Comments and corrections

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Abstract

This document contains additional remarks and errors for my papers that have not been corrected in public versions. Some corrections and remarks will appear in the final versions.

Contents

1 Shifted contact structures and their local theory [1]

1.1 Definitions 3.5 & 3.6.

Since $QCoh(\mathbf{X})$ is a stable ∞-category, we can simplify Definitions 3.5 & 3.6 in [1] and introduce/use the following ones instead:

Definition 1.1. Let $\mathbf{X}$ be a locally finitely presented derived (Artin) stack. A ***pre-k-shifted contact structure*** on $\mathbf{X}$ is given by a shifted line bundle $L[k]$ with a morphism $\alpha : \mathbb{T}_{\mathbf{X}} \to L[k]$. Denote such a structure by $(L[k], \alpha)$.

Note that we can consider a pre-k-shifted contact data as a perfect complex $\mathcal{K}$ and a line bundle L along with a morphism $\kappa : \mathcal{K} \to \mathbb{T}_{\mathbf{X}}$ such that $Cone(\kappa) \simeq L[k]$. Since $QCoh(\mathbf{X})$ is a stable ∞-category, we have a fiber-cofiber sequence $\mathcal{K} \to \mathbb{T}_{\mathbf{X}} \to L[k]$ in $QCoh(\mathbf{X})$, and hence, the cocone of $\mathbb{T}_{\mathbf{X}} \to L[k]$ is equivalent to $\mathcal{K}$. We then may denote a pre-k-shifted contact structure on $\mathbf{X}$ by $(\mathcal{K}, \kappa, L)$.

Definition 1.2. We say that a pre-k-shifted contact structure $(\mathcal{K}, \kappa, L)$ on $\mathbf{X}$ is a ***k-shifted contact structure*** if locally on $\mathbf{X}$, where L is trivial, the induced k-shifted 1-form[1] $\alpha : \mathbb{T}_{\mathbf{X}} \to \mathcal{O}_{\mathbf{X}}[k]$ is such that the map $d_{dR}\alpha|_{\mathcal{K}} \cdot := \kappa^{\vee}[k] \circ (d_{dR}\alpha \, \cdot) \circ \kappa : \mathcal{K} \to \mathcal{K}^{\vee}[k]$ is a weak equivalence.

In that case, we say the k-shifted 2-form $d_{dR}\alpha$ is ***non-degenerate on*** $\mathcal{K}$. Also, we call such local form a ***k-contact form***.

Remark 1.3. When $k \leq 0$, the triangle $\mathcal{K} \to \mathbb{T}_{\mathbf{X}} \to L[k]$ splits locally for any affine derived scheme (so, this also holds Zariski locally for any derived scheme)[2]. In fact, the nondegeneracy condition implies that $\mathcal{K}$ has Tor-amplitude $[0, -k]$ so that $\mathcal{K}[-k]$ is connective. Then the connecting homomorphism $L[k] \to \mathcal{K}[1]$ in the exact triangle is equivalently $L \to \mathcal{K}[1-k]$. Notice that $\mathcal{K}[1-k]$ is concentrated in degrees ≤ -1, so this morphism is automatically zero on any affine derived scheme, which implies the desired splitting.

These updates will appear in the final version of [2].

[1] We can locally identify the map α with the induced shifted one-form using the trivialization of $L^{\vee}[k]$.
[2] We thank the anonymous referee for this remark.

1.2 Example 3.10

There are some arguments requiring additional correction and comments. ***(Please notice the change in the generators.)*** Localizing if necessary, instead of the current generators given in the construction, we can consider the variables

$$x_1^{-i}, x_2^{-i}, \ldots, x_{m_i}^{-i} \qquad \text{in degree } -i \text{ for } i = 1, 2, \ldots, \ell, \tag{1.1}$$
$$y_1^{k+i}, y_2^{k+i}, \ldots, y_{m_i}^{k+i} \qquad \text{in degree } k+i \text{ for } i = 1, \ldots, \ell, \tag{1.2}$$
$$z^k, y_1^k, y_2^k, \ldots, y_{m_0}^k \qquad \text{in degree } k. \tag{1.3}$$

Here, we replace $\tilde{x}_1^0 \in A^0$ with $z^k \in A^k$ to fix the generator of the complex $Rest$, which is concentrated in $\deg -k$ and now generated by $\partial/\partial z^k$.

The current version of α in [1, Eqn. (3.14)] might be misleading. Instead, we first define the ***primitive element*** $\alpha \in (\Omega_A^1)^k$ by

$$\alpha = d_{dR}z^k + \sum_{i=0}^{\ell}\sum_{j=1}^{m_i} y_j^{k+i} d_{dR}x_j^{-i} \tag{1.4}$$

Note that $\alpha \in \Omega_A^1[k]$, but we also need it to be d-closed. We can actually make α d-closed by determining the differential d via the equations $d|_{A(0)} = 0$; $dx_j^{-i} = \partial H/\partial y_j^{k+i}$; $dy_j^{k+i} = \partial H/\partial x_j^{-i}$ for all i, j and $-kdz^k = H + d[\cdots]$ for the Hamiltonian $H \in A^{k+1}$. Then, the construction follows.

More details can be found in the final version of [2] and the revisited/latest version of [1] on arXiv.

1.3 Proof of Theorem 3.13

The generator for $Rest$ is incorrect. In fact, $Rest$ is a complex concentrated in degree $-k$ since we want the splitting $\mathbb{T}_A = \ker(\alpha) \oplus Rest$ over $\mathrm{spec} H^0(A)$, where $Rest = L[k]$ is a k-shifted line bundle. But it is said in the construction that $Rest$ is generated by the vector field $\partial/\partial\tilde{x}_1^0$, which is of degree 0. This is incorrect. So, instead of using $\tilde{x}_1^0$ in A^0, we must use a distinguished generator z^k in A^k in addition to the generators $y_1^k, y_2^k, \ldots, y_{m_0}^k$. Then, after suitable replacement, we should end up with the splitting $\mathbb{T}_A = \ker(\alpha) \oplus Rest$ over $\mathrm{spec} H^0(A)$ such that $\ker(\alpha)$ has Tor-amplitude $[0, -k]$ and $Rest$ is concentrated in $\deg -k$, and that $\ker(\alpha) \simeq {}^{\mathbb{T}_A}\!/_{Rest}$ with

$$\ker(\alpha)|_{\mathrm{spec} H^0(A)} = \langle \partial/\partial x_j^{-i}, \partial/\partial y_j^{k+i} \rangle,$$
$$Rest|_{\mathrm{spec} H^0(A)} = \langle \partial/\partial z^k \rangle,$$

where $0 \leq i \leq \ell,\ 1 \leq j \leq m_i$. Then we can get the desired k-shifted contact form as in Section 1.2.

We have misleading argument (although our intention was naively the opposite and we were trying to say d could be taken in that form by the results of Brav-Bussi-Joyce, etc... Clearly, the way of presenting this intention is incorrect.): "Imposing the differential"-type argument in the proof has been fixed. Instead, expanding the defining equations for the pair (H, ϕ), we arrive at the desired formulas for d, hence identify the cdga (A, d) exactly with the one above.

An adaptation of this proof to the case of derived Artin stacks will appear in the final version of [2]. Similar edits also available in the revisited version of [1] on arXiv.

1.4 Observations 2.27 and 3.17 revisited

For $k/2$ odd, the statements were incorrect; they are now fixed. See the latest version of [1] on arXiv or RG.

1.5 Uniqueness clarified

For the equations to be mentioned in this section, we refer to the ones in [1]. Assume that there is an element α' satisfying the desired properties listed in the proof of Theorem 3.13. Let us clarify how the corresponding conditions uniquely (up to interchange of x_j^{-i} and y_j^{k+i})[3] determine the

[3] The roles of x_j^{-i}, y_j^{k+i} are symmetric in (3.26) and (3.25), where $d_{dR}x_j^{-i}d_{dR}y_j^{k+i} = d_{dR}y_j^{k+i}d_{dR}x_j^{-i}$ for k odd.

representation in (3.27). We first observe that due to Eqn. (3.26) and the condition $\iota_{\partial/\partial z^k}\alpha' = 1$, any such element α' takes the form

$$\alpha' = d_{dR}z^k + \psi \quad \text{with} \quad \psi = \sum_{i,j} \left[a_j^{k+i} d_{dR} x_j^{-i} + b_j^{-i} d_{dR} y_j^{k+i} \right],$$

where $a_{\nu'}^{\mu'} \in A^{\mu'}, b_\nu^\mu \in A^\mu$ such that b_ν^μ's depend only on $x_{j'}^{-i'}$'s and $A(0)$ for degree reasons. From the condition (3.25), we take[4] $b_j^{-i} = 0$ for all i, j, leading to

$$d_{dR}\alpha' = d_{dR}\psi = \sum_{i,j} d_{dR} a_j^{k+i} d_{dR} x_j^{-i}.$$

Now, using the RHS of (3.26) for comparison, we have $a_j^{k+i} = y_j^{k+i}$ for all i, j. Replacing them accordingly, we obtain $\psi = \sum_{i,j} y_j^{k+i} d_{dR} x_j^{-i}$, which gives $\alpha' = d_{dR}z^k + \sum_{i,j} y_j^{k+i} d_{dR} x_j^{-i}$, hence the desired form (3.27).

See the latest version of [1] on arXiv or RG.

2 On shifted contact derived Artin stacks [2]

Similar modifications regarding the main definitions and some proofs in [2] will be included in the final version of [2], which is under review. Note that the need for additional corrections and remarks listed for [1] in the section 1 above has emerged during the reviewing process of [2].

I wish to warmly thank the anonymous referee(s) for their valuable comments and suggestions, which helped a lot and improved the quality of the original manuscript.

References

[1] K. İ. Berktav, Shifted contact structures and their local theory, arXiv:2209.09686. (To appear in the Ann. Fac. Sci. Toulouse, Math.)

[2] K. İ. Berktav, On shifted contact derived Artin stacks, arXiv:2401.03334.

[4] Instead, one may set $a_j^{k+i} = 0 \, \forall i, j$, giving rise to the form of α' with the roles of x_ν^μ, y_ν^σ interchanged.