

# Comments and corrections

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## Abstract

This document contains additional remarks and errors for my papers that have not been corrected in public versions. Some corrections and remarks will appear in the final versions.

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## 1 Shifted contact structures and their local theory [1]

### 1.1 Definitions 3.5 & 3.6 simplified

Since  $QCoh(\mathbf{X})$  is a stable  $\infty$ -category, we can simplify Definitions 3.5 & 3.6 in [1] and introduce/use the following ones instead:

**Definition 1.1.** Let  $\mathbf{X}$  be a locally finitely presented derived (Artin) stack. A *pre- $k$ -shifted contact structure* on  $\mathbf{X}$  is given by a shifted line bundle  $L[k]$  with a morphism  $\alpha : \mathbb{T}_{\mathbf{X}} \rightarrow L[k]$ . Denote such a structure by  $(L[k], \alpha)$ .

Note that we can consider a pre- $k$ -shifted contact data as a perfect complex  $\mathcal{K}$  and a line bundle  $L$  along with a morphism  $\kappa : \mathcal{K} \rightarrow \mathbb{T}_{\mathbf{X}}$  such that  $Cone(\kappa) \simeq L[k]$ . Since  $QCoh(\mathbf{X})$  is a stable  $\infty$ -category, we have a fiber-cofiber sequence  $\mathcal{K} \rightarrow \mathbb{T}_{\mathbf{X}} \rightarrow L[k]$  in  $QCoh(\mathbf{X})$ , and hence, the cocone of  $\mathbb{T}_{\mathbf{X}} \rightarrow L[k]$  is equivalent to  $\mathcal{K}$ . We then may denote a pre- $k$ -shifted contact structure on  $\mathbf{X}$  by  $(\mathcal{K}, \kappa, L)$ .

**Definition 1.2.** We say that a pre- $k$ -shifted contact structure  $(\mathcal{K}, \kappa, L)$  on  $\mathbf{X}$  is a  *$k$ -shifted contact structure* if locally on  $\mathbf{X}$ , where  $L$  is trivial, the induced  $k$ -shifted 1-form<sup>1</sup>  $\alpha : \mathbb{T}_{\mathbf{X}} \rightarrow \mathcal{O}_{\mathbf{X}}[k]$  is such that the map  $d_{dR}\alpha|_{\mathcal{K}} := \kappa^{\vee}[k] \circ (d_{dR}\alpha \cdot) \circ \kappa : \mathcal{K} \rightarrow \mathcal{K}^{\vee}[k]$  is a weak equivalence.

In that case, we say the  $k$ -shifted 2-form  $d_{dR}\alpha$  is *non-degenerate on  $\mathcal{K}$* . Also, we call such local form a  *$k$ -contact form*.

**Remark 1.3.** When  $k \leq 0$ , the triangle  $\mathcal{K} \rightarrow \mathbb{T}_{\mathbf{X}} \rightarrow L[k]$  splits locally for any affine derived scheme (so, this also holds Zariski locally for any derived scheme)<sup>2</sup>. In fact, the nondegeneracy condition implies that  $\mathcal{K}$  has Tor-amplitude  $[0, -k]$  so that  $\mathcal{K}[-k]$  is connective. Then the connecting homomorphism  $L[k] \rightarrow \mathcal{K}[1]$  in the exact triangle is equivalently  $L \rightarrow \mathcal{K}[1-k]$ . Notice that  $\mathcal{K}[1-k]$  is concentrated in degrees  $\leq -1$ , so this morphism is automatically zero on any affine derived scheme, which implies the desired splitting.

These updates will appear in the final version of [2].

<sup>1</sup>We can locally identify the map  $\alpha$  with the induced shifted one-form using the trivialization of  $L^{\vee}[k]$ .

<sup>2</sup>We thank the anonymous referee for this remark.

## 1.2 Example 3.10 clarified

There are some arguments requiring additional correction and comments. (*Please notice the change in the generators.*) Localizing if necessary, instead of the current generators given in the construction, we can consider the variables

$$x_1^{-i}, x_2^{-i}, \dots, x_{m_i}^{-i} \quad \text{in degree } -i \text{ for } i = 1, 2, \dots, \ell, \quad (1.1)$$

$$y_1^{k+i}, y_2^{k+i}, \dots, y_{m_i}^{k+i} \quad \text{in degree } k+i \text{ for } i = 1, \dots, \ell, \quad (1.2)$$

$$z^k, y_1^k, y_2^k, \dots, y_{m_0}^k \quad \text{in degree } k. \quad (1.3)$$

Here, we replace  $\tilde{x}_1^0 \in A^0$  with  $z^k \in A^k$  to fix the generator of the complex  $Rest$ , which is concentrated in  $\deg -k$  and now generated by  $\partial/\partial z^k$ .

The current version of  $\alpha$  in [1, Eqn. (3.14)] might be misleading. Instead, we first define  $\alpha \in (\Omega_A^1)^k$  by

$$\alpha = d_{dR} z^k + \sum_{i=0}^{\ell} \sum_{j=1}^{m_i} y_j^{k+i} d_{dR} x_j^{-i} \quad (1.4)$$

Note that  $\alpha \in \Omega_A^1[k]$ , but we also need it to be  $d$ -closed. We can actually make  $\alpha$   $d$ -closed by determining the differential  $d$  via the equations  $d|_{A(0)} = 0$ ;  $dx_j^{-i} = \partial H / \partial y_j^{k+i}$ ;  $dy_j^{k+i} = \partial H / \partial x_j^{-i}$  for all  $i, j$  and  $-kdz^k = H + d[\dots]$  for the Hamiltonian  $H \in A^{k+1}$ . Then, the construction follows.

More details can be found in the final version of [2] and the revisited/latest version of [1] on arXiv.

## 1.3 Proof of Theorem 3.13 edited

The generator for  $Rest$  is incorrect. In fact,  $Rest$  is a complex concentrated in degree  $-k$  since we want the splitting  $\mathbb{T}_A = \ker(\alpha) \oplus Rest$  over  $\text{spec} H^0(A)$ , where  $Rest = L[k]$  is a  $k$ -shifted line bundle. But it is said in the construction that  $Rest$  is generated by the vector field  $\partial/\partial \tilde{x}_1^0$ , which is of degree 0. This is incorrect. So, instead of using  $\tilde{x}_1^0$  in  $A^0$ , we must use a distinguished generator  $z^k$  in  $A^k$  in addition to the generators  $y_1^k, y_2^k, \dots, y_{m_0}^k$ . Then, after suitable replacement, we should end up with the splitting  $\mathbb{T}_A = \ker(\alpha) \oplus Rest$  over  $\text{spec} H^0(A)$  such that  $\ker(\alpha)$  has Tor-amplitude  $[0, -k]$  and  $Rest$  is concentrated in  $\deg -k$ , and that  $\ker(\alpha) \simeq \mathbb{T}_A / Rest$  with

$$\begin{aligned} \ker(\alpha)|_{\text{spec} H^0(A)} &= \langle \partial/\partial x_j^{-i}, \partial/\partial y_j^{k+i} \rangle, \\ Rest|_{\text{spec} H^0(A)} &= \langle \partial/\partial z^k \rangle, \end{aligned}$$

where  $0 \leq i \leq \ell$ ,  $1 \leq j \leq m_i$ . Then we can get the desired  $k$ -shifted contact form as in Section 1.2.

We have misleading argument (although our intention was naively the opposite and we were trying to say  $d$  could be taken in that form by the results of Brav-Bussi-Joyce, etc... Clearly, the way of presenting this intention is incorrect.): "Imposing the differential"-type argument in the proof has been fixed. Instead, expanding the defining equations for the pair  $(H, \phi)$ , we arrive at the desired formulas for  $d$ , hence identify the cdga  $(A, d)$  exactly with the one above.

An adaptation of this proof to the case of derived Artin stacks will appear in the final version of [2]. Similar edits also available in the revisited version of [1] on arXiv.

## 1.4 Observations 2.27 and 3.17 revisited

For  $k/2$  odd, the statements were incorrect; they are now fixed. See the latest version of [1] on arXiv or RG.

## 1.5 Uniqueness clarified

For the equations to be mentioned in this section, we refer to the ones in [1]. Assume that there is an element  $\alpha'$  satisfying the desired properties listed in the proof of Theorem 3.13. Let us clarify how the corresponding conditions uniquely (up to interchange of  $x_j^{-i}$  and  $y_j^{k+i}$ )<sup>3</sup> determine the

<sup>3</sup>The roles of  $x_j^{-i}, y_j^{k+i}$  are symmetric in (3.26) and (3.25), where  $d_{dR} x_j^{-i} d_{dR} y_j^{k+i} = d_{dR} y_j^{k+i} d_{dR} x_j^{-i}$  for  $k$  odd.

representation in (3.27). We first observe that due to Eqn. (3.26) and the condition  $\iota_{\partial/\partial z^k} \alpha' = 1$ , any such element  $\alpha'$  takes the form

$$\alpha' = d_{dR} z^k + \psi \quad \text{with} \quad \psi = \sum_{i,j} [a_j^{k+i} d_{dR} x_j^{-i} + b_j^{-i} d_{dR} y_j^{k+i}],$$

where  $a_{\nu'}^{\mu'} \in A^{\mu'}$ ,  $b_{\nu}^{\mu} \in A^{\mu}$  such that  $b_{\nu}^{\mu}$ 's depend only on  $x_{j'}^{-i}$ 's and  $A(0)$  for degree reasons. From the condition (3.25), we take<sup>4</sup>  $b_j^{-i} = 0$  for all  $i, j$ , leading to

$$d_{dR} \alpha' = d_{dR} \psi = \sum_{i,j} d_{dR} a_j^{k+i} d_{dR} x_j^{-i}.$$

Now, using the RHS of (3.26) for comparison, we have  $a_j^{k+i} = y_j^{k+i}$  for all  $i, j$ . Replacing them accordingly, we obtain  $\psi = \sum_{i,j} y_j^{k+i} d_{dR} x_j^{-i}$ , which gives  $\alpha' = d_{dR} z^k + \sum_{i,j} y_j^{k+i} d_{dR} x_j^{-i}$ , hence the desired form (3.27).

See the latest version of [1] on arXiv or RG.

## 2 On shifted contact derived Artin stacks [2]

Similar modifications regarding the main definitions and some proofs in [2] will be included in the final version of [2], which is under review. Note that the need for additional corrections and remarks listed for [1] in the section 1 above has emerged during the reviewing process of [2].

I wish to warmly thank the anonymous referee(s) for their valuable comments and suggestions, which helped a lot and improved the quality of the original manuscript.

## References

- [1] K. İ. Bertav, Shifted contact structures and their local theory, *Ann. Fac. Sci. Toulouse, Math.*, Serie 6, Vol. 33 (2024) No. 4, pp. 1019-1057. [arXiv:2209.09686](#).
- [2] K. İ. Bertav, On shifted contact derived Artin stacks, [arXiv:2401.03334](#).

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<sup>4</sup>Instead, one may set  $a_j^{k+i} = 0 \forall i, j$ , giving rise to the form of  $\alpha'$  with the roles of  $x_{\nu'}^{\mu}$ ,  $y_{\nu}^{\sigma}$  interchanged.