Some notes on Serre's formula

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Abstract

This is a collection of notes from a seminar the Author gave a long time ago, in a galaxy far, far away. The main body of the text contains key ideas and standard results that are gathered from the literature. For technicalities, we just give a modest list of references. Be vigilant for typos and errors.

Contents

1 Intro

Given subvarieties $C, C' \subseteq \mathbb{P}^n_{\mathbb{K}}$ of degrees m, ℓ and dimensions a, b, respectively, we want to understand the content of the following formula (and that of (1.9) in general) that is used to compute the *intersection multiplicity* along the irreducible components P of $C \cap C'$:

$$\mathfrak{m}_P(C, C') = \sum_i (-1)^i \dim_{\mathbb{K}} \mathrm{Tor}_i^{\mathcal{O}_{\mathbb{P}^n,P}}(\mathcal{O}_{C,P}, \mathcal{O}_{C',P}), \tag{1.1}$$

such that $[C] \cdot [C'] = \sum_P \mathfrak{m}_P(C, C')[P]$, where the sum is over irred. comp.'s P of $C \cap C'$.

Classically, we have Bézout's theorem to deal with the intersection problem of certain plane curves. More precisely, we have:

Theorem 1.1. (Bézout's theorem) *Let $C, C' \subseteq \mathbb{P}^2_{\mathbb{K}}$ be two smooth, projective algebraic curves of $\deg n$ and m. If C, C' meet transversely, then $C \cap C'$ has precisely $n \cdot m$ points; i.e., $\#(C \cap C') = n \cdot m$.*

For the *non-transverse* intersections, we may use the *scheme-theoretical intersection* rather than a naive set-theoretical approach. This amounts to compute the *dimension of the local ring $\mathcal{O}_{C \cap C',p}$*, which records the *multiplicity* at p. Let us start with some terminology and examples.

Definition 1.2. Let $C, C' \subseteq \mathbb{P}^2_{\mathbb{K}}$ be two plane curves as above (meeting either transversely or non-transversely). The ***intersection multiplicity*** $m_p(C, C')$ at $p \in C \cap C'$ is defined by

$$m_p(C, C') = \dim_{\mathbb{K}} \mathcal{O}_{C \cap C',p} = \dim_{\mathbb{K}} \big(\mathcal{O}_{C,p} \otimes_{\mathcal{O}_{\mathbb{P}^2,p}} \mathcal{O}_{C',p}\big). \tag{1.2}$$

Example 1.3. Let $C = \{x = 0\}$ and $C' = \{y = 0\}$ in $\mathbb{A}^2_{\mathbb{C}}$. Then $C \cap C' = \{p = (0, 0)\}$ and $\#(C \cap C') = 1$ (as Bézout's theorem suggested). We may obtain the same result using the scheme-theoretical approach: As $C = \mathrm{Spec}(\mathbb{C}[x, y]/(x))$ and $C' = \mathrm{Spec}(\mathbb{C}[x, y]/(y))$ and $C \cap C'$ is the *fiber product*, we have $C \cap C' = \mathrm{Spec}\big(\mathbb{C}[x, y]/(x) \otimes_{\mathbb{C}[x,y]} \mathbb{C}[x, y]/(y)\big)$. Thus we compute

$$\begin{aligned} m_p(C, C') &= \dim_{\mathbb{C}} \big(\mathbb{C}[x, y]/(x) \otimes_{\mathbb{C}[x,y]} \mathbb{C}[x, y]/(y)\big) \\ &= \dim_{\mathbb{C}} \big(\mathbb{C}[x, y]/(x, y)\big) \\ &= \dim_{\mathbb{C}}(\mathbb{C} \cdot 1) = 1. \end{aligned} \tag{1.3}$$

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Example 1.4. Let $C = \{y = x^2\}$ and $C' = \{y = 0\}$ in $\mathbb{A}^2_{\mathbb{C}}$. Now, we still have $C \cap C' = \{p = (0,0)\}$, which is zero-dimensional as a scheme. But, notice that the curves meet non-transversely with an obvious multiplicity (which cannot be detected by Bezout's theorem). As before, we compute

$$\begin{aligned} m_p(C,C') &= \dim_{\mathbb{C}} \big(\mathbb{C}[x,y]/(y-x^2) \otimes_{\mathbb{C}[x,y]} \mathbb{C}[x,y]/(y)\big) \\ &= \dim_{\mathbb{C}} \big(\mathbb{C}[x,y]/(y-x^2,y)\big) \\ &= \dim_{\mathbb{C}} \big(\mathbb{C}[x]/(x^2)\big) \\ &= \dim_{\mathbb{C}} \big(\mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot x\big) = 2. \end{aligned} \tag{1.4}$$

Remark 1.5. Scheme-theoretical intersections are still *not* good enough to deal with certain situations. Let us mention two important cases.

1. **Self-intersections:** Let $C = C' = \{x = 0\} = \mathrm{Spec}(\mathbb{C}[x,y]/(x))$. Then we get

$$\begin{aligned} C \cap C' &= \mathrm{Spec}\big(\mathbb{C}[x,y]/(x) \otimes_{\mathbb{C}[x,y]} \mathbb{C}[x,y]/(x)\big) \\ &= \mathrm{Spec}\big(\mathbb{C}[y]\big) \\ &= \mathbb{A}^1_{\mathbb{C}}, \end{aligned} \tag{1.5}$$

   which is of $\dim 1$ as a scheme (but the expected dimension is 0). The problem is that the defining equations for C, C' are not independent. In this framework, "setting $x = 0$ *twice*" is equivalent to "setting $x = 0$ *once*". Thus, we need a new formalism sensitive enough to deal with these kinds of issues. That is, it should be such a formalism in which imposing some equation more than once must be regarded as being inequivalent to imposing it once.

2. **Intersections in higher dimensions:** Consider two 2-planes X_1, X_2 in $(\mathbb{P}^4_k; [v : w : x : y : z])$ given by

$$\begin{aligned} X_1 &: w = x = 0, \\ X_2 &: y = z = 0. \end{aligned} \tag{1.6}$$

   In an affine chart ($v \neq 0$), we have

$$\begin{aligned} X_1 &= \mathrm{Spec}\big(k[w,x,y,z]/(x,w)\big) \\ X_2 &= \mathrm{Spec}\big(k[w,x,y,z]/(y,z)\big), \end{aligned} \tag{1.7}$$

   such that we set $X := X_1 \cup X_2$ as a scheme coreprenseted by $k[w,x,y,z]/(xy,xz,wy,wz)$. Using the coordinates above, let P be another plane given by $w = y,\ x = z$ such that $\#(X \cap P) = 1$, with the point of intersection $[1:0:0:0:0]$, as in Figure 1 below.

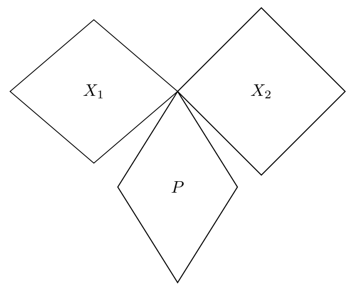

Figure 1: Triple intersection in $\mathbb{P}^4_k$

   Now, let us compute the scheme-theoretical intersection number:

$$\begin{aligned} m_{[1:0:0:0:0]}(X,P) &= k[w,x,y,z]/(xy,xz,wy,wz) \otimes_{k[w,x,y,z]} k[w,x,y,z]/(w-y,x-z) \\ &= k[w,x,y,z]/(xy,xz,wy,wz,w-y,x-z) \\ &= k[y,z]/(y^2,yz,z^2) = k\cdot 1 \oplus k\cdot y \oplus k\cdot z, \end{aligned} \tag{1.8}$$

   which has $\dim_k = 3$. However, the expected dimension must be 2 due to the Moving Lemma [2, App. A]. See also Figure 2.

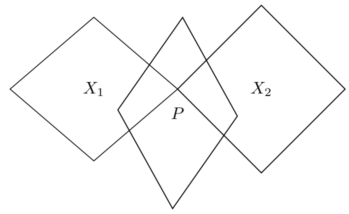

Figure 2: Perturbing non-transversal intersections

What is happening and how to handle this situation will be clear later (see Example 3.1). However, this example shows that the scheme-theoretical formulation of the intersection multiplicity may fail to capture the right number for the case of higher dimensional subvarieties (meeting non-transversely). This is where Serre's formula comes into play. The upshot is that the formula encodes the scheme-theoretical intersection as its *leading term* along with certain correction terms (cf. [2]). More precisely, we have:

Theorem 1.6. *Suppose $A, B \subset X$ are dimensionally transverse*[1] *subschemes of a smooth scheme X and Z is an irred. component of $A \cap B$. The* **intersection multiplicity of A and B along Z** *is given by*

$$\mathfrak{m}_Z(A,B) = \sum_{i=0}^{\dim X} (-1)^i lenght_{\mathcal{O}_{A\cap B},Z}\big(\mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z})\big). \tag{1.9}$$

Observation 1.7.

1. When we have k-vector spaces, $lenght_{\mathcal{O}_{A\cap B},Z}(\cdots) = \dim_k(\cdots)$. (cf. [2])
2. The leading term is precisely the scheme-theoretical multiplicity of Z in $A \cap B$:

$$lenght_{\mathcal{O}_{A\cap B},Z}\big(\mathrm{Tor}_0^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{B,Z})\big) = lenght_{\mathcal{O}_{A\cap B},Z}\big(\mathcal{O}_{A,Z} \otimes_{\mathcal{O}_{X,Z}} \mathcal{O}_{B,Z}\big). \tag{1.10}$$

   Therefore, in the case of k-vector spaces, we may write the intersection multiplicity as

$$\mathfrak{m}_Z(A,B) = \dim_k \big(\mathcal{O}_{A,Z} \otimes_{\mathcal{O}_{X,Z}} \mathcal{O}_{B,Z}\big) + \text{ correction/higher terms} \tag{1.11}$$

   Note that in the language of derived algebraic geometry, we also have an analogous notation. In fact, we introduce and use the *derived tensor product* $\cdot \otimes^{\mathbb{L}} \cdot$ instead of the naive one so that the intersection multiplicity becomes

$$\mathfrak{m}_Z(A,B) = \dim_k \big(\mathcal{O}_{A,Z} \otimes^{\mathbb{L}}_{\mathcal{O}_{X,Z}} \mathcal{O}_{B,Z}\big). \tag{1.12}$$

3. All higher terms vanish in the Cohen-Macaulay case (in which all intersections have the expected dimension). See [2].

2 Preliminaries

Some remarks on $\cdot \otimes^{\mathbb{L}}_{\mathcal{O}_X} \cdot$ We briefly discuss $\cdot \otimes^{\mathbb{L}}_{\mathcal{O}_X} \cdot$, and for details, we refer to Chapter 0 of [4] or Intro. of [3]. Let R be a commutative ring, B a R-module. Then the *derived tensor product* $\cdot \otimes^{\mathbb{L}}_R B$ arises from the construction of the *left-derived functor* associated to the right-exact functor

$$\cdot \otimes_R B : Mod_R \to Mod_R. \tag{2.1}$$

Let A, B be two commutative algebras over R. Then the definition of $A \otimes^{\mathbb{L}}_R B$ naturally appears in the construction of the i^{th} Tor groups $\mathrm{Tor}_i^R(A,B)$ given by the i^{th} homology of the ***tensor product complex*** $(P_\bullet \otimes_R B, \mathrm{d}')$:

$$\cdots \longrightarrow P_2 \otimes_R B \longrightarrow P_1 \otimes_R B \xrightarrow{\mathrm{d}'} P_0 \otimes_R B \longrightarrow 0, \tag{2.2}$$

where $P_\bullet$ is a projective resolution of A equipped with a differential d such that $(P_\bullet, \mathrm{d})$ becomes a commutative dg-algebra over R and $\mathrm{d}' = \mathrm{d} \otimes_R id_B$.

[1] By which, we mean $\mathrm{codim}\ Z = \mathrm{codim}\ A + \mathrm{codim}\ B$ for all irred. comps. Z of $A \cap B$.

Since B is a commutative R-algebra, the tensor product complex inherits the structure of a commutative dg-algebra over R as well, and we denote this tensor product complex by $A \otimes_R^{\mathbb{L}} B$. That is, we set

$$A \otimes_R^{\mathbb{L}} B := (P_\bullet \otimes_R B, \mathrm{d}'). \tag{2.3}$$

Remark 2.1. The resulting commutative dg-algebra $A \otimes_R^{\mathbb{L}} B$ is independent of the choice of $(P_\bullet \otimes_R B, \mathrm{d}')$ up to a quasi-isomorphism.

Construction of the Tor groups. Let A be a commutative ring, and M, N be two A-modules. Define the ith Tor group $\mathrm{Tor}_i^A(M, N)$ as follows:

1. Take any resolution R of N by free modules

$$R: \quad \cdots \to A^{\oplus n_2} \to A^{\oplus n_1} \to A^{\oplus n_0} \to N \to 0. \tag{2.4}$$

   Recall that we can get such a resolution by choosing generators of N and getting a surjective map $A^{\oplus n_0} \twoheadrightarrow N$; and next choosing generators of $\ker(A^{\oplus n_0} \twoheadrightarrow N)$; and so on...

2. Truncate the resolution and apply $M \otimes_A -$:

$$R': \quad \cdots \to A^{\oplus n_2} \to A^{\oplus n_1} \to A^{\oplus n_0} \to 0, \tag{2.5}$$
$$R'': \quad \cdots \to M^{\oplus n_2} \to M^{\oplus n_1} \to M^{\oplus n_0} \to 0, \tag{2.6}$$

   where $M \otimes_A A^{\oplus n_i} = M^{\oplus n_i}$. Here, tensoring with M does not preserve the exactness.

3. Define ***the ith Tor group*** by

$$\mathrm{Tor}_i^A(M, N)_R := H_i(R''),$$

   where $d_i : M^{\oplus n_{i+1}} \to M^{\oplus n_i}$ for $i \geq 0$.

Let us list some facts about the Tor groups. More details can be found in [1].

Observation 2.2. 1. $\mathrm{Tor}_0^A(M, N)_R \simeq M \otimes_A N$.

2. $\mathrm{Tor}_i^A(M, -)$ does not depend on the choice of resolution R.

3. $\mathrm{Tor}_i^A(M, -)$ is a covariant functor $Mod_A \to Mod_A, N \mapsto \mathrm{Tor}_i^A(M, N)$.

4. $\mathrm{Tor}_i^A(M, -)$ provides a *necessary condition* for flatness: If M is a flat A-module, then $\mathrm{Tor}_i^A(M, N) = 0$ for $i > 0$.

5. $\mathrm{Tor}_i^A(M, -)$ extends a short exact sequence of A-modules

$$0 \to N' \to N \to N'' \to 0$$

   to a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Tor}_i^A(M, N') \to \mathrm{Tor}_i^A(M, N) \to \mathrm{Tor}_i^A(M, N'') \\ \cdots \to \mathrm{Tor}_1^A(M, N') \to \mathrm{Tor}_1^A(M, N) \to \mathrm{Tor}_1^A(M, N'') \\ \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'', \end{aligned}$$

   where $\mathrm{Tor}_0^A(M, -) = M \otimes_A -$. (Recall: this part is only right exact.)

6. Let M be an A-module. TFAE:

   (a) M is flat.

   (b) $\mathrm{Tor}_i^A(M, N) = 0$ for all $i > 0$ and A-modules N.

   (c) $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N.

3 Some computations

Example 3.1. Let $I = (xy, xz, wz, wy)$, $J = (w-y, x-z)$, and $R = k[w,x,y,z]$. Write down a resolution for R/J:

$$0 \to k[w,x,y,z] \xrightarrow{\psi} k[w,x,y,z]^{\oplus 2} \xrightarrow{\phi} k[w,x,y,z] \xrightarrow{\pi} k[w,x,y,z]/(w-y, x-z) \to 0,$$

where π is the quotient map; ϕ maps the basis elements e_1, e_2 to $e_1 \mapsto w-y$ and $e_2 \mapsto x-z$; and ψ sends the basis element $f_1 \mapsto (x-z)e_1 + (y-w)e_2$. Note that $\ker\phi = \langle (x-z)e_1 + (y-w)e_2 \rangle$ is of $\dim 1$ and ψ is injective.

After truncating from the last term, where $\mathrm{Tor}_0^R(R/I, R/J) = R/I \otimes_R R/J = k[y,z]/(y^2, yz, z^2)$, apply $R/I \otimes_R -$ to the truncated resolution and get

$$0 \to R/I \xrightarrow{\overline{\psi}} (R/I)^{\oplus 2} \xrightarrow{\overline{\phi}} R/I \to 0,$$

where $\overline{\phi}$ maps $\overline{e_1} \mapsto w-y+I$ and $\overline{e_2} \mapsto x-z+I$; and $\overline{\psi}$ sends the basis element $\overline{f_1} \mapsto (x-z+I)\overline{e_1} + (y-w+I)\overline{e_2}$.

Observe that since $\overline{\psi}$ is injective, it follows that $\mathrm{Tor}_2(\cdots) = 0$ (same for the higher ones). Let us compute $\mathrm{Tor}_1(\cdots)$. Notice that

$$\phi(xe_1 - we_2) = wz - xy \in I$$
$$\phi(ze_1 - ye_2) = zw - xy \in I$$

Thus, both $\xi := xe_1 - we_2$ and $\eta := ze_1 - ye_2$ are in $\ker\phi$. Also, $\xi - \eta = (x-z)e_1 - (w-y)e_2 \in \mathrm{Im}\phi$. So, $[\xi] = [\eta]$ in $\ker\phi/\mathrm{Im}\psi = \mathrm{Tor}_1^R(R/I, R/J)$. We left details to the reader. In brief, we get:

Lemma 3.2. *Let R, I, J be as in Exampe 3.1. Then $\mathrm{Tor}_1^R(R/I, R/J) = \langle [xe_1 - we_2] \rangle$, and hence $\dim_k \mathrm{Tor}_1(\cdots) = 1$. Moreover, $\mathrm{Tor}_i(\cdots) = 0$ for all $i > 1$.*

Computing the correct intersection multiplicity. Let us revisit Remark 1.5/Case-2 and correct the corresponding intersection multiplicity using the Serre formula in (1.9) and Exampe 3.1.

Let X, X_1, X_2, P be as in Remark 1.5/Case-2 and $Z := [1:0:0:0:0]$ the point of intersection. Then using Lemma 3.2, we have

$$\begin{aligned}
\mathfrak{m}_Z(X,P) &= \sum_{i=0}^{\dim X} (-1)^i \mathit{lenght}_{\mathcal{O}_{X\cap P},Z}\left(\mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{A,Z}, \mathcal{O}_{P,Z})\right) \\
&= \dim_k \mathrm{Tor}_0^R(R/I, R/J) - \dim_k \mathrm{Tor}_1^R(R/I, R/J) + 0 \\
&= \dim_k \left(k[y,z]/(y^2, yz, z^2)\right) - 1 \\
&= \dim_k \left(k \cdot 1 \oplus k \cdot y \oplus k \cdot z\right) - 1 \\
&= 3 - 1 = 2,
\end{aligned}$$

which is the desired multiplicity.

References

[1] Weibel CA. *An Introduction to Homological Algebra*. Cambridge University Press; 1994.

[2] Eisenbud D, Harris J. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press; 2016.

[3] J. Lurie. *Derived algebraic geometry*, Ph.D. Thesis, Massachusetts Institute of Technology, 2004.

[4] J. Lurie, *Spectral algebraic geometry*, available at author's website.