Math 491
2 Dimensional Topological Quantum Field Theories

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Outline

Push-outs

Definition (Informal)

A category is a collection of objects together with the maps between them, obeying the associativity and identity laws for the composition.

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Definition

For objects X, Y, Z and maps i_X, i_Y in a category, an (unique) object called 'the' *push-out* can be obtained via the diagram

$$\begin{tikzcd} Z \arrow[r, "i_X"] \arrow[d, "i_Y"] & X \arrow[d] \\ Y \arrow[r] & X \amalg_Z Y. \end{tikzcd}$$

Maps

Definition

A *functor* $F : \mathcal{A} \to \mathcal{B}$ is a mapping of objects and morphisms such that:

- $F : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$
- $F : \mathrm{Mor}(A_1, A_2) \to \mathrm{Mor}(F(A_1), F(A_2))$,

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Definition

A *natural transformation* between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is the data of maps $\{\alpha_c\}_{c \in \mathcal{C}}$ compatible with the functors.

Essentials of a category

Definition

For a category $\mathcal{C}$, we have the following definitions:

1. *Skeleton:* A subcategory $\mathcal{S}$ consisting of only one object from each isomorphism class.
2. *Generating set:* A set $\mathcal{G}$ is such that all the maps in $\mathcal{C}$ can be obtained from compositions of the maps in $\mathcal{G}$.

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Example

[1] For the category $f.d.\mathbf{Vect}_{\mathbb{R}}$, its skeleton is the set $\{\mathbb{R}^n\}_{n\in\mathbb{N}}$ with all the linear maps between them. A generating set can be obtained by the decomposition theorems.

Monoidal Categories

Definition

[1] “A (strict) *monoidal category* is a category $\mathbf{V}$ together with two functors

$$\begin{aligned} \mu : \mathbf{V} \times \mathbf{V} &\to \mathbf{V}, & \nu : \mathbf{1} &\to \mathbf{V} \\ (A, B) &\mapsto A \Box B, & * &\mapsto I \end{aligned}$$

satisfying the associativity axiom and the neutral object axiom”, where $\mathbf{1}$ denotes the empty cartesian product of the category $\mathbf{V}$.

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A monoidal category is called *symmetric* if we have a *twist map* $(A \square B \to B \square A)$ subject to the desired properties, and a functor is called symmetric if it sends a twist map to twist map. A *symmetric monoidal category* is then a 4−tuple $(\mathbf{V}, \square, I, \tau)$.

Monoidal Functors & Natural Transformations

Definition

[1] "A (strict) *monoidal functor* between two (strict) monoidal categories $(\mathbf{V}, \Box, I)$ and $(\mathbf{V'}, \Box', I')$ is a functor $F : \mathbf{V} \to \mathbf{V'}$ that commutes with all the structure".

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Definition

Let $F, G : (\mathbf{V}, \square, I, \tau) \longrightarrow (\mathbf{V'}, \square', I', \tau')$ be symmetric monoidal functors. A natural transformation $u : F \implies G$ is called a *monoidal natural transformation* if it respects the monoidal structure.

Some Symmetric Monoidal Categories

Example

1. $(\mathbf{Sets}, \coprod, \emptyset, \tau)$,
2. $(\mathbf{Sets}, \times, 1, \sigma)$,
3. $(\mathbf{Vect}_k, \otimes, k, \sigma)$,
4. $(\mathbf{2Cob}, \coprod, \emptyset_1, T)$,
5. $(\mathbf{FA}_k, \otimes, k, T)$.

Cobordisms

Definition

Given two closed $(n-1)$-manifolds Σ_0, Σ_1, a *cobordism* is an n-manifold M whose in-boundary is Σ_0 and out-boundary is Σ_1. To indicate the orientation, we write $M : \Sigma_0 \Longrightarrow \Sigma_1$.

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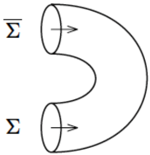
Convention

- Σ : a closed $n-$manifold,
- M : a cobordism,
- $\emptyset_n$ is a closed $n-$manifold!

2-Cobordism Examples

Example

1. U-tube: $S^1 \coprod \overline{S^1} \Rightarrow \emptyset_1$



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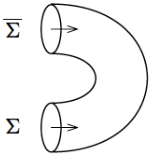
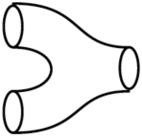


2. "Pair of pants": $S^1 \coprod S^1 \Rightarrow S^1$



2-Cobordism Examples

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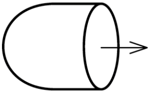
3. "Birth": $\emptyset_1 \Rightarrow S^1$



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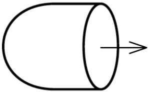
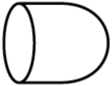


4. "Death": $S^1 \Rightarrow \emptyset_1$



Equivalent Cobordisms

Definition

Let $M, M' : \Sigma_0 \to \Sigma_1$. We call them equivalent ($\sim$) if the following diagram with the diffeomorphism $\psi : M \to M'$ commutes:

$$\begin{tikzcd}
& M' & \\
\Sigma_0 \arrow[ur, hook] \arrow[dr, hook] & & \Sigma_1 \arrow[ul, hook'] \arrow[dl, hook'] \\
& M \arrow[uu, "\psi"] &
\end{tikzcd}$$

Nonequivalent Cobordism Example

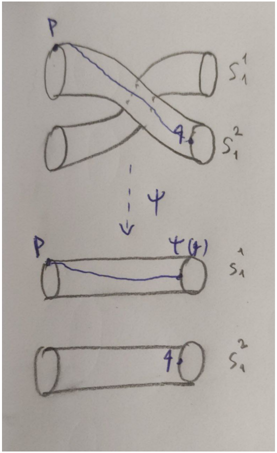


Figure: Two diffeomorphic manifolds which are not equivalent as cobordisms

nCob

Definition (Gluing)

Let $\psi : \Sigma_1 \to \Sigma_1$ be a diffeomorphism. We define $M_0 \coprod_{\Sigma_1} M_1$ to be the topological space $M_0 \coprod M_1$ under the equivalence relation $x \sim \psi(x), x \in \Sigma_1$.

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With this definition, $M_0 \coprod_{\Sigma_1} M_1$ is the push-out as shown in the diagram:

$$\begin{tikzcd} & M_0 \coprod_{\Sigma_1} M_1 & \\ M_0 \arrow[ur] & & M_1 \arrow[ul] \\ & \Sigma_1 \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

nCob

Definition (**nCob**)

$$\mathbf{nCob}\begin{cases} Ob(\mathbf{nCob}) := \{\Sigma : \Sigma \text{ is a closed } (n-1)\text{−manifold}\} \\ Mor(\Sigma_0, \Sigma_1) := \{[M] | M : \Sigma_0 \Longrightarrow \Sigma_1\}. \end{cases}$$

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Remark

With this definition, identity morphisms are the classes of cylinders over the objects.

nCob

The associativity of the gluing is proven via the following diagram:

$$\begin{tikzcd}
 & \Sigma_1 \arrow[dl, hook] \arrow[dr, hook] & & \Sigma_2 \arrow[dl, hook] \arrow[dr, hook] & \\
M_0 \arrow[dr, hook] \arrow[ddrrr, hook] & & M_1 \arrow[dl, hook] \arrow[dr, hook] \arrow[ddl, dashed] \arrow[ddr, dashed] & & M_2 \arrow[dl, hook] \arrow[ddlll, hook] \\
 & M_0 \coprod_{\Sigma_1} M_1 \arrow[d, hook] \arrow[drr, dotted] & & M_1 \coprod_{\Sigma_2} M_2 \arrow[d, hook] \arrow[dll, dotted] & \\
 & (M_0 \coprod_{\Sigma_1} M_1) \coprod_{\Sigma_2} M_2 & & M_0 \coprod_{\Sigma_1} (M_1 \coprod_{\Sigma_2} M_2) &
\end{tikzcd}$$

2Cob

For $n = 2$, we have a nice representation of **2Cob**.

Notation

We will take the skeleton of **2Cob** which is the set $\{\mathbf{m} := \coprod_{i=0}^{m} S^1\}$ with all the cobordisms, to be **2Cob**.

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With taking disjoint unions and usual twist maps, **2Cob** is a symmetric monoidal category.

Normal Form of a 2−cobordism

Theorem ([1])

"Two connected, compact oriented surfaces with oriented boundary are diffeomorphic if and only if they have the same genus and the same number of in-boundaries and the same number of out-boundaries."

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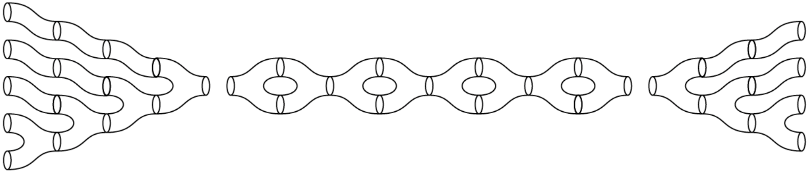
Claim

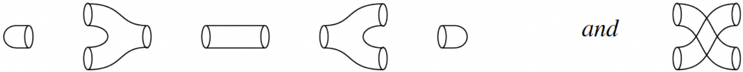
Any connected 2−cobordism can be written as



Generators of **2Cob**

Theorem ([1])

"The monoidal category **2Cob** *is generated under composition (serial connection) and disjoint union (parallel connection) by the following six cobordisms":*



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We can write bunch of relations between the generators which can immediately be proven by the classification theorem.

TQFTs

Definition

A *n-TQFT* is a symmetric monoidal functor from $(\mathbf{nCob}, \coprod, \emptyset_{n-1}, T)$ to $(\mathbf{Vect}_k, \otimes, k, \sigma)$.

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Original formulation [Atiyah]

A TQFT is an assignment $\mathcal{A}$ mapping the closed manifolds to vector spaces and cobordisms to the linear maps between them, subject to the following axioms:

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Original formulation [Atiyah]

A TQFT is an assignment $\mathcal{A}$ mapping the closed manifolds to vector spaces and cobordisms to the linear maps between them, subject to the following axioms:

A1. If $M \sim M'$, then $\mathcal{A}(M) = \mathcal{A}(M')$.

A2. $\mathcal{A}(\Sigma \times I) = id_{\mathcal{A}(\Sigma)}$.

A3. Given a decomposition $M = M'M''$, $\mathcal{A}M = \mathcal{A}M'' \circ \mathcal{A}M'$.

A4. $\mathcal{A}(\Sigma_0 \coprod \Sigma_1) = \mathcal{A}(\Sigma_0) \otimes \mathcal{A}(\Sigma_1)$.

A5. $\mathcal{A}\emptyset_{n-1} = k$.

(Co-)algebras

Definition (Co-algebra)

As a dual definition to algebras, we have a $k-$*co-algebra* A as a vector space with these two $k-$linear maps:

$$\delta : A \to A \otimes A \text{ and } \varepsilon : A \to k$$

called *co-multiplication* and *co-unit*, satisfying the following axioms:

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A \otimes A \arrow[ur, "\delta \otimes id"] & & A \otimes A \arrow[ul, "id \otimes \delta"'] \\
 & A \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
k \otimes A & A \otimes A \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & k \otimes A \\
 & A \arrow[ul] \arrow[u, "\delta"] \arrow[ur] &
\end{tikzcd}$$

Frobenious algebras

Definition

$\beta : V \otimes W \to k$ is called a *pairing*. If there exists another map (called *co-pairing*) $\gamma : k \to W \otimes V$ such that the composition of the below maps is the identity,

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Definition (Frobenious algebra)

A pair (A, β) where A is an algebra and β is a non-degenerate pairing over A is called a *Frobenious algebra*.

Frobenious algebras

Theorem

The followings hold good of a Frobenious algebra (A, β)*:*

- *There exists* $\varepsilon \in A^*$ *with its null-space containing no non-trivial left ideals. Such an* ε *is called a Frobenious form.*

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Relation between β and ε

$\beta(a \otimes b) := \varepsilon(a.b)$ and $\varepsilon(a) := \beta(1_A \otimes a)$.

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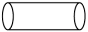
Example

All $n \times n$ real matrices form a Frobenious algebra over $\mathbb{R}$ with the usual matrix multiplication as multiplication and the trace map as both the non-degenerate pairing and the Frobenious form.

Graphical calculus

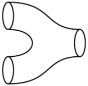
η

unit



id_A

identity



μ

multiplication

Graphical calculus

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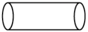

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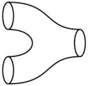

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Frobenius form

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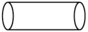

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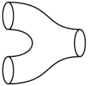

μ

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ε

Frobenius form

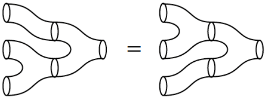
β

Frobenius pairing

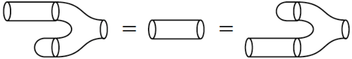
Co-multiplication δ

Axioms stated graphically

1. Associativity and unitality yield these relations:



associativity



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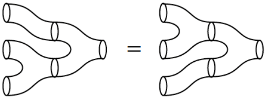
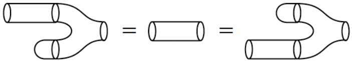
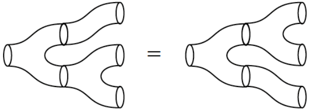


associativity



unit axiom

2. Co-associativity of the co-multiplication is encoded in the below figure:



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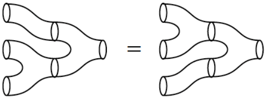

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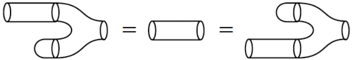

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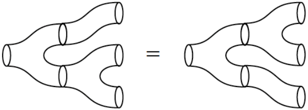
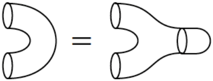


3. $\beta = \varepsilon\mu$.



$\mathbf{FA}_k$

Definition

A *Frobenious homomorphism* is an algebra homomorphism $\phi : (A, \varepsilon) \to (A', \varepsilon')$ such that

$$(\phi \otimes \phi) \circ \delta_1 = \delta_2 \circ \phi$$

where $\delta_{1,2}$ are the induced co-multiplication maps, and

$$\varepsilon' \phi = \varepsilon.$$

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With arrows defined as the Frobenious homomorphisms, we get the category $\mathbf{FA}_k$ (and $\mathbf{cFA}_k$ for the commutative Frobenious algebras' category).

$\mathbf{FA}_k$

Definition

A *Frobenious homomorphism* is an algebra homomorphism $\phi : (A, \varepsilon) \to (A', \varepsilon')$ such that

$$(\phi \otimes \phi) \circ \delta_1 = \delta_2 \circ \phi$$

where $\delta_{1,2}$ are the induced co-multiplication maps, and

$$\varepsilon' \phi = \varepsilon.$$

With arrows defined as the Frobenious homomorphisms, we get the category $\mathbf{FA}_k$ (and $\mathbf{cFA}_k$ for the commutative Frobenious algebras' category). Taking tensor products yields a natural Frobenious algebra structure. Hence, we constructed $(\mathbf{FA}_k, \otimes, k, T)$.

The Equivalence

Definition ($\mathbf{nTQFT}_k$)

$\mathbf{nTQFT}_k$ is the category whose objects are the symmetric monoidal functors from $(\mathbf{nCob}, \coprod, \emptyset_{n-1}, T)$ to $(\mathbf{Vect}_k, \otimes, k, \sigma)$ and the maps are the monoidal natural transformation between $n-$TQFTs.

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Theorem ([1])

There is a canonical equivalence of categories

$$\mathbf{2TQFT}_k \simeq \mathbf{cFA}_k.$$

The Equivalence

By monoidality, a 2−TQFT is fully characterized by its image of S^1 and the images of the 6 generators.

$$\mathbf{2Cob} \longrightarrow \mathbf{Vect}_{\Bbbk}$$

$$\mathbf{1} \longmapsto A$$

$$\mathbf{n} \longmapsto A^n$$

 $\longmapsto [\mathrm{id}_A : A \to A]$

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 $\longmapsto [\eta : \Bbbk \to A]$

 $\longmapsto [\mu : A^2 \to A]$

 $\longmapsto [\varepsilon : A \to \Bbbk]$

 $\longmapsto [\delta : A \to A^2].$

where $A^n = A \otimes \cdots \otimes A$.

Concluding Remarks

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    - Toy model for quantum gravity

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