-inertial elements estimated model

Computed Torque Method

$$\tau_a' = \ddot{q}_u^d + K_v\left(\dot{q}_u^d - \dot{q}_u\right) + K_p\left(q_u^d - q_u\right) \tag{5}$$

$q_u^d, \dot{q}_u^d, \ddot{q}_u^d$ are desired trajectory, desired velocity and desired acceleration of controlled joints respectively

K_v and K_p are $n \times n$ diagonal matrices with positive elements

$$e = q_u^d - q_u \tag{6}$$

(4), (5), (6) $\longrightarrow$ $\ddot{e} + K_v\dot{e} + K_p e = \bar{M}_{est}^{-1}(q)\left[\left(\bar{M}(q) - \bar{M}_{est}(q)\right)\ddot{q} + \bar{b}(q,\dot{q}) - \bar{b}_{est}(q,\dot{q})\right]$ (7)

Computed Torque Method

$$\ddot{e} + K_v\dot{e} + K_pe = \bar{M}_{est}^{-1}(q)\left[\left(\bar{M}(q) - \bar{M}_{est}(q)\right)\ddot{q} + \bar{b}(q,\dot{q}) - \bar{b}_{est}(q,\dot{q})\right] \xrightarrow{\text{adding disturbances}}$$

$$\ddot{e} + K_v\dot{e} + K_pe = \bar{M}_{est}^{-1}(q)\left[\left(\bar{M}(q) - \bar{M}_{est}(q)\right)\ddot{q} + \bar{b}(q,\dot{q}) - \bar{b}_{est}(q,\dot{q}) + \bar{d}_{est}(q,\dot{q})\right] \tag{7}$$

we can cross-out the right hand side by: $\bar{M}(q) = \bar{M}_{est}(q)$, $\bar{b}(q,\dot{q}) = \bar{b}_{ext}(q,\dot{q})$, $\bar{d}_{ext}(q,\dot{q}) = 0$

- we reach an ideal system with absolute precision

Computed Torque Method

$$\ddot{e} + K_v\dot{e} + K_pe = \bar{M}_{est}^{-1}(q)\left[\left(\bar{M}(q) - \bar{M}_{est}(q)\right)\ddot{q} + \bar{b}(q,\dot{q}) - \bar{b}_{est}(q,\dot{q})\right] \xrightarrow{\text{adding disturbances}}$$

$$\ddot{e} + K_v\dot{e} + K_pe = \cancel{\bar{M}_{est}^{-1}(q)\left[\left(\bar{M}(q) - \bar{M}_{est}(q)\right)\ddot{q} + \bar{b}(q,\dot{q}) - \bar{b}_{est}(q,\dot{q}) + d_{est}(q,\dot{q})\right]} \tag{7}$$

$$\ddot{e} + K_v\dot{e} + K_pe = 0 \tag{8}$$

Computed Torque Method

The linear state space system for the nominal plant controlled with $u(t)$ is :

$$\begin{bmatrix} \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K_p & -K_v \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \quad , \quad x = \begin{bmatrix} e \\ \dot{e} \end{bmatrix} = \begin{bmatrix} q^d - q \\ \dot{q}^d - \dot{q} \end{bmatrix}$$

$$\downarrow$$

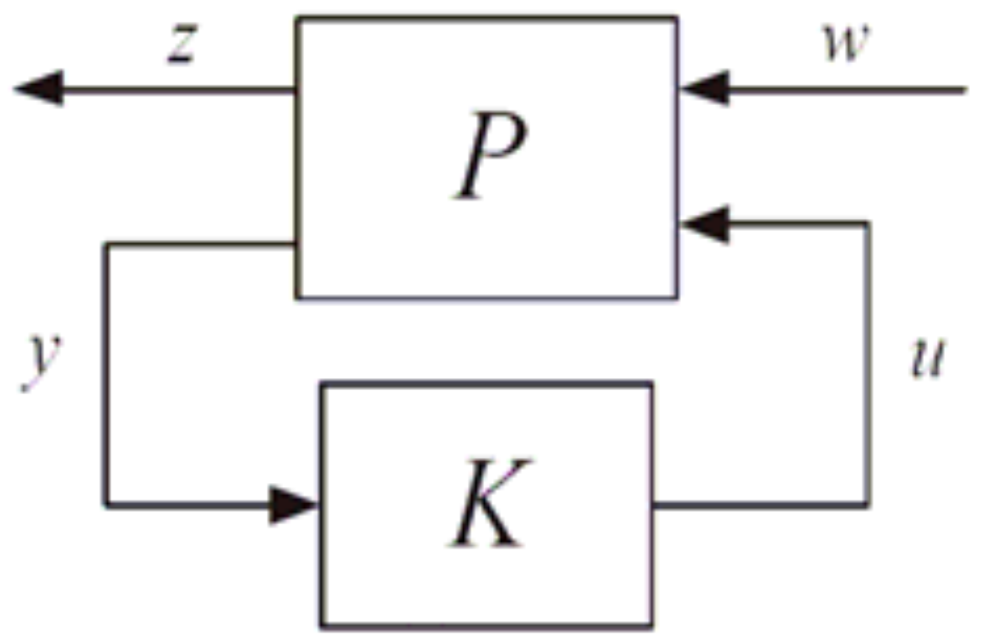
$$A = \begin{bmatrix} 0 & I \\ -K_p & -K_v \end{bmatrix} \quad B_g = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad C_g = \begin{bmatrix} I & 0 \end{bmatrix}$$

Robust Control Configuration

We need to form the MK standard form for this system :

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

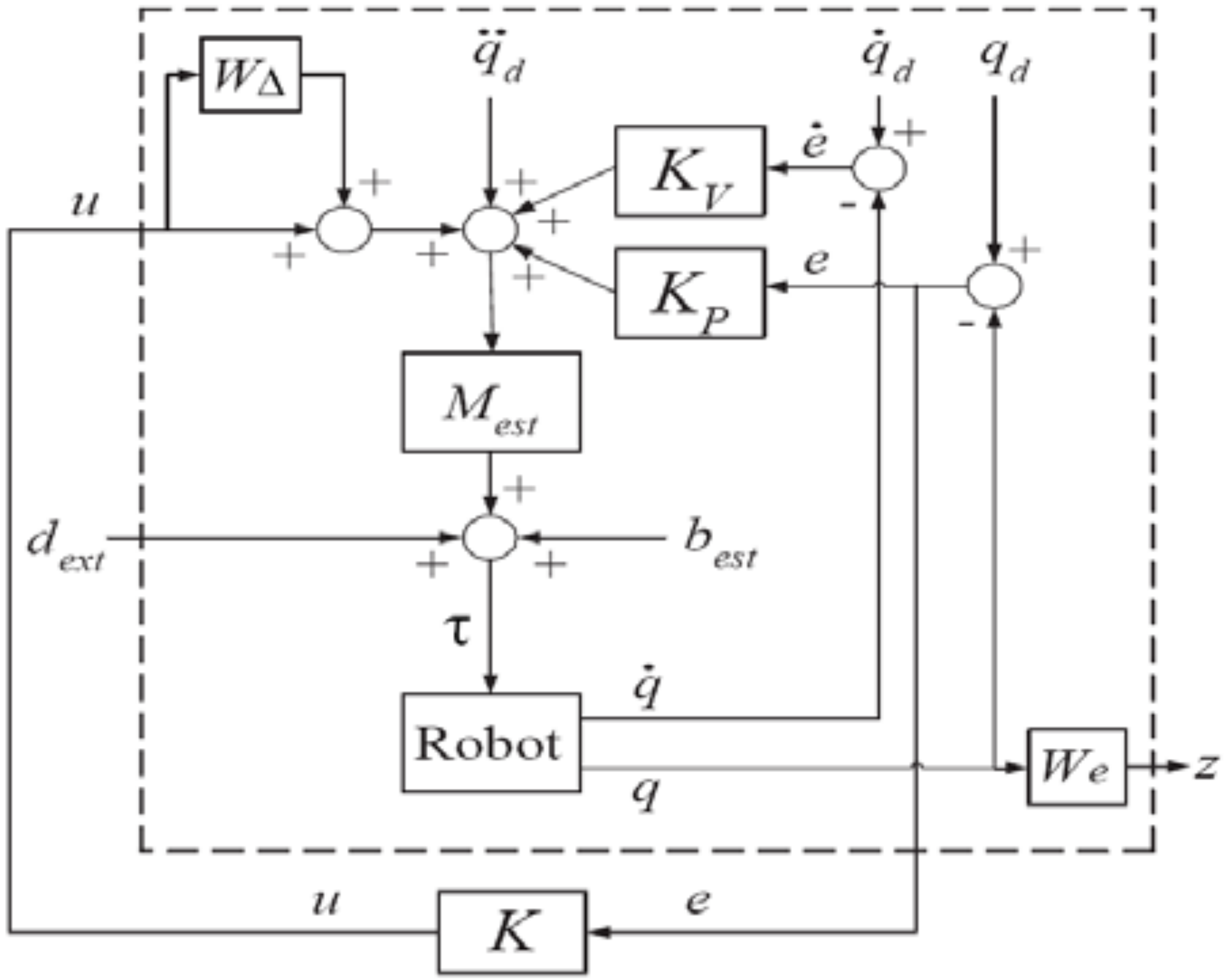
$$z = P_{11}w + P_{12}u \quad y = P_{21}w + P_{22}u \quad u = Ky$$



$$\text{LFT} : T_{wz}(s) = F_1(P, K) = P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1} P_{21}$$

Robust Control Configuration

The augmented system is shown bellow :



Robust Control Configuration

Two performance weighting transfer functions are defined :

- $W_e(s)$: To Control the frequency response of the sensitivity function $S(s) = (I + P(s)K(s))^{-1}$
- $W_\Delta(s)$: To shape the multiplicative unstructured uncertainties in the input of the plant

$$|W_e(j\omega)|^{-1} \geq |S(j\omega)|$$

$$|W_\Delta(j\omega)|^{-1} \geq |K(j\omega)S(j\omega)|$$

Robust Control Configuration

$$W_e(s) = \text{diag}\left\{F_{e,1}(s), \ldots, F_{e,n}(s)\right\} F_{e,i}(s) = \frac{\frac{s}{M_s} + \omega_b}{s + \omega_b \varepsilon}$$

M_s : Peak sensitivity

ω_b : The bandwidth

ε : The damping ratio

Robust Control Configuration

$$W_\Delta(s) = \text{diag}\left\{F_{\Delta,1}(s), \ldots, F_{\Delta,n}(s)\right\} F_{\Delta,i}(s) = \frac{s + \frac{\omega_{bc}}{M_u}}{\varepsilon_1 s + \omega_{bc}}$$

M_u : Maximum gain of $K(s)S(s)$

ω_{bc} : The controller bandwidth

ε_1 : a small positive value

Robust Control Configuration

The state space of the augmented system is given bellow :

$$A = \begin{bmatrix} 0 & I & 0 & 0 \\ -K_p & -K_v & 0 & 0 \\ 0 & 0 & A_{W_\Delta} & 0 \\ B_{W_e} & 0 & 0 & A_{W_e} \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & B_{W_e} \end{bmatrix} B_2 = \begin{bmatrix} 0 \\ I \\ B_{W_\Delta} \\ 0 \end{bmatrix} \quad C_1 = \begin{bmatrix} 0 & 0 & C_{W_\Delta} & 0 \\ D_{W_e} & 0 & 0 & C_{W_e} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} I & 0 & 0 & 0 \end{bmatrix} \quad D_{11} = \begin{bmatrix} 0 & 0 \\ 0 & D_{W_e} \end{bmatrix} \quad D_{12} = \begin{bmatrix} D_{W_\Delta} \\ 0 \end{bmatrix}$$

$$D_{21} = \begin{bmatrix} 0 & I \end{bmatrix} D_{22} = \begin{bmatrix} 0 \end{bmatrix}$$

The state space realization of the weighting function are augmented in these matrices

Robust Control Configuration

The augmented P matrix is given by :

$$P = \left[\begin{array}{cccc|ccc} 0 & I & 0 & 0 & 0 & 0 & 0 \\ -K_p & -K_v & 0 & 0 & I & 0 & I \\ 0 & 0 & A_{W_\Delta} & 0 & 0 & 0 & B_{W_\Delta} \\ B_{W_e} & 0 & 0 & A_{W_e} & 0 & B_{W_e} & 0 \\ \hline 0 & 0 & C_{W_\Delta} & 0 & 0 & 0 & D_{W_\Delta} \\ D_{W_e} & 0 & 0 & C_{W_e} & 0 & D_{W_e} & 0 \\ I & 0 & 0 & 0 & 0 & I & 0 \end{array}\right]$$

Simulation Parameters

The parameters needed for simulating are in the following tables :

Link	m_i (kg)	I_i (kgm^2)	l_i (m)	l_{ci} (m)
1	0.850	0.0075	0.203	0.096
2	0.850	0.0075	0.203	0.096
3	0.625	0.0060	0.203	0.077

Configuration	K_p	K_v
APA, phase 1	[20]	[20]
APA, phase 2	$\begin{bmatrix} 10 & 0 \\ 0 & 20 \end{bmatrix}$	$\begin{bmatrix} 5 & 0 \\ 0 & 20 \end{bmatrix}$

Simulation Parameters

Configuration	M_s	ω_b	ε
APA, phase 1	1.1	1	0.01
APA, phase 2	1.5	1	0.01

Configuration	M_u	ω_{bc}	ε_1
APA, phase 1	20	10^6	1
APA, phase 2	50	10^6	1

In phase 2 the weighting functions should be designed (2×2) and diagonally.

Simulation with H_∞

The MATLAB function for this synthesis is :

```
[K_Hinf, sys_CL_Hinf, gamma, INFO_Hinf] = hinfsyn(P,Kp_n,Kp_n)
```

Kp_n : number of controlled joints (phase 1 : 1 , phase 2 : 2)

P : the augmented (state space) system defined earlier

Simulation with H_∞

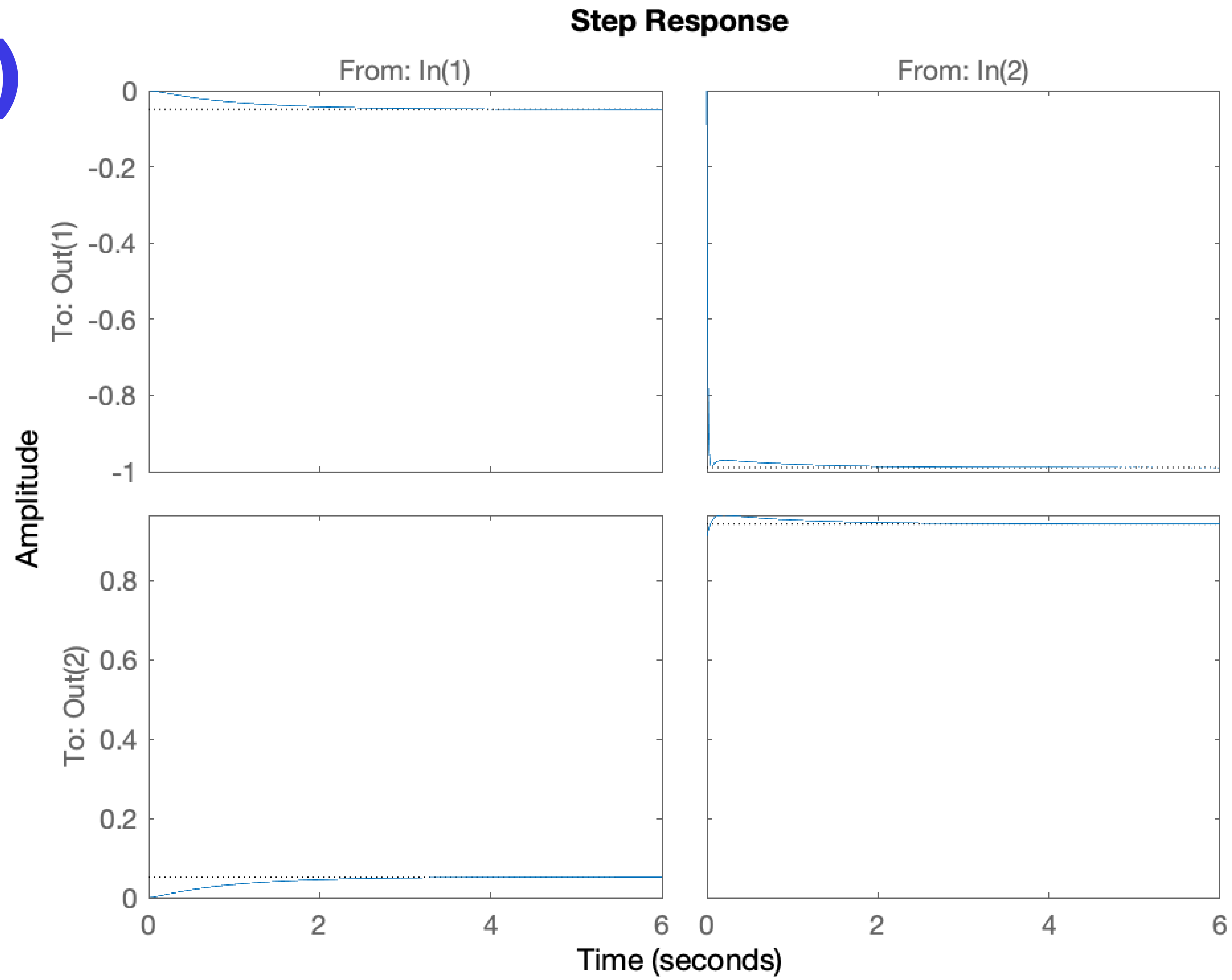
Synthesis Results (phase 1)

gamma =

1.3678

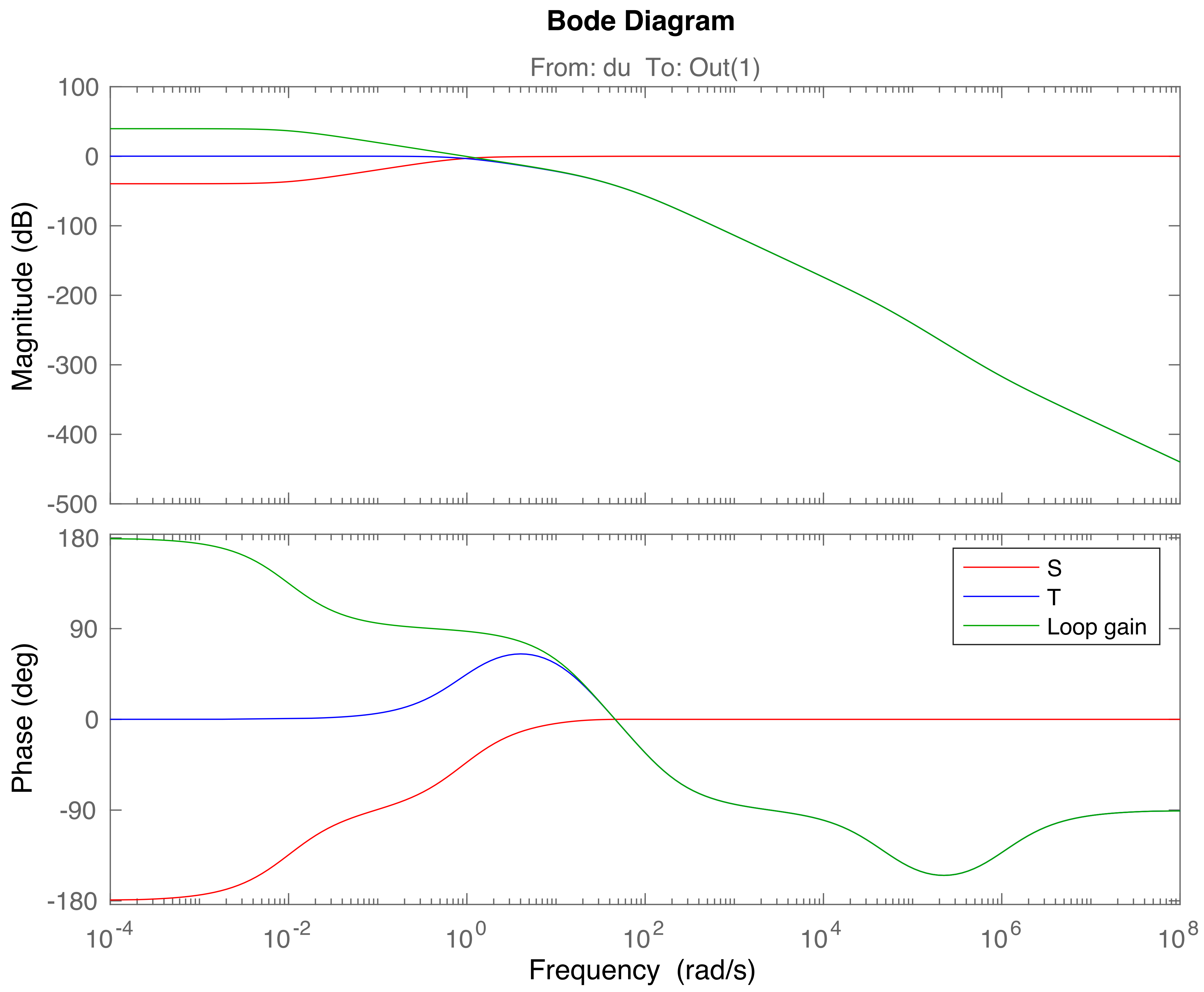
K_Hinf_tf =

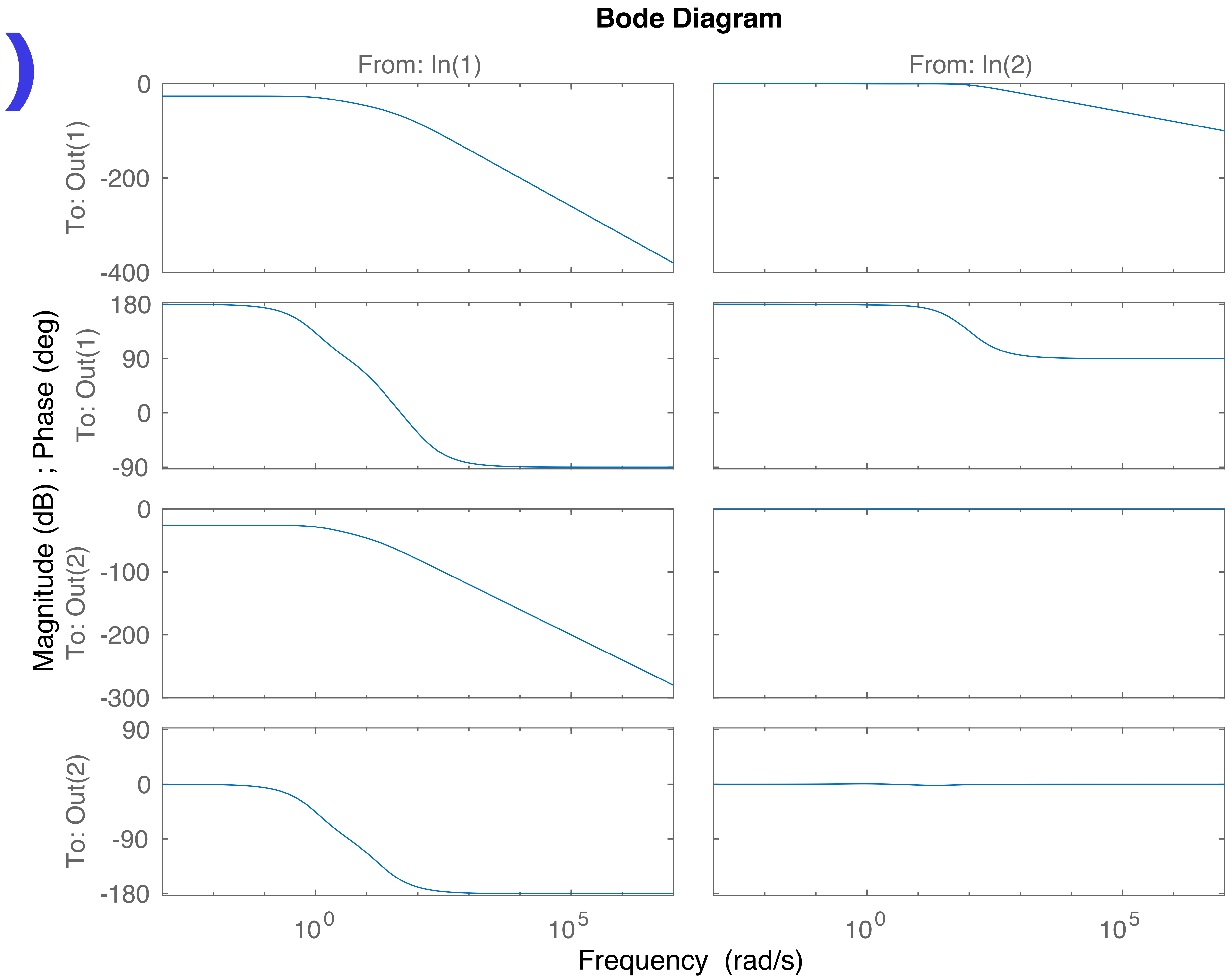
$$\frac{-101.9\,s^3 - 1.019e08\,s^2 - 2.038e09\,s - 2.038e09}{s^4 + 5.012e04\,s^3 + 6.033e06\,s^2 + 1.062e08\,s + 1.062e06}$$



Simulation with H_∞

Synthesis Results (phase 1)





Simulation with H_∞

Synthesis Results (phase 1)

The results show :

- stability (no comment on robustness)
- good performance
- best performance in lower frequencies

Simulation with H_∞

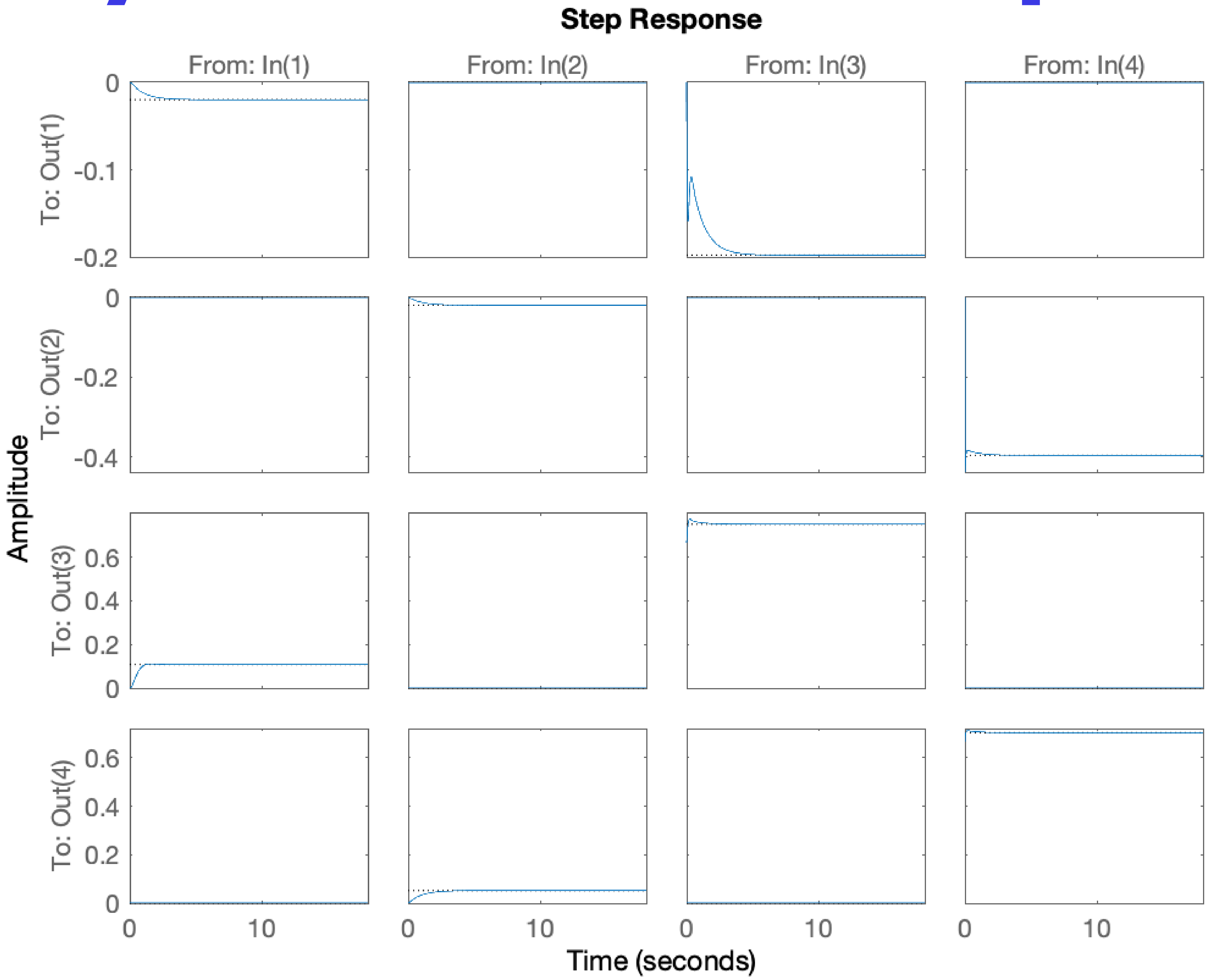
Synthesis Results (phase 2)

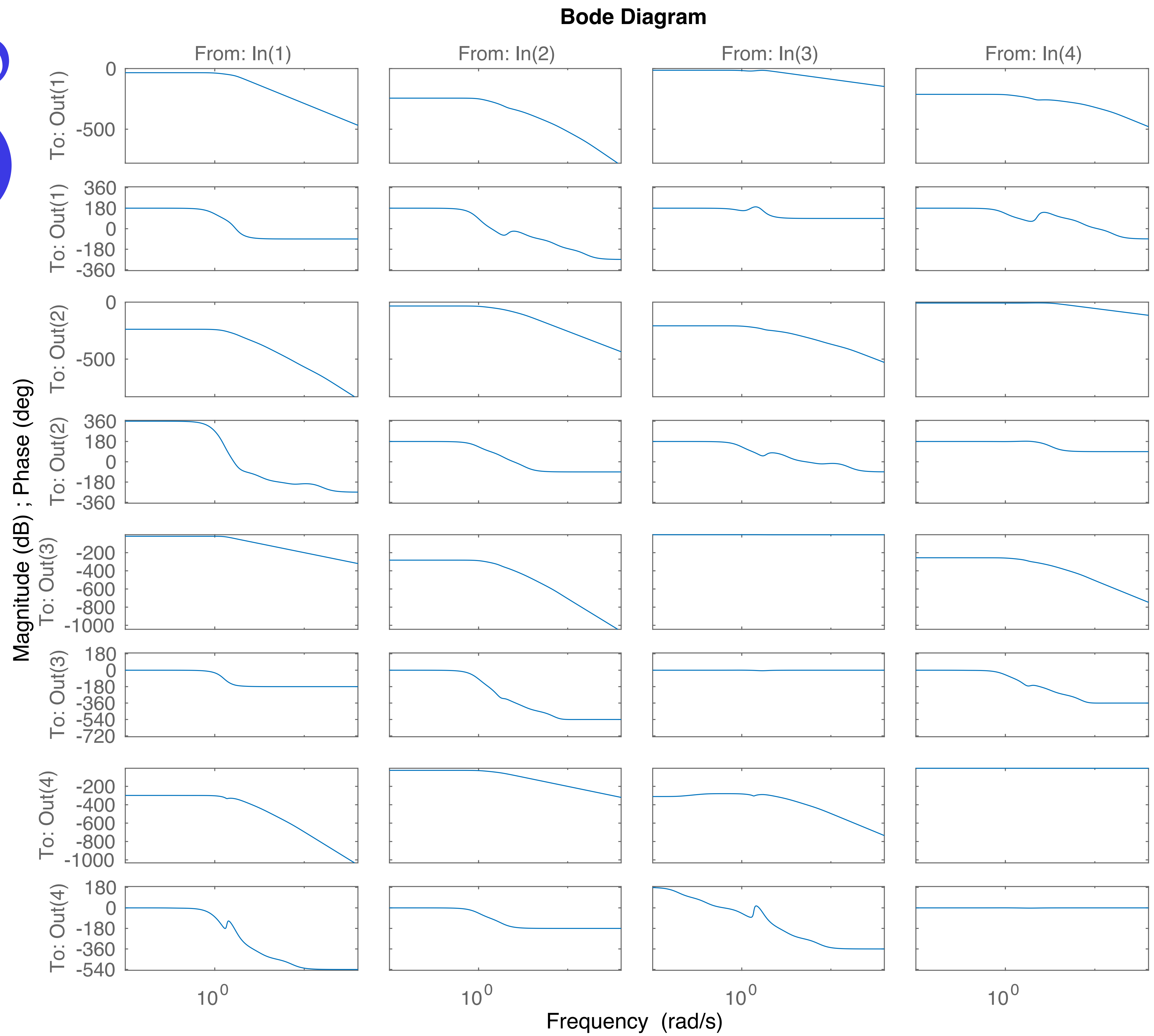
gamma =

0.8084

```
K_Hinf_tf =

  From input 1 to output...
       -4.038 s^7 - 4.12e06 s^6 - 8.237e10 s^5 - 3.176e13 s^4 - 8.595e14 s^3 - 3.829e15 s^2

                                                           - 7.001e15 s - 6.963e13

   1:  -------------------------------------------------------------------------------------
         s^8 + 4.041e04 s^7 + 4.164e08 s^6 + 1.649e11 s^5 + 7.117e12 s^4 + 1.135e14 s^3

                                             + 7.632e14 s^2 + 1.523e13 s + 7.609e10

       1.669e-05 s^6 + 0.3317 s^5 + 4.398 s^4 - 98.17 s^3 - 1773 s^2 - 1.282e04 s - 128

   2:  ---------------------------------------------------------------------------------
       s^8 + 4.041e04 s^7 + 4.164e08 s^6 + 1.649e11 s^5 + 7.117e12 s^4 + 1.135e14 s^3

                                           + 7.632e14 s^2 + 1.523e13 s + 7.609e10

  From input 2 to output...
         4.419e-06 s^6 + 0.1868 s^5 + 4.989 s^4 + 106 s^3 + 222.6 s^2 - 1346 s - 13.47

   1:  ---------------------------------------------------------------------------------
       s^8 + 4.041e04 s^7 + 4.164e08 s^6 + 1.649e11 s^5 + 7.117e12 s^4 + 1.135e14 s^3

                                           + 7.632e14 s^2 + 1.523e13 s + 7.609e10

       -166.5 s^7 - 1.699e08 s^6 - 3.338e12 s^5 - 1.428e14 s^4 - 2.318e15 s^3

                                           - 1.613e16 s^2 - 1.475e16 s - 1.458e14

   2:  ---------------------------------------------------------------------------------
       s^8 + 4.041e04 s^7 + 4.164e08 s^6 + 1.649e11 s^5 + 7.117e12 s^4 + 1.135e14 s^3

                                           + 7.632e14 s^2 + 1.523e13 s + 7.609e10
```

Simulation with H_∞ Synthesis Results (phase 2)

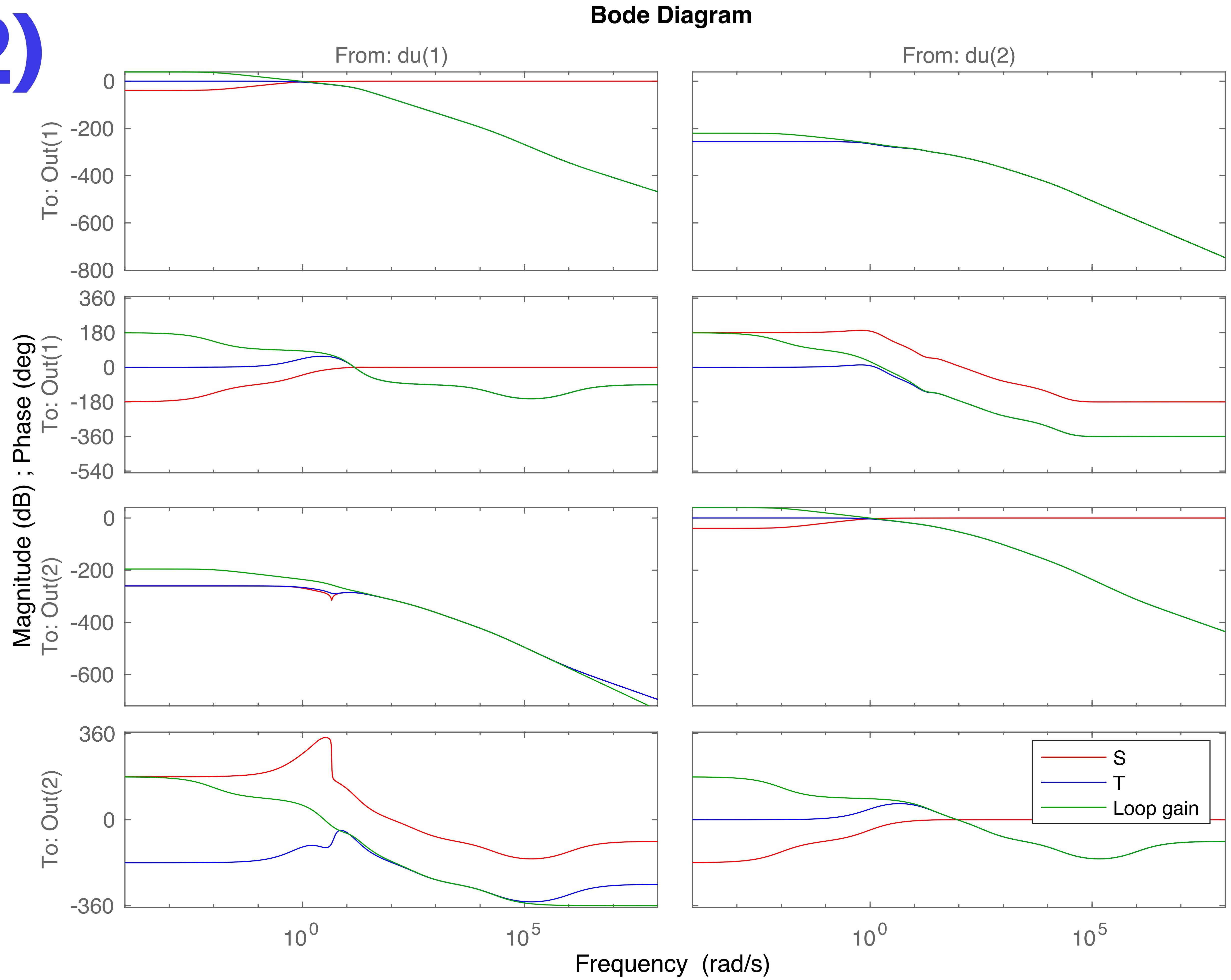




Simulation with H_∞

Synthesis Results (phase 2)

- The Sensitivity function is lower in more frequencies than phase 1
- From i2o1 and i1o2 we can see that both S(s) and T(s) are declining as frequency increases



Simulation with H_∞

Synthesis Results (phase 2)

The results show :

- robust stability
- very good performance
- best performance in lower frequencies

Simulation with H_2

The MATLAB function for this synthesis is :

```
[K_H2, sys_CL_H2, gamma_H2, INFO_H2] = hinfsyn(P,Kp_n,Kp_n)
```

Kp_n : number of controlled joints (phase 1 : 1 , phase 2 : 2)

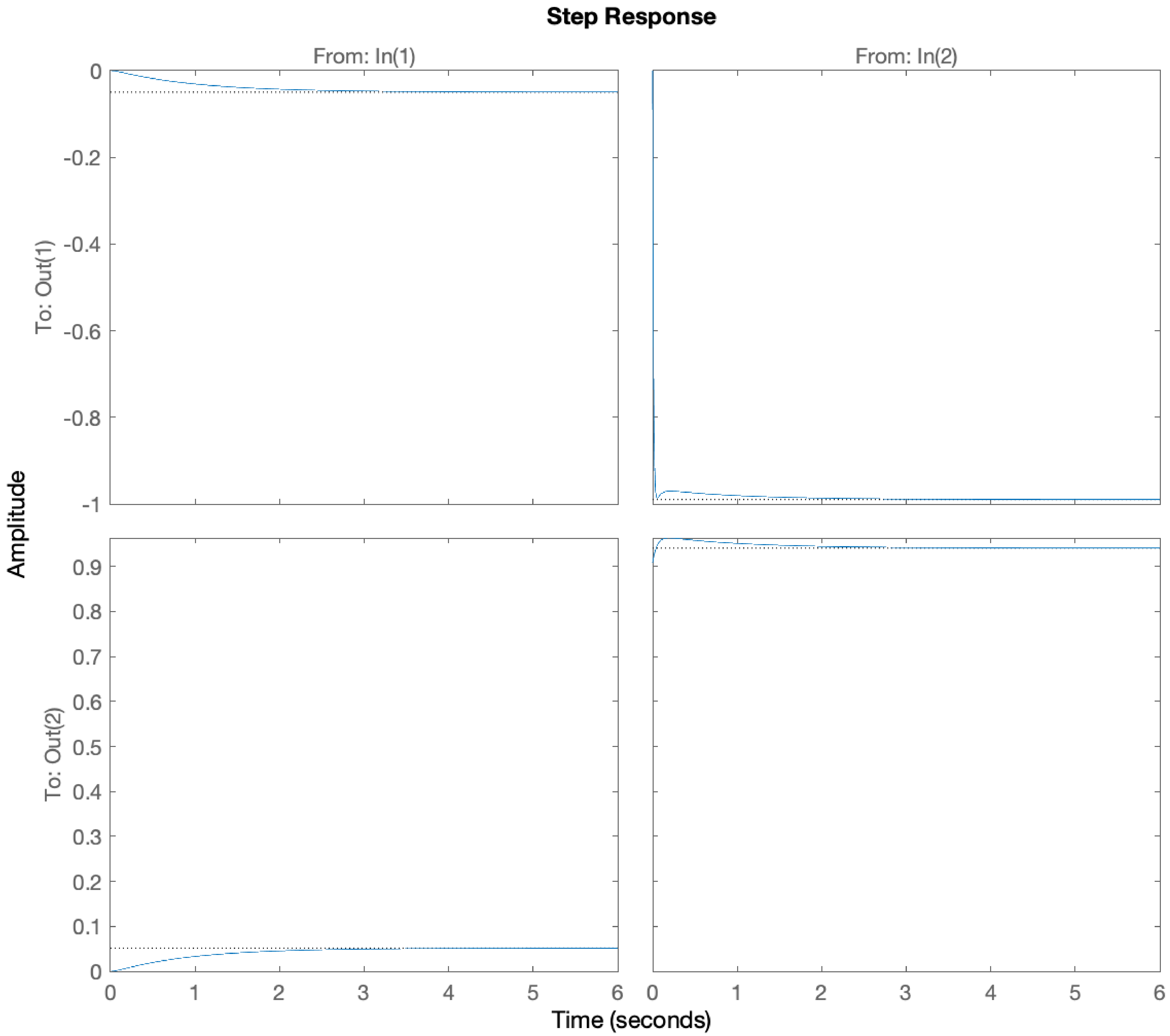
P : the augmented (state space) system defined earlier

Simulation with H_2

Synthesis Results (phase 1)

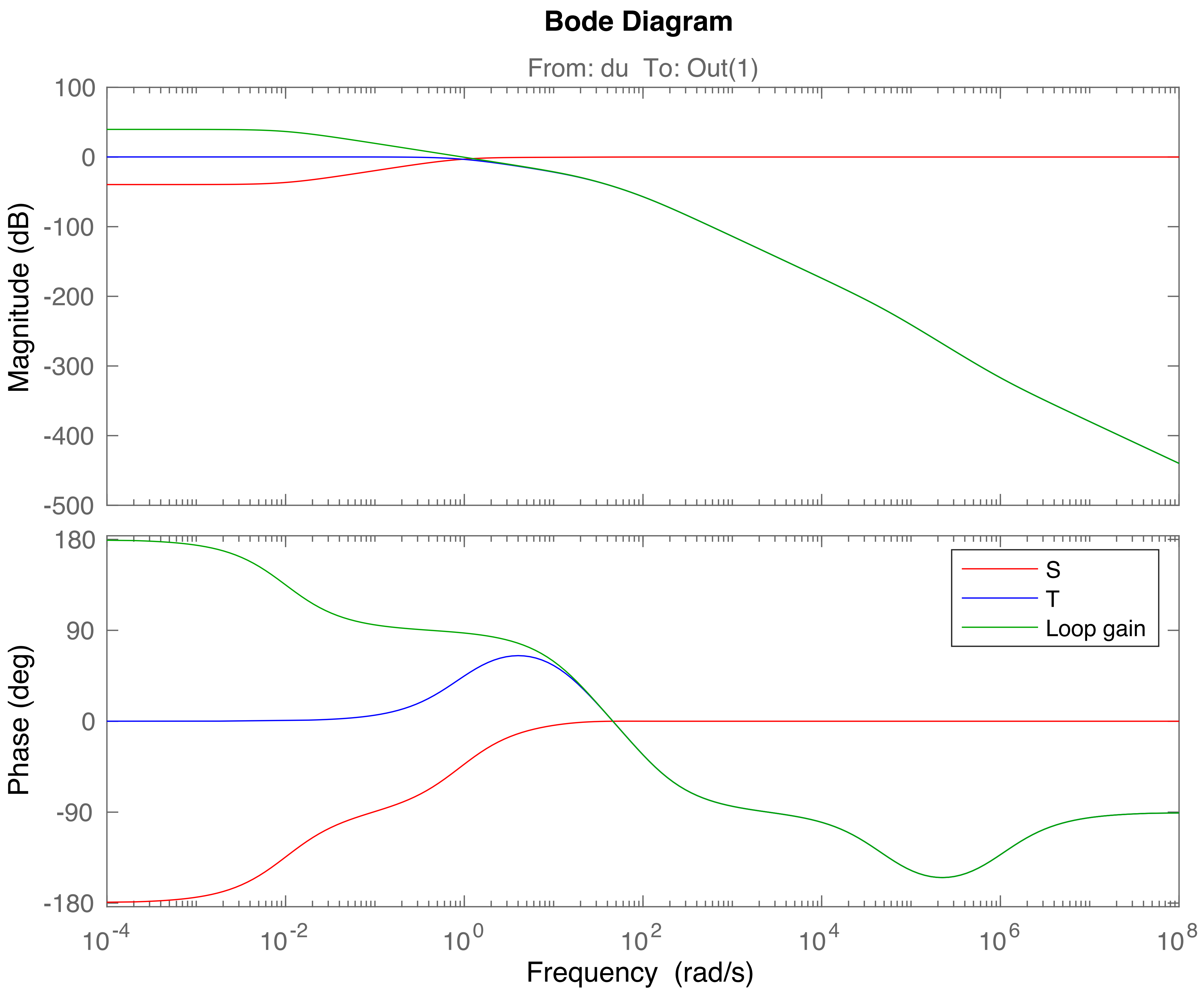
gamma_H2 =

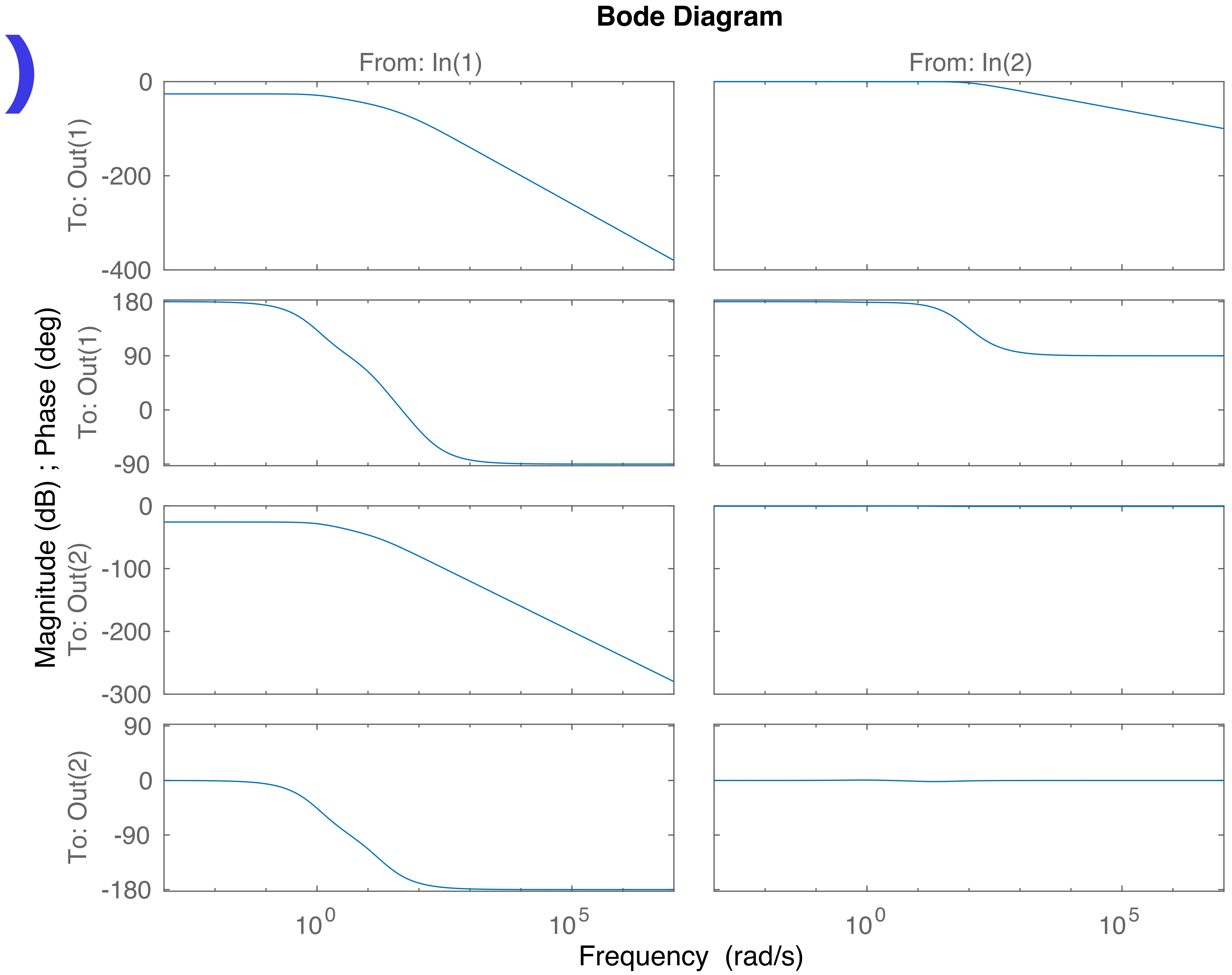
1.3678



Simulation with H_2

Synthesis Results (phase 1)





Simulation with H_2

Synthesis Results (phase 1)

The results show :

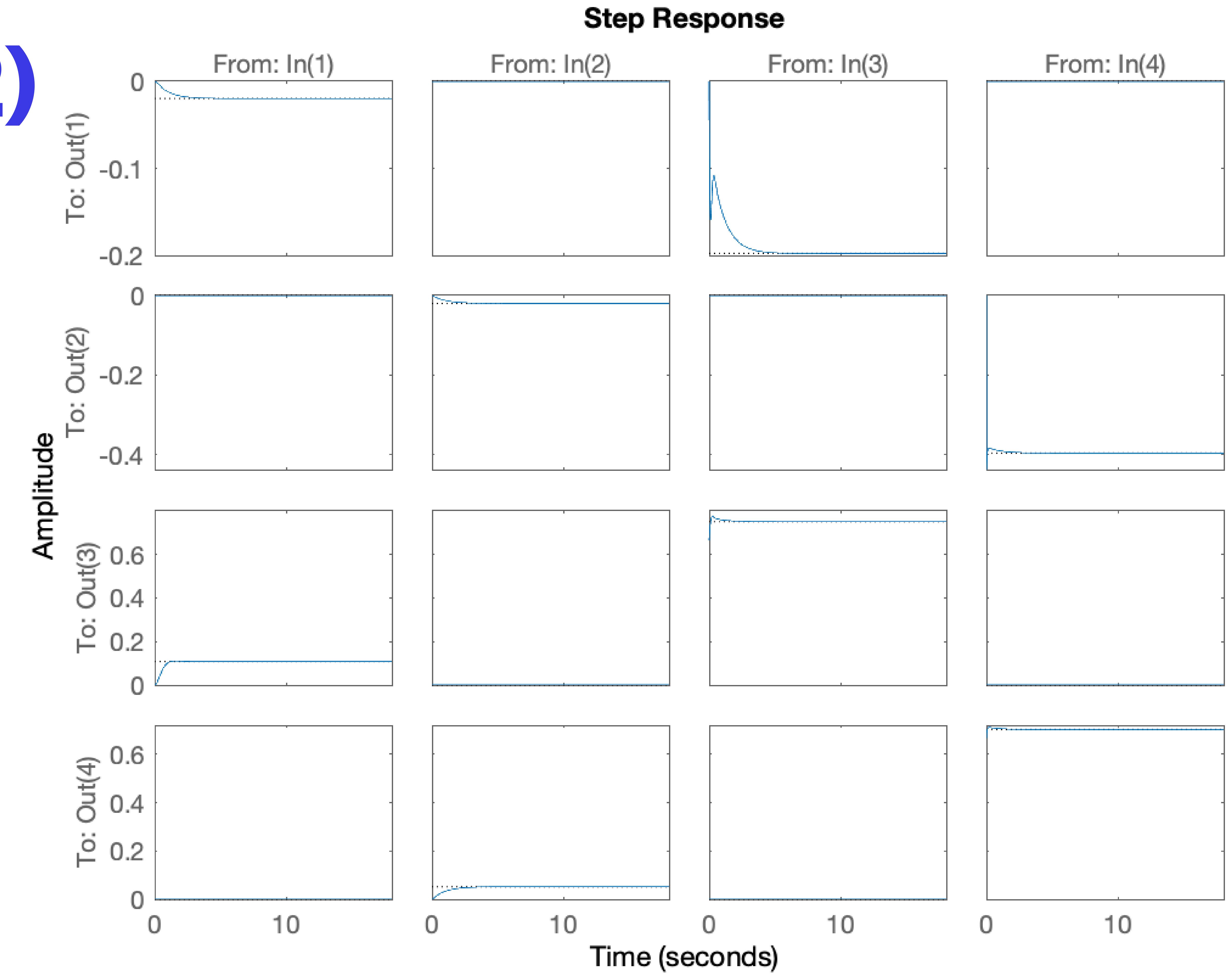
- stability (no comment on robustness)
- good performance
- best performance in lower frequencies

Simulation with H_2

Synthesis Results (phase 2)

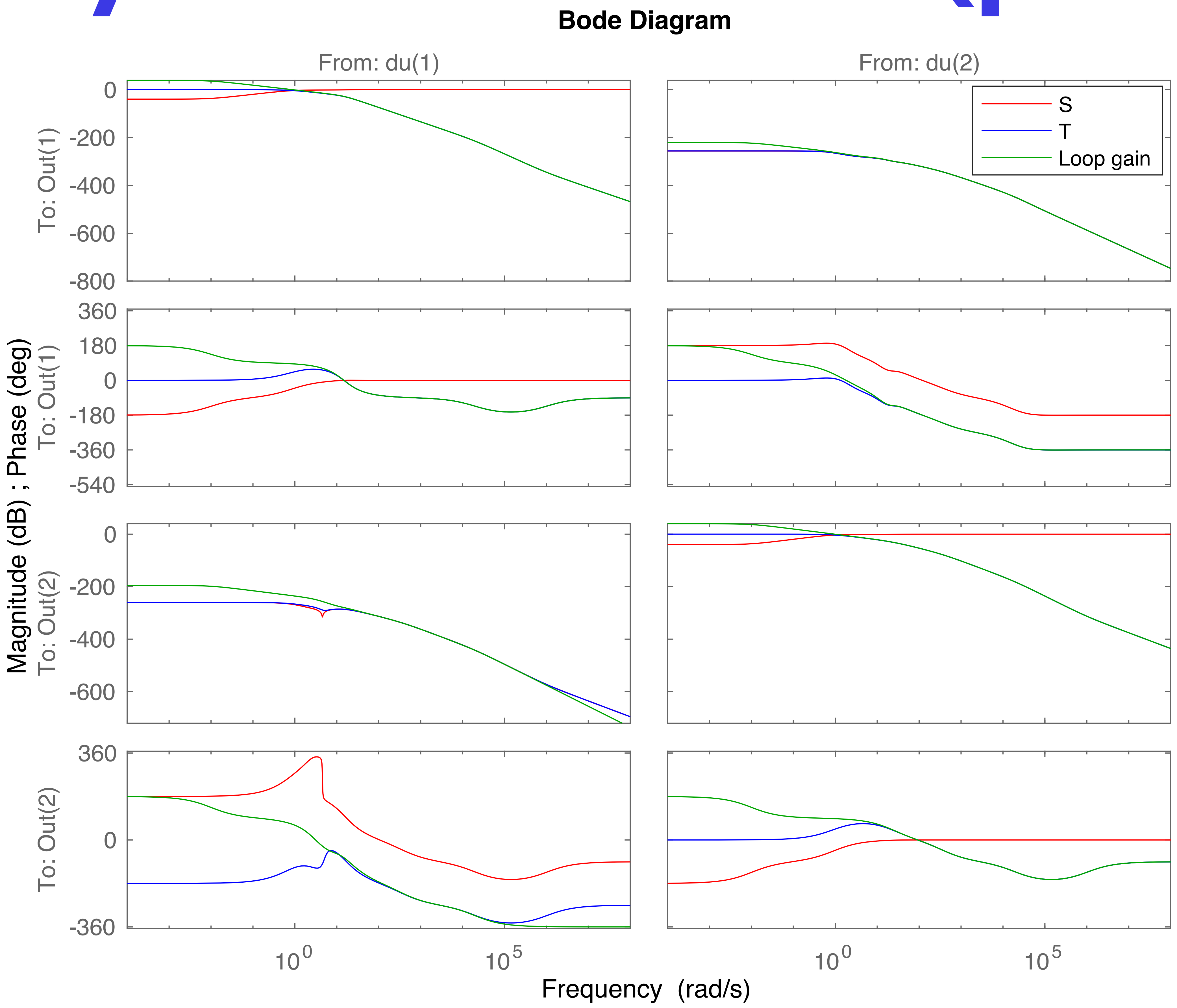
gamma_H2 =

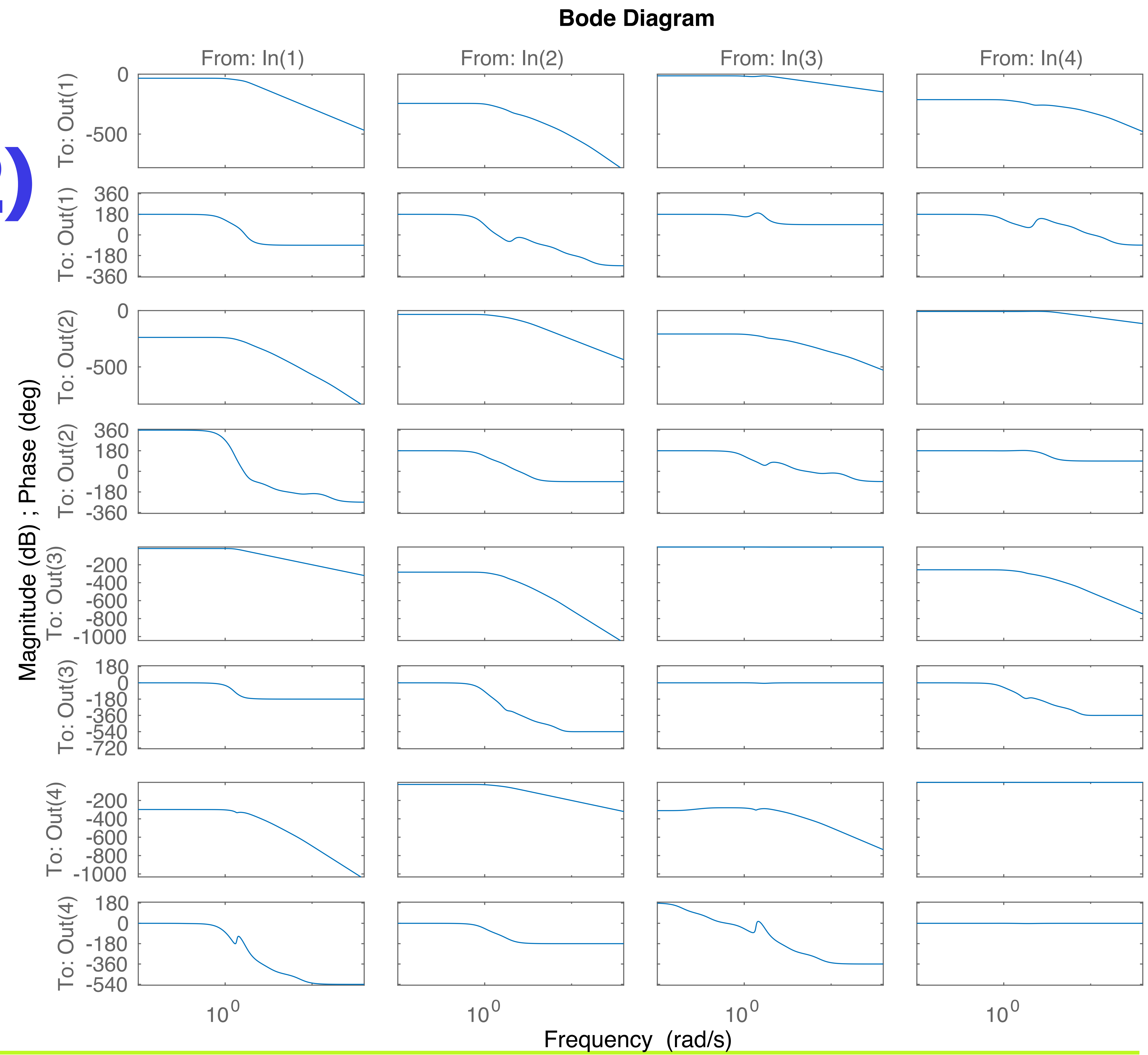
0.8084



Simulation with H_2

Synthesis Results (phase 2)





Simulation with H_2

Synthesis Results (phase 2)

The results show :

- stability (but no comment on robustness because of H2 sythesis)
- very good performance
- best performance in lower frequencies

Simulation with H_2/H_∞

Synthesis Results (phase 1 & 2)

The results show :

- infinite H2 - This method would not work
- very low gammas in an unacceptable way
- stable but very bad performance

Simulation with μ

- This simulation was made using the D-K iteration method
- The input system to the D-K method should be uncertain
- The uncertain model of the augmented system was made by putting a multiplicative uncertainty in the input of the plant with weighting function W_Δ

Simulation with μ

The MATLAB function for this synthesis is :

```
fmu = logspace(-2,4,60);
opt =
dksynOptions('FrequencyVector',fmu,'NumberofAutoIterations',5,'DisplayWhileAutoIter','on','MixedMU','on');
[K_DK, sys_CL_DK,bnd,INFO_DK] = dksyn(P_uss,Kp_n,Kp_n,opt)
```

Kp_n : number of controlled joints (phase 1 : 1 , phase 2 : 2)

P : the augmented (state space) system defined earlier

Simulation with μ

Synthesis Results (phase 1)

Iteration Summary

Iteration #	1	2	3	4	5
Controller Order	4	4	6	6	6
Total D-Scale Order	0	0	2	2	2
Gamma Acheived	1.659	0.982	0.924	0.922	0.922
Peak mu-Value	1.380	0.981	0.924	0.922	0.922

bnd =

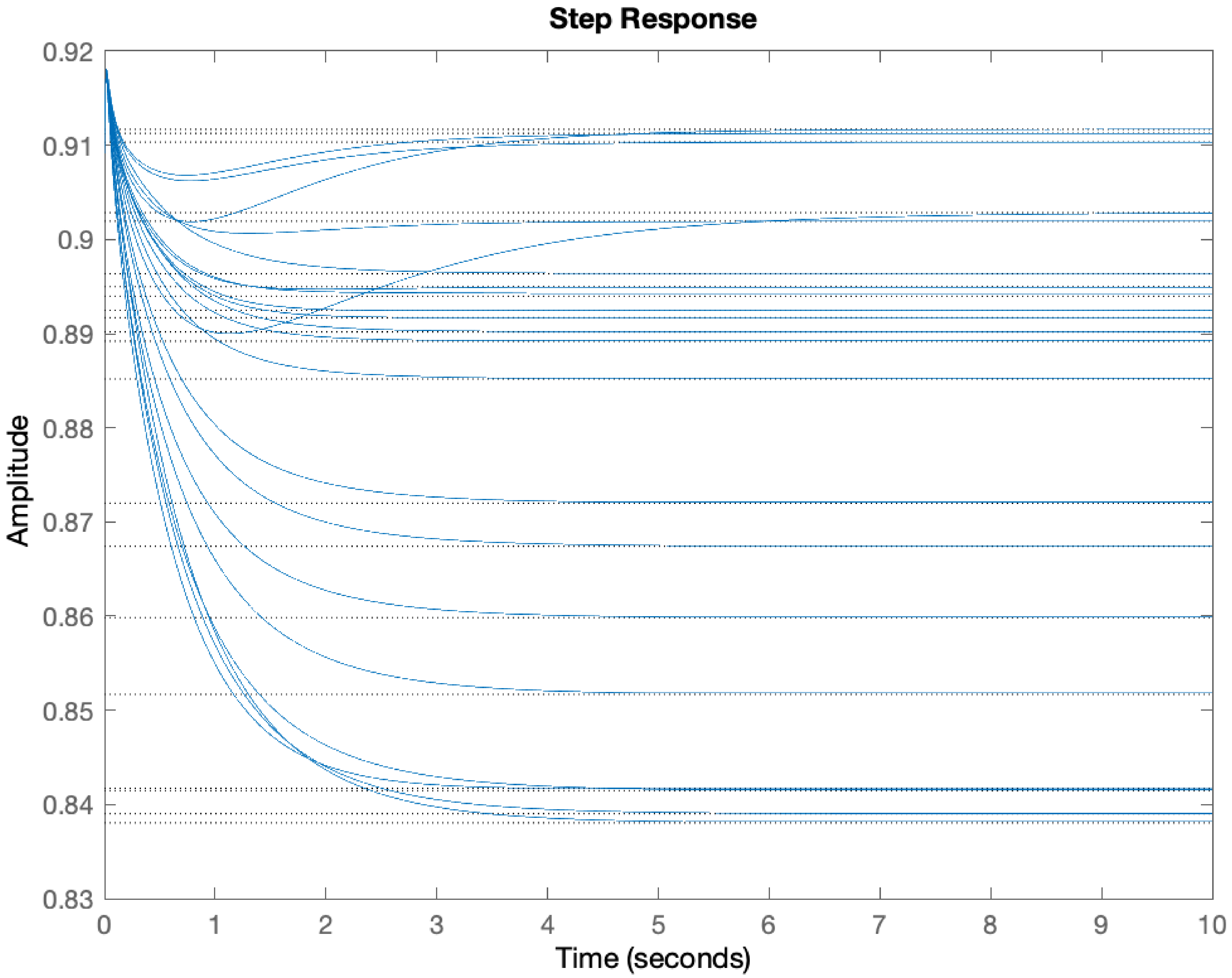
0.9216

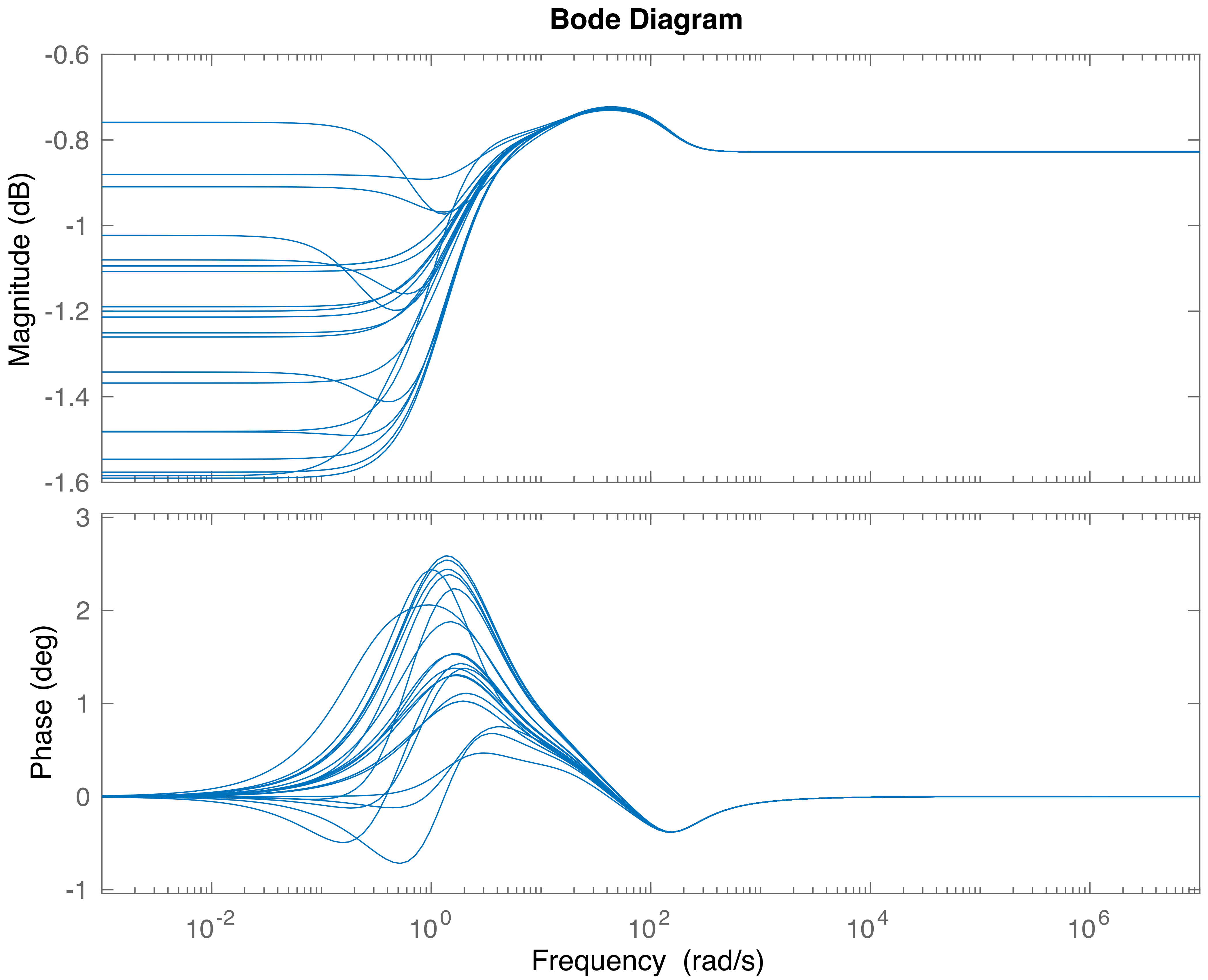
K_DK_tf =

$$\frac{-1685 s^5 - 1.685e09 s^4 - 8.934e10 s^3 - 1.194e12 s^2 - 3.028e12 s - 1.97e12}{s^6 + 5.031e04 s^5 + 1.545e07 s^4 + 1.816e09 s^3 + 4.6e10 s^2 + 8.713e10 s + 8.667e08}$$

Simulation with μ

Synthesis Results (phase 1)

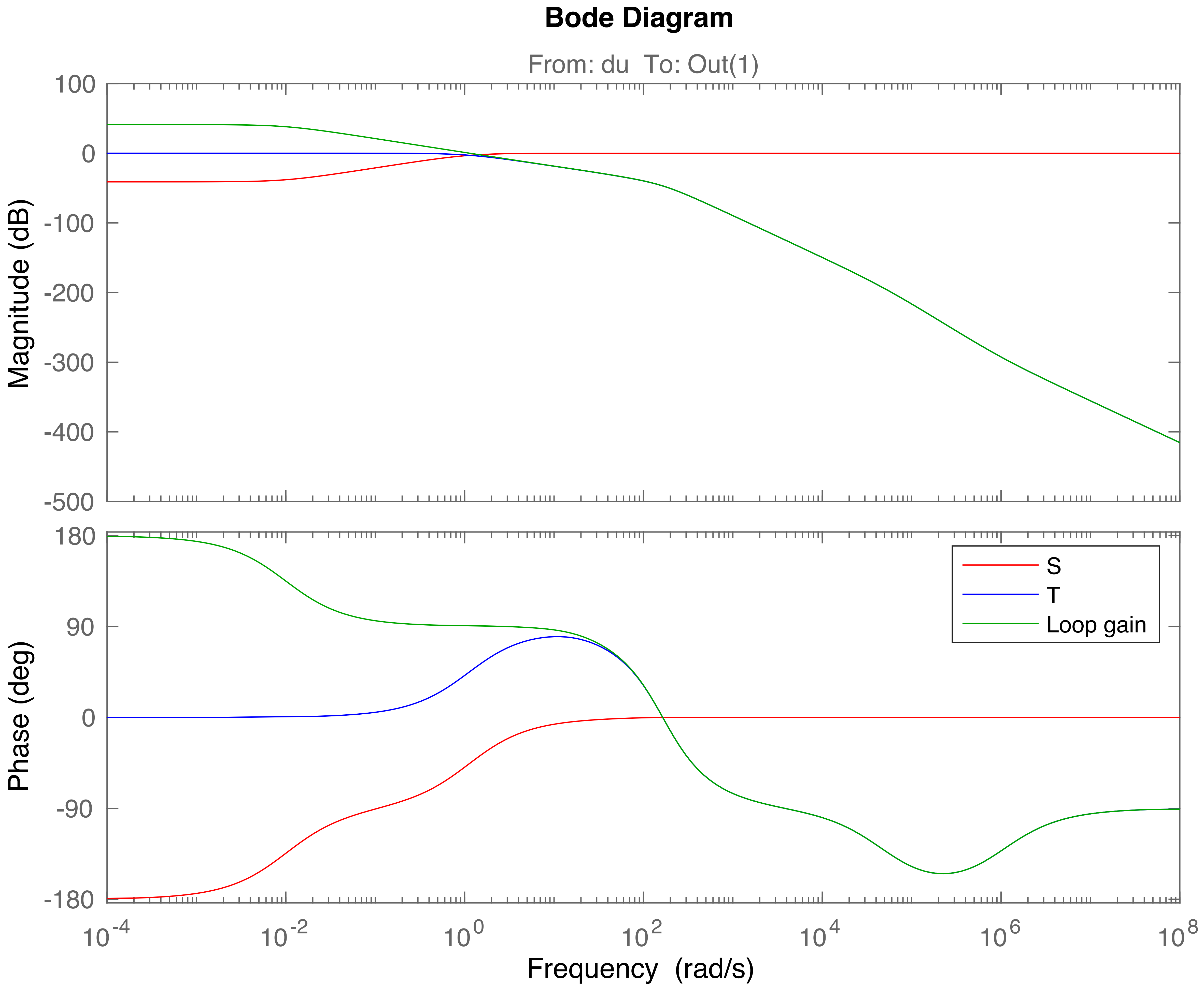




Simulation with μ

Synthesis Results (phase 1)

- Low sensitivity in lower frequencies
- sensitivity value goes to 1 as sensitivity complement and loop gain descend towards zero



Simulation with μ

Synthesis Results (phase 1)

The results show :

- Robust stability
- very good performance
- best performance in lower frequencies
- perfect for handling uncertainty

Simulation with μ

Synthesis Results (phase 2)

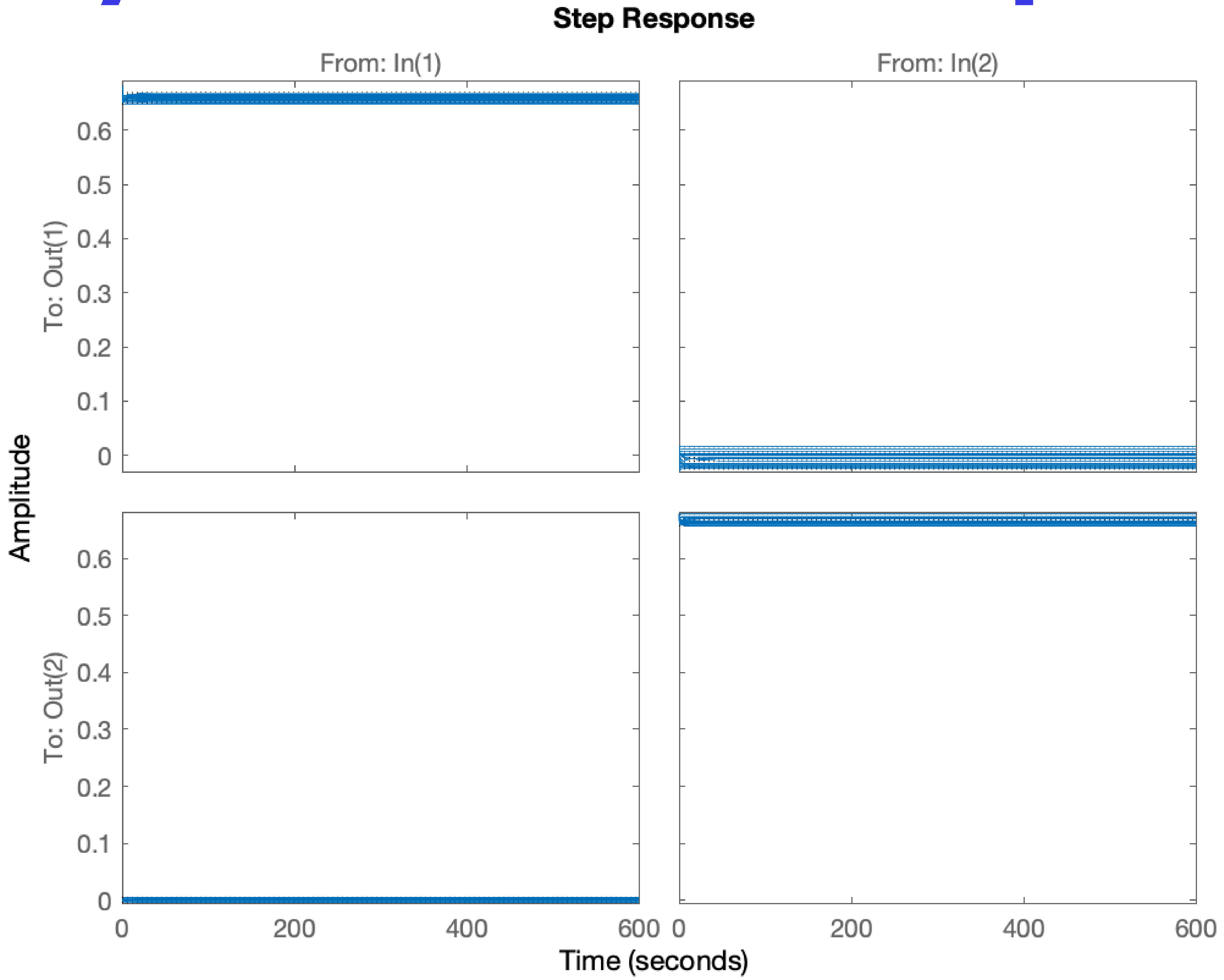
```
Iteration Summary
---------------------------------------------------------------------
Iteration #                  1         2         3         4         5
Controller Order             8         8        12        12        12
Total D-Scale Order          0         0         4         4         4
Gamma Acheived           1.053     0.738     0.698     0.690     0.689
Peak mu-Value            0.997     0.737     0.698     0.690     0.688
```

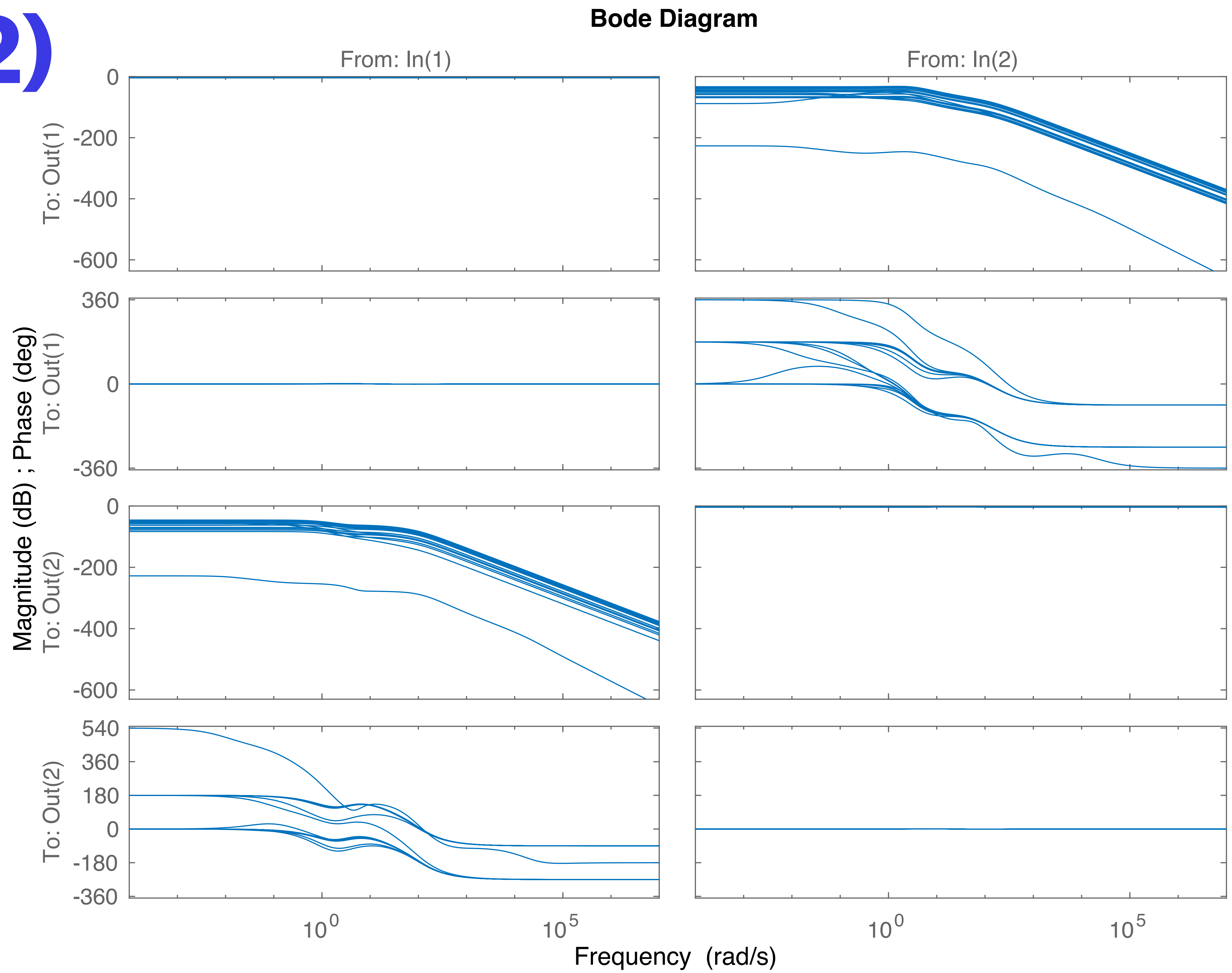
bnd =

0.6885

Simulation with μ

Synthesis Results (phase 2)

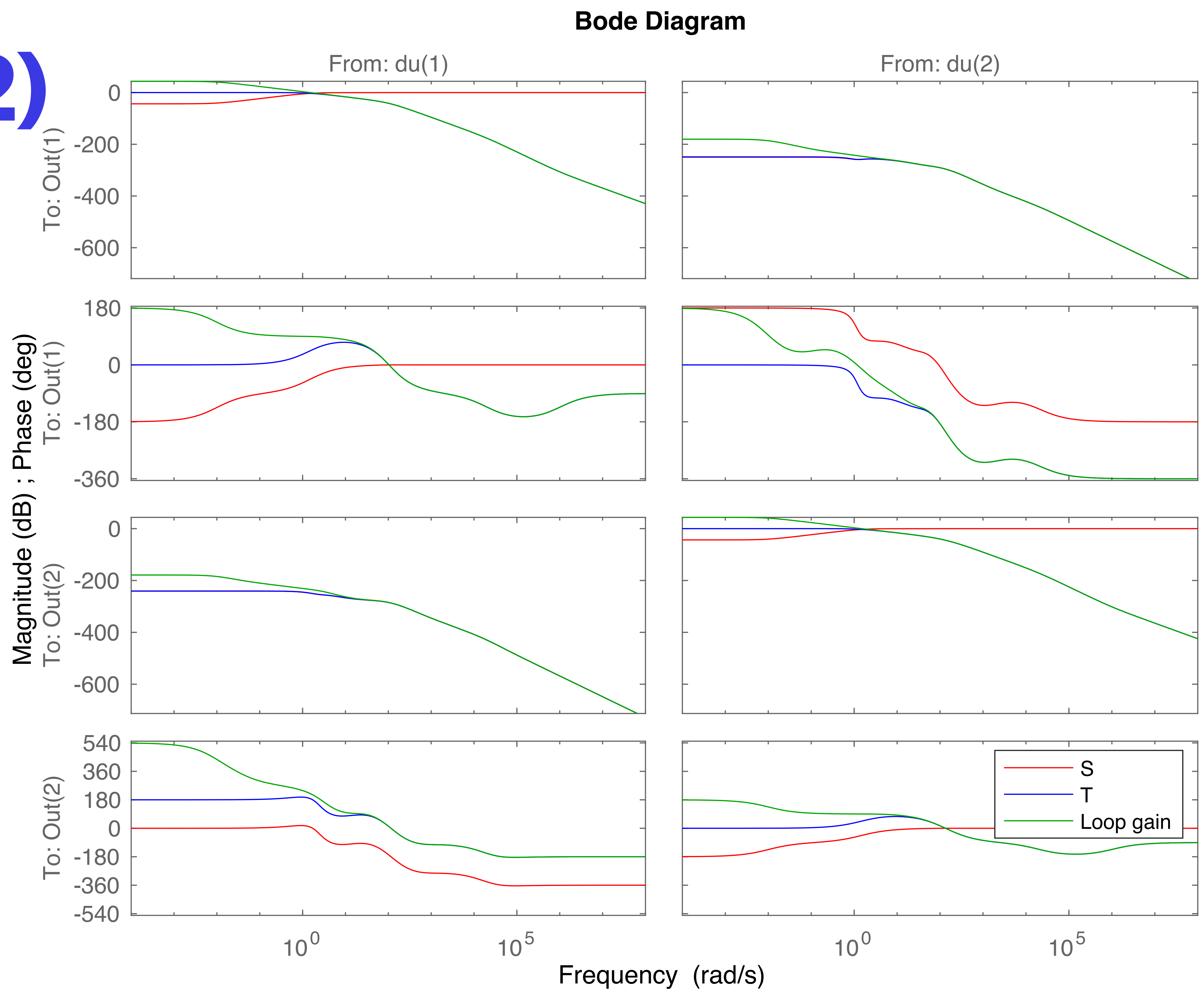




Simulation with μ

Synthesis Results (phase 2)

- Low sensitivity in lower frequencies
- sensitivity value goes to 1 as sensitivity complement and loop gain descend towards zero in the diagonal io selects
- On the contrary, non diagonal io selects can have all of them going towards zero with no trade-off



Simulation with μ

Synthesis Results (phase 2)

The results show :

- Robust stability (even better than phase 1 and the best in all the methods)
- very good performance
- best performance in lower frequencies
- perfect for handling uncertainty

Simulation Results Comparison

The γ_{obt} acquired for all 4 robust methods are compared in this table :

Configuration	H_∞	H_2	H_2/H_∞	μ
APA, phase 1	1.3678	1.3678	7.1364e-04	0.922
APA, phase 2	0.8084	0.8084	0.0010	0.689

The H_2/H_∞ gammas are not acceptable due to H_2 being infinite.

We can clearly see that phase 2 has lower gammas because of more controlled inputs and the μ synthesis method provides the best gammas, all lower than 1.

Simulation Results Comparison

- Best robust control method for all phases is μ synthesis which follows by H_∞ synthesis by about 22% to 40% weaker in robustness.
- Best phase for controlling the manipulator is phase 2 and that is due to having redundancy and more actuators (control Inputs) and less singularity.
- Therefore, it can be said that the number of actuators in a manipulator is related to the robustness of the system.

Conclusions

- Planar 3 DOF manipulator with 2 active joints and 1 passive joint
- Dynamics found with torque control method leading to linear state space system
- MK standard form for robust stability analysis
- Best gamma was achieved in μ synthesis method following by H_∞ synthesis
- H_2 had excellent performance but could not prove robustness
- H_2/H_∞ was not acceptable in any way because of very low gamma and infinite H_2
- Phase 2 had better robustness because of more actuators

Thank You

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