ME 505 EdX – Conduction Equation Sheet

Constants

Stefan-Boltzmann Constant: $\sigma = 5.67\times10^{-8}\ \dfrac{\text{W}}{\text{m}^2\text{K}^4}$

g = **gravitational acceleration** = 9.807 m/s^2

Geometry

Cylinder: $A = 2\pi r l$ $V = \pi r^2 l$

Sphere: $A = 4\pi r^2$ $V = \dfrac{4}{3}\pi r^3$

Triangle: $A = bh/2$ b: base h: height

Conservation Laws

Control Volume Energy Balance: $\dot{E}_{in} - \dot{E}_{out} + \dot{E}_{gen} = \dot{E}_{st}$ $\dot{E}_{st} = mC_p\, dT/dt$ $\dot{E}_{gen} = \dot{q}V$

Surface Energy Balance: $\dot{E}_{in} - \dot{E}_{out} = 0$

Conduction

Fourier's Law: $q''_{cond,x} = -k\dfrac{\partial T}{\partial x}$ $q''_{cond,n} = -k\dfrac{\partial T}{\partial n}$ $q_{cond} = q''_{cond}A$

Heat Flux Vector: $\vec{q''} = q''_x\vec{i} + q''_y\vec{j} + q''_z\vec{k} = -k\left[\dfrac{\partial T}{\partial x}\vec{i} + \dfrac{\partial T}{\partial y}\vec{j} + \dfrac{\partial T}{\partial z}\vec{k}\right]$

Thermal Diffusivity: $\alpha = \dfrac{k}{\rho c_p}$

Heat Diffusion Equation:

Rectangular Coordinates: $$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho C_p\frac{\partial T}{\partial t}$$

Cylindrical Coordinates: $$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho C_p\frac{\partial T}{\partial t}$$

Spherical Coordinates:
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right) + \dot{q} = \rho C_p\frac{\partial T}{\partial t}$$

Thermal Resistance Concepts:

Conduction Resistance: $R_{t,cond} \overset{plane\,wall}{=} \dfrac{L}{kA}$ $R_{t,cond} \overset{cylinder}{=} \dfrac{\ln(r_o/r_i)}{2\pi lk}$ $R_{t,cond} \overset{sphere}{=} \dfrac{(1/r_i)-(1/r_o)}{4\pi k}$

Conduction Shape Factor: $R_{t,cond} \overset{2D}{=} \dfrac{1}{Sk}$

Convection Resistance: $R_{t,conv} = \dfrac{1}{h_{conv}A_s}$

Radiation Resistance: $R_{t,rad} = \dfrac{1}{h_{rad}A_s}$ where $h_{rad} = \varepsilon\sigma(T_s + T_{sur})(T_s^2 + T_{sur}^2)$

Contact Resistance: $R_{t,c} = \dfrac{R''_{t,c}}{A}$

Extended Surfaces:

TABLE 3.4 Temperature distribution and heat loss for fins of uniform cross section

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b		Fin Heat Transfer Rate q_f	
A	Convection heat transfer: $h\theta(L) = -kd\theta/dx\|_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	(3.75)	$M\dfrac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$	(3.77)
B	Adiabatic: $d\theta/dx\|_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$	(3.80)	$M \tanh mL$	(3.81)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$	(3.82)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$	(3.83)
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx}	(3.84)	M	(3.85)

$\theta \equiv T - T_\infty$ $\quad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\quad M \equiv \sqrt{hPkA_c}\,\theta_b$

Fin Effectiveness: $\varepsilon_f = \dfrac{q_f}{hA_{c,b}\theta_b}$

Fin Efficiency:

Single fin: $\eta_f = \dfrac{q_f}{hA_f\theta_b}$ $\qquad \eta_f \overset{adiabatic}{=} \dfrac{\tanh(mL)}{mL}$ $\qquad L_c = L + \dfrac{A_c}{P}$ $\qquad \eta_f = \dfrac{\tanh(mL_c)}{mL_c}$

Surface: $\eta_o = \dfrac{q_{total}}{hA_{total}\theta_b} = 1 - \dfrac{NA_f}{A_{total}}\left(1-\eta_f\right)$

Fin Resistance: $R_{t,cond-fin} = \dfrac{1}{\eta_f hA_f}$ $\qquad R_{t,cond-fin\,array} = \dfrac{1}{\eta_o hA_{total}}$

Transient Conduction: $Fo = \dfrac{\alpha t}{L_c^2}$ $\qquad \alpha = \dfrac{k}{\rho c_p}$

Lumped System Analysis: $Bi = \dfrac{R_{t,inside}}{R_{t,outside}} = \dfrac{L_c / kA_S}{R_{t,outside}}$ $\qquad L_c = \dfrac{V}{A_s}$ $\qquad \tau_t = R_t C_t$

Considering only convection losses with constant h_{conv}:

$$Bi = \frac{R_{t,cond}}{R_{t,conv}} = \frac{h_{conv}L_c}{k_{solid}}$$

$$\frac{\theta}{\theta_i} = \frac{T-T_\infty}{T_i-T_\infty} = \exp\left(-\frac{t}{\tau_t}\right) = \exp\left[-\left(\frac{h_{conv}L_c}{k_{solid}}\right)\left(\frac{\alpha t}{L_c^2}\right)\right] = \exp\left[-(Bi)(Fo)\right]$$

$$\tau_t = \frac{\rho V c_p}{h_{conv}A_s} = \frac{\rho c_p L_c}{h_{conv}} = \left(\frac{1}{h_{conv}A_s}\right)\rho V c_p = R_t C_t$$

$$\frac{Q}{Q_o} = \left[1-\exp\left(-\frac{t}{\tau_t}\right)\right] \quad \text{where } Q_o = mC_p\left(T_i - T_\infty\right) = \rho V C_p\left(T_i - T_\infty\right)$$

Semi-Infinite Media:

$$\frac{T(x,t)-T_s}{T_i - T_s} = erf\left(\frac{x}{2\sqrt{\alpha t}}\right)$$ Constant surface temperature

Ordinary Differential Equations

Exponential Function:

$\frac{df}{dx} \pm cf = 0$ has solutions of the form $f(x) = a_1 \exp(\mp cx)$

$\frac{d}{dx}\left(\exp(\pm cx)\right) = \pm c \exp(\pm cx)$

$\exp(0) = 1$

Trigonometric Functions:

$\frac{d^2 f}{dx^2} + \lambda^2 f = 0$ has solutions of the form $f(x) = a_1 \sin(\lambda x) + a_2 \cos(\lambda x)$

$\frac{d}{dx}\left(\sin(\lambda x)\right) = \lambda \cos(\lambda x)$ $\frac{d}{dx}\left(\cos(\lambda x)\right) = -\lambda \sin(\lambda x)$

$\sin(0) = 0$ $\cos(0) = 1$

Hyperbolic Functions:

$\frac{d^2 f}{dx^2} - \lambda^2 f = 0$ has solutions of the form $f(x) = a_1 e^{\lambda x} + a_2 e^{-\lambda x} = c_1 \cosh(\lambda x) + c_2 \sinh(\lambda x)$

$\frac{d}{dx}\left(\sinh(\lambda x)\right) = \lambda \cosh(\lambda x)$ $\frac{d}{dx}\left(\cosh(\lambda x)\right) = \lambda \sinh(\lambda x)$

$\sinh(0) = 0$ $\cosh(0) = 1$

Bessel's Functions of Order 0:

$r^2 \frac{\partial^2 f}{\partial r^2} + r \frac{\partial f}{\partial r} + \lambda^2 r^2 f = 0$ has solutions of the form $f(r) = a_1 J_0(\lambda \mathrm{r}) + a_2 Y_0(\lambda \mathrm{r})$

$\frac{d}{dr}\left(J_o(\lambda r)\right) = -\lambda J_1(\lambda r)$ $\frac{d}{dr}\left(Y_o(\lambda r)\right) = -\lambda Y_1(\lambda r)$

$J_0(0) = 1$ $Y_0(x \to 0) \to -\infty$

Modified Bessel's Functions of Order 0:

$r^2 \frac{\partial^2 f}{\partial r^2} + r \frac{\partial f}{\partial r} - \lambda^2 r^2 f = 0$ has solutions of the form $f(r) = a_1 I_0(\lambda \mathrm{r}) + a_2 K_0(\lambda \mathrm{r})$

$\frac{d}{dr}\left(I_o(\lambda r)\right) = \lambda I_1(\lambda r)$ $\frac{d}{dr}\left(K_o(\lambda r)\right) = -\lambda K_1(\lambda r)$

$I_0(0) = 1$ $K_0(x \to 0) \to \infty$

Any tables or charts you need for error function, Bessel's functions, etc. will be provided.