Comments and corrections

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Abstract

This document contains additional remarks and errors for my papers that have not been corrected in public versions. Some corrections and remarks will appear in the final versions.

Contents

1 Shifted contact structures and their local theory [1]

1.1 Definitions 3.5 & 3.6 simplified

Since $QCoh(\mathbf{X})$ is a stable ∞-category, we can simplify Definitions 3.5 & 3.6 in [1] and introduce/use the following ones instead:

Definition 1.1. Let $\mathbf{X}$ be a locally finitely presented derived (Artin) stack. A ***pre-k-shifted contact structure*** on $\mathbf{X}$ is given by a shifted line bundle $L[k]$ with a morphism $\alpha : \mathbb{T}_{\mathbf{X}} \to L[k]$. Denote such a structure by $(L[k], \alpha)$.

Note that we can consider a pre-k-shifted contact data as a perfect complex $\mathcal{K}$ and a line bundle L along with a morphism $\kappa : \mathcal{K} \to \mathbb{T}_{\mathbf{X}}$ such that $Cone(\kappa) \simeq L[k]$. Since $QCoh(\mathbf{X})$ is a stable ∞-category, we have a fiber-cofiber sequence $\mathcal{K} \to \mathbb{T}_{\mathbf{X}} \to L[k]$ in $QCoh(\mathbf{X})$, and hence, the cocone of $\mathbb{T}_{\mathbf{X}} \to L[k]$ is equivalent to $\mathcal{K}$. We then may denote a pre-k-shifted contact structure on $\mathbf{X}$ by $(\mathcal{K}, \kappa, L)$.

Definition 1.2. We say that a pre-k-shifted contact structure $(\mathcal{K}, \kappa, L)$ on $\mathbf{X}$ is a ***k-shifted contact structure*** if locally on $\mathbf{X}$, where L is trivial, the induced k-shifted 1-form[1] $\alpha : \mathbb{T}_{\mathbf{X}} \to \mathcal{O}_{\mathbf{X}}[k]$ is such that the map $d_{dR}\alpha|_{\mathcal{K}} \cdot := \kappa^{\vee}[k] \circ (d_{dR}\alpha \,\cdot) \circ \kappa : \mathcal{K} \to \mathcal{K}^{\vee}[k]$ is a weak equivalence.

In that case, we say the k-shifted 2-form $d_{dR}\alpha$ is ***non-degenerate on*** $\mathcal{K}$. Also, we call such local form a ***k-contact form***.

Remark 1.3. When $k \leq 0$, the triangle $\mathcal{K} \to \mathbb{T}_{\mathbf{X}} \to L[k]$ splits locally for any affine derived scheme (so, this also holds Zariski locally for any derived scheme)[2].

These variations (with minor edits and clarifications) will appear in the **sequel** [2] and the **latest arXiv/ResearchGate (RG) versions** of [1].

[1]We can locally identify the map α with the induced shifted one-form using the trivialization of $L^{\vee}[k]$.

[2]We thank the anonymous referee for this remark.

1.2 Example 3.11 clarified

There are some arguments requiring additional correction and comments. ***(Please notice the change in the distinguished generator.)*** Localizing if necessary, instead of the current generators given in the construction, we can consider the variables

$$x_1^{-i}, x_2^{-i}, \ldots, x_{m_i}^{-i} \qquad \text{in degree } -i \text{ for } i = 1, 2, \ldots, \ell, \tag{1.1}$$

$$y_1^{k+i}, y_2^{k+i}, \ldots, y_{m_i}^{k+i} \qquad \text{in degree } k+i \text{ for } i = 1, \ldots, \ell, \tag{1.2}$$

$$z^k, y_1^k, y_2^k, \ldots, y_{m_0}^k \qquad \text{in degree } k. \tag{1.3}$$

Here, we replace $\tilde{x}_1^0 \in A^0$ with $z^k \in A^k$ to fix the generator of the complex $Rest$, which is concentrated in $\deg -k$ and now generated by $\partial/\partial z^k$.

The current version of α in [1, Eqn. (3.14)] might be misleading. Instead, it is better to define $\alpha \in (\Omega_A^1)^k$ by

$$\alpha = d_{dR} z^k + \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{dR} x_j^{-i}, \tag{1.4}$$

where $z^k \in A^k$ is now the *distinguished generator* of degree k. Note that $\alpha \in \Omega_A^1[k]$, but we also need it to be d-closed. We can actually make α d-closed by determining the differential d via the equations $d|_{A(0)} = 0$; $dx_j^{-i} = \partial H / \partial y_j^{k+i}$; $dy_j^{k+i} = \partial H / \partial x_j^{-i}$ for all i, j and $-kdz^k = H + d[\cdots]$ for the Hamiltonian $H \in A^{k+1}$ and a suitable term $d[\cdots]$. Then the construction follows in a similar way.

For details, see the final version of [2] or the latest arXiv/RG versions of [1].

1.3 Proof of Theorem 3.14 revisited

The generator for $Rest$ is incorrect. In fact, $Rest$ is a complex concentrated in degree $-k$ since we want the splitting $\mathbb{T}_A = \ker(\alpha) \oplus Rest$ over $\mathrm{spec} H^0(A)$, where $Rest = L[k]$ is a k-shifted line bundle. But it is said in the construction that $Rest$ is generated by the vector field $\partial/\partial \tilde{x}_1^0$, which is of degree 0. This is incorrect. So, instead of using $\tilde{x}_1^0$ in A^0, we must use a distinguished generator z^k in A^k in addition to the generators $y_1^k, y_2^k, \ldots, y_{m_0}^k$ in degree k. Then, after suitable replacement, we should end up with the splitting $\mathbb{T}_A = \ker(\alpha) \oplus Rest$ over $\mathrm{spec} H^0(A)$ such that $\ker(\alpha)$ has Tor-amplitude $[0, -k]$ and $Rest$ is concentrated in $\deg -k$, and that $\ker(\alpha) \simeq {}^{\mathbb{T}_A}\!/_{Rest}$ with

$$\ker(\alpha)|_{\mathrm{spec} H^0(A)} = \langle \partial/\partial x_j^{-i}, \partial/\partial y_j^{k+i} \rangle,$$

$$Rest|_{\mathrm{spec} H^0(A)} = \langle \partial/\partial z^k \rangle,$$

where $0 \leq i \leq \ell$, $1 \leq j \leq m_i$, leading to the desired k-shifted contact form as in Section 1.2 above.

We may have a misleading argument (although our intention was naively the opposite and we were trying to say d could be taken in that form by the results of Brav-Bussi-Joyce, etc... Clearly, the way of presenting this intention is incorrect.): "Imposing the differential"-type argument in the proof has been clarified. The upshot is that expanding the defining equations for the pair (H, ϕ), we arrive at the desired formulas for d, hence identify the cdga (A, d) exactly with the one above.

An adaptation of this proof to the case of derived Artin stacks will appear in the final version of [2]. Similar edits are also available in the latest arXiv/RG versions of [1].

1.4 Remarks 2.27 and 3.18 fixed

For $k/2$ odd, the statements were incorrect; they are now fixed. See the latest version of [1] on arXiv or RG. The latest enumerations might be shifted by 1 or 2.

1.5 Uniqueness clarified

For the equations (and page numbers) to be mentioned in this section, we refer to the ones in [1]. This section highlights the uniqueness of Darboux representation for shifted contact forms.

Assume that there is an element α' satisfying the desired properties listed in the proof of Theorem 3.14.

Let us clarify how the corresponding conditions uniquely (up to interchange of x_j^{-i} and y_j^{k+i})[3] determine the representation on Pg. 1047 of [1]. We first observe that due to Eqn. (3.23) and the condition $\iota_{\partial/\partial z^k}\alpha' = 1$, any such element α' takes the form

$$\alpha' = d_{dR}z^k + \psi \quad \text{with} \quad \psi = \sum_{i,j} \left[a_j^{k+i} d_{dR}x_j^{-i} + b_j^{-i} d_{dR}y_j^{k+i} \right],$$

where $a_{\nu'}^{\mu'} \in A^{\mu'}$, $b_\nu^\mu \in A^\mu$ such that b_ν^μ's depend only on $x_{j'}^{-i'}$'s and $A(0)$ for degree reasons. From the splitting condition, we take[4] $b_j^{-i} = 0$ for all i, j, leading to

$$d_{dR}\alpha' = d_{dR}\psi = \sum_{i,j} d_{dR}a_j^{k+i} d_{dR}x_j^{-i}.$$

Now, using the RHS of (3.23) for comparison, we have $a_j^{k+i} = y_j^{k+i}$ for all i, j. Replacing them accordingly, we obtain $\psi = \sum_{i,j} y_j^{k+i} d_{dR}x_j^{-i}$, which gives $\alpha' = d_{dR}z^k + \sum_{i,j} y_j^{k+i} d_{dR}x_j^{-i}$, hence the desired form. For details, see the latest arXiv/RG versions of [1].

2 On shifted contact derived Artin stacks [2]

Just as for [1], similar revisions will appear in the final version of [2], which has been under review (pre-accepted now, to appear in *Higher Structures*).

Note that the need for additional corrections and remarks listed in the section 1 above has emerged during the reviewing process of [2].

I wish to warmly thank the anonymous referee(s) for their valuable comments and suggestions, which helped a lot and improved the quality of the original manuscript(s).

References

[1] K. İ. Berktav, Shifted contact structures and their local theory, ***Ann. Fac. Sci. Toulouse***, Math., Serie 6, Vol. 33 (2024) No. 4, pp. 1019-1057. arXiv:2209.09686 (see the latest version including the edits mentioned above).

[2] K. İ. Berktav, On shifted contact derived Artin stacks, arXiv:2401.03334.

[3] The roles of x_j^{-i}, y_j^{k+i} are symmetric in (3.23), where $d_{dR}x_j^{-i} d_{dR}y_j^{k+i} = d_{dR}y_j^{k+i} d_{dR}x_j^{-i}$ for k odd.

[4] Instead, one may set $a_j^{k+i} = 0 \; \forall i, j$, giving rise to the form of α' with the roles of x_ν^μ, y_ν^σ interchanged.