Stacks in Mathematical Physics

Kadri İlker Berktav

Department of Mathematics
Bilkent University

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- Key notions: higher spaces and categorifications

Outline

1. Field theories and their phase spaces
2. Why higher categorical objects emerge
   - In what sense are they useful in geometry and physics?
3. A glimpse to higher structures in physics
   - Stacks in gauge theory
   - Stacks in gravity

Field Theories and Their Phase Spaces

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$$crit(\mathcal{S}) = \{p \in F_M : \mathrm{d}\mathcal{S}|_p = 0\}/\sim, \tag{2}$$

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- Want to understand EOMs for each open $U \subset M$ and organize this data coherently...

$$\boxed{\text{Classical field theories} \equiv \text{(Sheaves of) Moduli spaces}}$$

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Examples...

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- EOM: $m\ddot{q} = -grad\ V(q)$.
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- EOM: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$.
- $crit(\mathcal{S}) := \{g \in F_M : R_{\mu\nu} = 0\}/\sim$ *(Ricci-flat)*.

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   which may not exist in the same category!!

   [Bad intersections]

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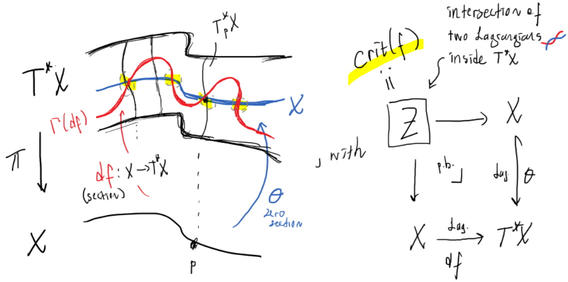

Figure: crit(f) as the intersection of two Lagrangians in T^*X

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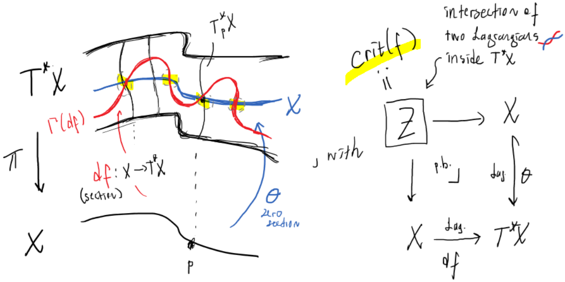

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1. crit($\mathcal{S}$), as the *fibered product/pullback* may not exist in the same category!! **[Bad intersections]**
2. $crit(\mathcal{S}) = \text{crit}(\mathcal{S})/G$ may not exist in the same category!! **[Bad quotients]**

Some issues with $crit(\mathcal{S})$

Hidden problem: Ordinary categories of spaces (manifolds, varieties, schemes,...) are **not** structured enough!!
e.g. existence of pullbacks/pushouts (limits/colimits):

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$$\rightsquigarrow \textbf{Higher spaces + Derived geometry}$$

(∞-categories of) Higher Spaces: stacks, higher stacks, derived schemes, derived stacks

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- Higher spaces as *homotopy sheaves*:

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- RHS encodes higher symmetries/equivalences/relations (more non-trivial higher morphisms are emerged).
- Get an infinite tower of equivalences

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    2. The *quotient stack* defined by

$$[X/G] = \operatorname{colim}\left(X \leftleftarrows X \times G \,\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\, X \times G \times G \,\substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow}\, \cdots \right),$$

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  1. Symplectic, Poisson (well-known)
  2. Contact (uncharted area)

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3. Why important? New formulations, tools and methods...

Cost: new dictionary, different levels of abstraction, new math...

Thank you!