Shifted Geometric Structures

TMD

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Outline

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Motivation and Setup

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How to describe higher spaces?
Answer: *Derived algebraic geometry (DAG)*

DAG:= A.G. + homotopy theory + higher category theory

In the context of DAG, using Yoneda's embedding we have

$$\begin{tikzcd}
CAlg_{\mathbb{K}} \arrow[r, "\text{Schemes}"] \arrow[dr, "\text{stacks}"] \arrow[ddr, "n\text{-stacks}"] \arrow[dd, "\alpha"'] & Sets \arrow[d] \\
 & Grpds \arrow[d] \\
cdga_{\mathbb{K}}^{\leq 0} \arrow[r, "\text{derived stacks}"'] & Ssets
\end{tikzcd}$$

Figure: Higher Spaces

$$Spaces \hookrightarrow Stacks \hookrightarrow 2\text{-}stacks \hookrightarrow \cdots \hookrightarrow \infty\text{-}stacks$$

Punchlines:

- DAG $\sim$ *a higher categorical/homotopy theoretical refinement of classical algebraic geometry*. It offers a new way of organizing information for various purposes.
- DAG essentially provides a new setup to deal with non-generic situations in geometry. E.g. non-transversal intersections and “bad” quotients.
- DAG also offers generalized versions of certain familiar geometric structures and studies their properties.

Geometric Structures on Higher Spaces

Example I: Derived symplectic geometry

PTVV's shifted symplectic structures and their local models:

1. T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted symplectic structures*,(2013)
2. C. Brav, V. Bussi and D. Joyce, *A Darboux theorem for derived schemes with shifted symplectic* structure (v1 2015, 2019)
3. D. Joyce, P. Safronov, *A Lagrangian Neighbourhood Theorem for shifted symplectic derived schemes.*(2019)

Shifted symplectic structures

1. For $A^\bullet \in cdga_{\mathbb{K}}$, define *the de Rham algebra of* $A^\bullet$ as a certain double complex

$$DR(A^\bullet) = \bigoplus_{p=0}^{\infty} \bigoplus_{k=-\infty}^{0} \Big(\bigwedge^{p} \Omega^1_{A^\bullet} \Big)^k [p]$$

with $d_{tot} = d + d_{dR}$.

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2. Generalized geometric objects on derived schemes/stacks:
   1. $\mathcal{A}^p(\mathbf{X}, k) :=$ the space of *p-forms of degree k* on $\mathbf{X}$.
   2. $\mathcal{A}^{(cl,p)}(\mathbf{X}, k) :=$ the space of *closed p-forms of degree k*.

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   3. A *non-degeneracy condition* (ND) for $\omega \in \mathcal{A}^2(\mathbf{X}, k)$: When the induced map $\omega\cdot : \mathbb{T}|_{A^\bullet} \to \mathbb{L}|_{A^\bullet}$ is a quasi-isomorphism.

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    4. $\mathcal{A}^{(cl,2)}(\mathbf{X}, k) \oplus \mathrm{ND} \rightsquigarrow$ *k-shifted symplectic forms* on $\mathbf{X}$

Shifted symplectic structures

Some other relevant concepts and key results:

1. Isotropic and Lagrangian structures $\mathbf{L} \to (\mathbf{X}, \omega)$ in derived symplectic geometry
2. Lagrangian intersection theorem & shifted symplectic structures on mapping stacks

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4. A Darboux-type theorem for shifted symplectic derived stacks
5. A Lagrangian neighborhood theorem for shifted symplectic derived stacks
6. ...

Example II: Shifted Poisson Geometry

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- Poisson structures can also be defined in the context of DAG: k-shifted Poisson structures on derived stacks
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- Poisson structures can also be defined in the context of DAG: k-shifted Poisson structures on derived stacks
- Coisotropics $\mathbf{Y} \to \mathbf{X}$ in $(\mathbf{X}, \pi)$
- A Darboux-type theorem
- Some classical interactions between symplectic and Poisson geometries can be extended to the shifted case:

Theorem

The spaces of k-shifted symplectic structures ω and non-degenerate Poisson structures π on $\mathbf{X}$ are equivalent.

1. D. Calaque, T. Pantev, B. Toën, M. Vaquié, and G. Vezzosi, *Shifted Poisson structures and deformation quantization* (2017)

Our work:
Shifted Contact Structures and Their Local Theory

Key ideas and results

Shifted Contact Geometry

Key ideas and results

1. A *k-shifted contact structure* on a derived Artin stack $\mathbf{X}$ consist of a morphism of $\mathcal{K} \to \mathbb{T}_{\mathbf{X}}$ of perfect complexes, a line bundle L, and a locally defined k-shifted 1-form $\alpha : \mathbb{T}_{\mathbf{X}} \to \mathcal{O}_{\mathbf{X}}[k]$ satisfying a non-degeneracy condition.

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2. A *Darboux-like theorem* for shifted contact derived schemes
3. *Symplectification* of a shifted contact derived scheme

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2. A *Darboux-like theorem* for shifted contact derived schemes
3. *Symplectification* of a shifted contact derived scheme
4. *Legendrians* in derived contact geometry and a neighborhood theorem
5. Stacky generalizations

Thank you!