

# Shifted Geometric Structures

## TMD

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Kadri İlker Berktaş

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## Outline

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How to describe higher spaces?

Answer: *Derived algebraic geometry (DAG)*

DAG := A.G. + homotopy theory + higher category theory

In the context of DAG, using Yoneda's embedding we have

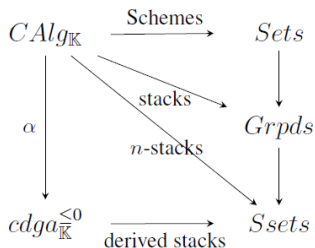


Figure: Higher Spaces

$$Spaces \hookrightarrow Stacks \hookrightarrow 2-stacks \hookrightarrow \cdots \hookrightarrow \infty-stacks$$

## Punchlines:

- DAG  $\sim$  a *higher categorical/homotopy theoretical refinement of classical algebraic geometry*. It offers a new way of organizing information for various purposes.
- DAG essentially provides a new setup to deal with non-generic situations in geometry. E.g. non-transversal intersections and “bad” quotients.
- DAG also offers generalized versions of certain familiar geometric structures and studies their properties.

# Geometric Structures on Higher Spaces

# Example I: Derived symplectic geometry

PTVV's shifted symplectic structures and their local models:

- 1 T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*,(2013)
- 2 C. Brav, V. Bussi and D. Joyce, *A Darboux theorem for derived schemes with shifted symplectic structure* (v1 2015, 2019)
- 3 D. Joyce, P. Safronov, *A Lagrangian Neighbourhood Theorem for shifted symplectic derived schemes*.(2019)

# Shifted symplectic structures

- 1 For  $A^\bullet \in cdga_{\mathbb{K}}$ , define *the de Rham algebra of  $A^\bullet$*  as a certain double complex

$$DR(A^\bullet) = \bigoplus_{p=0}^{\infty} \bigoplus_{k=-\infty}^0 \left( \bigwedge^p \Omega_{A^\bullet}^1 \right)^k [p]$$

with  $d_{tot} = d + d_{dR}$ .

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- 2 Generalized geometric objects on derived schemes/stacks:
  - 1  $\mathcal{A}^p(\mathbf{X}, k) :=$  the space of *p-forms of degree k* on  $\mathbf{X}$ .
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  - 3 A *non-degeneracy condition* (ND) for  $\omega \in \mathcal{A}^2(\mathbf{X}, k)$ : When the induced map  $\omega \cdot : \mathbb{T}|_{A^\bullet} \rightarrow \mathbb{L}|_{A^\bullet}$  is a quasi-isomorphism.

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  - 4  $\mathcal{A}^{(cl,2)}(\mathbf{X}, k) \oplus \text{ND} \rightsquigarrow k$ -*shifted symplectic forms* on  $\mathbf{X}$

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Some other relevant concepts and key results:

- 1 Isotropic and Lagrangian structures  $\mathbf{L} \rightarrow (\mathbf{X}, \omega)$  in derived symplectic geometry
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- 4 A Darboux-type theorem for shifted symplectic derived stacks
- 5 A Lagrangian neighborhood theorem for shifted symplectic derived stacks
- 6 ...

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- Coisotropics  $\mathbf{Y} \rightarrow \mathbf{X}$  in  $(\mathbf{X}, \pi)$
- A Darboux-type theorem
- Some classical interactions between symplectic and Poisson geometries can be extended to the shifted case:

### Theorem

*The spaces of  $k$ -shifted symplectic structures  $\omega$  and non-degenerate Poisson structures  $\pi$  on  $\mathbf{X}$  are equivalent.*

- 1 D. Calaque, T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted Poisson structures and deformation quantization* (2017)



**Our work:**  
**Shifted Contact Structures and Their Local Theory**

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## Key ideas and results

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- 1 A *k-shifted contact structure* on a derived Artin stack  $\mathbf{X}$  consist of a morphism of  $\mathcal{K} \rightarrow \mathbb{T}_{\mathbf{X}}$  of perfect complexes, a line bundle  $L$ , and a locally defined *k-shifted 1-form*  $\alpha : \mathbb{T}_{\mathbf{X}} \rightarrow \mathcal{O}_{\mathbf{X}}[k]$  satisfying a non-degeneracy condition.

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- 2 A *Darboux-like theorem* for shifted contact derived schemes
- 3 *Symplectification* of a shifted contact derived scheme
- 4 *Legendrians* in derived contact geometry and a neighborhood theorem
- 5 Stacky generalizations

**Thank you!**