

Comments and corrections

Kadri Berktav

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Abstract

This document contains additional remarks and errors for my papers that have not been corrected in public versions. Some corrections and remarks will appear in the final versions.

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1 Shifted contact structures and their local theory [1]

1.1 Definitions 3.5 & 3.6.

Since $QCoh(\mathbf{X})$ is a stable ∞ -category, we can simplify Definitions 3.5 & 3.6 in [1] and introduce/use the following ones instead:

Definition 1.1. Let \mathbf{X} be a locally finitely presented derived (Artin) stack. A *pre- k -shifted contact structure* on \mathbf{X} is given by a shifted line bundle $L[k]$ with a morphism $\alpha : \mathbb{T}_{\mathbf{X}} \rightarrow L[k]$. Denote such a structure by $(L[k], \alpha)$.

Note that we can consider a pre- k -shifted contact data as a perfect complex \mathcal{K} and a line bundle L along with a morphism $\kappa : \mathcal{K} \rightarrow \mathbb{T}_{\mathbf{X}}$ such that $Cone(\kappa) \simeq L[k]$. Since $QCoh(\mathbf{X})$ is a stable ∞ -category, we have a fiber-cofiber sequence $\mathcal{K} \rightarrow \mathbb{T}_{\mathbf{X}} \rightarrow L[k]$ in $QCoh(\mathbf{X})$, and hence, the cocone of $\mathbb{T}_{\mathbf{X}} \rightarrow L[k]$ is equivalent to \mathcal{K} . We then may denote a pre- k -shifted contact structure on \mathbf{X} by (\mathcal{K}, κ, L) .

Definition 1.2. We say that a pre- k -shifted contact structure (\mathcal{K}, κ, L) on \mathbf{X} is a *k -shifted contact structure* if locally on \mathbf{X} , where L is trivial, the induced k -shifted 1-form¹ $\alpha : \mathbb{T}_{\mathbf{X}} \rightarrow \mathcal{O}_{\mathbf{X}}[k]$ is such that the map $d_{dR}\alpha|_{\mathcal{K}} := \kappa^{\vee}[k] \circ (d_{dR}\alpha \cdot) \circ \kappa : \mathcal{K} \rightarrow \mathcal{K}^{\vee}[k]$ is a weak equivalence.

In that case, we say the k -shifted 2-form $d_{dR}\alpha$ is *non-degenerate on \mathcal{K}* . Also, we call such local form a *k -contact form*.

Remark 1.3. When $k \leq 0$, the triangle $\mathcal{K} \rightarrow \mathbb{T}_{\mathbf{X}} \rightarrow L[k]$ splits locally for any affine derived scheme (so, this also holds Zariski locally for any derived scheme)². In fact, the nondegeneracy condition implies that \mathcal{K} has Tor-amplitude $[0, -k]$ so that $\mathcal{K}[-k]$ is connective. Then the connecting homomorphism $L[k] \rightarrow \mathcal{K}[1]$ in the exact triangle is equivalently $L \rightarrow \mathcal{K}[1-k]$. Notice that $\mathcal{K}[1-k]$ is concentrated in degrees ≤ -1 , so this morphism is automatically zero on any affine derived scheme, which implies the desired splitting.

These updates will appear in the final version of [2].

¹We can locally identify the map α with the induced shifted one-form using the trivialization of $L^{\vee}[k]$.

²We thank the anonymous referee for this remark.

1.2 Example 3.10

There are some arguments requiring additional correction and comments. (*Please notice the change in the generators.*) Localizing if necessary, instead of the current generators given in the construction, we can consider the variables

$$x_1^{-i}, x_2^{-i}, \dots, x_{m_i}^{-i} \quad \text{in degree } -i \text{ for } i = 1, 2, \dots, \ell, \quad (1.1)$$

$$y_1^{k+i}, y_2^{k+i}, \dots, y_{m_i}^{k+i} \quad \text{in degree } k+i \text{ for } i = 1, \dots, \ell, \quad (1.2)$$

$$z^k, y_1^k, y_2^k, \dots, y_{m_0}^k \quad \text{in degree } k. \quad (1.3)$$

Here, we replace $\tilde{x}_1^0 \in A^0$ with $z^k \in A^k$ to fix the generator of the complex $Rest$, which is concentrated in $\deg -k$ and now generated by $\partial/\partial z^k$.

The current version of α in [1, Eqn. (3.14)] might be misleading. Instead, we first define the *primitive element* $\alpha \in (\Omega_A^1)^k$ by

$$\alpha = d_{dR} z^k + \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{dR} x_j^{-i} \quad (1.4)$$

Note that $\alpha \in \Omega_A^1[k]$, but we also need it to be d -closed. We can actually make α d -closed by determining the differential d via the equations $d|_{A(0)} = 0$; $dx_j^{-i} = \partial H / \partial y_j^{k+i}$; $dy_j^{k+i} = \partial H / \partial x_j^{-i}$ for all i, j and $-kdz^k = H + d[\dots]$ for the Hamiltonian $H \in A^{k+1}$. Then, the construction follows.

More details can be found in the final version of [2] and the revisited/latest version of [1] on arXiv.

1.3 Proof of Theorem 3.13

The generator for $Rest$ is incorrect. In fact, $Rest$ is a complex concentrated in degree $-k$ since we want the splitting $\mathbb{T}_A = \ker(\alpha) \oplus Rest$ over $\text{spec} H^0(A)$, where $Rest = L[k]$ is a k -shifted line bundle. But it is said in the construction that $Rest$ is generated by the vector field $\partial/\partial \tilde{x}_1^0$, which is of degree 0. This is incorrect. So, instead of using \tilde{x}_1^0 in A^0 , we must use a distinguished generator z^k in A^k in addition to the generators $y_1^k, y_2^k, \dots, y_{m_0}^k$. Then, after suitable replacement, we should end up with the splitting $\mathbb{T}_A = \ker(\alpha) \oplus Rest$ over $\text{spec} H^0(A)$ such that $\ker(\alpha)$ has Tor-amplitude $[0, -k]$ and $Rest$ is concentrated in $\deg -k$, and that $\ker(\alpha) \simeq \mathbb{T}_A / Rest$ with

$$\begin{aligned} \ker(\alpha)|_{\text{spec} H^0(A)} &= \langle \partial/\partial x_j^{-i}, \partial/\partial y_j^{k+i} \rangle, \\ Rest|_{\text{spec} H^0(A)} &= \langle \partial/\partial z^k \rangle, \end{aligned}$$

where $0 \leq i \leq \ell$, $1 \leq j \leq m_i$. Then we can get the desired k -shifted contact form as in Section 1.2.

We have misleading argument (although our intention was naively the opposite and we were trying to say d could be taken in that form by the results of Brav-Bussi-Joyce, etc... Clearly, the way of presenting this intention is incorrect.): "Imposing the differential"-type argument in the proof has been fixed. Instead, expanding the defining equations for the pair (H, ϕ) , we arrive at the desired formulas for d , hence identify the cdga (A, d) exactly with the one above.

An adaptation of this proof to the case of derived Artin stacks will appear in the final version of [2]. Similar edits also available in the revisited version of [1] on arXiv.

1.4 Observations 2.27 and 3.17 revisited

For $k/2$ odd, the statements were incorrect; they are now fixed. See the latest version of [1] on arXiv or RG.

1.5 Uniqueness clarified

For the equations to be mentioned in this section, we refer to the ones in [1]. Assume that there is an element α' satisfying the desired properties listed in the proof of Theorem 3.13. Let us clarify how the corresponding conditions uniquely (up to interchange of x_j^{-i} and y_j^{k+i})³ determine the

³The roles of x_j^{-i}, y_j^{k+i} are symmetric in (3.26) and (3.25), where $d_{dR} x_j^{-i} d_{dR} y_j^{k+i} = d_{dR} y_j^{k+i} d_{dR} x_j^{-i}$ for k odd.

representation in (3.27). We first observe that due to Eqn. (3.26) and the condition $\iota_{\partial/\partial z^k} \alpha' = 1$, any such element α' takes the form

$$\alpha' = d_{dR} z^k + \psi \quad \text{with} \quad \psi = \sum_{i,j} [a_j^{k+i} d_{dR} x_j^{-i} + b_j^{-i} d_{dR} y_j^{k+i}],$$

where $a_{\nu'}^{\mu'} \in A^{\mu'}$, $b_{\nu}^{\mu} \in A^{\mu}$ such that b_{ν}^{μ} 's depend only on $x_{j'}^{-i}$'s and $A(0)$ for degree reasons. From the condition (3.25), we take⁴ $b_j^{-i} = 0$ for all i, j , leading to

$$d_{dR} \alpha' = d_{dR} \psi = \sum_{i,j} d_{dR} a_j^{k+i} d_{dR} x_j^{-i}.$$

Now, using the RHS of (3.26) for comparison, we have $a_j^{k+i} = y_j^{k+i}$ for all i, j . Replacing them accordingly, we obtain $\psi = \sum_{i,j} y_j^{k+i} d_{dR} x_j^{-i}$, which gives $\alpha' = d_{dR} z^k + \sum_{i,j} y_j^{k+i} d_{dR} x_j^{-i}$, hence the desired form (3.27).

See the latest version of [1] on arXiv or RG.

2 On shifted contact derived Artin stacks [2]

Similar modifications regarding the main definitions and some proofs in [2] will be included in the final version of [2], which is under review. Note that the need for additional corrections and remarks listed for [1] in the section 1 above has emerged during the reviewing process of [2].

I wish to warmly thank the anonymous referee(s) for their valuable comments and suggestions, which helped a lot and improved the quality of the original manuscript.

References

- [1] K. İ. Berktav, Shifted contact structures and their local theory, [arXiv:2209.09686](https://arxiv.org/abs/2209.09686). (To appear in the Ann. Fac. Sci. Toulouse, Math.)
- [2] K. İ. Berktav, On shifted contact derived Artin stacks, [arXiv:2401.03334](https://arxiv.org/abs/2401.03334).

⁴Instead, one may set $a_j^{k+i} = 0 \forall i, j$, giving rise to the form of α' with the roles of $x_{\nu'}^{\mu}$, y_{ν}^{σ} interchanged.