Notes on What is DAG

22 Temmuz 2024 Pazartesi 13:27

" $DAG = A.G \underset{\infty\text{-cat}}{\oplus} H.T.$ " ← a short way of encoding DAG.

(if no idea about each, no worries { They are all powerful and rich theories with interesting problem. In this talk, no need to know exactly)

a framework combining classical alg-geo with homotopy theory (via higher category theory)

allows us to do homotopy theory in the context of A.G.

ideas from different math. domains

" homotopical refinement of AG "

OR

higher categorical

- A: Motivation
- B: Some terminology
- C: Some directions

Outline

- Try to understand what kind of framework we're talking about
- Why we care? Do we really need it?
- Tools / approach and hot topics in derived community

- no technicalities
- only key ideas to be mentioned
- explain the way of thinking "our mind setting" towards DAG.

Why important?

Key ideas:

Punchlines:

- DAG ~ *a higher categorical/homotopy theoretical refinement of classical algebraic geometry*. It offers a new way of organizing information for various purposes.
- DAG essentially provides a new setup to deal with non-generic situations in geometry. E.g. non-transversal intersections and "bad" quotients.
- DAG also offers generalized versions of certain familiar geometric structures and studies their properties.

{ e.g. moduli theory, intersection theory in physics

important requirement

Plan (Recall)

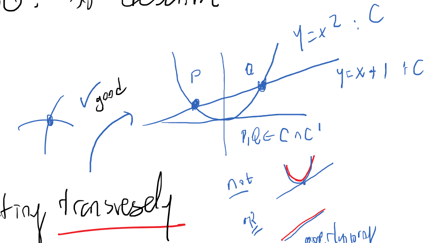
- A: Motivation
- B: Some terminology
- C: Some directions

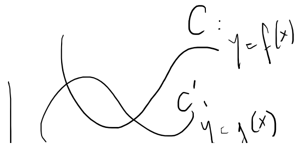
what if our underlying space is highly singular?

Classical, on a smooth mfld, ∃ interesting geometric strcs, like Riemannian, Symplectic, Complex, Contact, Poisson, etc...

A Motivation(s): "bad" intersections and quotients.

(1) Intersection theory: ∃ classical methods in A.G. to describe



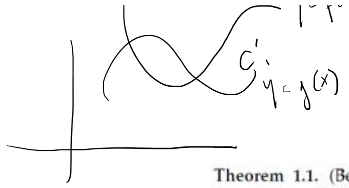
naive case { 

plane curves meeting transversely

$\Rightarrow C \cap C'$ has $\deg(f)\cdot\deg(g)$ points.

overlapping

naive case for curves (plane)



$\Rightarrow C \cap C'$ has $\deg(f) \cdot \deg(g)$ points.

Theorem 1.1. (Bézout's theorem) *Let $C, C' \subseteq \mathbb{P}^2_{\mathbb{K}}$ be two smooth, projective algebraic curves of deg n and m. If C, C' meet transversely, then $C \cap C'$ has precisely $n \cdot m$ points; i.e., $\#(C \cap C') = n \cdot m$.*

For the *non-transverse* intersections, we may use the *scheme-theoretical intersection* rather than a naive set-theoretical approach. This amounts to compute the *dimension of the local ring* $\mathcal{O}_{C \cap C', p}$, which records the *multiplicity* at p. Let us start with some terminology and examples.

- Scheme-theoretical intersection. (requires more algebra, such as local rings)
  - ↳ encodes non-trans. case as well.
  - ↳ But fails in higher dimensions (intersection of higher dim's varieties.)
- Serre's formula (= scheme-theo. intersection nb. ⊕ correction/higher terms)
  - ↳ a general formula including the prev. cases with certain correction/higher terms
  - ⇝ gives the correct int. nb. !

Rmk
- ! Combines ideas/tools from different math. domains (i.e. homological / commutative algebra)
- ! leads to the notion of "derived intersection"

"set of equivalence classes of objects."

(2) Quotient spaces :

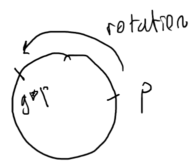
(useful in the context of moduli theory / moduli problems)

about "classifying spaces"
e.g. Classification of vector bundles (up to ---)
elliptic curves (up to ---)
fin. dim. vectorspaces (up to --)
⋮

$G \curvearrowright M$: $G \times M \longrightarrow M$, $(g, p) \longmapsto g \triangleright p$

t-X mfld; a group. some action

e.g.



may define an equ. relation

$p \sim q$ iff $\exists\, g \in G$ s.t. $q = g \triangleright p$

here
21 classes
fin. dim'l $\mathbb{K}$-v.s. up to lin.-isom.

Define $M/_\sim := M/G$

$M \xrightarrow{\pi} M/_\sim$, $p \longmapsto [p]$.

- $n \mapsto \mathbb{R}^n$
- for every n-dim v.s. V, $V \cong \mathbb{R}^n$.

Question : Is $M/_\sim$ a mfld? No, in general. (just have the str. of a set)

Question: Is M/G a mfld? No in general (just have the strc. of a set) i.e. lost info.

When does M/G have the strc. of a mfld? But, we want to "preserve the strc." that we begin with

↳ A suff. cond: when G acts "freely" say G is nice enough

Using categorical terms;

the resulting obj M/G may not live in the same cat, { Due to lack of certain properties of the cat we're working within.

↳ But still, it may be a well-defined obj. in somewhere else ! ⇝ i.e. some cat may not good. But, Real problem might be to work with "categories" need something bigger?

need: higher categories and higher spaces

Terminology:

- *Spaces:* topological spaces, manifolds, varieties, schemes
- *Higher spaces:* stacks, derived schemes/stacks

idea: Instead of naive quotient, we view $[M/G]$ as an "orbifold" or more generally, as a "stack"

Depending on the problem, each requires different level of abstraction and tools.

Key: orbifolds; stacks; higher / derived stack

Need: a setup extending / enlarging ordinary categories so that the resulting objs (under intersection / quotient operations) will lie in the same place !

Conc: from (1)-(2)

⇝ extend the classical framework in "derived and stacky directions"

(not telling you how to do this; too technical ! But this is the motivation and mind setting)

leads to "DAG" (or HAG) ← abstract way of doing Homotopy theory for arbit. cat. (rather than just for Top)

some keywords: model cat, derived cats, ∞-cats, derived schemes, derived stacks, ...

Q3: How to define higher spaces?

Key: view Higher spaces as certain functors

Thanks to Yoneda's emb.

B] : How to define higher spaces?

Key: "Higher spaces as certain functors"

tech: "homotopy sheaves"

Q: How do describe? (∃ many equivalent ways!)

"Functor of points approach to define certain geometric objects"

Slogan: "One recovers an object in $\mathcal{C}$ by knowing all morphisms into it!" *(Yoneda lemma/embedding)*

Rmk
- ∃ Yoneda emb. $\mathcal{C} \xrightarrow{h} \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$, $u \longmapsto (h_u : \mathcal{C}^{op} \to \mathrm{Sets})$, $= y \to \mathrm{Hom}(y,u)$
- Yoneda's lemma: h is fully faithful ($\sim$ inj.)
- But <u>not</u> essentially surjective

- Using Yoneda's emb; "functorial perspective" (*)

A is analogue of $C^\infty(M)$ in AG. "Algebra of funcs"

$$\mathrm{CAlg}_{\mathbb{k}} \xrightarrow{\text{schemes}} \mathrm{Sets}$$

"affine scheme"

$A \in$ e.g. $X = \mathrm{spec}\, A$

$0 \to 0 \to 0 \xrightarrow{d} A,\ d = 0$ — cdga — viewed as a cdga concentrated in deg 0 with $d = 0$.

move in this direction

play with the source: change the local algebraic. $\sim$ about interpreting the algebra of funcs.

stacks — "2-functor" — Grpds [2-cat. of groupoids] $=$

higher stacks

$\mathrm{Sets} =$ $\downarrow$ Grpds $\downarrow$ Ssets [∞-Cat of simplicial sets]

$$\mathrm{cdga}^{\le 0}_{\mathbb{k}} \xrightarrow{\text{derived}} \mathrm{Ssets}$$

like (CW) $\{X_n\}_{n \in \mathbb{N}}$ X_n: the set of n-simplexs

points are the same as $\tau(X)$. Scheme strc. are different.

$X = \mathrm{spec}\, A$ affine "derived" scheme

- grading + ∃ a truncation func. $\tau : \mathrm{dSt} \to \mathrm{Sch}$.

play with the target: move from top to buttom.

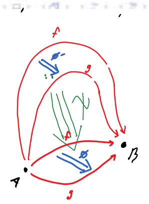
- strc. of points. {points of a scheme form a SET ...
- non trivial higher morphisms
- (notions like morphisms between morphism. " " " between mp)
- inf - tower of equivalence

need to employ higher (∞-) category theory (as the amount of info. about two objs. are related increases dramatically)

1. LHS is about local algebraic models.
2. RHS is about the structure of the points.
3. RHS encodes higher symmetries/equivalences/relations (more non-trivial higher morphisms are allowed).
4. Infinite tower of equivalences changes the way one should think of two objects being the same.

slogan: different layers of the simplicial strc. encode different levels of symm.

- weaker notions of equivalences, such as "homotopy"
- more subtle notion of "sameness"
- keep track of all different ways in which two objects are equal ("same")



"Simplicial structure"

0-simplex $\longleftrightarrow$ object (e.g. A, B)

1-simplex $\longleftrightarrow$ morphism (e.g. $f, g : A \to B$)

2-simplex $\longleftrightarrow$ 2-morphism ($\phi : f \Rightarrow g$, $\phi' : f \Rightarrow g$)

3-simplex $\longleftrightarrow$ 3-morphism (e.g. $\psi : \phi \Rrightarrow \phi'$)

⋮

One may ask
- when points A, B are equiv? (∃ a path f)
- when two paths are equiv? (∃ ϕ - homotopy)
- when two homotopy are equiv? (if ∃ ψ ....).

⋮

$\Longrightarrow$ Build an "infinite tower of equivalences" between equivalences

also shows how we combine ideas from homotopy theory and category theory to manifest DAG.

C] Hot topics in DAG

Finally: (What people do:)

"~~Important~~ some" topic(s) in derived geometry

(at least I'm interested in...)

(a biased and non-complete list of topics)

(1)! Derived symplectic geometry

- Can define (shifted) symp. strc. on derived stacks

extending classical symp. strc. on smooth manifolds / schemes. sit inside "0-shifted" level.

- Classical results (in symp Geo)

sit inside "0-shifted" ...

- Classical results (in symp Geo) happen to be hard to prove in the derived context (due to the flexibility of this setup)

Still, we have:

Advanced Rmk:

Some other relevant concepts and key results:

1. Isotropic and Lagrangian structures $L \to (X, \omega)$ in the derived context
2. PTVV's Lagrangian intersection theorem & shifted symplectic structures on mapping stacks
3. A Darboux-type theorem for shifted symplectic derived stacks (C. Brav, V. Bussi and D. Joyce)
4. A Lagrangian neighborhood theorem for shifted symplectic derived stacks (D. Joyce, P. Safronov)
5. ...

(2) Derived Poisson geometry and deformation theory

(3) Derived geometric formulations in physics.

OR Stacky

Note

my thesis: ~ Stacks in GR

Project: Development of derived
ongoing. contact geometry.
(some results available)

↳ for gauge theories

↳ functorial field theories (e.g. TQFTs)

↳ Deformation quantization

↳ Algebraic QFT

⋮

other stories and mathematical domains

(4) can see "derived" or higher categorical approaches in other mathematical domains (e.g. number theory, logic, foundations, etc...)