Stacks in Mathematical Physics

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Outline

1. Field theories and their phase spaces
2. Why higher categorical objects emerge
3. A glimpse to higher structures in physics
    - Stacks in gauge theory
    - Stacks in gravity

Field Theories and Their Phase Spaces

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- Want to understand EOMs for each open $U \subset M$ and organize this data coherently...

$$\boxed{\text{Classical field theories} \equiv \text{(Sheaves of) Moduli spaces}}$$

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Examples...

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- EOM: $m\ddot{q} = -grad\ V(q)$.
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   which may not exist in the same category!!

   [Bad intersections]

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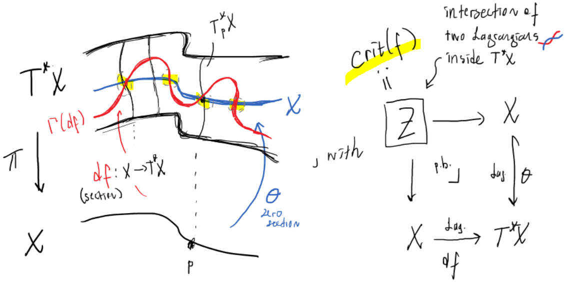

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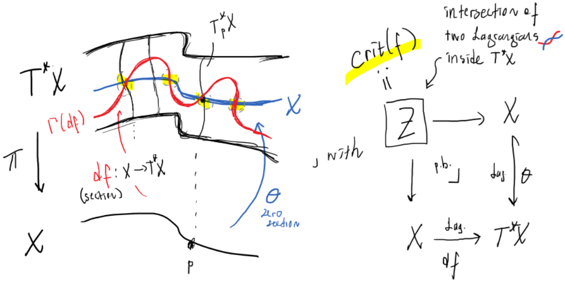

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2. $crit(\mathcal{S}) = \text{crit}(\mathcal{S})/G$ may not exist in the same category!! **[Bad quotients]**

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Hidden problem: Ordinary categories of spaces (manifolds, varieties, schemes,...) are **not** structured enough!!
e.g. existence of pullbacks/pushouts (limits/colimits):

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$\rightsquigarrow$ **Higher spaces + Derived geometry**

(∞-categories of) Higher Spaces: stacks, higher stacks, derived schemes, derived stacks

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- RHS encodes higher symmetries/equivalences/relations (more non-trivial higher morphisms are emerged).
- Get an infinite tower of equivalences

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    2. The *quotient stack* defined by

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  1. Symplectic, Poisson (well-known)
  2. Contact (uncharted area)

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3. Why important? New formulations, tools and methods...

   Cost: new dictionary, different levels of abstraction, new math...

Thank you!