Department of Mathematics

2 Dimensional Topological Quantum Field Theories

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Course: Math 491

Semester: Fall 2024

Contents

1 Abstract

In this report, we will present the theory of the *Topological Quantum Field Theories* (TQFTs) in dimension 2. In this regard, we will first give the necessary category theoretical background and then provide the formal definition of TQFTs. Later, we will focus on 2-TQFTs as a particular case and briefly discuss their characterization by means of *Frobenius algebras*. At the end, we present the main theorem of this report: $\mathbf{2TQFT}_k \simeq \mathbf{cFA}_k$.

2 Introduction

The flow of inspiration has been mainly from mathematics to physics, throughout the last century. This report is on one of the few counter-examples: the Topological Quantum Field Theories. Though, our interest is in the (very limited) mathematical aspects only, we believe it is always beneficial to mention the motivation of the concepts we use.

In physics, TQFTs can be used as a "baby model" to test some hypothesis in the search for a quantum gravity theory, as TQFTs "possess certain features one expects" from such a theory. In mathematics, they are used to produce manifold invariants (such as Jones polynomial) ([4], p.4). None of these motivations will be mentioned after this point. Now, let us give an outline of the report before starting to talk about categories.

In the first part, we will give the necessary category theoretical definitions. These will provide us with a concise and formal way to define TQFTs. The following section will be on the cobordism category (**nCob**), starting with the definitions regarding manifolds and cobordisms. Then, the first (informal) definition of TQFTs is stated: an assignment of a vector space and a linear map to a manifold and cobordism, respectively, satisfying certain axioms, which will turn out to yield the second and formal definition of TQFTs: symmetric monoidal functor from **nCob** to $\mathbf{Vect}_k$. Before giving this definition, we informally investigate the words 'symmetric' and 'monoidal structure.' Afterwards, we turn our attention to **2Cob**, which is our primary interest in this report and list its building blocks (generators). With this list, we see that all 2-cobordisms have a nice representation. Further, we list some relations between the generators in order to see, the (once) hidden link between TQFTs and Frobenius algebras. In order to see this relation in its proper context, we define symmetric monoidal categories and show that the category of TQFTs and the category of commutative Frobenius algebras are equivalent.

To have it recorded at the beginning, we announce that there is no original result in this report but the presentation style of the material.

3 Background in Category Theory

In this first section, we summarize the necessary knowledge of category theory in order to define the *Topological Quantum Field Theories* as a symmetric monoidal functor from **nCob** to $\mathbf{nVect}_k$ (all these will be defined in the next section.) Throughout this section, I followed [7], [2], and [6]; diagrams are drawn by me.

3.1 Definition of a Category

A **category** $\mathcal{C}$ consists of the following data:

- A collection of *objects*, denoted as $\mathrm{Ob}(\mathcal{C})$.
- For each pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set of *morphisms* (also called *arrows*) from A to B, denoted $\mathrm{Mor}_{\mathcal{C}}(A, B)$.
- For every object $A \in \mathrm{Ob}(\mathcal{C})$, there exists an identity morphism $\mathrm{id}_A \in \mathrm{Mor}_{\mathcal{C}}(A, A)$.
- For each triple of objects $A, B, C \in \mathrm{Ob}(\mathcal{C})$, there exists a composition law:

$$\circ : \mathrm{Mor}_{\mathcal{C}}(B, C) \times \mathrm{Mor}_{\mathcal{C}}(A, B) \to \mathrm{Mor}_{\mathcal{C}}(A, C)$$

  such that for $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(B, C)$, we have $g \circ f \in \mathrm{Mor}_{\mathcal{C}}(A, C)$.

These must satisfy the following properties:

- **Associativity**: For $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Mor}_{\mathcal{C}}(B, C)$, and $h \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, we have:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- **Identity**: For every $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$, we have:

$$\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$$

3.2 Examples of Categories

Here are some examples of categories:

1. **Sets**, $\mathbf{Vect}_k$, **Grp**, **Top** etc. are the usual categories whose objects and morphisms are indicated by their names.

2. **The Poset Category:** Let $(S, \leq)$ be a partially ordered set. Define a category $\mathcal{C}$ where:
$$\mathrm{Ob}(\mathcal{C}) = S, \quad \mathrm{Mor}_{\mathcal{C}}(x, y) = \begin{cases} \{*\} & \text{if } x \leq y, \\ \emptyset & \text{otherwise.} \end{cases}$$
In this category, morphisms correspond to the order relation $\leq$ between elements of S.

3. **The Monoid Category:** A monoid $(M, \cdot, 1)$ can be viewed as a category with one object. Define a category $\mathcal{C}$ as follows:
$$\mathrm{Ob}(\mathcal{C}) = \{*\}, \quad \mathrm{Mor}_{\mathcal{C}}(*, *) = M.$$
The composition law is given by the monoid operation, and the identity morphism is the identity element 1 of the monoid.

4. **The Groupoid Category:** A groupoid is a category where all morphisms are isomorphisms. For example, if G is a group, we can view G as a category with:
$$\mathrm{Ob}(\mathcal{C}) = \{*\}, \quad \mathrm{Mor}_{\mathcal{C}}(*, *) = G,$$
where every morphism has an inverse.

5. **The Category of Relations:** Define a category **Rel** where the objects are sets and the morphisms are relations between sets. That is, for sets A and B, the morphisms $\mathrm{Mor}_{\mathbf{Rel}}(A, B)$ are subsets of the Cartesian product $A \times B$. Composition is given by relational composition.

6. $\mathbf{Mat}_{\mathbb{R}}$ is the category whose objects are natural numbers and a map between m, n is a $m \times n$ matrix.

7. **Prop** is the category whose objects are the collection of all (mathematical) propositions and if $P \implies Q$, then $Mor(P, Q) = \{*\}$ and if not, no morphism in between.

3.3 Constructions by Universal Properties

(This subsection is based on [7]). Given $A, B \in \text{Ob}(\mathcal{C})$, a new object can be created via solely its relations (maps to / from it) to all other objects. We will observe that an object satisfying the desired properties will necessarily be 'unique', if it exists. Uniqueness in Category theory means 'unique up to a unique isomorphism'. Here are some such constructions.

3.3.1 Product

Given objects M and N in $\text{Ob}(\mathcal{C})$, $M \times N$ is 'the' *product object* along with maps $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$, satisfying the following universal property:

For any object S with morphisms $f_1 : S \to M$ and $f_2 : S \to N$, there exists a unique morphism $h : S \to M \times N$ such that the following diagram commutes:

$$\begin{tikzcd}
 & M \times N \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
M & S \arrow[l, "f_1"'] \arrow[r, "f_2"] \arrow[u, dashed, "h"] & N
\end{tikzcd}$$

This diagram shows that $f_1 = \pi_1 \circ h$ and $f_2 = \pi_2 \circ h$.

Uniqueness: Suppose T is another object with morphisms $\mu_1 : T \to M$ and $\mu_2 : T \to N$ such that the following diagram also commutes. Then, by the universal property of the product, there are unique morphisms h, h' that make the diagram commute.

$$\begin{tikzcd}
 & M \times N \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd, dashed, bend right, "h'"'] & \\
M & & N \\
 & T \arrow[ul, "\mu_1"] \arrow[ur, "\mu_2"'] \arrow[uu, dashed, bend right, "h"'] &
\end{tikzcd}$$

We want to show that $h \circ h' = \text{id}_T$ and $h' \circ h = \text{id}_{M \times N}$. If this were not the case, then there would be two morphisms from products to themselves, which is not possible. Hence, $T \cong M \times N$.

Example 3.1. Some realizations of products in different categories.

- The **Cartesian product** is the product in **Sets**.
- The **Direct product** is the product in **Grp**.
- $P \wedge Q$ is the product in **Prop**.

Similarly, we can define *coproducts* by reversing the arrows in the defining diagram. As an illustrative example, given objects M and N in the category of sets, the coproduct $M \sqcup N$ is the disjoint union of the sets, equipped with canonical inclusion maps.

The coproduct is constructed by the universal property as such: for any set S with maps $f : M \to S$ and $g : N \to S$, there exists a unique map $h : M \sqcup N \to S$ making the diagram commute:

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "f"'] & M \sqcup N \arrow[d, dashed, "h"] & N \arrow[l] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Obviously, h is defined as the piecewise function of f, g.

3.3.2 Tensor Product

We are working in the category $\mathbf{Mod}_R$.

For modules M and N, the *tensor product* $M \otimes_R N$ is defined as the object equipped with a bilinear map $t : M \times N \to M \otimes_R N$ such that for any $S \in \mathrm{Mod}_R$ and any bilinear map $s : M \times N \to S$, there exists a unique linear map $f : M \otimes_R N \to S$ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "t"] \arrow[dr, "s"'] & M \otimes_R N \arrow[d, dashed, "f"] \\ & S \end{tikzcd}$$

Uniqueness: Assume (T, t') with t' satisfies the same property. Then, following the same argument used in the uniqueness of product, there exists a unique isomorphism $f : M \otimes_R N \to T$ making the following diagram commute:

$$\begin{tikzcd}
M \times N \arrow[r, "t"] \arrow[d, "t'"'] & M \otimes_R N \arrow[dl, dashed, bend right, "f"'] \\
T \arrow[ur, dashed, bend right, "f'"'] &
\end{tikzcd}$$

Thus, $(T, t') \cong (M \otimes_R N, t)$ in the category $\mathbf{Mod}_R$ of R-modules with bilinear maps.

Existence: The free R-module on $M \times N$ modulo the relations

(i) $(m_1 + m_2, n) \sim (m_1, n) + (m_2, n)$

(ii) $(m, n_1 + n_2) \sim (m, n_1) + (m, n_2)$

(iii) $(rm, n) \sim r(m, n) \sim (m, rn)$

satisfies the universality criteria.

Idea: The tensor product of two R-modules is the result of a minimal bilinear map (meaning it satisfies properties (i), (ii), and (iii)).

Exercise 3.2.

- Tensor product of finite fields:

$$\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\gcd(n, m)\mathbb{Z}$$

- The tensor product $- \otimes N$ is a right exact functor, i.e., for a short exact sequence:

$$0 \to M' \xhookrightarrow{i} M \xrightarrow{\pi} M'' \to 0$$

where π is surjective, i is injective and $i(M') = \ker(\pi)$, we have:

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is exact.

However, the tensor product is not left exact. For example, tensor the sequence of $\mathbb{Z}-$Modules:

$$0 \to \mathbb{Z} \xhookrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$$

with $\mathbb{Z}_2$.

We get,

$$0 \to \mathbb{Z} \otimes \mathbb{Z}_2 \xhookrightarrow{(\times 2, id)} \mathbb{Z} \otimes \mathbb{Z}_2 \to \mathbb{Z}_2 \otimes \mathbb{Z}_2 \to 0$$

The map $(\times 2, id)$ is not injective, since

$$a \otimes b \mapsto 2a \otimes b = a \otimes 2b = a \otimes 0 = 0.$$

Hence the sequence is not exact.

This demonstrates that the tensor product does not preserve left exactness.

- $M \otimes \left(\bigoplus_{i=1}^{k} N_i \right) \cong \bigoplus_{i=1}^{k} (M \otimes N_i)$.

  The isomorphism is given by

$$m \otimes (n_1, n_2, \ldots, n_k) \mapsto (m \otimes n_1, m \otimes n_2, \ldots, m \otimes n_k)$$

3.3.3 Fibered Products (Pullback)

For morphisms $\alpha : X \to Z$ and $\beta : Y \to Z$, the *fibered product* is an object denoted as $X \times_Z Y$, along with projections π_1 and π_2 such that:

$$\alpha \circ \pi_1 = \beta \circ \pi_2$$

For any object W with morphisms $\mu_1 : W \to X$ and $\mu_2 : W \to Y$ such that:

$$\alpha \circ \mu_1 = \beta \circ \mu_2$$

there exists a unique morphism $f : W \to X \times_Z Y$ making the following diagram commute:

$$\begin{tikzcd}
& W \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\mu_2"'] \arrow[ddr, bend left, "\mu_1"] & \\
& X \times_Z Y \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
Y \arrow[dr, "\beta"'] & & X \arrow[dl, "\alpha"] \\
& Z &
\end{tikzcd}$$

Example 3.3. 1. In sets, $X \times_Z Y = \{(x, y) \in X \times Y \mid \alpha(x) = \beta(y)\}$.

2. In the case of topological spaces, for U and V, $U \times_Z V = U \cap V$, the fibered product corresponds to the intersection of open sets.

Exercise 3.4. If Z is the final object, show that $X \times_Z Y = X \times Y$.

Proof. Since Z is the final object, there exist unique maps α, β. Since Z is the final object, $\alpha \circ \pi_1 = \beta \circ \pi_2$ necessarily. Therefore, the product $X \times Y$ satisfies the conditions of being $X \times_Z Y$.

$$\begin{tikzcd}
& W \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\mu_2"'] \arrow[ddr, bend left, "\mu_1"] & \\
& X \times Y \arrow[dl, "pr_2"'] \arrow[dr, "pr_1"] & \\
X \arrow[dr, "\alpha"'] & & Y \arrow[dl, "\beta"] \\
& Z &
\end{tikzcd}$$

□

Exercise 3.5. Consider the following commutative diagram for the two Cartesian product. The outer diagram is also a Cartesian product.

$$\begin{tikzcd}
W \arrow[drr, "g"] \arrow[ddrr, dashed, bend left, two heads, "vg"] \arrow[dr, dotted, "h'", "h"'] \arrow[ddr, dotted, "f"] \arrow[dddr, two heads] & & \\
& U \arrow[r] \arrow[d] & V \arrow[d, "v"] \\
& W \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}$$

This can be seen by following the unique maps (shown as dashed).

As common in the category theory, we have also the *dual* concept *fibred coproducts (pushout).* The definition is by the following universal diagram:

$$\begin{tikzcd} Z \arrow[r, "i_X"] \arrow[d, "i_Y"] & X \arrow[d] \\ Y \arrow[r] & X \amalg_Z Y \end{tikzcd}$$

With this definition, we can see the coproduct as the pushout diagram with the initial object.

3.3.4 Kernel & Cokernel

(This subsection is based on [2].)

Definition 3.6. Let $\mathcal{C}$ be a category and assume that there exists a zero object O. Given any pair of objects X, Y in $\mathcal{C}$, we call *zero morphism*, denoted by 0_{XY}, the unique morphism $X \to Y$ that factors through O. In a diagram:

$$\begin{tikzcd} X \arrow[rr, "0_{XY}"] \arrow[dr] & & Y \\ & O \arrow[ur] & \end{tikzcd}$$

Definition 3.7. Let $\mathcal{C}$ be a category and assume that there exists a zero object O in $\mathcal{C}$. Given a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we say that a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(K, X)$ is a *kernel* of f if

1. $f \circ \phi = 0_{KY}$;
2. For every other morphism $h \in \mathrm{Hom}_{\mathcal{C}}(W, X)$ such that $f \circ h = 0_{WY}$, there exists a unique morphism $\eta \in \mathrm{Hom}_{\mathcal{C}}(W, K)$ such that $\phi \circ \eta = h$.

$$\begin{tikzcd} K \arrow[dr, "\phi"] \arrow[drr, bend left, "0_{KY}"] & & \\ & X \arrow[r, "f"] & Y \\ W \arrow[uu, dotted, "!\exists\gamma"] \arrow[ur, "g"] \arrow[urr, bend right, "0_{WY}"'] & & \end{tikzcd}$$

Example 3.8. In **Ab**, given a group homomorphism $f : A \to B$, the inclusion map, $i : ker(f) \to A$ is the kernel. More familiarly, the 'subgroup' $ker(f)$ can be defined as the following pullback diagram:

$$\begin{tikzcd} \ker f \arrow[r] \arrow[d, "i"] & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Remark 3.9. In any category, any kernel of f is necessarily a monomorphism.

We can also define the *cokernel* of a map f as the following pushout diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & coker f \end{tikzcd}$$

Example 3.10. In the category **Ab**, $coker f = \frac{B}{imf}$.

3.4 Special Morphisms

(This subsection is based on [7].)

3.4.1 Monomorphism

A morphism $\pi : X \to Y$ is a *monomorphism* if for all $\mu_1, \mu_2 : Z \to X$,

$$\pi\mu_1 = \pi\mu_2 \implies \mu_1 = \mu_2$$

This property is known as *left cancellability.*

Example 3.11. In the category of sets, $\pi : S \to T$ is a monomorphism if and only if π is injective.

Proof. ($\Rightarrow$) Assume $\pi\mu_1 = \pi\mu_2$ for $\mu_1, \mu_2 : Z \to S$. Let $Z = \{*\}$. Let $\mu_1(*) = s_1$ and $\mu_2(*) = s_2$

$$\pi(s_1) = \pi(s_2) \implies \pi(\mu_1(*) = \pi(\mu_2(*)) \implies \mu_1 = \mu_2 \implies s_1 = s_2.$$

Hence, π is injective.

($\Leftarrow$) Let μ_1, μ_2 be two maps between Z and S.

$$\pi(\mu_1(z)) = \pi(\mu_2(z)) \implies \mu_1(z) = \mu_2(z)$$

Hence, π is a monomorphism. $\square$

Example 3.12 (A mapping that is a monomorphism but not injective ([7], p.37):)**.** Consider the map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ in the category of divisible groups. It is not injective, but still monomorphism.

Proof. Let D be a divisible group, i.e., for all $d \in D$, there exists $d' \in D$ such that $nd' = d$.

Assume $\pi\mu_1(d) = \pi\mu_2(d)$. Then,

$$\mu_1(d) - \mu_2(d) \in \mathbb{Z}(= \ker(\pi))$$

Since D is divisible, let $d = 2md'$. Therefore,

$$\mu_1(2md') - \mu_2(2md') = 2m(\mu_1(d') - \mu_2(d'))$$

Thus, either $\mu_1(d') = \mu_2(d')$ or the difference is $1/2$, which is not possible. $\square$

3.4.2 Epimorphisms

A morphism $e : S \to T$ is an *epimorphism* if for all $\mu_1, \mu_2 : T \to Z$,

$$\mu_1 e = \mu_2 e \implies \mu_1 = \mu_2$$

This property is known as *right cancellability.*

Exercise 3.13. A morphism $e : S \to T$ is an epimorphism if and only if e is surjective.

Proof. ($\Rightarrow$) Assume $\mu_1 e = \mu_2 e \implies \mu_1 = \mu_2$. Take $t \in T$. We look for a s st $e(s) = t$.

By epimorphismity, if $\mu_1(s) \neq \mu_2(s)$, then $\mu_1 e \neq \mu_2 e$. Assume there is no such s. Construct μ_1, μ_2 st they agree in all values but t. Then, $\mu_1 e = \mu_2 e$ but $\mu_1(s) \neq \mu_2(s)$. Contradiction.

($\Leftarrow$) Trivial. $\square$

3.5 Initial and Final Objects

An object $I \in \mathcal{C}$ is *initial* if for all objects $C \in \mathcal{C}$, there exists a unique morphism $I \to C$. Similarly, F is *final* if for all $C \in \mathcal{C}$, there exists a unique morphism $C \to F$. If an object is both initial and final, it is called *zero* object.

Example 3.14. In the category of rings, the initial object is $\mathbb{Z}$.

Proof. Assume $\varphi_1, \varphi_2 : \mathbb{Z} \to R$. Then,

$$1_R = \varphi_1(1) = \varphi_2(1) \implies \varphi_1(n) = \varphi_2(n)$$

Hence, $\varphi_1 = \varphi_2$. □

Example 3.15. 1. In **Set**, $\emptyset$ is the initial object and a singleton is 'the' final object.

2. Initial object in **Prop** is $\perp$, final object is T.

3.6 Functors

A *functor* $F : \mathcal{A} \to \mathcal{B}$ is a mapping of objects and morphisms such that:

- $F : \mathrm{Obj}(\mathcal{A}) \to \mathrm{Obj}(\mathcal{B})$
- $F : \mathrm{Mor}(A_1, A_2) \to \mathrm{Mor}(F(A_1), F(A_2))$.

For identity and composition:

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}, \quad F(f \circ g) = F(f) \circ F(g).$$

F is *faithful* if $F : \mathrm{Mor}_{\mathcal{A}}(A, A') \to \mathrm{Mor}_{\mathcal{B}}(F(A), F(A'))$ is injective, and *full* if it is surjective, *fully faithful* if both injective and surjective. Additionally, if $F(Obj(\mathcal{A}) = Obj(\mathcal{B})$, F is called *essentially surjective*.

Functors defined in this way are called *covariant*. Very similar definition can be made for *contravariant* functors, $F : \mathcal{A} \to \mathcal{B}$ as such:

- $F : \mathrm{Obj}(\mathcal{A}) \to \mathrm{Obj}(\mathcal{B})$
- $F : \mathrm{Mor}(A_1, A_2) \to \mathrm{Mor}(F(A_2), F(A_1))$.

where the composition is defined correspondingly as follows:

$$F(f \circ g) = F(g) \circ F(f).$$

Here are some functor examples:

Example 3.16 ([6], pp.13-15).

1. There are many functors called *forgetful functors*. These functors forget additional structure and map the objects to their underlying structure (usually sets).

2. The 'tensor' functor $\otimes_R : \mathrm{Mod}_R \times \mathrm{Mod}_R \to Ab$ is the tensor product (bi-)functor.

3. $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ is the functor taking a pointed topological space X and associating it to its fundamental group $\pi_1(X)$.

4. For any category $\mathcal{C}$ and any object $a \in \mathcal{C}$, we have a *Hom functor*:

$$\begin{aligned} \mathrm{Hom}(a, -) : \mathcal{C} &\to \mathbf{Sets} \\ b &\mapsto \mathrm{Mor}(a, b) \\ (f : b \to c) &\mapsto (\mathrm{Mor}(a, b) \to \mathrm{Mor}(a, c)) \\ u &\mapsto f \circ u. \end{aligned}$$

5. Let $(-)^\vee : \mathbf{Vect}_k \to \mathbf{Vect}_k$ be the dual taking functor. This is an example of a contravariant functor.

6. Let $(-)^{\vee\vee} : f.d\mathbf{Vect}_k \to f.d\mathbf{Vect}_k$ be the double dual operator on the finite dimensional vector spaces. This functor is covariant though.

7. We can see taking direct-images as a functor. $P : \mathbf{Sets} \to \mathbf{Sets}$, $P(A) = \mathcal{P}(A)$ and $P(f : A \to B)(A' \subset A) = f(A) \subset B$. We also have a sibling (contravariant) functor for pre-image taking.

8. Let $\mathbf{Euclid}_*$ be the pointed Euclid spaces of any dimension. $D : \mathbf{Euclid}_* \to \mathbf{Mat}_{\mathbb{R}}$, associating a space to its dimensions and a differentiable map $f : \mathbb{R}^m \to \mathbb{R}^n$ is assigned to its *Jacobian matrix* at a. Composition rule for functors justifies one of the main results of multivariable calculus, $D(gf) = DgDf$.

3.7 Natural transformations

> "It is not too misleading, at least historically, to say that categories are what one must define in order to define functors, and that functors are what one must define in order to define natural transformations." - Peter Freyd ([6], p.23).

Definition 3.17. A *natural transformation* between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is the data of maps $\{\alpha_c\}_{c\in\mathcal{C}}$ [1] such that for all maps $f : a \to b$ of $Mor(a, b)$, the following diagram commutes:

$$\begin{tikzcd} Fa \arrow[r, "Ff"] \arrow[d, "\alpha_a"] & Fb \arrow[d, "\alpha_b"] \\ Ga \arrow[r, "Gf"] & Gb \end{tikzcd}$$

If all α_c is isomorphism, then α is called *natural isomorphism.* Two categories $\mathcal{C}, \mathcal{D}$ are said to be *equivalent* if there exists two functors F, G such that $G \circ F = id_{\mathcal{C}}$ and $F \circ G = id_{\mathcal{D}}$, where identity functors are defined in the trivial sense.

Example 3.18 ($V \simeq V^{\vee\vee}$)**.** Let α_V be the isomorphism between a finite dimensional V and its double dual, sending $v \mapsto E_v$ such that for for $\phi \in V^\vee$, $E_v(\phi) = \phi(v)$. Then, the natural transformation α is indeed a natural isomorphism between the functors $id_V, (-)^{\vee\vee}$, by making the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "\alpha_V"] \arrow[d, "f"] & V^{\vee\vee} \arrow[d, "f^{\vee\vee}"] \\ W \arrow[r, "\alpha_W"] & W^{\vee\vee} \end{tikzcd}$$

3.8 Yoneda's lemma

From a category $\mathcal{C}$, one can construct a *functor category / functor of points* whose objects are (contravariant) functors from $\mathcal{C}$ to **Sets** and morphisms are natural transformations between these functors.

[1] The element-hood drawn here is an abuse of notation.

Theorem 3.19 (Yoneda's lemma)**.** *For a fixed $a \in \mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$, we have the following natural isomorphism:*

$$Mor(Hom(-, a), F) \simeq Fa.$$

There are some implications of this ubiquitous lemma of category theory. First one to mention is how limited the number of natural transformations between the hom-functor and any other functor, namely $|Fa|$ [5]. Another implication, which is seen as the 'natural' interpretation of the lemma is this big claim: *maps to an object determine uniquely the object.* We can see this if we chose $f = Hom(b, -)$.

3.9 Equivalence of categories

Just like every branch of mathematics, we will regard some different categories as the same, called *equivalent categories.* Two categories $\mathcal{C}, \mathcal{D}$ are said to be equivalent if there exists two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that

$$FG \simeq 1_{\mathcal{D}} \text{ and } GF \simeq 1_{\mathcal{C}}.$$

Theorem 3.20 ([6], p.31)**.** *Any fully faithful and essentially surjective function defines an equivalence of categories (assuming the Axiom of Choice).*

A subcategory can also be equivalent to the 'super-category'. A special case is (in a reasonable sense) the (again, up to isomorphism) minimum such subcategory, called the *skeleton.* It is constructed by taking only one object from each isomorphism class.

We have one more 'fining' method for a category, considering only the 'essential' maps between objects. A *generating set* for a category is the class of maps such that every map can be written as composition of some of these maps ([4], p. 59). To illustrate these concepts, let us give an example:

Example 3.21 ([4], p.59)**.** We investigate $f.d.\mathbf{Vect}_k$. As we know from linear algebra, any vector space of dimension n is isomorphic to $\mathbb{R}^n$. So, $\{\mathbb{R}^n\}$ is a skeleton (with all the maps between spaces included). We also have a decomposition of a $m \times n$ matrix as ADB where A, B are invertible matrices and "D is a matrix consisting of zeros, except for an r-by-r minor identity matrix". Also, any invertible matrix can be written as compositions of elementary matrices. Hence, a generating set for $f.d.\mathbf{Vect}_k$ is all the elementary matrices and matrices of the form D.

We will use these definitions to have a better understanding of the cobordism category, to-be-defined. Equipped with the tools of category theory, we can move to the cobordism theory and then, TQFTs.

4 Introduction to Cobordisms and TQFTs

In this section, we briefly define cobordisms and then define TQFTs as a rule and formal definition, a functor from the cobordism category to vector fields category, will come in the next sections. Throughout this section, I will be closely following and using the images of [4], unless they are hand-drawn.

4.1 Some definitions from Manifold theory

Intuitively, a *n-dimensional manifold* is an object which resembles $\mathbb{R}^n$ at each point. Formally, it is a topological space which is locally homeomorphic to $\mathbb{R}^n$, second countable (has a countable basis) and Hausdorff. The latter two properties are just to avoid pathological examples that we will not deal with.

A *coordinate chart* over a manifold M is an homeomorphism $f_\alpha : U_\alpha \to \mathbb{R}^n$, here U_α is called the *chart domain* and a collection of coordinate charts whose coordinate domains cover M is called an *atlas*. For two intersecting U_α, U_β, the restricted version of the maps $f_\alpha \circ f_\beta^{-1}$ should be *smooth*, i.e., of class C^∞, in order for the manifold (with this particular atlas on it) to be a smooth.

A manifold M is said to be a *manifold with boundary* if it has a sub-manifold of dimension n-1 for which as a sub-manifold, it is locally homeomorphic to $\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R} | x_n \geq 0\}$. The *boundary* of M is denoted by ∂M. In this report, we will think all manifolds as manifolds with (possibly empty) boundary. We will also think the empty set $\emptyset_n$ as a $n-$dimensional manifold!

A manifold is called *compact* if it is compact as a topological space and *closed* if it is compact without boundary / with empty boundary. Throughout the report (as Koch does in [4]), we will denote the n-manifold with boundary with M and $(n-1)$-manifold without boundary with Σ.

The *orientation* of a (smooth) manifold is induced from the orientations of its tangent spaces as vector spaces. An orientation of a vector space is an assignment of a sign (+ or −) to an ordered basis such that the change of basis matrix has positive determinant if two bases have the same sign. Orientation of the manifold is the 'smooth' choice of basis for the tangent spaces, i.e., the change of basis matrix between two tangent spaces at two points that lay on the same open set should have positive determinant. The oppositely oriented manifold is denoted by $\overline{M}$ for an *orientable* (admitting an orientation) manifold M. All manifolds we will deal with will be orientable.

Let Σ be a connected boundary component of M. Σ is said to be *in-boundary*, if the *positive normal vector* (a vector that when added to the ordered basis of a tangent space of a point in Σ, the new basis will be a positive basis for the tangent space of the same point but now as an element of M) points inwards, otherwise Σ is said to be an *out-boundary.*

4.2 Cobordisms

Cobordisms can be thought as a generalizations of manifolds or special cases of manifolds with boundaries, depending on your point of view. Given two n-manifolds (without boundary) Σ_0, Σ_1, a (unoriented) *cobordism* is the $(n+1)$-manifold whose boundary is $\Sigma_0 \coprod \Sigma_1$. If two manifolds yield a cobordism, then they are called *cobordant.* Our eventual goal is to form a category of cobordisms, to be called **nCob**, whose objects are $(n-1)$-manifolds and morphisms are cobordisms. Therefore, we need a notion of start/ end points for a cobordism.

This is achieved via the *oriented cobordism*, from Σ_0 to Σ_1, denoted as $\Sigma_0 \Longrightarrow \Sigma_1$. Here, Σ_0 is the in-boundary and Σ_1 is the out-boundary. There is a subtle difficulty here: we cannot have a cobordism from the same manifold to itself. This is achieved via this trick:

$$\Sigma_0 \hookrightarrow M \hookleftarrow \Sigma_1$$

where the arrows are orientation-preserving smooth inclusions mapping (diffeomorphicaly) the manifolds to the in and out-boundaries respectively.

4.2.1 Cylinders

Cylinder constructions are very common in topology. Their general form for a manifold Σ is $\Sigma \times I$ where I is the unit interval. First example is for $0-$manifolds, i.e. points.

Since the tangent space of a point is the trivial vector space, we have not many options to assign a sign to this (non-existing) basis: $+$ or $-$. Unit interval with its classical orientation (from 0 to 1) is positively assigned. We will use this information/ convention later.

We can see the unit interval as the cobordism from a point to itself:

$$p \hookrightarrow I \hookleftarrow p.$$

Similarly, for a manifold Σ,

$$\Sigma \xrightarrow{\sim} \Sigma \times \{0\} \hookrightarrow \Sigma \times I \hookleftarrow \Sigma \times \{1\} \xleftarrow{\sim} \Sigma$$

gives us the cylinder construction. A cobordism induced from a diffeomorphism $\psi : \Sigma_0 \xrightarrow{\sim} \Sigma_1$ is constructed as follows:

$$\Sigma_0 \xrightarrow{\psi} \Sigma_1 \xrightarrow{\sim} \Sigma_1 \times \{0\} \hookrightarrow \Sigma_1 \times I \hookleftarrow \Sigma_1 \times \{1\} \xleftarrow{\sim} \Sigma_1.$$

Furthermore, for any automorphic diffeomorphism on Σ induces $M \sim \Sigma \times I$, then M will also be a cobordism from Σ to itself. This tells us that there are (uncountably)[2] many cobordisms but how different they are?

4.2.2 Equivalent cobordisms

Let $M, M' : \Sigma_0 \to \Sigma_1$. We regard them as the same (denoted by the symbol $\sim$) if the following diagram commutes with the diffeomorphism $\psi : M \to M'$:

$$\begin{tikzcd}
& M' & \\
\Sigma_0 \arrow[ur, hook] \arrow[dr, hook] & & \Sigma_1 \arrow[ul, hook'] \arrow[dl, hook'] \\
& M \arrow[uu, "\psi"] &
\end{tikzcd}$$

[2]To see why uncountable, take any open interval instead of I.

Observe that we require the diffeomorphism to be compatible with the inclusions. This is import as seen in this example which are not / should not be equivalent:

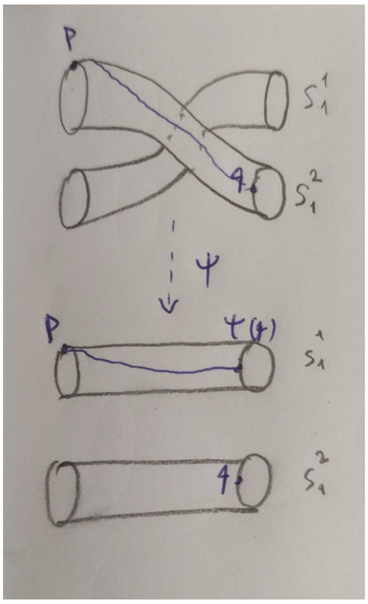


Figure 1: Two diffeomorphic manifolds which are not equivalent as cobordisms

In the figure above, first cylinder is mapped to the first one below (ones having a path). To see that they are not equivalent, observe that on the point q, direct inclusion and the inclusion composed with ψ do not coincide, first point lay on S_1^1 while the second one lying on S_2^1.

Before jumping to the decompositions of cobordisms, let us take a look at some 2−cobordism examples (of course, with images now).

Example 4.1. 1. Cobordisms over two diffeomorphic closed 1−manifolds which are equivalent to the cylinder.

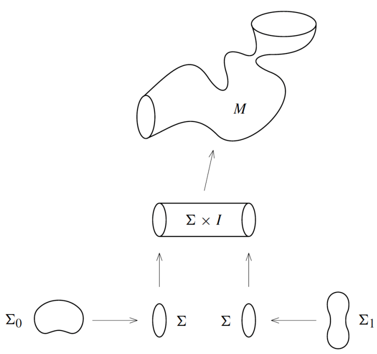


Figure 2: Cobordism equivalent to a cylinder.

2. U-tube: $S^1 \coprod \overline{S^1} \Rightarrow \emptyset_1$

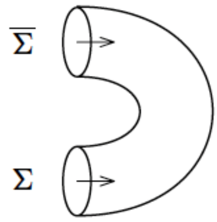


Figure 3: U-tube cobordism.

3. "Pair of pants": $S^1 \coprod S^1 \Rightarrow S^1$

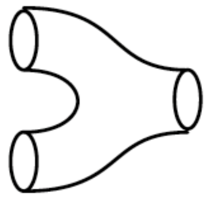


Figure 4: Cobordism called "Pair of pants".

4. "Birth": $\emptyset_1 \Rightarrow S^1$

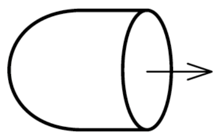


Figure 5: "Birth" cobordism: from the empty manifold to S^1.

5. "Death": $S^1 \Rightarrow \emptyset_1$

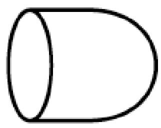


Figure 6: "Death" cobordism: from S^1 to the empty manifold.

4.2.3 Decomposition of cobordisms

Geometrically, when you draw a line vertically through a cobordism, most of the times, which will be made precise, you get two cobordisms. The result of the intersection of the cobordism with the line you drew is again a submanifold of co-dimension one and it will be closed. Giving it an appropriate orientation, we can see the cobordism as a composition / gluing of two cobordisms. Here is the formal description:

Instead of drawing lines, we will be more formal and for this purpose we will use *the Morse functions*. A Morse function for a manifold M is a smooth function $f : M \to I$ such that all if $df = 0$ at a point m (called a *critical point*) then the Hessian matrix at point m is non-singular and $f^{-1}(\partial I) = \partial M$. For a non-critical point, we say a *regular point* and for a point $t \in I$, if $f^{-1}(t)$ has no critical points, it is called a *regular value*.

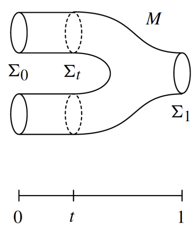


Figure 7: Decomposition of a cobordism

With this vocabulary, we define a decomposition of a cobordism M as a 'decomposition' of (one of) its Morse function $f : M \to I$. We take a regular value t and look at $f^{-1}([0,t])$ and $f^{-1}([t,1])$. These are two cobordisms, whose common boundaries (out-boundary for the first and in-boundary for the second) is the $\Sigma_t := f^{-1}(t)$, whose 'gluing' gives us M back .

4.3 TQFTs

In this subsection, we will describe the *Topological Quantum Field Theories* as a rule, $\mathcal{A}$, assigning each manifold a vector space and each cobordism a linear transformation. After defining **nCob**, we will come back to this definition and rewrite it as a functor from **nCob** to $\mathbf{Vect}_k$. Here are the axioms of this 'assignment' ([4], p.23).

A1. Two equivalent cobordisms should be assigned to the same transformation.

A2. The cylinder $\Sigma \times I$ should be sent to the identity map of $\mathcal{A}\Sigma$.

A3. Given a decomposition $M = M'M''$, $\mathcal{A}M = \mathcal{A}M'' \circ \mathcal{A}M'$.

A4. Disjoint union of manifolds is assigned to the tensor product of the corresponding images. Same is true for cobordisms.

A5. $\mathcal{A}\emptyset_{n-1} = k$.

These rules should sound familiar: functoriality! Now, we move towards the category of cobordisms, in order to formally define TQFTs.

4.4 nCob

We want to work in a categorical setting. The objects of our category are the closed $(n-1)$-manifolds and the maps between them are the equivalence classes of (oriented) cobordisms. We already defined what these equivalence classes and what the identity morphisms are. To have a category, we need to define the composition and prove the associativity.

4.4.1 Gluing of cobordisms

Let $M_0 : \Sigma_0 \Rightarrow \Sigma_1$ and $M_1 : \Sigma_1 \Rightarrow \Sigma_2$ be two cobordisms. We want to define a cobordism $M := M_0M_1 = \Sigma_0 \Rightarrow \Sigma_2$, i.e. a composition of two cobordisms. We will define M as a topological space made from gluing of two manifold and state that it also has a (smooth) manifold structure (although, not in a canonical way). For a discussion of details concerning differential topology, see our main source ([4]).

We will use the same notation with ([4]) but a slightly different (but more standard in the field of differential topology) definition for the gluing space.

Let $\psi : \Sigma_1 \to \Sigma_1$ be a diffeomorphism. We define $M_0 \coprod_{\Sigma_1} M_1$ to be the topological space $M_0 \coprod M_1$ under the equivalence[3] relation $x \sim \psi(x), x \in \Sigma_1$. Although this is not a complicated definition, we have simpler approach if we use the tools of category, namely the *fibered coproduct*. We define $M_0 \coprod_{\Sigma_1} M_1$ to be the following push-out:

[3]To be a 'real' equivalence relation, let $m \sim m$, for $m \in M_0 \coprod M_1 \setminus \Sigma_1 \coprod \Sigma_1$.

$$
\begin{tikzcd}
& M_0 \coprod_{\Sigma_1} M_1 & \\
M_0 \arrow[ur] & & M_1 \arrow[ul] \\
& \Sigma_1 \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

It is not hard to prove that $M_0 \coprod_{\Sigma_1} M_1$ in our first definition (in fact, construction) indeed satisfies the universal property. In this way, we obtained a cobordism $M := M_0 \coprod_{\Sigma_1} M_1 : \Sigma_0 \Rightarrow \Sigma_2$.

A cobordism $M : \Sigma_0 \Rightarrow \Sigma_1$ is said to be *invertible* if there exists a cobordism $M' : \Sigma_1 \Rightarrow \Sigma_0$ such that MM' and $M'M$ are equivalent to the identity cobordisms of Σ_1, Σ_0 respectively.

In order to have a category, we need to check the associativity, i.e, whether $(M_0M_1)M_2 \sim M_0(M_1M_2)$.

4.4.2 Associativity of gluing

The existence of a boundary-respecting diffeomorphism $\psi : (M_0M_1)M_2 \xrightarrow{\sim} M_0(M_1M_2)$ follows from the universal property shown in the diagram

$$
\begin{tikzcd}
& \Sigma_1 \arrow[dl, hook] \arrow[dr, hook] & & \Sigma_2 \arrow[dl, hook] \arrow[dr, hook] & \\
M_0 \arrow[dr, hook] \arrow[ddrrr, hook] & & M_1 \arrow[dl] \arrow[dr, hook] \arrow[ddl, dashed] \arrow[ddr, dashed] & & M_2 \arrow[dl, hook] \arrow[ddlll, hook] \\
& M_0 \coprod_{\Sigma_1} M_1 \arrow[d, hook] \arrow[drr, dotted] & & M_1 \coprod_{\Sigma_2} M_2 \arrow[d, hook] \arrow[dll, dotted] & \\
& (M_0 \coprod_{\Sigma_1} M_1) \coprod_{\Sigma_2} M_2 & & M_0 \coprod_{\Sigma_1} (M_1 \coprod_{\Sigma_2} M_2) &
\end{tikzcd}
$$

Hence, we have a category called **nCob**, whose objects are closed $(n-1)$-manifolds and maps are the equivalence classes of cobordisms. Composition is defined to be the gluing of two representatives. In the following of the paper, we will be dealing with **2Cob**.

4.5 2Cob

In this subsection, our aim is to understand the category **2Cob** via its 'building blocks'. In general, **nCob** is a huge category. This holds true when $n = 2$ but, we have a strong weapon in that case: 'classification of surfaces'. For the case $n = 3$, we have some classifications, though very complicated, and for $n > 3$, it is shown that there cannot be a 'reasonable' classification ([3]). We start reducing the complexity of **2Cob** by considering only the essence of the category, namely its skeleton.

We know that every closed 1−manifold is diffeomorphic to disjoint unions of S^1. Hence, the skeleton of **2Cob** consists of disjoint unions of S^1. Henceforth, **2Cob** will mean its skeleton (abuse of notation) together with the generator set. We will denote the object consisting disjoint union of n circles as **n** (here, **0** denotes the 2 dimensional empty manifold).

Finding the skeleton was the easy part. Now, we will find a generator set for this skeleton. Luckily, maps of this category are 2-manifolds with boundary and we have a classification theorem in this dimension.

Let us state the theorem (by [4], p.64):

Theorem 4.2. *"Two connected, compact oriented surfaces with oriented boundary are diffeomorphic if and only if they have the same genus and the same number of in-boundaries and the same number of out-boundaries."*

We will make use of this theorem in a second. Now, we should refine our understanding of a generator set in order to have a really nice set (with just 6 elements!). Here we define the monoidal category, as it is called.

4.5.1 Monoidal structure

Eventually, as spoilered in the previous section, we are to define TQFTs as functors from **nCob** to $\mathbf{Vect}_k$. In the latter category, we have the concept of *tensor product* and disjoint unions are sent to tensor products. It is natural to ask whether taking disjoint product is 'similar' to taking tensor product. The answer is yes, though the formal definition of what is called a *monoidal structure* has to wait until the end of this report. For now, we will just illustrate it with the case of sets with the operation of disjoint union.

Any two set S, T can be 'put' together. The usual way to do it is to take the union. But in this case, we loose information: if there happens to be a common element, we do not know where it came from. To keep this information; which is important for us, since we need $S^1 \coprod S^1$ in order to have **2**, not $S^1 \cup S^1$ which is just **1**; we use the disjoint union. A standard way to define the disjoint union is:

$$S \coprod T := \{(s_1|s \in S\} \cup \{(t_2|t \in T\}.$$

With this definition, the category **Sets** is closed under the operation of taking disjoint unions. In the previous section, we already showed that the resulting set is the 'coproduct' in **Sets**. We also have a symmetry with definition: $S \coprod T$ and $T \coprod S$ are *canonically* isomorphic via the *twist map* $\tau : s_1 \mapsto s_2; t_2 \mapsto t_1$.

Another example of a monoidal structure is the $\mathbf{Vect}_k$ with the tensor product. The twist map in this case is $\sigma : V \otimes W \to W \otimes V$ with the most natural definition.

Turning back to our case, the monoidal operation, called informally "paralleling" by [4], is again the disjoint union. In terms of maps, i.e. cobordisms, paralleling is just paralleling, putting one on the top of the other. As an example:

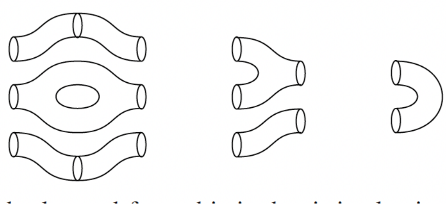


Figure 8: Paralleling and gluing

In this figure, three cobordisms are to be glued together. First two cobordisms can be seen as disjoint union of 3 and 2 connected cobordisms, respectively.

Corresponding to τ and σ, we have a *twist* cobordism drawn as

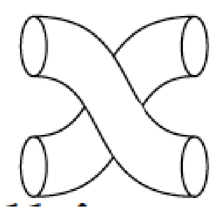


Figure 9: Twist cobordism

As shown in figure (1), this cobordism is not equivalent to the paralleling of two cylinders. With these 'twistings', all categories we mentioned become *symmetric* categories, hence, *symmetric monoidal categories.*

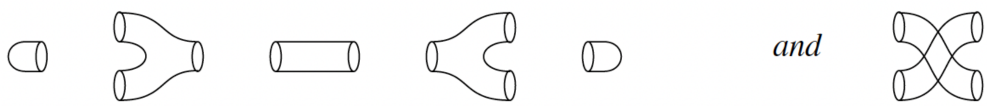
Now, with two cobordism operations, gluing and paralleling, we can refine the **2Cob** even further. Before that, let us mean what we mean by refinement, specifically for monoidal categories. We already gave a definition of a generating set for an arbitrary category in the previous section. Now, we enlarge the word 'composition' of two maps for the monoidal categories. A *generating set for a monoidal category* is the set of maps such that every map can be obtained by compositions and parallelings of these maps.

We go back to our category **2Cob** and find its generating set.

4.6 Generators of 2Cob

Let us directly state and prove the main theorem of this section.

Theorem 4.3 ([4], p.62)**.** *"The monoidal category **2Cob** is generated under composition (serial connection) and disjoint union (parallel connection) by the following six cobordisms":*

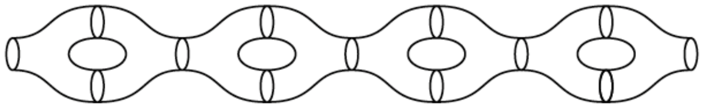


We will use theorem 4.2 to prove this theorem. Take a map $M : \mathbf{m} \Rightarrow \mathbf{n}$. First, we deal with the case where M is connected. Then, the unconnected case will be decomposed into its connected components (with a little trick).

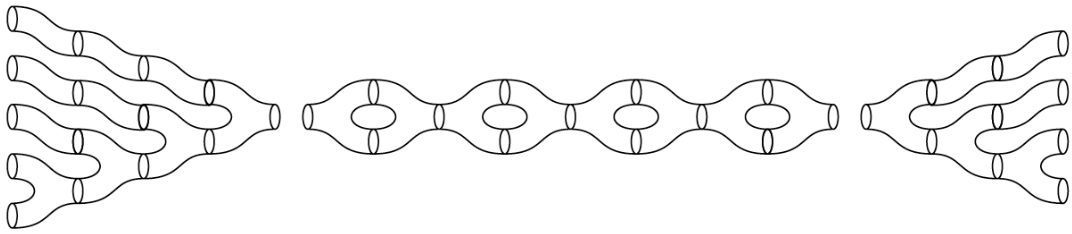
Proof.

i. (M is connected.) We try to find an algorithmic way to decompose M into its being glued parts. It is difficult and here we use 4.2 in order to deal with a more standard manifold which is equivalent to M, called the "normal form". Let M be of genus number g. Then, we can obtain these 'holes' by gluing g many reverse[4] and normal pants as such:

[4] 4th figure in the theorem.



By adding parallelings of cylinders and pants, we can reach the desired number of in and out boundaries. Without describing the procedure formally, let us give an example for the case ($m = 5, n = 4, g = 4$):



If m or n is zero, then just add a cap at the beginning or end. With this algorithm, we can decompose every cobordism into the 6 pieces in the theorem. Now, we go to the unconnected case.

ii. (M is not connected.) For simplicity, assume M has two connected components, first one being $\mathbf{p} \Rightarrow \mathbf{q}$ where $p \leq m, q \leq n$. If $\mathbf{p}$ is the first p circles, then we just parallel these two connected cobordisms. But if not, what do we do? Before answering this, let us give two lemmas which will turn out to be quite helpful.

Lemma 4.4. *A 2-cobordism $N : \Sigma_0 \Rightarrow \Sigma_1$ is invertible if [and only if, but we will not use this part] it is induced from a diffeomorphism $\Sigma_0 \xrightarrow{\sim} \Sigma_1$ via the cylinder construction.*

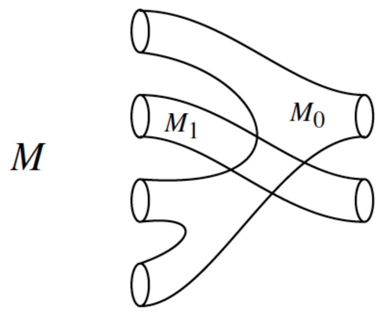
The proof of the lemma follows from the cylinder construction and inverse of a diffeomorphism being (naturally) a diffeomorphism such that when composed, giving the identity map.

Now, we can produce a cobordism $S : \mathbf{m} \Rightarrow \mathbf{m}$ induced from a diffeomorphism such that it maps the circles belonging to $\mathbf{p}$ to the first p circles of $\mathbf{m}$. By the lemma above, we keep in mind that we have S^{-1} too.

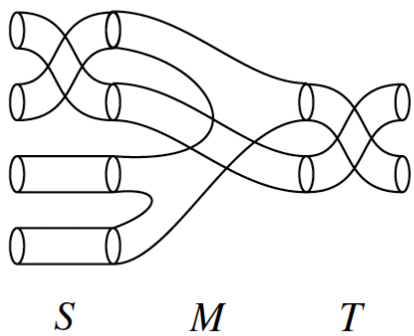
Similarly, we obtain $T : \mathbf{n} \Rightarrow \mathbf{n}$ mapping $\mathbf{q}$ to first q circles of $\mathbf{n}$. Now, the cobordism SMT can be written as a paralleling of two cobordisms, first being the cobordism mapping first p circles to first q circles. We apply the algorithm (i) and write it as a gluing of some generators. But, SMT is not the same cobordism, i.e. not necessarily belonging to the same class, with M. We need to glue it with the inverses S and T, so that $M = S^{-1}(SMT)T^{-1}$. Can we write these inverses as gluings and parallelings of the generators? If you have realized, we have not use the twist cobordism. Here comes the claim: any *permutation* cobordism can be written as composition of twist maps and cylinders. In ([4], p.57), this claim is justified by the Moore's theorem stating that every symmetric group of order ≥ 4 is generated by the 'twisting' cycle, (i, i+1). For a formal proof of this claim, we refer to the source mentioned.

Hence, both S^{-1}, T^{-1} can be written as compositions of twist cobordisms and cylinders. All together, we have shown that $M = S^{-1}(SMT)T^{-1}$ can be decomposed using the 6 generators in the theorem. Here is an illustration of the procedure.

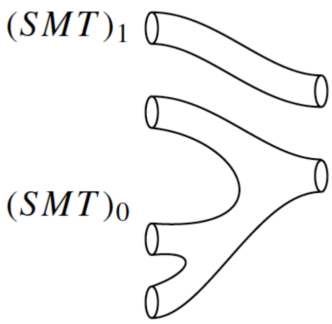
Let M be the following cobordism (of genus 0 for simplicity):



We compose S, T to separate M:



At the end, we have this figure:



Decompositions of S^{-1}, T^{-1} can be seen from the figure above and we are done! $\square$

4.6.1 Relations

In this section, we will list some equivalence relations between the generators. These are *obviously* true, but we need to list them down in order to observe the main surprise: equivalence between the (symmetric) Frobenius algebras and TQFTs. Here are some of the relations, for elaboration and the whole list, see [4]:

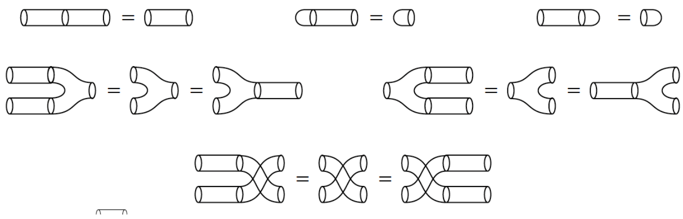


Figure 10: Identity relations

Figure 11: Cap relations

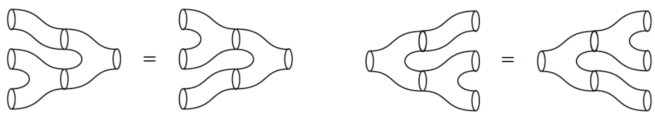


Figure 12: (Co-)associativity relations

Figure 13: (Co-)commutativity relations

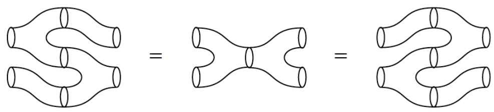


Figure 14: Frobenius relations

All these can be proven via the theorem 4.2. The namings will be clear in the Frobenius algebras section.

Like every group is characterized (up to isomorphism) by its generators and their relations, similarly, our category (**2Cob**) can be defined by its generating set and their relations with each other, from an algebraic point of view. It is natural to ask 'are these (ones listed in [4]) relations enough?' The answer is yes, indeed more than enough! See ([4], p. 74) for the proof.

Now, we jump to a Frobenius algebras, a seemingly unrelated topic for now.

5 Frobenius algebras and TQFTs

(This section is based on [4]) We define the Frobenius algebras and some of its equivalent definitions. Then, we by graphically representing the axioms of Frobenius algebras, we show the link between TQFTs and Frobenius algebras.

5.1 (Co-)algebras

> "Similar to the way modules generalize abelian groups by adding the operation of taking non-integer multiples, an $[k]$-algebra can be thought of as a generalization of a ring S, where the operation of taking integer multiples (seen as iterated addition) has been extended to taking arbitrary multiples with coefficients in $[k]$. In the trivial case, a $\mathbb{Z}$-algebra is simply a ring" ([1], names of structures changed for consistency and marked with '[ ]').

Formally, a $k-$*algebra* A is a vector space over k together with two maps $\mu : A \otimes A \to A, \mu(a \otimes b) = a.b$, called *multiplication*, and $\nu : k \to A$ called the *unit* satisfying the usual associativity and unital axioms:

$$(a.b).c = a.(b.c) \text{ and } \mu(k).a = ka.$$

Dually, we have a $k-$*co-algebra* A as a vector space with these two $k-$linear maps:

$$\delta : A \to A \otimes A \text{ and } \varepsilon : A \to k$$

called *co-multiplication* and *co-unit*, satisfying the axioms of *co-commutativity* and *co-unitality* shown below as the commuting diagrams (since writing down the algebraic expressions are hard):

$$\begin{tikzcd} & A \otimes A \otimes A & \\ A \otimes A \arrow[ur, "\delta \otimes id"] & & A \otimes A \arrow[ul, "id \otimes \delta"'] \\ & A \arrow[ul, "\delta"] \arrow[ur, "\delta"'] & \end{tikzcd}$$

and,

$$\begin{tikzcd} k \otimes A & A \otimes A \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"] & k \otimes A \\ & A \arrow[ul] \arrow[u, "\delta"] \arrow[ur] & \end{tikzcd}$$

Example 5.1. Here are some examples to (co-)algebras

1. (Algebra) The ground filed k is itself a $k-$algebra by the normal multiplication and unit.

2. (Algebra) Complex numbers over $\mathbb{R}$ is an algebra with the unit $(1, 0) = 1 + i.0$.

3. (Algebra) All $n \times n$ real matrices form an algebra over $\mathbb{R}$ with the usual matrix multiplication.

4. (Co-algebra) [([4], p.107), originally from Sweedler] "Let V be a $2-$dimensional

vector space [over $\mathbb{R}$] with basis S, C". Define a co-multiplication by

$$C \mapsto C \otimes C - S \otimes S,$$

$$S \mapsto C \otimes S + S \otimes C,$$

and co-unit by

$$C \mapsto 1, S \mapsto 0.$$

The logic behind these definitions is the classical trigonometric identities.

Now, we will use these terminology to define the Frobenius algebras.

5.2 Frobenius algebras

A *Frobenius algebra* is an algebra (co-algebra) which is at the same time a co-algebra (algebra). Original definition is different and the one we gave is a theorem with that definition. In fact, we have bunch of equivalent definitions which some of them are useful for our purposes.

Some definitions before stating these definitions are necessary. A map $\beta : V \otimes W \to k$ is called a *pairing*, where V, W are vector spaces over k. If there exists another map (called *co-pairing*) $\gamma : k \to W \otimes V$ such that the composition of the below maps is the identity,

$$\begin{tikzcd}
V \arrow[d] & (V \otimes W) \otimes V \arrow[r, "\beta\otimes id"] & k \otimes V \arrow[d] \\
V \otimes k \arrow[r, "id\otimes\gamma"] & V \otimes (W \otimes V) \arrow[u] & V
\end{tikzcd}$$

β is said to be *non-degenerate* in V. If β is non-degenerate in both V, W, it is just said to be non-degenerate. This is a strong condition and in particular, implies finite dimensionality of both vector spaces:

Lemma 5.2. *Non-degeneracy of the pairing β implies finite dimensionality of W (and of V too).*

Proof. ([4], p.83) By non-degeneracy, we have such a γ mentioned above. Let $\gamma(1_k) = \sum_{i=1}^{n} w_i \otimes v_i$ for some n. Take a vector $x \in W$ and put it into the composition drawn above in the non-degeneracy diagram. Since this composition gives the identity map, we have:

$$\sum_{i=1}^{n} w_i \otimes \beta(v_i \otimes x) = x.$$

Remember that $\beta(v_i \otimes x)$ is just a scaler and taking tensor product just means the scaler multiplication. Since this equality holds for each vector, the set $\{w_i\}$ spans W. □

Now, let us state these equivalent definitions of Frobenius algebras (see [4] for a proof):

Theorem 5.3. *The followings hold true for a $k-$structure (algebra, co-algebra, etc.) F* [5]*:*

1. *If F has both an algebra (F, μ, ν) and co-algebra (F, δ, ε) , it is a Frobenius algebra (together with the maps defined in those structures).*

2. *(F, ε) is a Frobenius algebra where $\varepsilon \in F^*$ with a null space not including any (left) ideals. Here, ε is called a Frobenius form.*

3. *(F, β) is a Frobenius algebra where β is a non-degenerate pairing defined on $F \otimes F$ (F is taken as a vector space). This pairing is called a Frobenius pairing.*

Actually, the relation between (2) and (3) can be seen explicitly. Given ε, we can define $\beta(a \otimes b) := \varepsilon(a.b)$ and given the pairing, $\varepsilon(a) := \beta(1_F \otimes a)$ yields a Frobenius form. Also, ε in the definition of co-algebra in the first definition yields a Frobenius form.

A Frobenius algebra is said to be *symmetric* if $\varepsilon(a.b) = \varepsilon(b.a)$ for all $a, b \in F$.

Example 5.4. Here are some Frobenius algebra examples ([4], p.99).

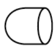
1. $\mathbb{C}$ as a vector field over $\mathbb{R}$ with the usual complex multiplication and the Frobenius form $\varepsilon(a + ib) = a$ is a Frobenius algebra.

2. All $n \times n$ real matrices together with the matrix multiplication and the trace being the Frobenius form, form a Frobenius algebra.

[5] "Being Frobenius algebra is a a structure". It makes no sense to say F is a Frobenius algebra without specifying the necessary maps, see ([4], p.98).

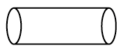
5.2.1 Graphical representation

Now, by theorem (5.3), we can refer to a Frobenius algebra as a 6-tuple $(A, \mu, \nu, \varepsilon, \delta, \beta)$ where the first two operations represent the multiplication and unit maps and the second two represent the Frobenius form (also co-unit) and the co-multiplication map and the last one being the Frobenius pairing. Indeed, by the same theorem, this much information is redundant. We stated above that actually co-algebra structure can be obtained (in fact in a unique way, see ([4], p.118)) from the algebra structure and the Frobenius form. Now, we will represent these graphically, via what is called the "Graphical calculus", which is common in knot theory ([4], p.108):



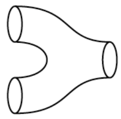
η

unit



id_A

identity



μ

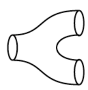
multiplication

ε

Frobenius form

β

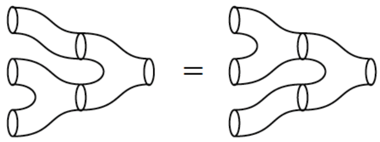
Frobenius pairing



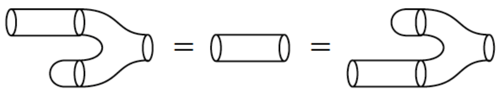
Co-multiplication δ

Just like in cobordisms, in-boundaries represent the tensor products of A (instead of disjoint union) as the domain and out-boundaries represent co-domain. Empty manifold is replaced by the 0 'times' A, i.e. only the ground field. The similarities do not end here: the relations between these maps actually are the same with the relations between the generators! Let us show some of them in order to be convinced:

1. Associativity and unitality yield these relations:

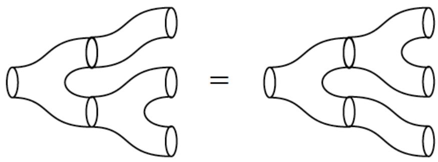


associativity

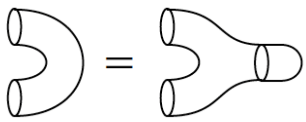


unit axiom

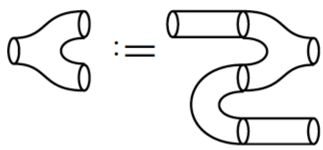
2. Co-associativity of the co-multiplication is encoded in the below figure:



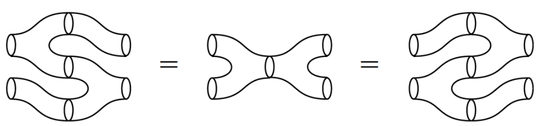
3. We mentioned in theorem (5.3) how to define the pairing given the form and vice versa. Below, second image represents $\varepsilon\mu$.



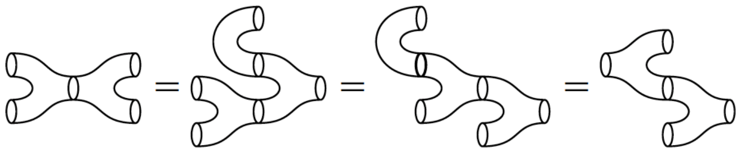
4. As said above, given a Frobenius algebra, there exists a (unique) co-multiplication δ. Forgetting it is given, we could have constructed it as such:



where the 'inverse U' is the associated co-pairing $\gamma : k \to A \otimes A$. With this definition and some other relations, we can actually prove (now, it is a proposition to be proven, not a mere representation) (2). With this definition, we will prove (LHS of) what was called 'Frobenius relations' in the previous section:

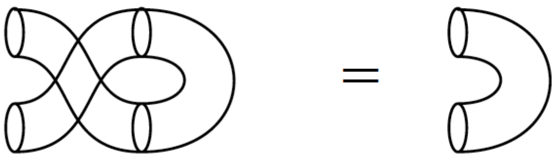


We ([4], p.117) use the definition of the co-multiplication and associativity of the below maps:



.

Among the generators of **2Cob**, only cobordism we have not assigned a map is the twist cobordism. Not surprisingly, we represent the natural twist map $\sigma : V \otimes V \to V \otimes V$ defined as $v \otimes v' \mapsto v' \otimes v$ as the twist cobordism. Similar relations for this map could also be observed. With this representation, we can see that for a symmetric Frobenius algebra A, this relation holds:



So far, we talked about Frobenius algebras as individuals. As general, we want to collect al these structures under a category. The missing part for that is a mapping between these structures. Now, we define the category of Frobenius algebras, namely $\mathbf{FA}_k$.

5.3 $\mathbf{FA}_k$

A *Frobenius homomorphism* is an algebra homomorphism $\phi : (A, \varepsilon) \to (A', \varepsilon')$ (meaning $\phi(a.b) = \phi(a).\phi(b)$ and $\phi(1_A) = 1_{A'}$) such that

$$(\phi \otimes \phi) \circ \delta_1 = \delta_2 \circ \phi$$

where $\delta_{1,2}$ are the induced co-multiplication maps, and

$$\varepsilon'\phi = \varepsilon.$$

With arrows defined as the Frobenius homomorphisms, we get the category $\mathbf{FA}_k$ (and $\mathbf{cFA}_k$ for the commutative Frobenius algebras' category).

Similar with the vector spaces, we have a tensor product defined for two Frobenius algebras $(A, \varepsilon) \otimes (A', \varepsilon')$. The algebra structure is given as

$$(A \otimes A') \otimes (A \otimes A') \to (A \otimes A')$$
$$(x \otimes x') \otimes (y \otimes y') \mapsto (xy \otimes x'y')$$

for the multiplication and

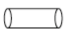
$$k \to A \otimes A'$$
$$1 \mapsto 1_A \otimes 1_{A'}$$

for the unit map. Co-algebra structure is given similarly in a natural way. Hence, $\mathbf{FA}_k$ is closed under tensor products and $(\mathbf{FA}_k, \otimes, k)$ is a monoidal category. Not surprisingly, with the twist map, this is actually a symmetric monoidal category!

5.4 The hidden relation

Let us observe first the 'weak relation', as I call it, between the Frobenius algebras and TQFTs. To begin with, by the monoidal structure of **2Cob** and the axioms of TQFT, given where $\mathbf{1} = S^1$ is sent, call this vector space A, we know where other objects are sent. Cobordisms are sent to linear maps between then tensor products of A. Since all the cobordisms are generated by the six generators, it is enough to determine where these are sent to. We have this picture:

$$\textbf{\textit{2Cob}} \longrightarrow \textbf{\textit{Vect}}_{\Bbbk}$$
$$\mathbf{1} \longmapsto A$$
$$\mathbf{n} \longmapsto A^n$$

 $\longmapsto [\mathrm{id}_A : A \to A]$

 $\longmapsto [\sigma : A^2 \to A^2].$

 $\longmapsto [\eta : \Bbbk \to A]$

 $\longmapsto [\mu : A^2 \to A]$

 $\longmapsto [\varepsilon : A \to \Bbbk]$

 $\longmapsto [\delta : A \to A^2]$.

where $A^n = A \otimes \cdots \otimes A$. The naming is not random and it is not a coincidence to be same with the names of the maps of a Frobenius algebra. Because of the (topological) relations between cobordisms, it is not hard to see that these maps has to satisfy relations given in the previous section. Hence, given a TQFT $\mathcal{A}$, we have automatically a Frobenius algebra: $\mathcal{A}(\mathbf{1})$. Indeed, it is a commutative Frobenius algebra because of the symmetry of the pants cobordism. Obviously a TQFT yields a unique Frobenius algebra.

Conversely, given a commutative Frobenius algebra $(A, \mu, \nu, \varepsilon, \delta)$, β is obtained from ε as stated before; we have TQFT $\mathcal{A}$ such that $\mathcal{A}(\mathbf{1}) = A$ and the generators are sent to the maps in the natural way. Again, one commutative Frobenius algebra yields unique TQFT. Therefore, we have a one-to-one correspondence! But, between what do we have a bijection? These structures form sets but actually we have more than just a bijection. In order to state the hidden relation in its full power, we need some more definitions.

5.5 Monoidal Categories

We introduced TQFTs as functors. Naturally, there are natural transformations between these functors. There are some criteria that we want these natural transformations to satisfy. After giving the criteria, all put together, we will have the category of 2−TQFTs, called **2TQFT**. To present the ideas in the most elegant way, it is better to talk about more general setting and then come back to what interests us: **2TQFT** and $\mathbf{cFA}_k$.

We already talked (informally) a bit about monoidal structure in the previous section where we gave 'tensor product' and 'paralleling' as examples. Now, we define the *monoidal categories* formally.

Definition 5.5 ([4], p.150)**.** "A (strict)[6] monoidal category is a category $\mathbf{V}$ together with two functors

$$\mu : \mathbf{V} \times \mathbf{V} \to \mathbf{V}, \;\; \nu : \mathbf{1} \to \mathbf{V}$$

[6] Our categories will not be strict but this is just a technical difference for us. See ([4], p.154) for a discussion.

satisfying the associativity axiom and the neutral object axiom", where $\mathbf{1}$ denotes the empty cartesian product of the category $\mathbf{V}$.

Unlike monoids, we do not define the multiplication in terms of the elements but via a map (functor) and the unit element is represented by ν, since in category theory we do not like elements! Axioms mentioned in the definition are same with the algebra axioms replaced with a category instead of a vector space.

We should note that μ and ν are functors. Let us represent the so-called multiplication/ paralleling operation by $\Box$. For us, $\Box$ will be tensor product, disjoint union or cartesian product. With this notation, we have

$$\mathbf{V} \times \mathbf{V} \xrightarrow{\mu} \mathbf{V}$$
$$(X, Y) \mapsto X \Box Y$$
$$(f, g) \mapsto f \Box g.$$

for the functor μ. For ν, let $\nu(\mathbf{1}) = I$ be the identity object. So, we have

$$I \Box X = X = X \Box I, id_I \Box f = f = f \Box id_I.$$

Hence, we refer to a monoidal category by the triples $(\mathbf{V}, \mu, \nu)$ or $(\mathbf{V}, \Box, I)$.

We will use an abbreviation to shorten the next definition. Let $\mathbf{V}^n$ be the $n-$ary cartesian product of $\mathbf{V}$s. Let $\mu^{(n)} = \mathbf{V}^n \to \mathbf{V}$ be the induced map for $n \geq 2$, for instance $\mu^{(2)} = \mu$ and $\mu^{(3)} = \mu \circ (\mu \times id)$ etc. Additionally, let $\mu^{(1)} := id_V$ and $\mu^{(0)} = \nu$.

Now, let us continue with monoidal functors.

Definition 5.6 ([4], p.153)**.** "A (strict) monoidal functor between two (strict) monoidal categories $(\mathbf{V}, \Box, I)$ and $(\mathbf{V'}, \Box', I')$ is a functor $F : \mathbf{V} \to \mathbf{V'}$ that commutes with all the structure", which is encoded with the below diagram

$$\begin{tikzcd}
\mathbf{V}^n \arrow[r, "F^n"] \arrow[d, "\mu^{(n)}"] & \mathbf{V'}^n \arrow[d, "\mu'^{(n)}"] \\
\mathbf{V} \arrow[r, "F"] & \mathbf{V'}
\end{tikzcd}$$

Let us give some (already known) examples to monoidal categories (which none is strict): $(\mathbf{Sets}, \coprod, \emptyset)$, $(\mathbf{Sets}, \times, 1)$, $(\mathbf{Vect}_k, \otimes, k)$, $(\mathbf{2Cob}, \coprod, \emptyset_1)$, $(\mathbf{FA}_k, \otimes, k)$. Informally (but intuitively), we call a monoidal category *symmetric* if we have a twist map subject to the desired properties, and a functor is called symmetric if it sends the twist map to twist map. We represent a *symmetric monoidal category* as a 4−tuple $(\mathbf{V}, \square, I, \tau)$.

Last definition before revealing the hidden relation.

Definition 5.7. Take two symmetric monoidal functors $F, G : (\mathbf{V}, \square, I, \tau) \longrightarrow (\mathbf{V'}, \square', I', \tau')$. A natural transformation $u : F \implies G$ is called a *monoidal natural transformation* if the diagram

$$\begin{tikzcd} F(X\square Y) \arrow[r, "u_{X\square Y}"] \arrow[d, Rightarrow] & G(X\square Y) \arrow[d, Rightarrow] \\ FX\square FY \arrow[r, "u_X\square u_Y"] & GX\square GY \end{tikzcd}$$

commutes and $u_I = id_{I'}$.

Now, we have the power to formally define $\mathbf{nTQFT}_k$: the category whose objects are the symmetric monoidal functors from $(\mathbf{nCob}, \coprod, \emptyset_{n-1}, T)$ to $(\mathbf{Vect}_k, \otimes, k, \sigma)$ and the maps are the monoidal natural transformation between TQFTs.

6 The canonical equivalence

Here is the main theorem of this report:

Theorem 6.1 ([4], p.173)**.** *"There is a canonical equivalence of categories*

$$\boldsymbol{2TQFT}_k \simeq \boldsymbol{cFA}_k."$$

We have stated the correspondence between the objects. Now, to be convinced to this equivalence, we need to show the correspondence between the maps of each categories. A map between two TQFTs is a monoidal natural transformation. It should be sent to a Frobenius homomorphism.

Let $\mathcal{U} : \mathcal{A} \implies \mathcal{B}$ be a monoidal natural transformation. By monoidality, it is a collection of maps $\{u_n\}$ where $u_n := u^{(n)}$ and $u := u_1 : \mathcal{A}(\mathbf{1})(=: (A, \mu_A, \nu_A, \varepsilon_A, \delta_A)) \to \mathcal{B}(\mathbf{1})(=: (B, \mu_B, \nu_B, \varepsilon_B, \delta_B))$. We send $\mathcal{U}$ to u. To see why this is a Frobenius homomorphism, we need to justify three things

1. u is an algebra homomorphism
2. $\varepsilon_B u = \varepsilon_A$
3. $(u \otimes u)\delta_A = \delta_B u$

We will justify the first claim only, rest can be done in a similar fashion. By $\mathcal{U}$ being a (monoidal) natural transformation, we have this following diagram:

$$\begin{tikzcd}
A^2 \arrow[rrr, "u_2=u\otimes u"] \arrow[ddd, "\mu_A"] & & & B^2 \arrow[ddd, "\mu_B"] \\
& a_1 \otimes a_2 \arrow[r, mapsto] \arrow[d, mapsto] & u(a_1) \otimes u(a_2) \arrow[d, mapsto] & \\
& a_1.a_2 \arrow[r, mapsto] & u(a_1).u(a_2) = u(a_1.a_2) & \\
A \arrow[rrr, "u"] & & & B
\end{tikzcd}$$

Maps between A^2, A and B^2, B are the multiplication maps, since these are the images of the pair of pants cobordism ($S^1 \coprod S^1 \Rightarrow S^1$). Hence, u is an algebra homomorphism. Rest can be checked by following the other generating cobordisms (see ([4], p.173) for details).

In the other direction, given a Frobenius homomorphism $\phi : A \to B$, we need to assign it a monoidal natural transformation $\mathcal{U}$. We claim $\{\phi^{(n)} : A^n \to B^n\}$ is such a transformation. To see why, again, we need to check whether the naturality squares (in the definition of the natural transformations) commute for the generators only. We only show for the case inverse pair of pants, which is sent to $\delta_{A,B} : A, B \to A^2, B^2$.

$$\begin{tikzcd}
A^2 \arrow[r, "\phi^{(2)}=\phi\otimes\phi"] \arrow[d, "\delta_A"] & B^2 \arrow[d, "\delta_B"] \\
A \arrow[r, "\phi"] & B
\end{tikzcd}$$

This diagram commutes, since ϕ is a Frobenius homomorphism (see section 5.3 for the definition).

Hence, we have defined (implicitely) two functors whose compositions gives us back the identity functors, meaning our theorem has been proven!

7 Concluding remarks

We now have achieved our goal in characterizing 2-TQFTs as Frobenius algebras. In order to even define the terms mentioned in this goal, we needed to talk about Category theory (functors, natural transformations, equivalence of two categories, skeletons, generators in the first part and symmetric monoidal categories/ functors later), cobordism & manifold theory (including bunch of unmotivated definitions at the beginning of the chapter) and (co-)algebras to define Frobenius algebras. With these machinery in hand, we first showed the 'weak' correspondence between TQFTs and Frobenius algebras, and then, with the power of monoidal functors, we unfolded the full relation stated in the above section.

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