On a new collection of stochastic linear programming test problems

K. A. Ariyawansa* and Andrew J. Felt†

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Abstract

The purpose of this paper is to introduce a new test problem collection for stochastic linear programming that the authors have recently begun to assemble. While there are existing stochastic programming test problem collections, our new collection has three features that distinguish it from existing collections. First, our collection is web-based with free public access, and we intend to enrich it as new test problems become available. Indeed, we encourage submissions of new test problems. Second, for each test problem class we provide a short description, a mathematical problem statement and a notational reconciliation to a standard format. Third, for each test problem instance in our collection we provide numerical data in the SMPS [4] format. In a companion effort we have developed a data structure for implementing algorithms for stochastic linear programs, and C-routines that convert data in SMPS format to that data structure. These routines will also be freely available from the authors soon.

*Department of Pure and Applied Mathematics, Washington State University, Pullman, WA 99164-3113. Email: ari@wsu.edu. The work of this author was supported in part by the U.S. Army Research Office under Grant DAAD19-00-1-0465

†Department of Mathematics and Computing, University of Wisconsin-Stevens Point, Stevens Point, WI 54481-3897. Email: afelt@uwsp.edu. This author was supported in part by Washington State University under a research assistantship during Summer 2000.

1 Introduction

Stochastic programming has grown in importance in recent decades, because it allows the modeler to accurately represent planning under uncertainty. With strong interest in solving such problems and in finding more efficient solution techniques, there has arisen a need for a test set of stochastic programming problems.

One of the most popular forms of stochastic programming problems is the multistage stochastic linear program with recourse (MSSLP). See Section 2, for a precise statement of the MSSLP problem. While MSSLPs are growing in popularity, many of the applications are proprietary, and therefore the models are not publicly available. Test set collections of MSSLPs exist [11, 12]. However, they need to be enriched with newer applications. In fact, in many cases, the application associated with the existing test case is not known. Also, it would be helpful if the original applications are described in the notation of the original model, and related to a unified notation such as in Section 2.

In addition, it would be desirable if the data for the test problems is available in SMPS [4], the (emerging) standard for specifying input to software for MSSLPs.

To address the above needs, we have collected a group of eleven problem classes from a variety of settings. They are all MSSLPs, but of various structures and sizes, with randomness occuring in different parts in different problems. In some cases, problem instances were explicitly stated in the literature. In other cases, we created the problems based solely on the problem description in the literature, and in some cases, there is not yet any sample problem.

For each model application, we present a problem description, a concise problem statement, and, if available, a numerical example given by the model authors. We have attempted to stay as close to the authors' notation as possible in these subsections. Additionally, where feasible, we present a notational reconciliation, which shows how to transform the notation of the problem into that in Section 2.

Each problem class may be used to generate one or more instances of MSSLPs. We have created 15 such instances. The data for these 15 test problems, as well as six other test problems that we did not create, are available in SMPS format [4] from the authors at `http://www.uwsp.edu/math/afelt/slptestset.html`.

Also available at the web site is a chapter of problem descriptions. Each section covers a single problem class. At the beginning of each section, we give a citation to the original application, a brief description of the problem structure, and if applicable, the names of the SMPS files for the associated problem instances.

It is the intention of the authors to update the classes of applications and the test problem instances as new application areas become prominent, and to make the information that we present for each application area, as well as SMPS inputs for each test problem instance freely available to the stochastic programming research community.

In that spirit, we encourage colleagues to submit new problem data with an accompanying description. Such submissions should include the following:

1. description of the application and problem notation,
2. problem statement, in the same notation,
3. numerical example, if practical,
4. reconciliation to the notation of Section 2,
5. data files in SMPS format for each instance, and
6. optimal solutions for each instance and example.

We end this section by mentioning, that the initial motivation for the present work came from the need to carefully test the new polynomial interior cutting plane algorithms for stochastic programs developed in [1]. The design of our test problem collection was motivated and influenced by the elegant work of Moré et al. [17, 2] on test problem collections for several classes of (deterministic) nonlinear optimization problems, and the impact such collections have had on the development of software for such problem classes [18, 14, 3]. As part of testing the algorithms developed in [1], we have also developed C-routines that convert SMPS data into a data structure suitable for implementing algorithms for stochastic programs. These routines will soon be made available as open source from the authors.

2 Notation for multistage stochastic linear programs

In this section we state a generic form of the multistage stochastic linear program (MSSLP). Our notation here is motivated by the implementation of algorithms for the MSSLP, especially those based on cutting plane notions.

We begin by describing the underlying probability structure. We have N sequential discrete time stages with stage 1 representing the present. Time stages $2, 3, \ldots, N$ occur in the future sequentially in that order, at which realizations of random variables[1] $\boldsymbol{\xi_2}, \boldsymbol{\xi_3} \ldots, \boldsymbol{\xi_N}$ become available respectively.

The random variable $\boldsymbol{\xi_2}$ has a known discrete distribution with a finite number of realizations. At stage 2, a realization of $\boldsymbol{\xi_2}$ becomes available, and the system moves forward to stage 3 at which a realization of $\boldsymbol{\xi_3}$ becomes available. The conditional distribution of $\boldsymbol{\xi_3}$ given that ξ_2 has been observed is discrete with a finite number of realizations, and is known. Note that the pair ξ_2, ξ_3 may be termed a *partial scenario* since they represent only the realizations from the stages 2 and 3 of the N-stage process. We shall write $\sigma_3 := (\xi_2, \xi_3)$, to indicate a partial scenario with realizations up to stage 3 consisting of realizations ξ_2 and ξ_3 at stages 2 and 3 respectively. Note that we may write $\sigma_2 := \xi_2$.

Now suppose that we are at stage $t-1$ ($3 \leq t \leq N$) and that realizations $\xi_2, \xi_3, \ldots, \xi_{t-1}$ have become available in stages 2 through $t-1$ respectively. Let $\sigma_{t-1} := (\xi_2, \xi_3, \ldots, \xi_{t-1})$. The system now moves forward to stage t at which a realization of $\boldsymbol{\xi_t}$ becomes available. The conditional distribution of $\boldsymbol{\xi_t}$ given that the partial scenario σ_{t-1} has been observed is discrete with a finite number of realizations, and is known.

Before proceeding further we pause for some comments. Let ξ_N be the realization of $\boldsymbol{\xi_N}$ observed in stage N and let $\sigma_N := (\xi_2, \xi_3, \ldots, \xi_N)$. We call σ_N a *scenario*. We let $\mathcal{S}_t$ be the set of partial scenarios with realizations up to stage t for $t = 2, 3, \ldots, N$.

An MSSLP is a mathematical formulation of the decision process we now describe in association with the above probability structure. In the linear case that leads to MSSLPs, x_1 is a decision that has to be chosen in stage 1 from the set $\{x_1 \in \mathbb{R}^{n_1} : A_1 x_1 = b_1, x_1 \geq 0\}$ at a direct cost $c_1^\top x_1$, where $c_1 \in \mathbb{R}^{n_1}$, $b_1 \in \mathbb{R}^{m_1}$, and $A_1 \in \mathbb{R}^{m_1 \times n_1}$ constitute the first-

[1] In this paper, random variables will be represented in boldface. The expected value with respect to $\boldsymbol{x}$ will be written $\underset{\{\mathbf{x}\}}{E}[\cdot]$, and the conditional expected value (conditioned on y) will be written $\underset{\{\mathbf{x}|y\}}{E}[\cdot]$.

stage deterministic data. In addition, depending on the decision x_1 taken at present and the realizations of $\boldsymbol{\xi_2}, \boldsymbol{\xi_3}, \ldots, \boldsymbol{\xi_N}$ that would become available in the future, there would be an indirect cost due to *recourse* actions that may become necessary. In an MSSLP the objective function at stage 1 is to minimize the sum of the direct cost $c_1^{\mathsf{T}} x_1$ and the expectation $\mathcal{Q}_2(x_1)$ of this indirect cost. The computation of $\mathcal{Q}_2(x_1)$ requires recursion and is as specified below.

In an MSSLP, $\boldsymbol{\xi}_2 := (\boldsymbol{c_2}, \boldsymbol{b_2}, \boldsymbol{A_2}, \boldsymbol{T_2})$ and the distribution of $\boldsymbol{\xi_2}$ is such that $\boldsymbol{c_2}$, $\boldsymbol{b_2}$, $\boldsymbol{A_2}$ and $\boldsymbol{T_2}$ have realizations in $\mathbb{R}^{n_2}$, $\mathbb{R}^{m_2}$, $\mathbb{R}^{m_2 \times n_2}$ and $\mathbb{R}^{m_2 \times n_1}$ respectively. If the realizatation observed is $\xi_2 := (c_2, b_2, A_2, T_2)$ then the recourse decision x_2 is chosen from the set $\{x_2 \in \mathbb{R}^{n_2} : A_2 x_2 = b_2 - T_2 x_1, x_2 \geq 0\}$. The direct cost of this recourse action is $c_2^{\mathsf{T}} x_2$. The conditional expected cost (conditioned on ξ_2) of the recourse action is the sum of $c_2^{\mathsf{T}} x_2$ and the expectation $\mathcal{Q}_{3,\xi_2}(x_2)$ of the indirect cost of future recourse actions, and $\mathcal{Q}_2(x_1)$ is the expectation of this sum over $\boldsymbol{\xi_2}$.

Now suppose that we are at stage $t-1$ $(3 \leq t \leq N-1)$ and that the partial scenario that has been observed is $\sigma_{t-1} := (\xi_2, \xi_3, \ldots, \xi_{t-1})$. In an MSSLP, $\boldsymbol{\xi}_t := (\boldsymbol{c_t}, \boldsymbol{b_t}, \boldsymbol{A_t}, \boldsymbol{T_t})$ and the distribution of $\boldsymbol{\xi_t}$ given σ_{t-1} is such that $\boldsymbol{c_t}$, $\boldsymbol{b_t}$, $\boldsymbol{A_t}$ and $\boldsymbol{T_t}$ have realizations in $\mathbb{R}^{n_t}$, $\mathbb{R}^{m_t}$, $\mathbb{R}^{m_t \times n_t}$ and $\mathbb{R}^{m_t \times n_{t-1}}$ respectively. If the realization to be observed at stage t is $\xi_t = (c_t, b_t, A_t, T_t)$ then the recourse decision x_t is chosen from the set $\{x_t \in \mathbb{R}^{n_t} : A_t x_t = b_t - T_t x_{t-1}, x_t \geq 0\}$. The direct cost of this recourse action is $c_t^{\mathsf{T}} x_t$. The conditional expected cost (conditioned on σ_{t-1}) of the recourse action is the sum of $c_t^{\mathsf{T}} x_t$ and the expectation $\mathcal{Q}_{t+1,\sigma_t}(x_t)$ of the indirect cost of future recourse actions, where $\sigma_t := (\sigma_{t-1}, \xi_t)$. The value $\mathcal{Q}_{t,\sigma_{t-1}}(x_{t-1})$ is the expectation of this sum over partial scenarios in $\mathcal{S}_{t-1}$.

Now suppose that we are at stage $N-1$, and that we have observed the partial scenario $\sigma_{N-1} := (\xi_2, \xi_3, \ldots, \xi_{N-1})$. In an MSSLP, $\boldsymbol{\xi_N} := (\boldsymbol{c_N}, \boldsymbol{b_N}, \boldsymbol{A_N}, \boldsymbol{T_N})$ and the distribution of $\boldsymbol{\xi_N}$ given σ_{N-1} is sucth that $\boldsymbol{c_N}, \boldsymbol{b_N}, \boldsymbol{A_N}$, and $\boldsymbol{T_N}$ have realizations in $\mathbb{R}^{n_N}, \mathbb{R}^{m_N}, \mathbb{R}^{m_N \times n_N}$ and $\mathbb{R}^{m_N \times n_{N-1}}$, respectively. The function $\mathcal{Q}_{N,\sigma_{N-1}}$ is specified by the statements in previous paragraphs with t replaced by N, and by setting $\mathcal{Q}_{N+1,\sigma_N} \equiv 0$, since the process has only N stages. This recursion can be used to specify functions $\mathcal{Q}_{t,\sigma_{t-1}}$ for $t = 2, 3, \ldots, N$.

The preceding description leads to the following statement of the multistage stochastic linear program with recourse.

$$\begin{array}{rl} \text{Minimize} & Z(x_1) := c_1^{\mathsf{T}} x_1 + \mathcal{Q}_2(x_1) \\ \text{subject to} & A_1 x_1 \qquad\qquad = b_1 \\ & \quad x_1 \qquad\qquad \geq 0, \end{array}$$

where

$$\begin{aligned}
\mathcal{Q}_2(x_1) &:= \underset{\{\boldsymbol{\xi_2}\}}{E} \left[Q_2(x_2, \boldsymbol{\xi_2}) \right], \\
Q_2(x_1, \xi_2) &:= \inf_{x_2 \in \mathbb{R}^{n_2}} \left\{ c_2^{\mathsf{T}} x_2 + \mathcal{Q}_{3,\sigma_2}(x_2) : A_2 x_2 = b_2 - T_2 x_1, x_2 \geq 0 \right\} \\
&\quad (\text{with } \sigma_2 := \xi_2), \\
\mathcal{Q}_{t,\sigma_{t-1}}(x_{t-1}) &:= \underset{\{\boldsymbol{\xi_t} | \sigma_{t-1}\}}{E} \left[Q_t(x_{t-1}, \boldsymbol{\xi_t}) \right] \text{ for } t = 3, 4, \ldots, N, \\
Q_t(x_{t-1}, \xi_t) &:= \inf_{x_t \in \mathbb{R}^{n_t}} \left\{ c_t^{\mathsf{T}} x_t + \mathcal{Q}_{t+1,\sigma_t}(x_t) : A_t x_t = b_t - T_t x_{t-1}, x_t \geq 0 \right\} \\
&\quad (\text{with } \sigma_t := (\sigma_{t-1}, \xi_t)) \text{ for } t = 3, 4, \ldots, N-1,
\end{aligned}$$

and

$$Q_N(x_{n-1}, \xi_N) := \inf_{x_N \in \mathbb{R}^{n_N}} \left\{ c_N^{\mathsf{T}} x_N : A_N x_N = b_N - T_N x_{N-1}, x_N \geq 0 \right\}. \tag{1}$$

Note that the data for the MSSLP above consist of:

- first stage deterministic data c_1, b_1, A_1,
- the distribution of $(\boldsymbol{c_2}, \boldsymbol{b_2}, \boldsymbol{A_2}, \boldsymbol{T_2})$, and
- for all $\sigma_{t-1} \in \mathcal{S}_{t-1}$ the conditional distribution of $(\boldsymbol{c_t}, \boldsymbol{b_t}, \boldsymbol{A_t}, \boldsymbol{T_t})$, conditioned on σ_{t-1}, for $t = 3, 4, \ldots, N$.

We refer to the case where the distribution of $\boldsymbol{\xi_t}$ is independent of σ_{t-1} for $t = 3, 4, \ldots, N$ as the *independent* case. In the independent case (1) simplifies to the following form, which we state for convenient reference.

$$\begin{array}{rll} \text{Minimize} & Z(x_1) := c_1^\top x_1 + \mathcal{Q}_2(x_1) & \\ \text{subject to} & A_1 x_1 & = b_1 \\ & x_1 & \geq 0, \end{array}$$

where

$$\mathcal{Q}_t(x_{t-1}) := \underset{\{\boldsymbol{c_t},\boldsymbol{b_t},\boldsymbol{A_t},\boldsymbol{T_t}\}}{E} [Q_t(x_{t-1}, \boldsymbol{c_t}, \boldsymbol{b_t}, \boldsymbol{A_t}, \boldsymbol{T_t})] \text{ for } t = 2, 3, \ldots, N,$$

$$Q_t(x_{t-1}, c_t, b_t, A_t, T_t) := \inf_{x_t \in \mathbb{R}^{n_t}} \left\{ c_t^\top x_t + \mathcal{Q}_{t+1}(x_t) : A_t x_t = b_t - T_t x_{t-1}, x_t \geq 0 \right\} \quad t = 2, 3, \ldots, N-1,$$

and

$$Q_N(x_{N-1}, c_N, b_N, A_N, T_N) := \inf_{x_N \in \mathbb{R}^{n_N}} \left\{ c_N^\top x_N : A_N x_N = b_N - T_N x_{N-1}, x_N \geq 0 \right\}. \tag{2}$$

Note that in this independent case the data for the MSSLP consist of:

- first stage deterministic data c_1, b_1, A_1, and
- the distribution of $(\boldsymbol{c_t}, \boldsymbol{b_t}, \boldsymbol{A_t}, \boldsymbol{T_t})$ for $t = 2, 3, \ldots, N$.

The most common data input standard for problems of type (1) and (2) is the SMPS [4] standard. This standard requires that one realization or scenario of the *entire* problem (i.e. all stages) be specified first. The other realizations in the scenario tree may then be described in several formats, including description of independent realizations for scalars or vectors, and description of branching scenarios.

Such flexibility of input format has advantages and disadvantages. While problems of type (1) or (2) may be easily described, it is dificult to write computer routines for reading such flexible input. In a companion effort, the authors will realease open source routines for reading SMPS data and placing the data into appropriate internal computer data structures.

2.1 Airlift operations scheduling

Due to J.L. Midler and R.D. Wollmer [16]
(2 stage, mixed integer linear stochastic problem)

`/airlift/AIRL.cor`, `/AIRL.tim`, $\begin{cases} \texttt{/AIRL.sto.first} \\ \texttt{/AIRL.sto.second} \end{cases}$

2.1.1 Description

In scheduling monthly airlift operations, demands for specific routes can be predicted. Actual requirements will be known in the future, and they may not agree with predicted requirements. Recourse actions are then required to meet the actual requirements. The actual requirements are expressed in tons, or any other appropriate measure, and they can be represented by a random variable. Aircraft of several different types are available for service. Each of these types of aircraft has its own restriction on number of flight hours available during the month.

The recourse actions available include allowing available flight time to go unused, switching aircraft from one route to another, and buying commercial flights. Each of these has its associated cost, depending on the type(s) of aircraft involved.

Let F_i be the maximum number of flight hours for aircraft of type i available during the month, and let a_{ij} be the number of flight hours required for an aircraft of type i to complete one flight of route j. Then if x_{ij} is the number of flights originally planned for route j using aircraft of type i, the first stage constraint is

$$\sum_j a_{ij} x_{ij} \leq F_i, \qquad \forall i. \tag{3}$$

When taking recourse action, we are under the constraint that we cannot switch away more flight hours from aircraft of type i and from route j, than we have originally scheduled for such. This leads to a second stage constraint:

$$\sum_{k \neq j} a_{ijk} x_{ijk} \leq a_{ij} x_{ij}, \qquad \forall i, \forall j. \tag{4}$$

Here, x_{ijk} represents the increase in the number of flights for route k flown by aircraft type i, because of being switched from route j. Also, a_{ijk} is the number of flight hours required for aircraft of type i to fly route k, after having been switched from route j. Note that an increase of x_{ijk} flights for route k results in the cancellation of

$$\left(\frac{a_{ijk}}{a_{ij}}\right) x_{ijk}$$

flights for route j, since 'k flights' and 'j flights' are not necessarily equal units.

We also have the recourse constraint that the demand for each route must be met. Let b_{ij} be the carrying capacity (in tons) of a single flight of an aircraft of type i, flying route j. Then the load originally scheduled to be carried on route j (i.e. the 'best guess' of the demand) is

$$\sum_i b_{ij} x_{ij}. \tag{5}$$

The total carrying capacity switched away from route j in the recourse action is

$$\sum_{i,k \neq j} b_{ij} \left(\frac{a_{ijk}}{a_{ij}}\right) x_{ijk}. \tag{6}$$

Conversely, the carrying capacity switched to route j is

$$\sum_{i,k \neq j} b_{ij} x_{ikj}. \tag{7}$$

If we let y_j^+ be the demand for route j which is contracted commercially in the recourse, and y_j^- be the unused capacity assigned to route j, then we may combine expressions (5), (6), and (7) to form the demand constraint for the recourse[2]:

$$\sum_i b_{ij} x_{ij} - \sum_{i,k \neq j} b_{ij} \left(\frac{a_{ijk}}{a_{ij}}\right) x_{ijk} + \sum_{i,k \neq j} b_{ij} x_{ikj} + y_j^+ - y_j^- = \boldsymbol{d_j}. \tag{8}$$

Here, $\boldsymbol{d_j}$ is the random variable representing the demand for route j.

Finally, let c_{ij} be the cost for aircraft type i to be initially assigned and fly one flight of route j. Let c_{ijk} be the cost for aircraft type i to fly one flight of route k, after having been initially assigned route j. Let c_j^+ be the cost per ton of commercially contracted transport on route j, and let c_j^- be the cost per ton of unused capacity on route j.

[2] We believe a typographical error was made in [16, equation (2.3)]. Specifically, '$\sum_{i,k \neq j} b_{ij} x_{ijk} - y_j^+$' should read '$\sum_{i,k \neq j} b_{ij} x_{ikj} + y_j^+$'.

2.1.2 Problem statement

The problem statement combines equations (3), (4), and (8):

$$\begin{aligned} \text{minimize} \quad & \sum_{i,j} c_{ij} x_{ij} + \mathcal{Q}(\{x_{ij}\}) \\ \text{subject to} \quad & \sum_{j} a_{ij} x_{ij} \le F_i, \qquad \forall i \\ & x_{ij} \ge 0, \qquad \forall i, \forall j, \end{aligned}$$

where

$$\mathcal{Q}(\{x_{ij}\}) := \underset{\boldsymbol{d_j}}{E} \left\{ \min \left[\sum_{i,j,k \ne j} \left(c_{ijk} - c_{ij} \frac{a_{ijk}}{a_{ij}} \right) x_{ijk} + \sum_{j} c_j^+ y_j^+ + \sum_{j} c_j^- y_j^- \right] \right\}$$

subject to

$$\begin{aligned} \sum_{k \ne j} a_{ijk} x_{ijk} \le a_{ij} x_{ij}, & \qquad \forall i, \forall j \\ - \sum_{i,k \ne j} b_{ij} \left(\frac{a_{ijk}}{a_{ij}} \right) x_{ijk} + \sum_{i,k \ne j} b_{ij} x_{ikj} + y_j^+ - y_j^- = \boldsymbol{d_j} - \sum_{i} b_{ij} x_{ij}, & \forall j \\ x_{ijk}, y_j^+, y_j^- \ge 0, & \qquad \forall i, \forall j, \forall k. \end{aligned}$$

Note that the variables x_{ij} and x_{ijk} represent numbers of flights, and therefore should be integer valued. This is not specified in the problem statement, however. This is apparently an acceptable compromise to Midler and Wollmer [16] in order to simplify the problem, and they recommend that the user adopt his/her own rounding scheme.

2.1.3 Numerical example

Midler and Wollmer [16] provide a small numerical example, with two routes and two types of aircraft. The constants are given as follows:

Flying hours per round trip			
a_{11}	a_{21}	a_{12}	a_{22}
24	49	14	29

Carrying capacity (tons)			
b_{11}	b_{21}	b_{12}	b_{22}
50	20	75	20

Cost per flight ($)			
c_{11}	c_{21}	c_{12}	c_{22}
7200	7200	6000	4000

Penalty costs ($/ton)			
c_1^+	c_2^+	c_1^-	c_2^-
500	250	0	0

Flying hours - switched flights			
a_{112}	a_{121}	a_{212}	a_{221}
19	29	36	56

Costs per flight-switched ($)			
c_{112}	c_{121}	c_{212}	c_{221}
7000	8200	5500	8700

The total flying time available is $F_1 = F_2 = 7200$. The demand for route 1, $\boldsymbol{d_1}$, follows a lognormal distribution with parameters $\mu_1 = 1000$, $\sigma_1 = 50$, and $\boldsymbol{d_2}$ independently follows a lognormal distribution with parameters $\mu_2 = 1500$, $\sigma_2 = 300$.

The optimal solution of this problem,

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 16.5 & 23.2 \\ 6.7 & 0.0 \end{bmatrix},$$

was given by Midler and Wollmer [16], and is based on drawing a sample of 25 observations from each distribution of $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$. Midler and Wollmer [16] did not specify how the observations were drawn from the distributions. Therefore, we were not able to exactly replicate this problem. We have created two versions: `airlift.first` and `airlift.second`. These are intended to be as close as possible to the original problem stated here.

2.1.4 Notational reconciliation

To make this problem fit the notation of Problem (2), we make some minimal changes. Let I be the total number of aircraft types, J be the total number of routes, and set $n_1 := (I)(J) + I$. Set

$$x_1 := \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1J} \\ x_{21} \\ \vdots \\ x_{IJ} \\ s_1 \\ s_2 \\ \vdots \\ s_I \end{bmatrix}, \qquad c_1 := \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1J} \\ c_{21} \\ \vdots \\ c_{IJ} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad b_1 := \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_I \end{bmatrix},$$

and

$$A_1 :=$$
$$\left[\begin{array}{ccccccccc|c} a_{11} & \cdots & a_{1J} & & & & & & & \\ & & & a_{21} & \cdots & a_{2J} & & 0 & & \\ & 0 & & & & & \ddots & & & I^{I\times I} \\ & & & & & & a_{I1} & \cdots & a_{IJ} & \end{array}\right].$$

Note that the number of stages, N, is two. The recourse vectors are

$$x_2 := \begin{bmatrix} x_{112} \\ x_{113} \\ \vdots \\ x_{11J} \\ x_{121} \\ x_{123} \\ \vdots \\ x_{12J} \\ \vdots \\ x_{IJ(J-1)} \\ s_{11} \\ s_{12} \\ \vdots \\ s_{IJ} \\ y_1^+ \\ \vdots \\ y_J^+ \\ y_1^- \\ \vdots \\ y_J^- \end{bmatrix}, \qquad c_2 := \begin{bmatrix} c_{112} - c_{11}a_{112}/a_{11} \\ c_{113} - c_{11}a_{113}/a_{11} \\ \vdots \\ c_{11J} - c_{11}a_{11J}/a_{11} \\ c_{121} - c_{12}a_{121}/a_{12} \\ c_{123} - c_{12}a_{123}/a_{12} \\ \vdots \\ c_{12J} - c_{12}a_{12J}/a_{12} \\ \vdots \\ c_{IJ(J-1)} - c_{IJ}a_{IJ(J-1)}/a_{IJ} \\ 0 \\ 0 \\ \vdots \\ 0 \\ c_1^+ \\ \vdots \\ c_J^+ \\ c_1^- \\ \vdots \\ c_J^- \end{bmatrix},$$

and

$$\boldsymbol{b_2} := \begin{bmatrix} 0^{IJ\times 1} \\ \boldsymbol{d_1} \\ \boldsymbol{d_2} \\ \vdots \\ \boldsymbol{d_J} \end{bmatrix}.$$

The transition matrix is

$$T_2 := \left[\begin{array}{cccc|c} -a_{11} & & & 0 & 0^{IJ\times I} \\ & -a_{12} & & & \\ & & \ddots & & \\ 0 & & & -a_{IJ} & \\ \hline B_1 & B_2 & \cdots & B_I & 0^{J\times I} \end{array}\right],$$

where

$$B_i := \begin{bmatrix} b_{i1} & & & 0 \\ & b_{i2} & & \\ & & \ddots & \\ 0 & & & b_{iJ} \end{bmatrix}.$$

The matrix A_2 is defined by

$$A_2 := \left[\begin{array}{c|c|cc} \hat{A} & I^{IJ\times IJ} & \multicolumn{2}{c}{0^{IJ\times 2J}} \\ \hline \hat{B} & 0^{J\times IJ} & I^{J\times J} & -I^{J\times J} \end{array}\right],$$

where

$$\hat{A} := \begin{bmatrix} \hat{a}_{11}^{\mathsf{T}} & & & \\ & \hat{a}_{12}^{\mathsf{T}} & & 0 \\ 0 & & \ddots & \\ & & & \hat{a}_{IJ}^{\mathsf{T}} \end{bmatrix},$$

$$\hat{B} := \begin{bmatrix} \hat{b}_{111}^{\mathsf{T}} & \hat{b}_{121}^{\mathsf{T}} & \cdots & \hat{b}_{IJ1}^{\mathsf{T}} \\ \hat{b}_{112}^{\mathsf{T}} & \hat{b}_{122}^{\mathsf{T}} & \cdots & \hat{b}_{IJ2}^{\mathsf{T}} \\ \vdots & \vdots & \vdots & \vdots \\ \hat{b}_{11J}^{\mathsf{T}} & \hat{b}_{12J}^{\mathsf{T}} & \cdots & \hat{b}_{IJJ}^{\mathsf{T}} \end{bmatrix},$$

$$\hat{a}_{ij} := \sum_{k=1}^{J} a_{ijk}\hat{e}_{kj}$$

and

$$\hat{b}_{ikj} := \begin{cases} \sum_{p=1}^{J} -b_{ik}\left(a_{ikp}/a_{ik}\right)\hat{e}_{pj} & \text{if } j = k, \\ b_{ij}\hat{e}_{jk} & \text{if } j \neq k. \end{cases}$$

Here,

$$\hat{e}_{jk} := \begin{cases} e_j \in \mathbb{R}^{J-1} & \text{if } j < k \\ e_{j-1} \in \mathbb{R}^{J-1} & \text{if } j > k \\ 0 \in \mathbb{R}^{J-1} & \text{if } j = k. \end{cases}$$

2.2 Forest planning

Due to H. Gassmann [11]
(Multistage, linear stochastic problem)
`/stocfor1` `/stocfor1.cor`, `/stocfor1.tim`, `/stocfor1.sto`
`/stocfor2` `/stocfor2.cor`, `/stocfor2.tim`, `/stocfor2.sto`
`/stocfor3` `/stocfor3.cor`, `/stocfor3.tim`, `/stocfor3.sto`

2.2.1 Description

The job of a long range forest planner is to decide what parts of the forest will be harvested when. Important criteria for such a decision are the age of the trees, and the likelihood that trees left standing will be destroyed by fire.

Gassmann [11] creates a set of K age classifications of equal length (e.g. 20 years), and places each portion of the forest into one of the classes, according to the age of the trees within. He also divides the future planning horizon into T rounds, each with a time length *equal* to that of each age classification. That is, in one time round, any trees that are not destroyed or harvested will move to the next age class.

Let the vector $s_t \in \mathbb{R}^K$ represent the total amount of area of the forest in each age class 1 through K in round t, and let $x_t \in \mathbb{R}^K$ be the area of the forest harvested in each age class in round t. Obviously, we cannot harvest more trees of any age than currently exist. Therefore,

$$x_t \leq s_t, \quad t = 1, 2, \ldots, T. \tag{9}$$

Immediately replanting harvested land will cause an area increase of Qx_t in the next round, where

$$Q = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The area of unharvested trees in round t will be $s_t - x_t$. Of this area, a random proportion $\boldsymbol{p_t} = (\boldsymbol{p_{t1}}, \boldsymbol{p_{t2}}, \ldots, \boldsymbol{p_{tK}})^\mathsf{T} \in \mathbb{R}^K$ will be destroyed by fire in round t. Let

$$\boldsymbol{P_t} = \begin{bmatrix} \boldsymbol{p_{t1}} & \boldsymbol{p_{t2}} & \cdots & \boldsymbol{p_{tK-1}} & \boldsymbol{p_{tK}} \\ 1-\boldsymbol{p_{t1}} & 0 & \cdots & 0 & 0 \\ 0 & 1-\boldsymbol{p_{t2}} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1-\boldsymbol{p_{tK-1}} & 1-\boldsymbol{p_{tK}} \end{bmatrix}.$$

Then, assuming all burned areas are immediately replanted, and therefore wind up in age class 1,

$$s_{t+1} = \boldsymbol{P_t}(s_t - x_t) + Qx_t. \tag{10}$$

The material balance in (10), along with the availability limits in (9), will be constraints in the problem.

The last type of constraint that will be in the problem is of the form

$$\alpha y^\mathsf{T} x_{t-1} \leq y^\mathsf{T} x_t \leq \beta y^\mathsf{T} x_{t-1}, \quad t = 2, 3, \ldots, T. \tag{11}$$

Here $y \in \mathbb{R}^K$ represents the yield, and α and β are constants. This constraint might represent limits on how fast the timber industry can change its purchasing volume from one time period to the next.

The objective will be to maximize the value of timber, both cut and remaining after round T, subject to the constraints (9), (10), and (11). Since the time scale of the problem is quite large, Gassmann discounts monetary values in future round t to current monetary scales by multiplying by δ^{t-1}. For example, if each round is 20 years long, for interest (or inflation) rate i, $\delta = (1-i)^{20}$. Therefore, the present value of timber harvested in round t is

$$\delta^{t-1} y^\mathsf{T} x_t.$$

If $v \in \mathbb{R}^K$ is the value of the trees standing after round T, then the total value of trees left standing after round T and cut during rounds 1 through T is

$$\sum_{t=1}^{T} \delta^{t-1} y^\mathsf{T} x_t + \delta^T v^\mathsf{T} s_{T+1}.$$

2.2.2 Problem statement

We are given the vector $s_1 \in \mathbb{R}^K$, denoting the area of forest covered with timber in the K different age classes at the beginning of time period 1. We are also given $y \in \mathbb{R}^K$ the vector of yields (in units currency/hectacre of forest harvested), $v \in \mathbb{R}^K$ the value of standing timber after round T, the discount rate δ, and constants α, β.

With such information, the problem is then to

$$\begin{array}{rl} \text{maximize} & y^\mathsf{T} x_1 + \mathcal{Q}_2(x_1) \\ \text{subject to} & x_1 \le s_1 \\ & x_1 \ge 0, \end{array} \tag{12}$$

where

$$\begin{aligned}
\mathcal{Q}_t(x_{t-1}) \quad &:= \quad \underset{\{\boldsymbol{P_{t-1}}, \boldsymbol{P_t}, \ldots, \boldsymbol{P_T}\}}{E} \left\{ \max \left[\delta^{t-1} y^\mathsf{T} x_t + \mathcal{Q}_{t+1}(x_t) \right] : \right. \\
x_t \quad &\le \quad s_t \qquad (13) \\
s_t \quad &= \quad (Q - \boldsymbol{P_{t-1}}) x_{t-1} + \boldsymbol{P_{t-1}} s_{t-1} \qquad (14) \\
\alpha y^\mathsf{T} x_{t-1} \quad &\le \quad \left. y^\mathsf{T} x_t \le \beta y^\mathsf{T} x_{t-1} \right\}, \quad t = 2, \ldots, T, \qquad (15)
\end{aligned}$$

and

$$\mathcal{Q}_{T+1}(x_T) \quad := \quad \left\{ \delta^T v^\mathsf{T} s_{T+1} : s_{T+1} = (Q - \boldsymbol{P_T}) x_T + P_T s_T \right\}. \tag{16}$$

Equations (12), (13) and (16) have been changed slightly from the problem statement in [11], in order to more closely match the content of the example problems submitted to Netlib by Gassmann [10].

2.2.3 Numerical results

Gassmann [11] reported numerical results for many cases. In all cases, he assigned the values shown in Table 1. For the distribution of $\boldsymbol{P}$, Gassmann [11] used several different discretizations. The few included in Table 2 are called "upper bound discretizations" by Gassmann. In the first set of trials, Gassmann [11] found the constraints to be too severe. Therefore, he changed the problem as follows. Violations to the constraints

$$\alpha y^\mathsf{T} x_{t-1} \le y^\mathsf{T} x_t \le \beta y^\mathsf{T} x_{t-1}$$

were allowed, but penalized. These constraints were replaced with

$$\begin{aligned} \alpha y^\mathsf{T} x_{t-1} - y^\mathsf{T} x_t \quad &\le \quad p_{t1} \\ y^\mathsf{T} x_t - \beta y^\mathsf{T} x_{t-1} \quad &\le \quad p_{t2} \\ p_{t1}, p_{t2} \quad &\ge \quad 0, \end{aligned}$$

Table 1: Values of parameters used in Gassmann [11]

$T = 7$	$K = 8$	$\delta = 0.905$
$\alpha = 0.9$	$\beta = 1.1$	
$v = \begin{bmatrix} 320.3417 \\ 356.1874 \\ 398.4370 \\ 448.2349 \\ 506.9294 \\ 564.9294 \\ 587.9294 \\ 595.9294 \end{bmatrix}$	$s_1 = \begin{bmatrix} 241 \\ 125 \\ 1,404 \\ 2,004 \\ 9,768 \\ 16,385 \\ 2,815 \\ 61,995 \end{bmatrix}$	$y = \begin{bmatrix} 0 \\ 0 \\ 16 \\ 107 \\ 217 \\ 275 \\ 298 \\ 306 \end{bmatrix}$

Table 2: Discretizations used in Gassmann [11]

1 point discretization			
Fire Rate	0.06258		
Probability	1.000		
2 point discretization			
Fire Rate	0.08612	0.04240	
Probability	0.4616	0.5384	
3 point discretization			
Fire Rate	0.10499	0.07354	0.04240
Probability	0.1847	0.2769	0.5384

and the term

$$\sum_{t=1}^{T-1} -\delta^{t-1}\gamma(p_{t1} + p_{t2})$$

was added to the objective. In all the numerical results, $\gamma := 50$.

Results from Gassmann [11] are shown in Table 3. Here, a discretization structure of $i.jjj.kkk$ means that an i point discretization was used in the first round, a j point discretization was used in rounds two through four, and a k point discretization was used in rounds five through seven. The only nonzero component of the optimal x_1 was component 8.

Problem statements in MPS format may be found at Netlib, at `http://www.netlib.org/lp/data/` under the names stocfor1, stocfor2, and stocfor3

Table 3: Results from Gassmann [11]

	Discretization Structure					
	1.111.111	1.222.222	1.322.222	1.332.222	1.333.222	1.333.322
Obj. value	41,132.0	40,914.3	40,897.0	40,864.2	40,835.8	40,703.1
Opt. $x_1(8)$	20,495.8	20,047.9	20,076.9	19,952.8	19,947.4	19,726.6

[10].

2.2.4 Notational reconciliation

To express the problem in the notation of Problem (2), we define the slack variable $z_t := s_t - x_t$, which allows us to eliminate the variable s_t for $t > 1$. The vectors c_t and x_t are defined and *redefined*, respectively, as

$$c_t := \begin{bmatrix} -\delta^{t-1} y \\ 0^{K\times 1} \\ 0 \\ 0 \end{bmatrix}, \qquad x_t := \begin{bmatrix} x_t \\ z_t \\ l_t \\ m_t \end{bmatrix},$$

where $l_t, m_t \in \mathbb{R}$ are slack variables. This definition for c_t is not valid for $t = T$, when we have

$$\boldsymbol{c_T} := \begin{bmatrix} -\delta^{T-1}(y + \delta Q^{\mathsf{T}} v) \\ -\delta^T \boldsymbol{P_T}^{\mathsf{T}} v \\ 0 \\ 0 \end{bmatrix}$$

Let

$$A_1 := \begin{bmatrix} I^{K\times K} & I^{K\times K} & 0 & 0 \end{bmatrix}$$

and $b_1 := s_1$. Then for $t = 2, 3, \ldots, T$, we define

$$A_t := \begin{bmatrix} I^{K\times K} & I^{K\times K} & 0 & 0 \\ y^{\mathsf{T}} & 0 & 1 & 0 \\ -y^{\mathsf{T}} & 0 & 0 & 1 \end{bmatrix}, \qquad b_t := \begin{bmatrix} 0^{K\times 1} \\ 0 \\ 0 \end{bmatrix},$$

and

$$\boldsymbol{T_t} := \begin{bmatrix} -Q & -\boldsymbol{P_{t-1}} & 0 & 0 \\ -\beta y^{\mathsf{T}} & 0 & 0 & 0 \\ \alpha y^{\mathsf{T}} & 0 & 0 & 0 \end{bmatrix}.$$

Finally, setting $N := T$, we have expressed the problem in the format of (2).

2.3 Electrical investment planning

Due to Louveaux and Smeers [15]
(Two-stage, linear stochastic problem)
`/electric` `/LandS.cor`, `/LandS.tim`, `/LandS.sto`

2.3.1 Description

Louveaux and Smeers [15] consider the challenge of planning investments in the electricity generation industry. While the model is, in general, multi-stage, the specific example given in [15] is two-stage. The general N-stage stochastic model will be developed in this section and the next, with the specific example in the following section.

In each stage of planning, investments in n different technologies may be considered. Technology i has an associated random investment cost, $\boldsymbol{c_i}$, a random operating cost, $\boldsymbol{q_i}$, and an availability factor, a_i. The availability factor is the portion of time during which the technology may be operated.

For planned capacity, a distinction is made between capacity which was planned *before* time $t = 1$, and that which was planned *after* $t = 0$. (Here, t is an integer.) The former, g_i, includes capacity which exists on the ground at the start of the simulation, and new capacity that has already been planned. The latter is denoted by x_i. If we let s_i be the total capacity, both actual and on order, planned after $t = 0$, then we have,

$$s_i^1 = x_i^1$$

and

$$s_i^t = s_i^{t-1} + x_i^t - x_i^{t-L_i}, \quad i = 1, \ldots, n, \quad t = 2, \ldots, N.$$

Capacity in technology i also has a construction delay, Δ_i, and a finite lifetime, L_i, from planning to retirement. The total capacity for technology i at time t is then $(g_i^t + s_i^{t-\Delta_i})$.

Demand for electricity may come in k different modes, and the realization of the random demand in each mode must be met at each time stage. Therefore, if we let y_{ij} be the production of electricity in mode j from technology i and let $\boldsymbol{d_j}$ be the random demand variable for mode j electricity, we require

$$\sum_{i=1}^{n} y_{ij}^t = \boldsymbol{d_j^t}, \quad j = 1, \ldots, k, \quad t = 1, \ldots, N.$$

A production balance yields

$$\sum_{j=1}^{k} y_{ij}^t \leq a_i(g_i^t + s_i^{t-\Delta_i}), \quad i = 1, \ldots, n.$$

With the constraints listed so far, the problem does not have relatively complete recourse. To give it such, Louveaux and Smeers [15] add an additional constraint[3]. They assume there is a technology, which is always called technology n, which can always be called upon to meet demand in an immediate way. Therefore Δ_n is always zero. Typically, the investment cost for technology n is high. To simulate purchased electricity, one may simply let the lifetime $L_n = 1$. The added constraint is

$$a_n(g_n^t + s_n^t) \geq \sum_{j=1}^{k} \boldsymbol{d_j^t} - \sum_{1}^{n-1} a_i \left(g_i^t + s_i^{t-\Delta_i} \right), \quad t = 1, \ldots, N. \tag{17}$$

The objective is to minimize the expected value of the future cost, as represented by the operating and investment costs. The random variable is made up of the demands $(\boldsymbol{d_1}, \ldots, \boldsymbol{d_n})$, and the costs $(\boldsymbol{c_1}, \ldots, \boldsymbol{c_n})$ and $(\boldsymbol{q_1}, \ldots, \boldsymbol{q_n})$.

2.3.2 Problem statement

We have the following definitions:

n = number of available technologies (index i)
k = number of modes of electricity demand (index j)
N = number of time stages (index t)
g_i^t = capacity of i to exist at time t, decided upon before $t = 1$
x_i^t = new capacity of i, decided at time $t > 0$
s_i^t = total capacity, both actual and on order, planned after $t = 0$
$\boldsymbol{c_i^t}$ = unit investment cost of i at time t
$\boldsymbol{q_i^t}$ = unit production cost of i at time t
a_i = availability factor for i
L_i = life of i, from planning to retirement
Δ_i = construction time for i

[3]Constraint (17) is not in exactly the same form as in [15]. We have changed it, so that x_i^t may reflect electricity purchased from the so-called grid.

$\boldsymbol{d_j^t}$ = electricity demand in mode j at time t
y_{ij}^t = production rate from i for mode j at time t
T_j^t = duration of mode j at time t
$\boldsymbol{\xi^t}$ = random variable whose elements are $\{\boldsymbol{d_j^t}, \boldsymbol{c_i^t}, \boldsymbol{q_i^t}, \forall i, j, t\}$.

We are given all elements of g, T, a, L, and Δ, with $\Delta_n = 0$. The problem[4] is to choose s, x, and y to

$$
\begin{aligned}
\text{minimize} \quad & \mathop{E}_{\boldsymbol{\xi}} \left[\sum_{t=1}^{N} \sum_{i=1}^{n} \left(\boldsymbol{c_i^t} x_i^t + \sum_{j=1}^{k} \boldsymbol{q_i^t} T_j^t y_{ij}^t \right) \right] \\
\text{subject to} \quad & s_i^1 = x_i^1 \\
& s_i^t = s_i^{t-1} + x_i^t - x_i^{t-L_i}, \quad i = 1, \ldots, n, \quad t = 2, \ldots, N \\
& \sum_{i=1}^{n} y_{ij}^t = \boldsymbol{d_j^t}, \quad j = 1, \ldots, k, \quad t = 1, \ldots, N \\
& \sum_{j=1}^{k} y_{ij}^t \leq a_i (g_i^t + s_i^{t-\Delta_i}), \quad i = 1, \ldots, n \\
& a_n (g_n^t + s_n^t) \geq \sum_{j=1}^{k} \boldsymbol{d_j^t} - \sum_{i=1}^{n-1} a_i \left(g_i^t + s_i^{t-\Delta_i} \right), \quad t = 1, \ldots, N \\
& s_i^t, x_i^t, y_{ij}^t \geq 0, \quad i = 1, \ldots, n, \quad t = 1, \ldots, N, \quad j = 1, \ldots, k.
\end{aligned}
\tag{18}
$$

2.3.3 Numerical results

Louveaux and Smeers [15] present a two-stage example, with $k = 3$ operating modes and $n = 4$ available technologies. Their example differs from their general problem development in several ways. There is no immediate source of emergency electricity, as Δ_i is set to 1 for *all* i. Additionally, there is a budget constraint of 120 in stage 1. Also, c and q are not stochastic. All of the parameters are shown in Table 4.

Note that $x^2 = 0$ and $y^1 = 0$, and that $s^t = x^t$. Here $\boldsymbol{\xi} = (3, 5, 7)$ with probabilities $(0.3, 0.4, 0.3)$, respectively. Also note that with all technologies having a construction delay of 1, x^2 should be zero. If the world is ending,

[4] The term x_i^t in the objective function in (18) is written as s_i^t in [15]. We believe this to be a typographical error in [15].

there's no reason to build a power plant. To force this condition, c^2 may be chosen to be any positive vector. Therefore, the problem may be reduced to

$$
\begin{array}{ll}
\text{minimize} & \\
& 10x_1 + 7x_2 + 16x_3 + 6x_4 + \underset{\boldsymbol{\xi}}{E}\ [40y_{11} + 45y_{21} + 32y_{31} + 55y_{41} \\
& \qquad + 24y_{12} + 27y_{22} + 19.2y_{32} + 33y_{42} \\
& \qquad + 4y_{13} + 4.5y_{23} + 3.2y_{33} + 5.5y_{43}] \\
\text{subject to} & \sum_{i=1}^4 y_{i1} = \boldsymbol{\xi} \qquad\qquad \sum_{j=1}^3 y_{1j} \le x_1 \\
& \sum_{i=1}^4 y_{i2} = 3 \qquad\qquad \sum_{j=1}^3 y_{2j} \le x_2 \\
& \sum_{i=1}^4 y_{i3} = 2 \qquad\qquad \sum_{j=1}^3 y_{3j} \le x_3 \\
& \sum_{i=1}^4 x_i \ge 12 \qquad\qquad \sum_{j=1}^3 y_{4j} \le x_4 \\
& 10x_1 + 7x_2 + 16x_3 + 6x_4 \le 120 \\
& x, y \ge 0.
\end{array}
$$

Louveaux and Smeers [15] report the optimal solution to be

$$x = \left[\frac{8}{3}, 4, \frac{10}{3}, 2\right]^{\mathsf{T}}$$

with an objective value of 381.853.

2.3.4 Notational reconciliation

We make several changes to problem (18) to facilitate its transition into the format of problem (2). We specify c_i^1, q_i^1, and d_j^1 to be deterministic. Further, we force L_i to be a number larger than N for $i \ne n$, and $L_n := 1$. Also, let

$$\delta_i := \begin{cases} 1 & \text{if } i \ne n \\ 0 & \text{if } i = n \end{cases}.$$

Table 4: Parameters for the example from Louveaux and Smeers [15]

n	4
k	k
N	2
g^1 g^2	$[0,0,0,0]^\mathsf{T}$ $[0,0,0,0]^\mathsf{T}$
c^1 c^2	$[10,7,16,6]^\mathsf{T}$ $[1,1,1,1]^\mathsf{T}$
q^1 q^2	$[0,0,0,0]^\mathsf{T}$ $[4,4.5,3.2,5.5]^\mathsf{T}$
a	$[1,1,1,1]^\mathsf{T}$
L	$[2,2,2,2]^\mathsf{T}$
Δ	$[1,1,1,1]^\mathsf{T}$
d^1 $\boldsymbol{d^2}$	$[0,0,0]^\mathsf{T}$ $[\boldsymbol{\xi},3,2]^\mathsf{T}$
T^1 T^2	$[1,1,1]^\mathsf{T}$ $[10,6,1]^\mathsf{T}$

With these restrictions, we let

$$x_t := \begin{bmatrix} x_1^t \\ \vdots \\ x_n^t \\ s_1^t \\ \vdots \\ s_n^t \\ y_{11}^t \\ y_{12}^t \\ \vdots \\ y_{nk}^t \\ z_1^t \\ \vdots \\ z_n^t \\ w^t \end{bmatrix}, \qquad \boldsymbol{c_t} := \begin{bmatrix} \boldsymbol{c_1^t} \\ \vdots \\ \boldsymbol{c_n^t} \\ 0 \\ \vdots \\ 0 \\ \boldsymbol{q_1^t} T_1^t \\ \boldsymbol{q_1^t} T_2^t \\ \vdots \\ \boldsymbol{q_n^t} T_k^t \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

where z_i^t and w^t are slack variables. For $t = 1, 2, \ldots, N$, let

$$A_t := \begin{bmatrix} -I^{n\times n} & I^{n\times n} & 0^{n\times(nk)} & 0^{n\times n} & 0^{n\times 1} \\ 0^{k\times n} & 0^{k\times n} & \tilde{A} & 0^{k\times n} & 0^{k\times 1} \\ 0^{n\times n} & \left[\begin{array}{c} 0^{(n-1)\times n} \\ \hline -a_n e_n^\top \end{array}\right] & \hat{A} & I^{n\times n} & 0^{n\times 1} \\ 0 & a_n e_n^\top & 0^{1\times(nk)} & 0^{1\times n} & -1 \end{bmatrix},$$

where

$$\tilde{A} := \begin{bmatrix} I^{k\times k} & I^{k\times k} & \ldots & I^{k\times k} \end{bmatrix} \in \mathbb{R}^{k\times(nk)},$$

and

$$\hat{A} := \begin{bmatrix} 1^{1\times n} & 0^{1\times n} & \cdots & 0^{1\times n} \\ 0^{1\times n} & 1^{1\times n} & & 0^{1\times n} \\ \vdots & & \ddots & \vdots \\ 0^{1\times n} & 0^{1\times n} & \cdots & 1^{1\times n} \end{bmatrix}.$$

Additionally, for $t = 2, 3, \ldots, N$, let

$$T_t := \begin{bmatrix} \left[\begin{array}{c} 0^{(n-1)\times n} \\ \hline e_n^\top \end{array}\right] & -I^{n\times n} & 0^{n\times(nk)} & 0^{n\times n} & 0 \\ 0^{k\times n} & 0^{k\times n} & 0^{k\times(nk)} & 0^{k\times n} & 0 \\ 0^{n\times n} & \tilde{T} & 0^{n\times(nk)} & 0^{n\times n} & 0 \\ 0^{1\times n} & \bar{T} & 0^{1\times(nk)} & 0^{1\times n} & 0 \end{bmatrix},$$

where

$$\tilde{T} := \begin{bmatrix} -a_1 & & & & 0 \\ & -a_2 & & & \\ & & \ddots & & \\ & & & -a_{n-1} & \\ 0 & & & & 0 \end{bmatrix},$$

and

$$\bar{T} := \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 0 \end{bmatrix}.$$

Finally, letting

$$\boldsymbol{b_t} := \begin{bmatrix} 0^{n\times 1} \\ \boldsymbol{d_1^t} \\ \vdots \\ \boldsymbol{d_k^t} \\ a_1 g_1^t \\ \vdots \\ a_n g_n^t \\ \sum_{j=1}^k \boldsymbol{d_j^t} - \sum_{i=1}^n a_i g_i^t \end{bmatrix},$$

we have put the problem into the format of (2).

2.4 Selecting currency options

Due to Klaassen, Shapiro, and Spitz [13]
(Multistage, non-stair step, linear stochastic problem)

2.4.1 Description

The situation described by Klaassen, Shapiro, and Spitz [13] involves a U. S. multi-national corporation (MNC), which has significant forecasted revenues in a foreign currency (FC). If the *exchange rate*, $\boldsymbol{S}$ ($US/FC), goes down, the MNC would face declining revenue versus the forecast. To protect, or hedge, against this undesirable possibility, the MNC may choose to purchase *options* which guarantee a certain exchange rate at some point in the future. The guaranteed exchange rate is called the *strike price*, E.

The current time is given the value $t = 0$, and the time at which the forecasted revenue will be realized is $t = T$. The amount of said revenue is assumed to be known with certainty, and is scaled to be 1 FC. At times $t = 0, 1, \ldots, T-1$, the MNC may decide to purchase any of the available options. These options mature at time $t = T$. There are a total of J specific options packages available for purchase, each with a different strike price.

Of course, the exchange rates for $t = 1, 2, \ldots, T$ are unknown at time $t = 0$, but a suitable probability distribution can be constructed. We enumerate the possible exchange rate values at each time t as $S_t^1, S_t^2, \ldots, S_t^{N_t}$, for $t = 0, 1, \ldots, T$. Then the set of scenarios

$$\mathcal{S} := \\ \{\text{sequences } s = (S_0, S_1, \ldots, S_T) : 1 \le S_t \le N_t, \quad \forall t = 1, 2, \ldots, T\},$$

is the set of all possible realizations of a random variable $\boldsymbol{s} = (\boldsymbol{S_0}, \boldsymbol{S_1}, \ldots, \boldsymbol{S_T})$. Thus, the cardinality of $\mathcal{S}$ is $\prod_{t=0}^{T} N_t$. Each realization scenario s specifies the exchange rate at *each* time step, and has an associated probability, ρ_s. For scenario $s \in \mathcal{S}$, we denote by s_t the partial realization $(S_0, S_1, \ldots, S_t)$.

The decision to purchase options at any time t will depend on the current exchange rate, and on the type and quantity of options previously purchased. This, in turn, depends on historical exchange rates. Therefore, the decision variable $X_{s_t j}$ is the amount of option j purchased at time t, based on the partial realization $s_t = (S_0, S_1 \ldots, S_t)$. This purchase costs $P_{s_t j}$ per scaled unit of currency.

Because decisions to purchase options are only available through time $T-1$, there is no decision to be made at time T. This allows the constraints associated with time T to be rolled into time $T-1$.

For each exchange rate scenario $s \in \mathcal{S}$, the MNC must specify an acceptable *effective* exchange rate, Q_s, which would include effects of options purchased as well as the actual terminal exchange rate S_T. This leads to the constraint

$$\boldsymbol{S_T} + \sum_{t=0}^{T-1} \sum_{j=1}^{J} X_{s_t j} \left[\max\{E_j - \boldsymbol{S_T}, 0\} - \left(1 + i_{s_t}^{US}\right)^{T-t} \boldsymbol{P}_{s_t j} \right] \geq \boldsymbol{Q}_s, \quad \forall s \in \mathcal{S}, \tag{19}$$

where $i_{s_t}^{US}$ is the U.S. interest rate at time t for partial scenario s_t. The left side of inequality (19) includes the payoff from options which are active at S_T as well as the discounted cost of all options purchased. Since $\boldsymbol{Q}_s$ is a function of random variable $\boldsymbol{S}_T$, it is random as well.

Note that the $\boldsymbol{S_T}$ in (19) implies that these constraints are in the time stage associated with time T. However, there are no time T decision variables, so we can write these constraints in the time $T-1$ stage. At that stage, $\boldsymbol{S_T}$ is "still" random, as is $\boldsymbol{P}_{s_T j}$. So, the only stage in which constraint (19) occurs is the stage associated with time $T-1$, and moreover, there are N_T of these constraints.

A further restriction ensures that the MNC does not venture into the realm of foreign currency speculation. That is, the MNC should only be able to purchase options to cover a maximum of 100% of the forecasted revenues. This gives the constraint

$$\sum_{t=0}^{T-1} \sum_{j=1}^{J} X_{s_t j} \leq 1 \quad \forall s \in \mathcal{S}. \tag{20}$$

The objective of this exercise is to minimize the expected cost of all options purchased:

$$\sum_{j=1}^{J} X_{s_0 j} P_{s_0 j} + \underset{\boldsymbol{s}}{E} \left[\sum_{t=1}^{T-1} \left(\frac{1}{1 + i_{s_t}^{US}} \right)^t \sum_{j=1}^{J} X_{s_t j} \boldsymbol{P}_{s_t j} \right].$$

The parameters $i_{s_t}^{US}$ must be estimated, and for each realization S_t of $\boldsymbol{S_t}$, the coefficient $P_{s_t j}$ is calculated by the following formula, given in [13]:

$$P_{s_t j} := (1 - c_1) \left[\frac{e^{-\left[i_{s_t}^{US}(T-t)\right]}}{E_j} \right] - (1 - c_2) \left[\frac{e^{-\left[i_{s_t}^{FC}(T-t)\right]}}{S_t} \right], \tag{21}$$

where

$$c_1 := N\left\{\frac{\ln(E_j/S_t) + \left[i_{s_t}^{US} - i_{s_t}^{FC} - (T-t)\left(\frac{V_t^2}{2}\right)\right]}{V_t\sqrt{T-t}}\right\},$$

$$c_2 := N\left\{\frac{\ln(E_j/S_t) + \left[i_{s_t}^{US} - i_{s_t}^{FC} + (T-t)\left(\frac{V_t^2}{2}\right)\right]}{V_t\sqrt{T-t}}\right\},$$

and V_t is the volatility of the exchange rate, as measured by the instantaneous standard deviation of the spot rate as a percentage of the current spot rate. Here $N\{x\}$ is the cumulative standard normal distribution function. The foreign interest rate, $i_{s_t}^{FC}$ is calculated by

$$i_{s_t}^{FC} := \frac{S_t\left(1 + i_{s_t}^{US}\right)}{E\ \ [\boldsymbol{S_{t+1}}|\boldsymbol{S_t}]} - 1,$$

where the term in the denominator is a conditional expected value.

2.4.2 Problem statement

Given all elements of i^{US}, V, and E, and given a discrete probability distribution for $\boldsymbol{s}$ and corresponding minimum effective exchange rates Q_s, we calculate the value of each P_{s_tj} from (21). Then the problem is to minimize

$$\sum_{j=1}^{J} X_{s_0j}P_{s_0j} + \underset{\boldsymbol{s}}{E}\left[\sum_{t=1}^{T-1}\left(\frac{1}{1+i_{s_t}^{US}}\right)^t \sum_{j=1}^{J} X_{s_tj}\boldsymbol{P}_{s_tj}\right]$$

subject to

$$\sum_{t=0}^{T-1}\sum_{j=1}^{J} X_{s_tj} \le 1 \quad \forall s \in \mathcal{S}$$

$$\boldsymbol{S_T} + \sum_{t=0}^{T-1}\sum_{j=1}^{J} X_{s_tj}\left[\max\{E_j - \boldsymbol{S_T}, 0\} - \left(1 + i_{s_t}^{US}\right)^{T-t}\boldsymbol{P}_{s_tj}\right] \ge \boldsymbol{Q}_s, \quad \forall s \in \mathcal{S}$$

$$X_{s_tj} \ge 0 \quad \forall j = 1, \ldots, J; t = 0, 1, \ldots, T; s \in \mathcal{S}.$$

2.4.3 Numerical results

Klaassen, Shapiro and Spitz [13] present a four stage ($T = 4$) example, with $i^{US} = 0.10$ and $V = 0.11$ for all time periods and scenarios. At time stages $0, 1, 2$, and 3, ten different options are available, with strike prices

$$(E_1, E_2, \ldots, E_{10}) = (0.44, 0.50, 0.57, 0.63, 0.70, 0.76, 0.83, 0.89, 0.96, 1.02).$$

The scenario tree for the exchange rate $\boldsymbol{S_t}$ is given in Figure 1. Each branch of the tree has equal probability. Therefore we have the following probabilities:

$$\begin{aligned}
\rho_{s_0} &= 1 \\
\rho_{s_1} &= 1/3, \quad \forall s_1, \\
\rho_{s_2} &= 1/9, \quad \forall s_2, \\
\rho_{s_3} &= 1/27, \quad \forall s_3, \\
\rho_{s_4} &= 1/81, \quad \forall s_4.
\end{aligned}$$

The minimum acceptable effective exchange rates, Q_s are shown in Table 5, for each complete scenario. Results are given in Table 6. Results for different values of $\boldsymbol{Q}_s$, i^{US} and V are also given in [13].

Table 5: Minimum acceptable effective exchange rates

scenario ($t = 4$)	Target Exchange Rates Q_s
1	0.407
$2 - 5$	0.416
$6 - 15$	0.423
$16 - 31$	0.429
$32 - 50$	0.444
$51 - 66$	0.466
$67 - 76$	0.494
$77 - 80$	0.527
81	0.564

2.4.4 Notational reconciliation

This problem does not fit into form (2), because the last stage (that associated with time $T-1$) contains constraints of the form (19). These constraints

Table 6: Results for Klaassen, Shapiro, and Spitz [13]
All nonzero option purchases, X_{s_tj} are shown. Optimal objective value: 0.1057

		Option strike prices									
Year	Scenario	0.44	0.50	0.57	0.63	0.70	0.76	0.83	0.89	0.96	1.02
2	1									0.03	0.47
	2										0.04
3	1								0.09		
	3									0.49	
4	3				0.0004	0.09					
	4								0.06		
	7					0.21					
	9					0.31	0.25				
	10						0.56		0.03		

contain all the decision variables X_{s_tj}, not just $X_{s_{T-2}j}$. Therefore, we need "T-type" matrices to connect not only time $T-1$ to time $T-2$, but also time $T-1$ to each time $t < T-1$. We denote such matrices T_{Tt}.

In fact, the speculation constraint (20) also contains all the decision variables. However, with a trick, we can make these constraints fit into the stair step form of (2). We create a new variable X_{zt} and introduce the constraints

$$X_{z0} = \sum_{j=1}^{J} X_{s_0j},$$

and

$$X_{zt} = X_{z(t-1)} + \sum_{j=1}^{J} X_{s_tj}, \qquad t = 1, 2, \ldots, T-2.$$

Constraint (20) then may be written

$$X_{z(T-2)} + \sum_{j=1}^{J} X_{s_{T-1}j} \leq 1. \tag{22}$$

With these definitions, we can define, for $t = 0, 1, \ldots, T-2$,

$$x_{t+1} := \begin{bmatrix} X_{s_t1} \\ \vdots \\ X_{s_tJ} \\ X_{zt} \end{bmatrix}, \quad \boldsymbol{c}_{t+1} := \left[1/\left(1+i_{s_t}^{US}\right)\right]^t \begin{bmatrix} \boldsymbol{P}_{s_t1} \\ \vdots \\ \boldsymbol{P}_{s_tJ} \\ 0 \end{bmatrix},$$

for $t = 0, 1, \ldots, T-2$,

$$A_{t+1} := \begin{bmatrix} -1 & \cdots & -1 & 1 \end{bmatrix}, \qquad\qquad b_{t+1} := 0 \in \mathbb{R},$$

and for $t = 2, 3 \ldots, T-1$,

$$T_{t(t-1)} := \begin{bmatrix} 0 & \cdots & 0 & -1 \end{bmatrix} \in \mathbb{R}^{1\times(J+1)}.$$

In the last stage, that is the stage associated with time $T-1$, we will use $X_{z(T-1)}$ as a slack variable in (22).

In addition, we will need N_T surplus variables y_k, for $k = 1, 2, \ldots, N_T$. That is because, in the final stage we also have the N_T constraints (19). For $k = 1, 2, \ldots, N_T$, define

$$\begin{aligned} \gamma_{s_t jk} := \max\{E_j - \boldsymbol{S}_{Tk}, 0\} - \left(1 + i_{s_t}^{US}\right)^{T-t} \boldsymbol{P}_{s_t j}, \quad & t = 0, \ldots, T-1, \\ & j = 1, \ldots, J, \end{aligned}$$

where $\boldsymbol{S}_{Tk}$ is realization k of $\boldsymbol{S}_T$, given a partial realization s_{T-1}. Note that $\boldsymbol{S}_{Tk}$ is random (for $t < T-1$), because it is dependent on $\boldsymbol{s}_{T-1}$.

Let

$$\boldsymbol{T}_{T(t+1)} := \begin{bmatrix} 0 & 0 & \cdots & 0 & r(t) \\ \gamma_{s_t 11} & \gamma_{s_t 21} & \cdots & \gamma_{s_t J1} & 0 \\ \gamma_{s_t 12} & \gamma_{s_t 22} & \cdots & \gamma_{s_t J2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \gamma_{s_t 1N_T} & \gamma_{s_t 2N_T} & \cdots & \gamma_{s_t JN_T} & 0 \end{bmatrix},$$

for $t = 0, 1, \ldots, T-2$, where

$$r(t) := \begin{cases} 1 & t = T-2 \\ 0 & \text{otherwise.} \end{cases}$$

The matrix $\boldsymbol{A}_T$ is defined by

$$\boldsymbol{A}_T := \left[\begin{array}{ccccc|ccc} 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ \gamma_{s_{T-1}11} & \gamma_{s_{T-1}21} & \cdots & \gamma_{s_{T-1}J1} & 0 & & & \\ \gamma_{s_{T-1}12} & \gamma_{s_{T-1}22} & \cdots & \gamma_{s_{T-1}J2} & 0 & & -I^{J\times J} & \\ \vdots & \vdots & & \vdots & \vdots & & & \\ \gamma_{s_{T-1}1N_T} & \gamma_{s_{T-1}2N_T} & \cdots & \gamma_{s_{T-1}JN_T} & 0 & & & \end{array}\right],$$

and the right hand side for this stage is

$$\boldsymbol{b}_T := \begin{bmatrix} 1 \\ \boldsymbol{Q}_{s1} - \boldsymbol{S}_{T1} \\ \boldsymbol{Q}_{s2} - \boldsymbol{S}_{T2} \\ \vdots \\ \boldsymbol{Q}_{sN_T} - \boldsymbol{S}_{TN_T} \end{bmatrix}.$$

The decision and cost vectors for the final stage are

$$x_T := \begin{bmatrix} X_{s_{T-1}1} \\ \vdots \\ X_{s_{T-1}J} \\ X_{zT} \\ y_1 \\ y_2 \\ \vdots \\ y_J \end{bmatrix}, \quad \boldsymbol{c}_T := \left[1/\left(1+i^{US}_{s_{T-1}}\right)\right]^{T-1} \begin{bmatrix} \boldsymbol{P}_{s_{T-1}1} \\ \vdots \\ \boldsymbol{P}_{s_{T-1}J} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

With all of the definitions above, we have transformed the problem into a familiar format. It is a non-stair step version of (MSSLP),

$$\begin{array}{rll} \text{minimize} & Z(x_1) := c_1^\top x_1 + \mathcal{Q}_2(x_1) & \\ \text{subject to} & A_1 x_1 & = b_1 \\ & x_1 & \geq 0, \quad x_1 \in \mathbb{R}^{n_1}, \end{array}$$

where

$$\mathcal{Q}_t(x_{t-1}) := \underset{\{\boldsymbol{c_t}, \boldsymbol{b_t}, \boldsymbol{A_t}, \boldsymbol{T_t}\}}{E} \left[Q_t(x_{t-1}, \boldsymbol{c_t}, \boldsymbol{b_t}, \boldsymbol{A_t}, \boldsymbol{T_t})\right],$$

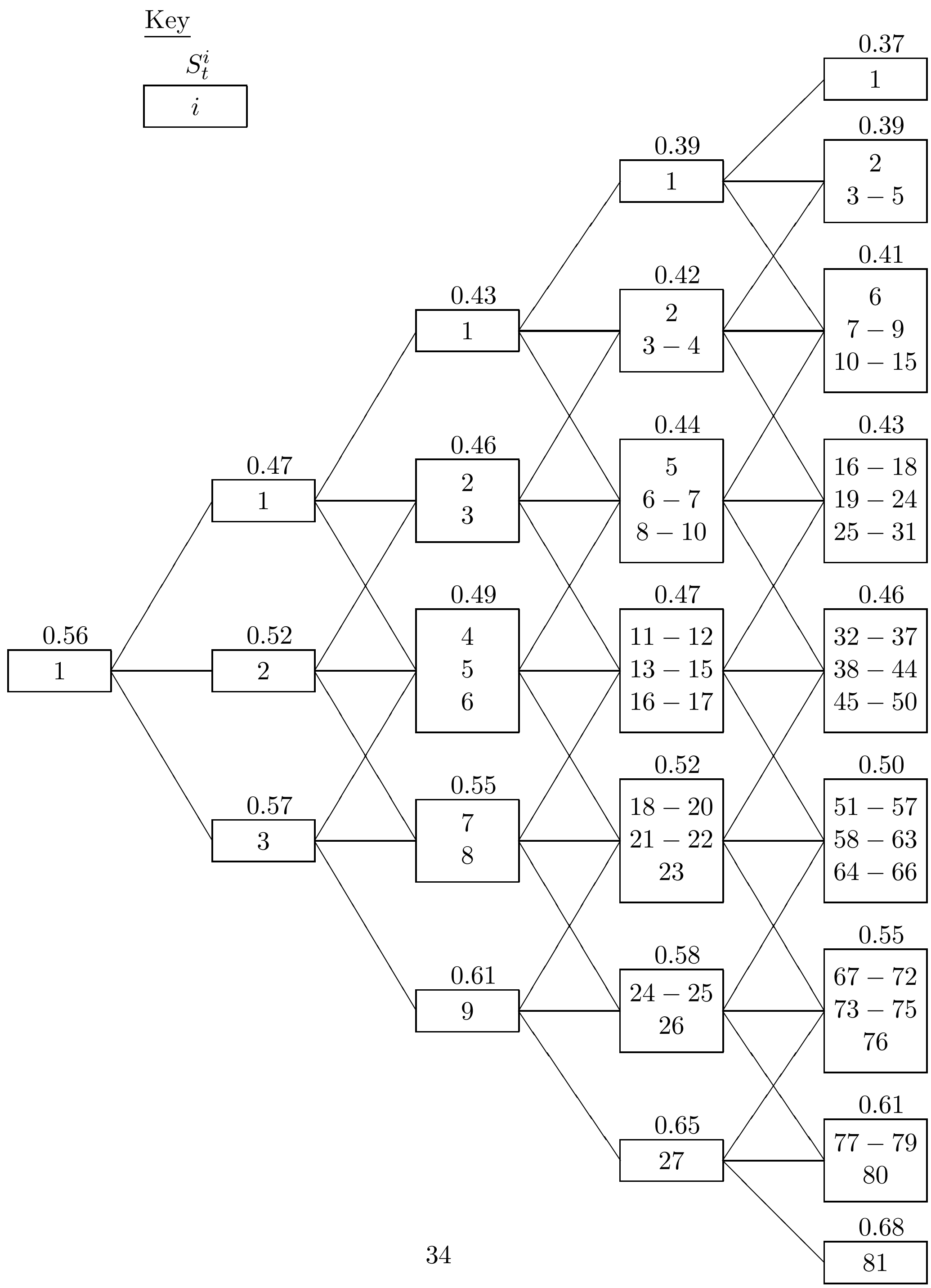
$$Q_t(x_{t-1}, c_t, b_t, A_t, T_{t(t-1)}) := \inf_{x_t \in \mathbb{R}^{n_t}} \{c_t^\top x_t + \mathcal{Q}_{t+1}(x_t) : A_t x_t = b_t - T_{t(t-1)} x_{t-1}, x_t \geq 0\} \qquad t = 2, 3, \ldots, N-1,$$

$$\mathcal{Q}_N(x_{N-1}) := \underset{\{\boldsymbol{c_N}, \boldsymbol{b_N}, \boldsymbol{A_N}, \boldsymbol{T_{N1}}, \boldsymbol{T_{N2}}, \ldots, \boldsymbol{T_{N(N-1)}}\}}{E} \left[Q_N(x_{N-1}, \boldsymbol{c_N}, \boldsymbol{b_N}, \boldsymbol{A_N}, \boldsymbol{T_{N1}}, \boldsymbol{T_{N2}}, \ldots, \boldsymbol{T_{N(N-1)}})\right],$$

and

$$Q_N(x_{N-1}, c_N, b_N, A_N, T_{N1}, T_{N2}, \ldots, T_{N(N-1)}) := \inf_{x_N \in \mathbb{R}^{n_N}} \{c_N^\top x_N : A_N x_N = b_N - \sum_{t=1}^{N-1} T_{Nt} x_t, x_t \geq 0\}.$$

Figure 1: Scenario tree for Klaassen, Shapiro and Spitz [13]



2.5 Financial planning model

Due to Cariño and Ziemba [7, 6]
(Multistage, linear stochastic problem)

2.5.1 Description

Cariño and Ziemba [7, 6] describe a model created for the Yasuda Fire and Marine Insurance Co., Ltd. (Yasuda Kasai) of Tokyo by the Frank Russell Company (Russell) of Takoma, Washington. The model is a comprehensive investment, liability, and risk planning tool. It is a multistage linear stochastic model with a steady-state condition imposed on the last stage.

The complexity of the model is such that it cannot be completely described in article format. The model presented here is therefore a simplification of the original [7], although it is much more detailed than the abbreviated model presented in an earlier paper [5, Appendix].

Yasuda Kasai offers many types of insurance policies, which differ in structure and in regulatory treatment. One type of policy is a traditional, non-savings insurance policy. Premiums from this type of policy go to the Yasuda Kasai general account. Other policies are called *savings* policies. These policies are really two policies in one, with part of the premium paying for insurance and the rest constituting a deposit for savings. The insurance portion of the premium goes into the general account, and the rest goes into one of many savings accounts. The savings accounts are separated based on regulations, but they are treated the same in this model. Therefore, one savings account is included in this model.

The general account is divided into a *general* allocatable account and a non-allocatable *exogenous* account. Funds in the exogenous account may not be invested. In this problem description, the superscript S will refer to quantities relating to the savings account, while G and E will refer to those relating to the general allocatable and exogenous accounts, respectively. Define V_t^S, V_t^G, and V_t^E as the market value of the savings, general and exogenous accounts, respectively, at the *end* of period t.

Fund allocations are not only classified by account, they are also classified by investment type and asset class. The investment type indicates how the funds are invested. Money in the savings and general accounts may be invested either directly (D) or indirectly. There are three possible indirect investment types: tokkin funds (T), capital to foreign subsidiaries (C), and loans to foreign subsidiaries (L). So, the four investment types are D, T, C and L.

In contrast, there are many asset classes, such as domestic bonds, foreign equity, and real estate. In theory, each combination of account, investment type, and asset class may have its own fund allocation. However, some of the combinations are prohibited by regulations. All of the permissible allocations are indexed, and X_{nt} is defined as the allocation of funds to combination n at the end of time stage t. The classifications are used in quite a flexible way, so that $n \in$ loans means the set of indexes for all combinations with an asset class which can be described as a loan, and $n \in S$ is the set of indexes for all combinations involving the savings account. Therefore the market value of the savings account can be expressed by the constraint

$$V_t^S - \sum_{n \in S} X_{nt} = 0. \tag{23}$$

The market value of the general account is written similarly, except that it includes v_t^G, the surplus income in the general account. The constraint is therefore

$$V_t^G - \sum_{n \in G} X_{nt} - v_t^G = 0.$$

The random variables in this model have dependence on various rates of return and other company projections. They are defined in Table 7. Each has a discrete probability distribution.

Table 7: Random variables in Russel-Yasuda Kasai model

$\boldsymbol{RI_{nt+1}}$	income return of allocation n from the end of t to the end of $t+1$
$\boldsymbol{RP_{nt+1}}$	price return of allocation n from the end of t to the end of $t+1$
$\boldsymbol{g_{t+1}}$	interest rate credited to policies from the end of t to the end of $t+1$
$\boldsymbol{F_{t+1}}$	deposit inflow from the end of t to the end of $t+1$
$\boldsymbol{P_{t+1}}$	principal payments from the end of t to the end of $t+1$
$\boldsymbol{I_{t+1}}$	interest payments from the end of t to the end of $t+1$
$\boldsymbol{L_t}$	total reserve liability at the end of t
$\boldsymbol{N_t}$	interest portion of L_t
$\boldsymbol{IG_{t+1}}$	income gap resulting from the difference between current market yields and existing loan portfolio cash flows

The savings and general accounts are modeled by several balance and flow equations. For example, investment income D_{t+1} is defined, for the savings account, by

$$D_{t+1}^S := \sum_{n\in SD} \boldsymbol{RI_{nt+1}} X_{nt} + \sum_{n\in SI} (\boldsymbol{RI_{nt+1}} + \boldsymbol{RP_{nt+1}}) X_{nt} - \boldsymbol{IG_{t+1}^S},$$

and for the general account by

$$D_{t+1}^G := \sum_{n\in GD} \boldsymbol{RI_{nt+1}} X_{nt} + \sum_{n\in GI} (\boldsymbol{RI_{nt+1}} + \boldsymbol{RP_{nt+1}}) X_{nt} - \boldsymbol{IG_{t+1}^G}.$$

Here, SD is the set of indices corresponding to direct type allocations from the savings account, SI is the set corresponding to indirect allocations from the savings account, and similarly for GD and GI from the general account. One of the properties of the indirect investment types is that all price returns are translated into income. This is not the case for the direct investment type. Therefore, capital gain

$$G_{t+1}^S := \sum_{n\in SD} \boldsymbol{RP_{nt+1}} X_{nt},$$

and

$$G_{t+1}^G := \sum_{n\in GD} \boldsymbol{RP_{nt+1}} X_{nt},$$

includes the price return from direct investments.

Let B_t^S be the income accumulated in the savings account through the end of t, and w_t^S be the amount transfered to the savings account from the general account at the end of t. If v_t^S is the amount transfered from the savings account to the general account at the end of t, then

$$B_{t+1}^S = B_t^S + D_{t+1}^S - \boldsymbol{I_{t+1}^S} + w_{t+1}^S - v_{t+1}^S.$$

The transfers, w_t^S and v_t^S, from and to the general account, are established as slack variables in a constraint. This constraint expresses the desire to keep accumulated income B_{t+1}^S greater than accrued interest liability N_{t+1}^S. Since $\boldsymbol{N_{t+1}^S} = \boldsymbol{N_t^S} + \boldsymbol{g_{t+1}^S L_t^S} - \boldsymbol{I_{t+1}^S}$, the constraint is

$$B_t^S + D_{t+1}^S + w_{t+1}^S - v_{t+1}^S = \boldsymbol{N_t^S} + \boldsymbol{g_{t+1}^S L_t^S}.$$

The excess accumulated income, v_{t+1}^S, is in general good, because it contributes to income before taxes. This constraint only occurs when $t+1$ is a fiscal year-end period.

In addition to the income constraint, there is a reserve constraint, which measures the total reserve shortfall z^S_{t+1} or surplus q^S_{t+1}. These are established by

$$V^S_t + G^S_{t+1} + D^S_{t+1} + z^S_{t+1} - q^S_{t+1} = (1 + \boldsymbol{g^S_{t+1}})\boldsymbol{L^S_t}.$$

A value of $z^S_{t+1} > 0$ represents the undesirable situation where the income cannot meet the required liability reserve. No funds need be transfered, but a penalty is assigned in the objective.

Another constraint, the cash flow constraint, addresses the unlikely event that net pay outs from the savings account, $\boldsymbol{P^S_{t+1}} + \boldsymbol{I^S_{t+1}} - \boldsymbol{F^S_{t+1}}$, exceed the market value of the savings account itself. A shortfall y^S_{t+1} would require a transfer from the general account, while the surplus u^S_{t+1} is a slack variable. The constraint is expressed as

$$V^S_t + G^S_{t+1} + D^S_{t+1} + w^S_{t+1} - v^S_{t+1} + y^S_{t+1} - u^S_{t+1} = \boldsymbol{P^S_{t+1}} + \boldsymbol{I^S_{t+1}} - \boldsymbol{F^S_{t+1}}. \quad (24)$$

Any surplus from constraint (24) is equal to the new market value of the savings account:

$$V^S_{t+1} = u^S_{t+1}.$$

Let B^G_{t+1} represent the income accumulated in the general account from the beginning of the fiscal year to the beginning of $t + 2$. Then

$$B^G_{t+1} := \begin{cases} 0 & \text{if } t+1 \text{ is a year-end stage,} \\ B^G_t + D^G_t & \text{otherwise,} \end{cases}$$

since the beginning of the fiscal year *is* at the beginning of $t + 2$.

Nonnegative income before taxes Y_{t+1} includes any income from the general account, and any transfers between the general and savings accounts. The calculation

$$Y_{t+1} - s_{t+1} = B^G_t + D^G_{t+1} + v^S_{t+1} - w^S_{t+1}$$

is made for fiscal year-end stages $t + 1$ only. Here, s_{t+1} is the non-positive income, should such a dreadful thing occur.

Of course, income should be sufficient to pay dividends to shareholders and taxes. To encourage such outcomes, let Γ_{t+1} be an income target, v^G_{t+1} be the income in excess of Γ_{t+1}, and w^G_{t+1} be the shortfall. The objective function will include w^G_{t+1}, along with a cost penalty, and any $v^G_{t+1} > 0$ will contribute directly to net worth according to (23). The required income constraint is

$$Y_{t+1} - s_{t+1} + w^G_{t+1} - v^G_{t+1} = \Gamma_{t+1}.$$

The amount w^G_{t+1} will need to be transferred from the exogenous account.

The net worth of the company before taxes and shareholder dividends is q^G_{t+1}. It is defined by the constraint

$$V^E_t + V^G_t + G^G_{t+1} + D^G_{t+1} + q^S_{t+1} - z^S_{t+1} + z^G_{t+1} - q^G_{t+1} = \boldsymbol{L^G_t},$$

where z^G_{t+1} represents a negative net worth, a dire situation.

Cash flow must also be balanced in the general account. Taxes are assumed to be a constant τ times income before taxes. Dividend payments to shareholders are included in $\boldsymbol{P^G_{t+1}}$. Therefore, we have the constraint

$$\begin{aligned} V^G_t + G^G_{t+1} + D^G_{t+1} - \tau Y_{t+1} + v^S_{t+1} - w^S_{t+1} - y^S_{t+1} + w^G_{t+1} + \\ y^G_{t+1} - u^G_{t+1} = \boldsymbol{P^G_{t+1}} - \boldsymbol{F^G_{t+1}}. \end{aligned}$$

A positive value for y^G_{t+1} would be very serious, as that amount would have to be transferred from the exogenous account to pay all the bills. The excess u^G_{t+1} is, as with the savings account, a slack variable which represents the accumulated market value of the general account. So,

$$V^G_{t+1} = u^G_{t+1}.$$

The accumulation constraint for the exogenous account includes k^E_{t+1}, the projected increase in the exogenous account:

$$V^E_{t+1} = V^E_t - w^G_{t+1} - y^G_{t+1} + k^E_{t+1}.$$

In addition to the flow, income and accumulation constraints, there are many other constraints in the model by Cariño and Ziemba [7]. These constraints find their origins in external and internal regulations and policies. For example, since loans are a particularly illiquid asset class, an internal policy limits the change in allocations to loan asset investments from one period to the next. With lb and ub being constants defined by the policy, the constraint

$$\begin{aligned} lb(1 + \boldsymbol{RI_{nt+1}} + \boldsymbol{RP_{nt+1}})X_{nt} \leq X_{nt+1} \leq \\ ub(1 + \boldsymbol{RI_{nt+1}} + \boldsymbol{RP_{nt+1}})X_{nt}, \quad n \in \text{loans} \end{aligned}$$

is added to the model.

The objective is to maximize the market value of the accounts at time T and minimize the costs involved with shortfalls, while meeting all the constraints. Let $c^S_{wt}, c^S_{yt}, c^S_{zt}, c^G_{wt}, c^G_{zt}$, and c^G_{yt} be the cost parameters associated with shortfalls $w^S_t, y^S_t, z^S_t, w^G_t, z^G_t$, and y^G_t, respectively. Let

$$C_t := c^S_{wt}w^S_t + c^S_{yt}y^S_t + c^S_{zt}z^S_t + c^G_{wt}w^G_t + c^G_{zt}z^G_t + c^G_{yt}y^G_t.$$

Then the objective is to minimize the expected value

$$E\left[-V_T^S - V_T^G - V_T^E + \sum_{t=2}^{T}(1+\gamma)^{N(t,T)}C_t + \alpha C_f\right],$$

where γ is the discount factor, the function $N(t,T)$ gives the number of years from stage t to stage T, α is the discount factor for the end-effects period, and C_f is the cost of shortfalls for the end-effects stage.

2.5.2 Problem statement

Given constants lb, ub, τ, k_t^E, and Γ_t, costs and given discrete distributions for the set of random variables

$$\boldsymbol{R} := \{\boldsymbol{RI_{nt}}, \boldsymbol{RP_{nt}}, \boldsymbol{IG_t^S}, \boldsymbol{F_t^S}, \boldsymbol{P_t^S}, \boldsymbol{I_t^S}, \boldsymbol{g_t^S}, \boldsymbol{L_t^S}, \boldsymbol{IG_t^G}, \boldsymbol{F_t^G}, \boldsymbol{P_t^G} : \\ t = 1, 2, \ldots, T; \text{for all } n\},$$

the problem is to

$$\text{minimize} \quad \mathop{E}_{\boldsymbol{R}}\left[-V_T^S - V_T^G - V_T^E + \sum_{t=2}^{T}(1+\gamma)^{N(t,T)}C_t + \alpha C_f\right] \tag{25}$$

subject to

$$
\begin{aligned}
V_t^S - \sum_{n\in S} X_{nt} &= 0 \\
V_t^G - \sum_{n\in G} X_{nt} - v_t^G &= 0 \\
D_{t+1}^S = \sum_{n\in SD} \boldsymbol{RI_{nt+1}} X_{nt} &+ \sum_{n\in SI}(\boldsymbol{RI_{nt+1}} + \boldsymbol{RP_{nt+1}})X_{nt} - \boldsymbol{IG_{t+1}^S} \\
D_{t+1}^G = \sum_{n\in GD} \boldsymbol{RI_{nt+1}} X_{nt} &+ \sum_{n\in GI}(\boldsymbol{RI_{nt+1}} + \boldsymbol{RP_{nt+1}})X_{nt} - \boldsymbol{IG_{t+1}^G} \\
G_{t+1}^S &= \sum_{n\in SD} \boldsymbol{RP_{nt+1}} X_{nt} \\
G_{t+1}^G &= \sum_{n\in GD} \boldsymbol{RP_{nt+1}} X_{nt} \\
B_{t+1}^S &= B_t^S + D_{t+1}^S - \boldsymbol{I_{t+1}^S} + w_{t+1}^S - v_{t+1}^S \\
B_t^S + D_{t+1}^S + w_{t+1}^S - v_{t+1}^S &= \boldsymbol{N_t^S} + \boldsymbol{g_{t+1}^S}\boldsymbol{L_t^S} \\
V_t^S + G_{t+1}^S + D_{t+1}^S + z_{t+1}^S - q_{t+1}^S &= (1 + \boldsymbol{g_{t+1}^S})\boldsymbol{L_t^S} \\
V_t^S + G_{t+1}^S + D_{t+1}^S + w_{t+1}^S - v_{t+1}^S + y_{t+1}^S - u_{t+1}^S &= \boldsymbol{P_{t+1}^S} + \boldsymbol{I_{t+1}^S} - \boldsymbol{F_{t+1}^S}
\end{aligned}
\tag{26}
$$

$$V_{t+1}^S = u_{t+1}^S \tag{27}$$

$$B_{t+1}^G = \begin{cases} 0 & \text{if } t+1 \text{ is a year-end stage,} \\ B_t^G + D_t^G & \text{otherwise,} \end{cases} \tag{28}$$

$$Y_{t+1} - s_{t+1} = B_t^G + D_{t+1}^G + v_{t+1}^S - w_{t+1}^S$$

$$Y_{t+1} - s_{t+1} + w_{t+1}^G - v_{t+1}^G = \Gamma_{t+1}$$

$$V_t^E + V_t^G + G_{t+1}^G + D_{t+1}^G + q_{t+1}^S - z_{t+1}^S + z_{t+1}^G - q_{t+1}^G = \boldsymbol{L_t^G}$$

$$\begin{aligned} V_t^G + G_{t+1}^G + D_{t+1}^G - \tau Y_{t+1} + v_{t+1}^S - w_{t+1}^S - y_{t+1}^S + w_{t+1}^G + \\ y_{t+1}^G - u_{t+1}^G = \boldsymbol{P_{t+1}^G} - \boldsymbol{F_{t+1}^G} \end{aligned} \tag{29}$$

$$V_{t+1}^G = u_{t+1}^G \tag{30}$$

$$V_{t+1}^E = V_t^E - w_{t+1}^G - y_{t+1}^G + k_{t+1}^E$$

$$\begin{aligned} lb(1 + \boldsymbol{RI_{nt+1}} + \boldsymbol{RP_{nt+1}})X_{nt} \leq X_{nt+1} \leq \\ ub(1 + \boldsymbol{RI_{nt+1}} + \boldsymbol{RP_{nt+1}})X_{nt}, \quad n \in \text{loans} \end{aligned} \tag{31}$$

$$X_{nt}, w_t^S, v_t^S z_t^S, q_t^S, y_t^S, u_t^S, Y_t, s_t, w_t^G, v_t^G, z_t^G, q_t^G, y_t^G, u_t^G \geq 0.$$

2.5.3 Numerical results

The model described here is too complex for us to create empirical data at this time. Further, the original creators of the model [7, 6] did not provide specific problem data.

2.5.4 Notational reconciliation

In order to put this problem in the notation of (2), we make a few changes to the problem:

1. Assume each period is a year-end stage. This assumption is not necessary, but we are required to state which stages are year-end, and which are not. The result of this supposition is the elimination from the problem of the variable B_t^G and the equation (28).

2. The end-effects stage is eliminated. This eliminates the term αC_f from the objective.

3. Equations (27) and (30) are eliminated by substituting V for u in equations (26) and (29).

4. The conditions (31) constraining the loans are eliminated.

Order the number of accounts in any way, and let M be the number of accounts. That is, the index n runs from 1 to M. Define the vectors

$$X_t := \begin{bmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{Mt} \end{bmatrix}, \qquad \boldsymbol{RP_t} := \begin{bmatrix} \boldsymbol{RP_{1t}} \\ \boldsymbol{RP_{2t}} \\ \vdots \\ \boldsymbol{RP_{Mt}} \end{bmatrix},$$

and

$$\boldsymbol{RI_t} := \begin{bmatrix} \boldsymbol{RI_{1t}} \\ \boldsymbol{RI_{2t}} \\ \vdots \\ \boldsymbol{RI_{Mt}} \end{bmatrix}.$$

Let $\Delta \in \mathbb{R}^{M \times M}$ be the diagonal matrix defined for sets $\Phi \in \{S, G, D, I\}$ by

$$(\Delta^{\Phi})_{jj} := \begin{cases} 1 & \text{if account } j \text{ is in set } \Phi \\ 0 & \text{otherwise.} \end{cases}$$

Then we may express the sums from the problem statement in Section 2.5.2 in matrix notation. For example,

$$\sum_{n \in GD} \boldsymbol{RI_{nt+1}} X_{nt} = (\boldsymbol{RI_{t+1}})^{\mathsf{T}} \Delta^G \Delta^D X_t.$$

To begin putting the problem into the notation of (2), set

$$x_1 := \begin{bmatrix} X_1 \\ V_1^S \\ V_1^G \end{bmatrix}, \quad c_1 := \begin{bmatrix} 0^{M \times 1} \\ 0 \\ 0 \end{bmatrix}, \quad b_1 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$A_1 := \begin{bmatrix} (-1^{1 \times M} \Delta^S) & 1 & 0 \\ (-1^{1 \times M} \Delta^G) & 0 & 1 \end{bmatrix}.$$

Then for $t = 2, 3, \ldots, T$, define

$$x_t := \begin{bmatrix} X_t \\ B_t^S \\ D_t^S \\ G_t^S \\ V_t^S \\ q_t^S \\ v_t^S \\ w_t^S \\ y_t^S \\ z_t^S \\ D_t^G \\ G_t^G \\ V_t^G \\ q_t^G \\ v_t^G \\ w_t^G \\ y_t^G \\ z_t^G \\ V_T^E \\ Y_t \\ s_t \end{bmatrix}, \text{ and } \qquad c_t := \begin{bmatrix} 0^{M\times 1} \\ 0 \\ 0 \\ 0 \\ -\delta_{tT} \\ 0 \\ 0 \\ (1-\gamma)^{T-t} c_{wt}^S \\ (1-\gamma)^{T-t} c_{yt}^S \\ (1-\gamma)^{T-t} c_{zt}^S \\ 0 \\ 0 \\ -\delta_{tT} \\ 0 \\ 0 \\ (1-\gamma)^{T-t} c_{wt}^G \\ (1-\gamma)^{T-t} c_{yt}^G \\ (1-\gamma)^{T-t} c_{zt}^G \\ -\delta_{tT} \\ 0 \\ 0 \end{bmatrix},$$

where

$$\delta_{tT} := \begin{cases} 1 & \text{if } t = T \\ 0 & \text{otherwise.} \end{cases}$$

The remaining assignments necessary are

$$A_t := \begin{bmatrix} A_t^S & A_t^G \end{bmatrix}, \qquad \boldsymbol{T_t} := \begin{bmatrix} \boldsymbol{T_t^S} & T_t^G \end{bmatrix},$$

and

$$
\boldsymbol{b_t} := \begin{bmatrix} 0 \\ 0 \\ -\boldsymbol{IG_t^S} \\ -\boldsymbol{IG_t^G} \\ 0 \\ 0 \\ -\boldsymbol{I_t^S} \\ (\boldsymbol{N_{t-1}^S} + \boldsymbol{g_t^S L_{t-1}^S}) \\ ((1+\boldsymbol{g_t^S})\boldsymbol{L_{t-1}^S}) \\ (\boldsymbol{P_t^S} + \boldsymbol{I_t^S} - \boldsymbol{F_t^S}) \\ 0 \\ \Gamma \\ \boldsymbol{L_{t-1}^G} \\ (\boldsymbol{P_t^G} - \boldsymbol{F_t^G}) \\ k^E \end{bmatrix},
$$

where $A_t^S, A_t^G, \boldsymbol{T_t^S}$, and T_t^G are defined in Figures 2, 3, 4 and 5, respectively.

Figure 2: Array A_t^S for Russell-Yasuda Kasai example

$$
A_t^S := \begin{bmatrix}
(-1^{(1\times M)}\Delta^S) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
(-1^{(1\times M)}\Delta^G) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & (-1) & 0 & 0 & 0 & 1 & (-1) & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & (-1) & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & (-1) & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & (-1) & 0 & (-1) & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (-1) & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & (-1) \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & (-1) & (-1) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Figure 3: Array A_t^G for Russell-Yasuda Kasai example

$$A_t^G := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & (-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & (-1) \\ 0 & 0 & 0 & 0 & (-1) & 1 & 0 & 0 & 0 & 1 & (-1) \\ 1 & 1 & 0 & (-1) & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & (-1) & 0 & 0 & 1 & 1 & 0 & 0 & (-\tau) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Figure 4: Array $\boldsymbol{T_t^S}$ for Russell-Yasuda Kasai example

$$\boldsymbol{T_t^S} := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \begin{matrix}[-\boldsymbol{RI_t}^\mathsf{T}\Delta^S\Delta^D \\ -(\boldsymbol{RI_t}+\boldsymbol{RP_t})^\mathsf{T}\Delta^S\Delta^I]\end{matrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \begin{matrix}[-\boldsymbol{RI_t}^\mathsf{T}\Delta^G\Delta^D \\ -(\boldsymbol{RI_t}+\boldsymbol{RP_t})^\mathsf{T}\Delta^G\Delta^I]\end{matrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\boldsymbol{RP_t}^\mathsf{T}\Delta^S\Delta^D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\boldsymbol{RP_t}^\mathsf{T}\Delta^G\Delta^D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Figure 5: Array T_t^G for Russell-Yasuda Kasai example

$$
T_t^G := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (-1) & 0 & 0
\end{bmatrix}
$$

2.6 Design of batch chemical plants

Due to Subrahmanyam, Pekny, and Reklaitis [23]
(Multistage, mixed integer linear stochastic problem)
`/chem/chem.cor`,`/chem.tim`,`chem.sto`

2.6.1 Description

Subrahmanyam, Pekny, and Reklaitis [23] describe the design of a batch type chemical plant to produce products for which we do not know the future demand. We present here only half of the problem given in [23], the "Design SuperProblem."

We must decide how many plants to build, of what type, when to build them, and how to operate them. Therefore the problem has some integer decision variables. Let n_{jt} be the number of new units of equipment type j to come online in time stage t. This must take only integer values. The cumulative number of units of type j at time t is defined as

$$N_{jt} := \sum_{\tau=1}^{t} n_{j\tau}, \quad \forall j, t.$$

The various plants can perform different tasks, which are indexed as $i = 1, \ldots, I$. Certain plants can perform more than one task. Let I_j^{tasks} be the set of tasks from 1 to I which can be performed by a plant of type j, and let I_i^{equip} be the set of plant types from 1 to J which can perform task i.

We must decide which tasks to perform on which equipment during each time period. Let y_{ijt} be the number of times task i is performed on equipment type j during time stage t. If p_{ij} is the processing time for task i with equipment type j, and if H_t is the length of stage t, then

$$\sum_{i \in I_j^{\text{tasks}}} p_{ij} y_{ijt} \leq H_t N_{jt}, \quad \forall j, t.$$

This constraint enforces the fact that time is limited in each stage. Something else which might limit the number of batches is the much feared operating expense budget. Let C_t^o be the total operating budget per plant for time t, and let c_{ijt}^o be the operating expense incurred for using equipment type j to perform task i in stage t. Then the operating expense constraint is

$$\sum_j \sum_{i \in I_j^{\text{tasks}}} c_{ijt}^o y_{ijt} \leq C_t^o \sum_j N_{jt}, \quad \forall t.$$

The material balance on the system includes inventory, production, consumption, sales, and purchasing effects. Let B_{ijt} be the amount of task i performed on plant type j in stage t, measured in somewhat arbitrary reaction units. If f_{si} is the stoichiometric ratio representing mass of resource s produced per unit of reaction i, then the amount of s produced in stage t is

$$\sum_i \sum_{j \in I_i^{\text{equip}}} f_{si} B_{ijt}.$$

Note that f_{si} is negative if resource s is consumed in task i. The mass of resource s in inventory at the end of stage t is A_{st}, with maximum limit $A_{st}^{\max}$. The material balance constraints are then

$$A_{st} = A_{s(t-1)} + \sum_i \sum_{j \in I_i^{\text{equip}}} f_{si} B_{ijt} - q_{st}^s + q_{st}^b, \quad \forall s, t,$$

and

$$A_{st} \leq A_{st}^{\max}, \quad \forall s, t,$$

where q_{st}^s is the mass of resource s sold in stage t, and q_{st}^b is that bought in stage t.

The relationship between B_{ijt} and y_{ijt} is

$$B_{ijt} \leq m_{ij} y_{ijt}, \quad \forall i, j, t.$$

Here, m_{ij} is the capacity of equipment type j to perform task i, measured in units of reaction per batch.

There are a couple ways to limit purchases. One is to simply impose a limit, as in

$$q_{st}^b \leq Q_{st}^b, \quad \forall s, t,$$

for some constant Q_{st}^b. Another is to limit capital expenditures to not exceed a constant MC_t, as in

$$\sum_j C_{jt} n_{jt} + \sum_s v_{st}^b q_{st}^b \leq MC_t, \quad \forall t.$$

The symbol C_{jt} is the capital investment cost for a plant of type j in stage t. The term v_{st}^b is the value of purchased resource s in stage t per unit mass.

One of the random variables in this problem is the demand, $\boldsymbol{Q_{skt}^s}$, for resource s at time t. The index k determines the scenario. The other random variable is $\boldsymbol{v_{skt}^s}$, the price per unit mass of resource s sold at or below the demand level $\boldsymbol{Q^s_{skt}}$ in stage t. We define a new random variable by $\boldsymbol{r_{skt}} := (\boldsymbol{Q^s_{skt}}, \boldsymbol{v^s_{skt}})$.

The recourse variables are q^{s0}_{skt}, the amount of s sold in stage t which does not exceed demand, and q^{s+}_{skt}, the amount which exceeds demand. Essentially, q^{s+}_{skt} is given away, rather than sold, as no credit toward profit may be taken for this quantity. The recourse variables are limited by the equation

$$q^s_{st} = q^{s0}_{skt} + q^{s+}_{skt}, \quad \forall s, k, t,$$

and the inequality

$$q^{s0}_{skt} \leq \boldsymbol{Q^s_{skt}}, \quad \forall s, k, t.$$

In some industries, it is important that the demand be met exactly. For such cases, define the variable

$$x_{kt} := \begin{cases} 1 & \text{if for each } s,\ q^{s0}_{skt} = \boldsymbol{Q^s_{skt}}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can set a guarantee index G_t, to serve as the minimum number of scenarios for which the demand may be met exactly. We get the constraint

$$\sum_k x_{kt} \geq G_t, \quad \forall t.$$

The objective function,

$$\sum_{t=1}^{T} \left\{ \mathop{E}_{r} \left[\sum_{s=1}^{S} (\boldsymbol{v^s_{skt}} q^{s0}_{skt} - v^b_{st} q^b_{st}) - \sum_{j=1}^{J} \left(n_{j(t+\delta)} C_{jt} + \sum_{i=1}^{I} c^o_{ijt} y_{ijt} \right) \right] \right\},$$

is the net present value of the facilities, and includes potential income, capital expenditures and operating expense. It contains no first stage objective terms, and should obviously be maximized. Note that $c^o_{ijt} = 0$ if we cannot perform task i with equipment j. Here, δ is the construction delay once the plant has been ordered.

2.6.2 Problem statement

We are given a discrete probability distribution

$$\{(P_{kt}, \boldsymbol{r_{skt}}) : k = 1, 2, \ldots, K\}.$$

In addition, we require constants c^o_{ijt}, v^b_{st}, C_{jt}, p_{ij}, H_t, C^o_t, f_{si}, Q^b_{st}, $A^{\max}_{st}$, m_{ij}, G_t, MC_t, and δ, and the index sets I^{tasks}_j and I^{equip}_i. Then our goal is

to
maximize

$$\sum_{t=1}^{T}\left\{\ \underset{r}{E}\ \left[\sum_{s=1}^{S}(\boldsymbol{v^s_{skt}}q^{s0}_{skt} - v^b_{st}q^b_{st}) - \sum_{j=1}^{J}\left(n_{j(t+\delta)}C_{jt} + \sum_{i=1}^{I}c^o_{ijt}y_{ijt}\right)\right]\right\},$$

subject to

$$\sum_{i\in I_j^{\text{tasks}}} p_{ij}y_{ijt} \le H_t N_{jt}, \quad \forall j,t$$

$$\sum_j \sum_{i\in I_j^{\text{tasks}}} c^o_{ijt}y_{ijt} \le C^o_t \sum_j N_{jt}, \quad \forall t \tag{32}$$

$$N_{jt} = \sum_{\tau=1}^{t} n_{j\tau}, \quad \forall j,t \tag{33}$$

$$A_{st} = A_{s(t-1)} + \sum_i \sum_{j\in I_i^{\text{equip}}} f_{si}B_{ijt} - q^s_{st} + q^b_{st}, \quad \forall s,t \tag{34}$$

$$A_{st} \le A^{\max}_{st}, \quad \forall s,t$$

$$B_{ijt} \le m_{ij}y_{ijt}, \quad \forall i,j,t$$

$$q^s_{st} = q^{s0}_{skt} + q^{s+}_{skt}, \quad \forall s,k,t$$

$$q^{s0}_{skt} \le \boldsymbol{Q^s_{skt}}, \quad \forall s,k,t$$

$$q^b_{st} \le Q^b_{st}, \quad \forall s,t$$

$$\sum_j C_{jt}n_{jt} + \sum_s v^b_{st}q^b_{st} \le MC_t, \quad \forall t \tag{35}$$

$$x_{kt} = \begin{cases} 1 & \text{if for each } s,\ q^{s0}_{skt} = \boldsymbol{Q^s_{skt}}, \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

$$\sum_k x_{kt} \ge G_t, \quad \forall t \tag{37}$$

$$A_{st}, q^b_{st}, q^s_{st} \ge 0, \quad \forall s,t$$

$$q^{s0}_{skt}, q^{s+}_{skt} \ge 0, \quad \forall s,k,t$$

$$B_{ijt} \ge 0, \quad \forall i,j,t$$

$$N_{jt}, n_{jt}, y_{ijt} \in \mathbb{Z}^+.$$

2.6.3 Numerical results

Subrahmanyam, Pekny, and Reklaitis [23] present a problem with $I = 4$ tasks, $S = 7$ resources, $J = 3$ equipment types, $T = 2$ time stages, and $K = 2$ scenarios per stage. Operating costs are neglected, so $c^o_{ijt} = 0$ for all cases, and (32) is removed from the problem. Also, the construction delay δ and constraints (35), (36), and (37) are not included in the problem.

The parameters for the problem are shown in Tables 8, 9, 10, and 11. Note that resources 1 and 2 must be purchased, resources 4 and 7 are sold, and the remainder are intermediate resources.

Table 8: Probability distribution for random variables (demands not shown are zero)

kt	P_{kt}	Q^s_{4kt}	Q^s_{7kt}	v^s_{4kt}	v^s_{7kt}
11	0.4	0	0	51	70
21	0.6	150	200	58	80
12	0.4	0	0	50	71
22	0.6	150	200	59	81

Table 9: Parameters for purchased resources

st	v^b_{st}	Q^b_{st}
11	23	200
12	24	200
21	25	250
22	26	250

Additionally,

$$A^{\max}_{st} = 400 \quad \forall s, t,$$

$$H_1 = H_2 = 80 \text{ days},$$

$$p_{11} = p_{41} = p_{12} = p_{42} = p_{23} = p_{33} = 4,$$

$$I^{\text{tasks}}_1 = I^{\text{tasks}}_2 = \{1, 4\},$$

and

$$I^{\text{tasks}}_3 = \{2, 3\}.$$

Table 10: Parameters for equipment types

j	C_{j1}	C_{j2}	$m_{ij} \forall i$
1	2500	2600	100
2	3000	3100	200
3	2800	2900	150

Table 11: Stoichiometric coefficients f_{si}

$i \backslash s$	1	2	3	4	5	6	7
1	-1	-1	1	0	0	0	0
2	0	0	-1	1	1	0	0
3	0	-1	0	0	-1	1	0
4	0	0	0	0	0	-1	1

The optimal objective value stated in [23] is 3300, with optimal values of $N_{21} = 1$ and $N_{31} = 1$.

The problem `chem` in our collection is an attempt at recreating this example. We have not been able to verify that we have succeeded in this attempt, as we have only run `chem` as a continuous model. The optimal objective value for `chem` as a continuous model is 13009.16667.

2.6.4 Notational reconciliation

For simplicity, equations (36) and (37) are removed from the problem, and δ is set to 0. The transition of notation is then quite straightforward. Define the diagonal matrix $\Delta_j^{\text{tasks}} \in \mathbb{R}^{I \times I}$ by

$$(\Delta_j^{\text{tasks}})_i i := \begin{cases} 1 & \text{if } i \in I_j^{\text{tasks}} \\ 0 & \text{otherwise.} \end{cases}$$

Then, we make the following definitions, and also introduce slack variables $u^1, u^2, \ldots, u^7$:

$$p_j := \begin{bmatrix} p_{1j} \\ \vdots \\ p_{Ij} \end{bmatrix}; \quad p := \begin{bmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_J \end{bmatrix};$$

$$y_{jt} := \begin{bmatrix} y_{1jt} \\ \vdots \\ y_{Ijt} \end{bmatrix}; \quad y_t := \begin{bmatrix} y_{1t} \\ \vdots \\ y_{Jt} \end{bmatrix}; \quad B_{jt} := \begin{bmatrix} B_{1jt} \\ \vdots \\ B_{Ijt} \end{bmatrix};$$

$$B_t := \begin{bmatrix} B_{1t} \\ \vdots \\ B_{Jt} \end{bmatrix}; \quad N_t := \begin{bmatrix} N_{1t} \\ \vdots \\ N_{Jt} \end{bmatrix}; \quad n_t := \begin{bmatrix} n_{1t} \\ \vdots \\ n_{Jt} \end{bmatrix};$$

$$u_t^i := \begin{bmatrix} u_{1t}^i \\ \vdots \\ u_{St}^i \end{bmatrix}, i = 3, 5, 6; \quad u_t^1 := \begin{bmatrix} u_{1t}^1 \\ \vdots \\ u_{Jt}^1 \end{bmatrix};$$

$$u_t^4 := \begin{bmatrix} u_{11t}^4 \\ u_{12t}^4 \\ \vdots \\ u_{IJt}^4 \end{bmatrix}; \quad c_{jt}^o := \begin{bmatrix} c_{1jt}^o \\ \vdots \\ c_{Ijt}^o \end{bmatrix}; \quad c_t^o := \begin{bmatrix} c_{1t}^o \\ \vdots \\ c_{Jt}^o \end{bmatrix};$$

$$C_t := \begin{bmatrix} C_{1t} \\ \vdots \\ C_{Jt} \end{bmatrix}; \quad \hat{A}_t := \begin{bmatrix} A_{1t} \\ \vdots \\ A_{St} \end{bmatrix}; \quad A_t^{\max} := \begin{bmatrix} A_{1t}^{\max} \\ \vdots \\ A_{St}^{\max} \end{bmatrix};$$

$$\boldsymbol{Q_t^s} := \begin{bmatrix} \boldsymbol{Q_{1kt}^s} \\ \vdots \\ \boldsymbol{Q_{Skt}^s} \end{bmatrix}; \quad Q_t^b := \begin{bmatrix} Q_{1t}^b \\ \vdots \\ Q_{St}^b \end{bmatrix}; \quad v_t^b := \begin{bmatrix} v_{1t}^b \\ \vdots \\ v_{St}^b \end{bmatrix};$$

$$\boldsymbol{v_t^s} := \begin{bmatrix} \boldsymbol{v_{1kt}^s} \\ \vdots \\ \boldsymbol{v_{Skt}^s} \end{bmatrix}; \quad q_t^s := \begin{bmatrix} q_{1t}^s \\ \vdots \\ q_{St}^s \end{bmatrix}; \quad q_t^{s0} := \begin{bmatrix} q_{1kt}^{s0} \\ \vdots \\ q_{Skt}^{s0} \end{bmatrix};$$

$$q_t^{s+} := \begin{bmatrix} q_{1kt}^{s+} \\ \vdots \\ q_{Skt}^{s+} \end{bmatrix}; \quad M := \begin{bmatrix} m_{11} & & & 0 \\ & m_{21} & & \\ & & \ddots & \\ 0 & & & m_{IJ} \end{bmatrix};$$

$$f_s := \begin{bmatrix} f_{s1} \\ \vdots \\ f_{sI} \end{bmatrix}; \quad F_s := f_s^{\mathsf{T}} \begin{bmatrix} \Delta_1^{\text{tasks}} & \Delta_2^{\text{tasks}} & \cdots & \Delta_J^{\text{tasks}} \end{bmatrix}$$

$$F := \begin{bmatrix} F_1 \\ \vdots \\ F_S \end{bmatrix}; \quad \Delta^{\text{tasks}} := \begin{bmatrix} \Delta_1^{\text{tasks}} & & & 0 \\ & \Delta_2^{\text{tasks}} & & \\ & & \ddots & \\ 0 & & & \Delta_J^{\text{tasks}} \end{bmatrix}.$$

Then, let

$$x_t := \begin{bmatrix} y_t \\ N_t \\ n_t \\ \hat{A}_t \\ B_t \\ q_t^s \\ q_t^b \\ q_t^{s0} \\ q_t^{s+} \\ u_t^1 \\ u_t^2 \\ u_t^3 \\ u_t^4 \\ u_t^5 \\ u_t^6 \\ u_t^7 \end{bmatrix}, \quad \text{and} \quad c_t := \begin{bmatrix} -c_t^o \\ 0^{J\times 1} \\ -C_t \\ 0^{S\times 1} \\ 0^{IJ\times 1} \\ 0^{S\times 1} \\ -v_t^b \\ \boldsymbol{v_t^s} \\ 0^{S\times 1} \\ 0^{J\times 1} \\ 0 \\ 0^{S\times 1} \\ 0^{IJ\times 1} \\ 0^{S\times 1} \\ 0^{S\times 1} \\ 0 \end{bmatrix}. \tag{38}$$

We assign A_t according to Figure 6. The vertical lines separate parts of the matrix according to (38). So, for example, the second partition of the matrix corresponds will be multiplied by N_t. All blanks are zero. Note that the double sum in equation (34) may be expressed as

$$\sum_j \sum_{i \in I_j^{\text{tasks}}} f_{si} B_{ijt}.$$

The transition matrix T_t is defined in Figure 7. There are only two nonzero entries. One is from (34), and the other is from (33), which may be rewritten

$$N_{jt} = N_{jt-1} + n_{jt}.$$

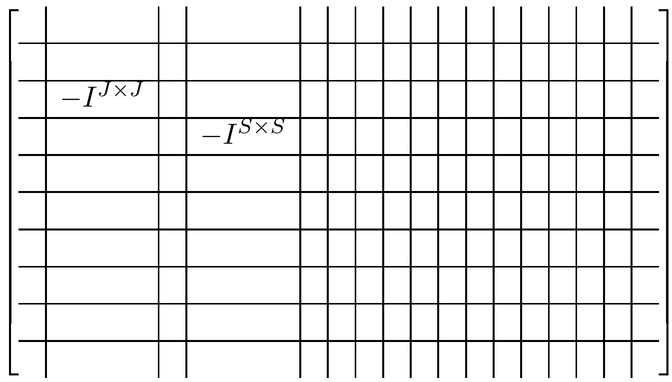
Our transition to the notation of Problem (2) is complete if we set

$$\boldsymbol{b_t} := \begin{bmatrix} 0^{J\times J} \\ 0 \\ 0^{J\times J} \\ 0^{S\times S} \\ A_t^{\max} \\ 0^{IJ\times IJ} \\ 0^{S\times S} \\ \boldsymbol{Q_t^s} \\ Q_t^b \\ MC_t \end{bmatrix}.$$

Figure 6: Array A_t for chemical design planning example

$$
A_t := \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
p^\intercal \Delta^{\text{tasks}} & -H_t I^{J\times J} & & & & & & & & I^{J\times J} & & & & & & \\ \hline
(c_t^o)^\intercal \Delta^{\text{tasks}} & -C_t^o 1^{1\times J} & & & & & & & & & 1 & & & & & \\ \hline
 & I^{J\times J} & -I^{J\times J} & & & & & & & & & & & & & \\ \hline
 & & & I^{S\times S} & -F & I^{S\times S} & -I^{S\times S} & & & & & & & & & \\ \hline
 & & & I^{S\times S} & & & & & & & & I^{S\times S} & & & & \\ \hline
-M & & & & I^{IJ\times IJ} & & & & & & & & I^{IJ\times IJ} & & & \\ \hline
 & & & & & I^{S\times S} & & -I^{S\times S} & -I^{S\times S} & & & & & & & \\ \hline
 & & & & & & & I^{S\times S} & & & & & & I^{S\times S} & & \\ \hline
 & & & & & & I^{S\times S} & & & & & & & & I^{S\times S} & \\ \hline
 & & C_t^\intercal & & & & (v_t^b)^\intercal & & & & & & & & & 1
\end{array}\right]
$$

Figure 7: Array T_t for chemical design planning example

$T_t :=$ 

3 Energy and environmental planning

Due to Fragnière [8]
(Multistage, linear stochastic problem)

$$\texttt{/environ/env.cor},\ \texttt{/env.tim},\ \begin{cases} \texttt{/env\_det.aggr} \\ \texttt{/env.sto.imp} \\ \texttt{/env.sto.loose} \\ \texttt{/env.sto.lrge} \\ \texttt{/env.sto.xlrge} \end{cases}$$

3.1 Description

The model by Fragnière [8] assists the Canton of Geneva in planning its energy supply infrastructure and policies. The model is based on the MARKAL (market allocation) model. This is quite an extensive model, containing a great degree of realism. Included is the possibility that emissions of greenhouse gases will be required to decrease. This possibility is expressed in a discrete random distribution.

The model includes equilibrium constraints, capacity expansion constraints, demand constraints, production constraints, and environmental constraints. Energy is supplied by many different technologies, including hydro power, cogeneration, fossil fuels, urban waste incineration, and imported electricity. Demands are also classified by technology. Examples are electricity for industrial use, gas furnaces in existing houses, and wood stoves in new houses. Variables expressed in upper case letters are decision variables.

An energy balance may be performed on the supply grid, for each energy type. For types k which are neither electricity nor low temperature heat, the balance yields

$$\begin{aligned} &\sum_{\substack{i \in TCH \\ i \notin DMD}} out_{ki}(t) P_i(t) + \sum_{i \in DMD} out_{ki}(t) cf_i(t) C_i(t) + \sum_s IMP_{ks}(t) \\ &\geq \sum_{\substack{i \in TCH \\ i \notin DMD}} inp_{ki}(t) P_i(t) + \sum_{i \in DMD} inp_{ki}(t) cf_i(t) C_i(t) + \sum_s EXP_{ks}(t), \\ &\qquad\qquad\qquad\qquad \forall k \in ENC, \forall t \in T, \quad (39) \end{aligned}$$

where the variables and index sets are defined in Table 12 and Table 13, respectively. Note that for $i \in DMD$, the term $C_i(t)$ refers to the installed *delivery* capacity, whereas for production type technologies, it refers to the installed *production* capacity.

For electricity and low temperature (district) heat, the energy balances are

$$\eta \left[\sum_{i \in ELA} P_{izy}(t) + \sum_{s} IMPELC_{szy}(t) \right] \geq \sum_{i \in PRC} inp_{ELC,i}(t) q_{zy} P_i(t) + \sum_{i \in DMD} inp_{ELC,i}(t) cf_i(t) fr_{j(i)zy} C_i(t) + \sum_{k} EXPELC_{kzy}(t) + \eta \sum_{\substack{i \in STG \\ \ni y=n}} e_i P_{izd}(t), \quad \forall z \in Z, \forall y \in Y, \forall t \in T,$$

and

$$\gamma \sum_{i \in HPL} P_{iz}(t) \geq \sum_{i \in DMD} inp_{LTH,i}(t) cf_i(t) C_i(t) \sum_{y \in Y} fr_{j(i)zy}, \qquad \forall z \in Z, \forall t \in T,$$

respectively.

Table 12: Variable and parameter definitions

$P_i(t)$	the activity, or utilization, of technology i, in period t
$P_{izy}(t)$	the production of electricity from technology i, in period t, season z, and part of the day y
$P_{iz}(t)$	the production of low temperature heat from technology i, in period t, season z
$C_i(t)$	the total installed capacity of technology i in period t
$M_{iz}(t)$	production lost due to regular maintenance of technology i in season z, period t
$IMP_{ks}(t)$	imported energy of type k in period t, from source s
$IMPELC_{szy}(t)$	imported electricity, from source s, in period t, season z, and part of the day y
$EXP_{ks}(t)$	exported energy of type k in period t, to destination s
$EXPELC_{szy}(t)$	exported electricity, to destination s, in period t, season z, and part of the day y
$out_{ki}(t)$	output of energy type k in period t, per unit activity from technology $i \notin DMD$, or per unit capacity from $i \in DMD$

(continued on the next page)

Variable definitions (continued)	
$out_{ik}(t)$	fraction of demand technology i which supplies utility demand $k \in DM$ in period t
$inp_{ki}(t)$	input of energy type k in period t, per unit activity from technology $i \notin DMD$, or per unit capacity from $i \in DMD$
$cf_i(t)$	mean utilization factor of the total installed capacity for technology $i \in DMD$ in period t
q_{zy}	the fraction of a year covered by season z, part of the day y
$j(i)$	utility demand category $j(i) \in DM$, for $i \in DMD$
$fr_{j(i)zy}$	fraction of the utility demand from category $j(i)$ which comes in season z, time of day y
e_i	the electricity input required at night to produce one unit of electricity in the daytime from technology $i \in STG$
η	efficiency coefficient for electrical distribution
ρ	efficiency coefficient for low temperature heat distribution
γ	efficiency coefficient for district (low temperature) heat distribution
l_i	duration of equipment i, in time stages
$I_i(t)$	new capacity purchased for technology i, starting in period t
$resid_i(t)$	capacity which existed at the beginning of the optimization problem
$demand_k(t)$	demand for utility $k \in DM$ in period t
$af_i(t)$	availability factor of technology i in period t
fo_i	the fraction of a year that technology i is lost for production, due to one unit of unavailability
u_i	conversion factor from units of capacity to units of production
er	reserve capacity necessary to cover daily peak demand for electricity
hr	reserve capacity necessary to cover daily peak demand for low temperature heat
$pk_i(t)$	fraction of installed capacity for production technology i, available to satisfy peak demand in period t
$epk_i(t)$	fraction of electrical consumption for production technology i, which corresponds to peak consumption in period t

(continued on the next page)

Variable definitions (continued)	
$elf_{j(i)}(t)$	fraction of capacity for demand technology i, which corresponds to the peak consumption in period t
bl	maximum fraction of nighttime electrical production from technologies $i \in BAS$
α	annual discount rate
n	number of years per period
$invcost_i(t)$	cost per unit investment in technology i, period t
$fixom_i(t)$	fixed annual operation and maintenance costs for technology i, period t, per unit capacity
$varom_i(t)$	variable annual operation and maintenance costs, per unit production, for non-demand technology i, period t
$cost_{ks}(t)$	unit cost of energy type k, purchased from source s in period t
$cost_{ELC,s}(t)$	unit cost of electricity, purchased from source s in period t
$price_{ks}(t)$	unit price of energy type k, sold to source s in period t
$price_{ks}(t)$	unit price of electricity, sold to source s in period t
$co2_i(t)$	carbon dioxide emissions per unit capacity, from technology i, period t
$\boldsymbol{limit_{CO2}}(t)$	limit imposed on carbon dioxide emissions in period t
$\boldsymbol{\delta}(t)$	probability of a law allowing imported electricity to count toward a CO_2 limit

The capacity of each technology was either installed after the beginning of the optimization problem, or it was there from the beginning. From this, we get the constraint

$$C_i(t) = \sum_{m=\text{Max}\{1,t-l_i+1\}}^{t} I_i(m) + resid_i(t), \quad \forall t \in T, \forall i.$$

We must meet the demand for each utility in each round. Thus,

$$\sum_{i \in DMD(k)} C_i(t) + \sum_{\substack{i \in DMD \\ i \notin DMD(k)}} out_{ik}(t) C_i(t) \geq demand_k(t),$$

$$\forall k \in DM, \forall t \in T.$$

Of course, we cannot produce more than the capacity. For general production technologies, this constraint is

$$P_i(t) \leq af_i(t) C_i(t), \quad \forall i \in PRC, \forall t \in T.$$

Table 13: Set definitions

ENC	the energy types, except electricity (ELC) and low temperature heat (LTH)
T	time periods
TCH	supply and demand technologies
DMD	demand technologies
$DMD(k)$	demand technologies which can only supply utility demand $k \in DM$
DM	utility demands
Y	parts of the day (d for daytime, n for nighttime)
Z	seasons of the year (w for winter, s for summer, i for intermediate)
ELA	technologies that produce electricity
PRC	energy production technologies
STG	technologies that effectively allow the storage of electricity
HPL	technologies which produce low temperature heat (LTH)
CON	technologies which produce electricity and/or low temperature heat
BAS	electrical production technologies which produce only at a steady rate, day and night
$CO2$	technologies which emit carbon dioxide

For technologies that produce electricity, the production constraint is

$$P_{izy}(t) + \left(\frac{q_{zy}}{q_{zd} + q_{zn}}\right) \leq u_i q_{zy} \left(1 - [1 - af_i(t)]\, fo_i\right) C_i(t),$$
$$\forall i \in ELA, \forall z \in Z, \forall y \in Y, \forall t \in T.$$

The second term is the production lost due to maintenance.

Similarly for technologies that produce low temperature heat,

$$P_{iz}(t) + M_{iz}(t) \leq u_i(q_{zd} + q_{zn}) \left(1 - [1 - af_i(t)]\, fo_i\right) C_i(t),$$
$$\forall i \in HPL, \forall z \in Z, \forall t \in T. \quad (40)$$

The following constraint pertains to maintenance.

$$\sum_{z \in Z} M_{ix}(t) \geq [1 - af_i(t)][1 - fo_i] u_i C_i(t), \quad \forall i \in CON, \forall t \in T.$$

On any given day, the peak demand level is, of course, higher than the daily average demand. The capacity for production of electricity must be sufficient to cover peak demands, which occur during the day in both winter and summer. The constant er sets how much higher than daily average demand levels the peak can be. The peak constraint for electricity is

$$\frac{\eta}{1 + er} \left[\sum_{i \in ELA} u_i pk_i(t) C_i(t) + \frac{1}{q_{zd}} \sum_s IMPELC_{szd}(t)\right] \geq$$
$$\sum_{i \in PRC} inp_{ELC,i}(t) epk_i(t) P_i(t) + \frac{1}{q_{zd}} \sum_s EXPELC_{szd}(t)$$
$$+ \sum_{i \in DMD} inp_{ELC,i}(t) elf_{j(i)}(t) cf_i(t) \left(\frac{fr_{j(i)zd}}{q_{zd}}\right) C_i(t),$$
$$\forall z \in \{w, s\}, \forall t \in T.$$

The peak demand constraint for district heat is

$$\frac{\rho}{1 + hr} \sum_{i \in HPL} u_i pk_i(t) C_i(t) \geq$$
$$\sum_{i \in DMD} inp_{LTH,i}(t) cf_i(t) \left(\frac{fr_{j(i)wd} + fr_{j(i)wn}}{q_{wd} + q_{wn}}\right) C_i(t), \quad \forall t \in T,$$

where hr is the analog to er for electricity.

Some types of electrical production technologies, here called BAS, can only operate at a constant production level, day and night. We may desire to limit the percentage of production from such technologies, since they do not give hour to hour operation flexibility. The upper bound, *bl* is used in the following constraint:

$$\begin{aligned}
&\sum_{i \in BAS} P_{izn}(t) + \sum_{s} \eta IMPELC_{szn}(t) - EXPELC_{szn}(t) \\
&\quad \leq bl \left[\sum_{i \in ELA} P_{izn}(t) + \sum_{s} \eta IMPELC_{szn}(t) - EXPELC_{szn}(t) \right], \\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall z \in Z, \forall t \in T.
\end{aligned}$$

Fragnière [8] states that the production of greenhouse gases is limited, but we were unable to find an explicitly stated constraint. Therefore, we propose our own of the form

$$\sum_{i \in CO2} co2_i(t) C_i(t) + \boldsymbol{\delta} \sum_{s} \sum_{z \in Z} \sum_{y \in Y} IMPELC_{szy}(t) \leq \boldsymbol{limit_{CO2}}(t), \quad \forall t \in T. \tag{41}$$

The second term on the left hand side represents the possibility of imported electricity counting toward the CO_2 limit. Random $\boldsymbol{\delta}(t) \in (0,1)$ represents the probability of such a rule. Of course, $\delta(1) = 0$ with probability one.

The objective is to minimize capital and operating costs, which can be expressed as

$$\begin{aligned}
&\sum_{t \in T} \frac{1}{(1+\alpha)^{n(t-1)}} \sum_{i \in TCH} invcost_i(t) I_i(t) + \left(\sum_{m=1}^{n} (1+\alpha)^{1-m} \right) \\
&\quad \sum_{t \in T} \frac{1}{(1+\alpha)^{n(t-1)}} \left[\sum_{i \in TCH} fixom_i(t) C_i(t) + \sum_{i \in PRC} varom_i(t) P_i(t) + \right. \\
&\qquad \sum_{i \in HPL} \sum_{z \in Z} varom_i(t) P_{iz}(t) + \sum_{i \in ELA} \sum_{z \in Z} \sum_{y \in Y} varom_i(t) P_{izy}(t) + \\
&\sum_{k \in ENC} \sum_{s} cost_{ks}(t) IMP_{ks}(t) + \sum_{s} \sum_{z \in Z} \sum_{y \in Y} cost_{ELC,s}(t) IMPELC_{szy}(t) - \\
&\qquad\qquad \sum_{k \in ENC} \sum_{s} price_{ks}(t) EXP_{ks}(t) - \\
&\qquad\qquad\qquad \left. \sum_{s} \sum_{z \in Z} \sum_{y \in Y} price_{ELC,s}(t) EXPELC_{szy}(t) \right].
\end{aligned}$$

3.2 Problem statement

We present a problem that is a reduced version that created by Fragnière [8]. Production constraints of the type (40) are not included. This problem statement corresponds to the numerical examples given in the "Numerical examples" section.

Minimize

$$\sum_{t\in T} \frac{1}{(1+\alpha)^{n(t-1)}} \sum_{i\in TCH} invcost_i(t)I_i(t) + \left(\sum_{m=1}^{n}(1+\alpha)^{1-m}\right)$$

$$\sum_{t\in T} \frac{1}{(1+\alpha)^{n(t-1)}} \left[\sum_{i\in TCH} fixom_i(t)C_i(t) + \sum_{i\in PRC} varom_i(t)P_i(t) + \right.$$

$$\sum_{i\in HPL}\sum_{z\in Z} varom_i(t)P_{iz}(t) + \sum_{i\in ELA}\sum_{z\in Z}\sum_{y\in Y} varom_i(t)P_{izy}(t) +$$

$$\sum_{k\in ENC}\sum_{s} cost_{ks}(t)IMP_{ks}(t) + \sum_{s}\sum_{z\in Z}\sum_{y\in Y} cost_{ELC,s}(t)IMPELC_{szy}(t) -$$

$$\sum_{k\in ENC}\sum_{s} price_{ks}(t)EXP_{ks}(t) -$$

$$\left.\sum_{s}\sum_{z\in Z}\sum_{y\in Y} price_{ELC,s}(t)EXPELC_{szy}(t)\right].$$

subject to

$$\begin{aligned}
&\sum_{\substack{i\in TCH\\ i\notin DMD}} out_{ki}(t)P_i(t) + \sum_{i\in DMD} out_{ki}(t)cf_i(t)C_i(t) + \sum_{s} IMP_{ks}(t)\\
\geq &\sum_{\substack{i\in TCH\\ i\notin DMD}} inp_{ki}(t)P_i(t) + \sum_{i\in DMD} inp_{ki}(t)cf_i(t)C_i(t) + \sum_{s} EXP_{ks}(t),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \forall k\in ENC, \forall t\in T,
\end{aligned}$$

$$\eta \left[\sum_{i \in ELA} P_{izy}(t) + \sum_{s} IMPELC_{szy}(t) \right] \geq \sum_{i \in PRC} inp_{ELC,i}(t) q_{zy} P_i(t)$$
$$+ \sum_{i \in DMD} inp_{ELC,i}(t) cf_i(t) fr_{j(i)zy} C_i(t) + \sum_{k} EXPELC_{kzy}(t)$$
$$+ \eta \sum_{\substack{i \in STG \\ \ni y=n}} e_i P_{izd}(t), \quad \forall z \in Z, \forall y \in Y, \forall t \in T,$$

$$\gamma \sum_{i \in HPL} P_{iz}(t) \geq \sum_{i \in DMD} inp_{LTH,i}(t) cf_i(t) C_i(t) \sum_{y \in Y} fr_{j(i)zy},$$
$$\forall z \in Z, \forall t \in T,$$

$$C_i(t) = \sum_{m=\text{Max}\{1, t-l_i+1\}}^{t} I_i(m) + resid_i(t), \quad \forall t \in T, \forall i,$$

$$\sum_{i \in DMD(k)} C_i(t) + \sum_{\substack{i \in DMD \\ i \notin DMD(k)}} out_{ik}(t) C_i(t) \geq demand_k(t),$$
$$\forall k \in DM, \forall t \in T,$$

$$P_i(t) \leq af_i(t) C_i(t), \quad \forall i \in PRC, \forall t \in T,$$

$$P_{izy}(t) + \left(\frac{q_{zy}}{q_{zd} + q_{zn}} \right) \leq u_i q_{zy} \left(1 - [1 - af_i(t)] fo_i \right) C_i(t),$$
$$\forall i \in ELA, \forall z \in Z, \forall y \in Y, \forall t \in T,$$

$$\sum_{z \in Z} M_{ix}(t) \geq [1 - af_i(t)][1 - fo_i] u_i C_i(t), \quad \forall i \in CON, \forall t \in T,$$

$$\frac{\eta}{1 + er} \left[\sum_{i \in ELA} u_i pk_i(t) C_i(t) + \frac{1}{q_{zd}} \sum_{s} IMPELC_{szd}(t) \right] \geq$$
$$\sum_{i \in PRC} inp_{ELC,i}(t) epk_i(t) P_i(t) + \frac{1}{q_{zd}} \sum_{s} EXPELC_{szd}(t)$$
$$+ \sum_{i \in DMD} inp_{ELC,i}(t) elf_{j(i)}(t) cf_i(t) \left(\frac{fr_{j(i)zd}}{q_{zd}} \right) C_i(t),$$
$$\forall z \in \{w, s\}, \forall t \in T,$$

$$\frac{\rho}{1+hr}\sum_{i\in HPL} u_i pk_i(t)C_i(t) \geq \sum_{i\in DMD} inp_{LTH,i}(t)cf_i(t)\left(\frac{fr_{j(i)wd}+fr_{j(i)wn}}{q_{wd}+q_{wn}}\right)C_i(t), \quad \forall t\in T,$$

$$\sum_{i\in BAS} P_{izn}(t)+\sum_{s}\eta IMPELC_{szn}(t)-EXPELC_{szn}(t) \leq bl\left[\sum_{i\in ELA} P_{izn}(t)+\sum_{s}\eta IMPELC_{szn}(t)-EXPELC_{szn}(t)\right], \quad \forall z\in Z, \forall t\in T,$$

$$\sum_{i\in CO2} co2_i(t)C_i(t)+\boldsymbol{\delta}(t)\sum_{s}\sum_{z\in Z}\sum_{y\in Y} IMPELC_{szy}(t) \leq \boldsymbol{limit_{CO2}}(t), \quad \forall t\in T$$

and for all $t \in T$,

$$\begin{gathered}
P_i(t)\geq 0,\ \forall i\in PRC, \quad I_i(t)\geq 0,\ \forall i\in TCH, \\
C_i(t)\geq 0,\ \forall i\in TCH, \quad P_{iz}(t)\geq 0,\ \forall i\in HPL, \forall z\in Z, \\
P_{izy}(t)\geq 0,\ \forall i\in ELA, \forall z\in Z, \forall y\in Y, \quad IMP_{ks}(t)\geq 0, \forall k\in ENC \forall s, \\
IMPELC_{szy}(t)\geq 0, \forall s, \forall z\in Z, \forall y\in Y, \quad EXP_{ks}(t)\geq 0, \forall k\in ENC\forall s, \\
EXPELC_{szy}(t)\geq 0, \forall s, \forall z\in Z, \forall y\in Y, \quad \delta(1)=0.
\end{gathered}$$

3.3 Numerical examples

The problem created by Fragnière [8] for the Canton of Geneva is extremely large and complex, and the input data format is not SMPS. Therefore, we have created our own sample problems of this kind. The numbers in this example are based on the authors' judgment, not actual economic data.

The example creates a situation similar to that experienced in the United States, where oil imports (OIL) are the largest source of energy. Other imports are coal (COL), natural gas (NGS), propane (PRO), nuclear fuel (NUF), and electricity (ELC). There are no exports in this example.

The energy types allowed are electricity (ELC), gasoline (GAS), coal (COL), heating oil and diesel (HOL), natural gas (NGS), propane/LPG (LPG), jet fuel (JET), and nuclear fuel (NUC). When one unit of oil is imported, the following portions of hydrocarbon based energy types are assumed to be gained: 0.45 gasoline, 0.25 heating oil/diesel, 0.10 natural gas, 0.10 jet fuel, and 0.10 propane/LPG. The inequalities of type (39) must take this into account. For example, the inequality balancing natural gas is

$$0.10 IMP_{\mathrm{OIL}}(t) + IMP_{\mathrm{NG}} \geq inp_{\mathrm{NGS,HNG}}(t) cf_{\mathrm{HNG}}(t) C_{\mathrm{HNG}}(t) \\ + inp_{\mathrm{NGS,NEL}}(t) cf_{\mathrm{NEL}}(t) C_{\mathrm{NEL}}(t).$$

The available technologies are listed in Table 17, along with their associated coefficients for the example problem. Other coefficients are listed in Table 14, Table 15 and Table 16.

There are several two stage versions of this problem in the test set. They differ in how stochasticity is introduced. The problem *env:loose*, using the stochastic file `env.sto.loose`, simply assumes very non-challenging (i.e. loose) CO_2 limits. The problem *env:aggressive* (`env.sto.aggr`) sets aggressive CO_2 limits. Each of these has five random realizations, and the parameter $\boldsymbol{\delta}(t)$ takes a value 0 with probability one.

The problem *env:import* (`env.sto.imp`) uses the aggressive CO_2 limits, and, in addition, considers the possibility that imported electricity (IMPELC) will be counted toward such limits in period two. That is, $\boldsymbol{\delta}(2)$ takes a nonzero value with nonzero probability. This problem has fifteen random realizations.

The problem *env:large* (`env.sto.lrge`) builds on *env:import* by making random the costs of various energy sources. The number of realizations is $8,232$. The problem *env:xlarge* (`env.sto.xlrge`) is a larger version still, mostly to test distributed memory capabilities of the solver.

Table 14: Example problem seasonal coefficients

	summer		winter	
	day	night	day	night
q_{zy}	0.60	0.40	0.40	0.60
$cost_{\mathrm{ELC}}$	5.2	5.0	4.8	4.6
$fr_{\mathrm{ELC},zy}$	0.35	0.25	0.10	0.30

Table 15: Example problem demands

k	$demand_k(1)$	$demand_k(2)$
ELC	170	230
HHO	30	30
NG	15	25
GAS	60	80
LPG	3	3
JET	10	20

Table 16: Example problem coefficients

α	0.05
n	5
η	0.80
e_{HYD}	0.10
er	0.20
$cost_{\mathrm{OIL}}$	0.8
$cost_{\mathrm{COAL}}$	0.7
$cost_{\mathrm{NG}}$	0.6
$cost_{\mathrm{PRO}}$	0.7
$cost_{\mathrm{NUF}}$	0.9

Table 17: Example problem technologies and associated coefficients

description	i	$cf_i(t)$	$inp_{ki}(t)^{\dagger}$	$\begin{pmatrix} resid_i(1): \\ resid_i(2) \end{pmatrix}$	$u_i(t)$	$pk_i(t)$	$invcost_i(1)^{\ddagger}$	$fixom_i(t)$	$varom_i(t)$	$co2_i(t)$
industrial electricity	ELI	0.8	3.0	(100 : 100)		0.1	30	3.0	0.1	0.0
domestic electricity	ELD	0.8	2.8	(110 : 110)		0.2	50	4.0	0.1	0.0
heating oil/diesel	HHO	0.4	1.0	(15 : 15)			20	2.0	0.5	1.4
household nat. gas	HNG	0.4	0.5	(25 : 25)			25	2.0	0.5	1.0
automobiles	CAR	0.5	0.7	(80 : 80)			40	2.0	0.5	1.5
household propane	HLP	0.4	0.6	(5 : 5)			30	2.5	0.8	1.1
elec. from coal	CEL	0.75	0.7	(110 : 100)	$\frac{1}{0.7}$	0.1	30	3.0	0.4	1.2
elec. from NG	NEL	0.7	0.4	(40 : 40)	$\frac{1}{0.4}$	0.3	35	2.0	0.5	0.9
jet fuel prod.	AIR	0.75	0.8	(15 : 15)			40	2.0	0.5	1.5
diesel trucks	TRK	0.75	1.0	(30 : 30)			40	2.0	05	1.7
elec. from nucl.	NUL	0.8	0.1	(40 : 20)	$\frac{1}{0.7}$	0.10	80	4	0.7	0.0
elec. from hydr.	HYD	0.8	0.8	(10 : 8)	$\frac{1}{0.8}$	0.2	80	4	0.1	0.0

†For the obvious k.

‡For $t = 2$, multiply value by 1.5.

3.4 Notational reconciliation

Because of the size of the problem, we reconcile only the problem from the Numerical examples section to the format of (2). In this section, we will use "ELI ... HYD" to denote the set of technologies listed in Table 17, in the order presented. We will also use "OIL ... ELC" to denote the imports, and "ELC ... JET" for the demands in Table 15. Additionally, "WD ... SN" will mean the sequence "WD, WN, SD, SN," and "CEL ... HYD" will stand for the sequence of electricity producers "CEL, NEL, NUL, HYD." These abbreviations will make our arrays smaller to print.

We will also use the notation e_i to mean the unit vector in the ith direction from the space $\mathbb{R}^{12}$.

For $t = 1, 2$, make the following definitions:

$$x_t := \left[\begin{array}{c} I_{\text{ELI}}(t) \\ \vdots \\ I_{\text{HYD}}(t) \\ \hline C_{\text{ELI}}(t) \\ \vdots \\ C_{\text{HYD}}(t) \\ \hline \begin{bmatrix} P_{\text{CEL,WD}}(t) \\ \vdots \\ P_{\text{CEL,SN}}(t) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} P_{\text{HYD,WD}}(t) \\ \vdots \\ P_{\text{HYD,SN}}(t) \end{bmatrix} \\ \hline IMP_{\text{OIL}}(t) \\ \vdots \\ IMP_{\text{NUF}}(t) \\ \hline IMPELC_{\text{WD}}(t) \\ \vdots \\ IMPELC_{\text{SN}}(t) \end{array}\right], \qquad c_t := \left[\begin{array}{c} r(t)invcost_{\text{ELI}}(t) \\ \vdots \\ r(t)invcost_{\text{HYD}}(t) \\ \hline s(t)fixom_{\text{ELI}}(t) \\ \vdots \\ s(t)fixom_{\text{HYD}}(t) \\ \hline \begin{bmatrix} s(t)varom_{\text{CEL}}(t) \\ \vdots \\ s(t)varom_{\text{CEL}}(t) \end{bmatrix} \\ \vdots \\ \begin{bmatrix} s(t)varom_{\text{HYD}}(t) \\ \vdots \\ s(t)varom_{\text{HYD}}(t) \end{bmatrix} \\ \hline s(t)cost_{\text{OIL}} \\ \vdots \\ s(t)cost_{\text{NUF}} \\ \hline s(t)cost_{\text{ELC,WD}} \\ \vdots \\ s(t)cost_{\text{ELC,SN}} \end{array}\right],$$

where $r(t) := (1+\alpha)^{-n(t-1)}$ and $s(t) := \left(\sum_{m=1}^{n} (1+\alpha)^{1-m}\right) r(t)$.

Define the following matrices:

$$BA(t) := \begin{bmatrix} inp_{\mathrm{GAS,CAR}}(t) cf_{\mathrm{CAR}}(t) e_5^\mathsf{T} \\ inp_{\mathrm{COL,CEL}}(t) cf_{\mathrm{CEL}}(t) e_7^\mathsf{T} + inp_{\mathrm{HOL,TRK}}(t) cf_{\mathrm{TRK}}(t) e_{10}^\mathsf{T} \\ inp_{\mathrm{HOL,HHO}}(t) cf_{\mathrm{HHO}}(t) e_3^\mathsf{T} + inp_{\mathrm{NGS,NEL}}(t) cf_{\mathrm{NEL}}(t) e_8^\mathsf{T} \\ inp_{\mathrm{NGS,HNG}}(t) cf_{\mathrm{HNG}}(t) e_4^\mathsf{T} \\ inp_{\mathrm{LPG,HLP}}(t) cf_{\mathrm{HLP}}(t) e_6^\mathsf{T} \\ inp_{\mathrm{JET,AIR}}(t) cf_{\mathrm{AIR}}(t) e_9^\mathsf{T} \\ inp_{\mathrm{NUC,NUL}}(t) cf_{\mathrm{NUL}}(t) e_{11}^\mathsf{T} \end{bmatrix},$$

$$BB(t) := \begin{bmatrix} -0.45 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -0.25 & 0 & 0 & 0 & 0 & 0 \\ -0.10 & 0 & 0 & 0 & 0 & 0 \\ -0.10 & 0 & 0 & 0 & 0 & 0 \\ -0.10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix},$$

$$BC(t) := -\eta \begin{bmatrix} I^{4\times4} & I^{4\times4} & I^{4\times4} & I^{4\times4} \end{bmatrix}, \qquad BD(t) := -\eta \left[I^{4\times4} \right]$$

$$BE(t) :=$$

$$\left[\begin{array}{cc|c} inp_{\mathrm{ELC,ELI}}(t) cf_{\mathrm{ELI}}(t) fr_{\mathrm{ELC,WD}} & inp_{\mathrm{ELC,ELD}}(t) cf_{\mathrm{ELD}}(t) fr_{\mathrm{ELC,WD}} & \\ \vdots & \vdots & 0^{4\times10} \\ inp_{\mathrm{ELC,ELI}}(t) cf_{\mathrm{ELI}}(t) fr_{\mathrm{ELC,SN}} & inp_{\mathrm{ELC,ELD}}(t) cf_{\mathrm{ELD}}(t) fr_{\mathrm{ELC,SN}} & \end{array} \right],$$

$$BF(t) := \eta e_{\mathrm{HYD}} \begin{bmatrix} 0^{4\times12} & I^{4\times4} \end{bmatrix}, \qquad BH(t) := -\begin{bmatrix} (e_1 + e_2)^\mathsf{T} \\ (e_3 + e_{10})^\mathsf{T} \\ e_4^\mathsf{T} \\ e_5^\mathsf{T} \\ e_6^\mathsf{T} \\ e_9^\mathsf{T} \end{bmatrix},$$

$$BM := \begin{bmatrix} \left(\frac{1}{q_{\mathrm{WD}}}\right) e_1^\mathsf{T} \\ \left(\frac{1}{q_{\mathrm{SD}}}\right) e_2^\mathsf{T} \end{bmatrix}, \qquad BN := [co2_{\mathrm{ELI}} \cdots co2_{\mathrm{HYD}}],$$

$$
BK := -\begin{bmatrix} \begin{bmatrix} u_{\mathrm{CEL}} q_{\mathrm{WD}} e_7^{\mathsf{T}} \\ \vdots \\ u_{\mathrm{CEL}} q_{\mathrm{SN}} e_7^{\mathsf{T}} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} u_{\mathrm{HYD}} q_{\mathrm{WD}} e_{12}^{\mathsf{T}} \\ \vdots \\ u_{\mathrm{HYD}} q_{\mathrm{SN}} e_{12}^{\mathsf{T}} \end{bmatrix} \end{bmatrix}, \qquad bk := -\begin{bmatrix} q_{\mathrm{WD}}/(q_{\mathrm{WD}} + q_{\mathrm{WN}}) \\ q_{\mathrm{WN}}/(q_{\mathrm{WD}} + q_{\mathrm{WN}}) \\ q_{\mathrm{SD}}/(q_{\mathrm{SD}} + q_{\mathrm{SN}}) \\ q_{\mathrm{SN}}/(q_{\mathrm{SD}} + q_{\mathrm{SN}}) \end{bmatrix},
$$

$$
\begin{aligned}
BL(t) := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{n}{1+er} \right) [-u_{\mathrm{CEL}} pk_{\mathrm{CEL}}(t) e_7 - \\
u_{\mathrm{NEL}} pk_{\mathrm{NEL}}(t) e_8 - u_{\mathrm{NUL}} pk_{\mathrm{NUL}}(t) e_{11} - u_{\mathrm{HYD}} pk_{\mathrm{HYD}}(t) e_{12}]^{\mathsf{T}} + \\
cf_{\mathrm{ELI}}(t) \begin{bmatrix} elf_{\mathrm{ELC,WD}}(t) fr_{\mathrm{ELC,WD}}/q_{\mathrm{WD}} \\ elf_{\mathrm{ELC,SD}}(t) fr_{\mathrm{ELC,SD}}/q_{\mathrm{SD}} \end{bmatrix} (inp_{\mathrm{ELC,ELI}}(t) e_1^{\mathsf{T}} + \\
inp_{\mathrm{ELC,ELD}}(t) e_2^{\mathsf{T}}),
\end{aligned}
$$

and the random

$$
\boldsymbol{BP}(t) := \begin{bmatrix} \boldsymbol{\delta}(t) & \boldsymbol{\delta}(t) & \boldsymbol{\delta}(t) & \boldsymbol{\delta}(t) \end{bmatrix}.
$$

Then, finally, we can assign $\boldsymbol{A_t}$, for $t = 1, 2$, and T_2 in blocks. Let

$$
\boldsymbol{A_t} := \left[\begin{array}{c|c|c|c|c}
 & BA(t) & & BB(t) & \\ \hline
 & BE(t) & (BC(t) + BF(t)) & & BD(t) \\ \hline
-I^{12\times 12} & I^{12\times 12} & & & \\ \hline
 & BH & & & \\ \hline
 & BK & I^{16\times 16} & & \\ \hline
 & BL(t) & & & BM \\ \hline
 & BN & & & \boldsymbol{BP}
\end{array} \right],
$$

and

$$
T_2 := \left[\begin{array}{c}
0^{7\times 50} \\ \hline
0^{4\times 50} \\ \hline
\begin{array}{c|c|c|c|c} \; & -I^{12\times 12} & \; & \; & \; \end{array} \\ \hline
0^{6\times 50} \\ \hline
0^{16\times 50} \\ \hline
0^{2\times 50} \\ \hline
0^{1\times 50}
\end{array} \right].
$$

We define the random right hand side as

$$
\boldsymbol{b}_t := \begin{bmatrix} 0^7 \\ 0^4 \\ resid_{\mathrm{ELI}}(t) \\ \vdots \\ resid_{\mathrm{HYD}}(t) \\ -demand_{\mathrm{ELC}}(t) \\ -demand_{\mathrm{HHO}}(t) \\ -demand_{\mathrm{NGS}}(t) \\ -demand_{\mathrm{GAS}}(t) \\ -demand_{\mathrm{LPG}}(t) \\ -demand_{\mathrm{JET}}(t) \\ b_k \\ b_k \\ b_k \\ b_k \\ 0^2 \\ \boldsymbol{limit_{CO2}}(t) \end{bmatrix}.
$$

If the user then appends slack variables in the blocks corresponding to $BA(t)$, $BE(t)$, $BH(t)$, $BK(t)$, $BL(t)$ and $BN(t)$, we will have the problem in the form of (2).

3.5 Network model for asset or liability management

Due to J. M. Mulvey and H. Vladimirou [21]
See also Mulvey and Ruszczyński [20].
(Two-stage, linear stochastic problem)

`/assets/assets.cor`,`/assets.tim`, $\begin{cases}\texttt{/assets.sto.small}\\ \texttt{/assets.sto.large}\end{cases}$

3.5.1 Description

The management of assets or liabilities can be looked at as a network problem, where the asset categories are represented by nodes, and transactions are represented by arcs. The purchase or sale of an asset usually has fixed, deterministic associated costs, while the return on an investment from one stage to the next is usually unknown.

Let the set of nodes be $\mathcal{N}$, and let $\mathcal{A}$ be the set of arcs. There exists a set of terminal arcs $\mathcal{T} \subset \mathcal{A}$, over which the objective value will be calculated. Define the following notation:

$\mathcal{A}_1$=the subset of arcs associated with deterministic multipliers and first stage decisions
$\mathcal{A}_2$=the subset of arcs associated with stochastic multipliers and first stage decisions
$\mathcal{A}_3$=the subset of arcs associated with second stage decisions
$\mathcal{N}_1$=the subset of nodes with deterministic balance equations
$\mathcal{N}_2$=$\mathcal{N} \backslash \mathcal{N}_1$
D_n^+=the set of outgoing arcs at node n
D_n^-=the set of incoming arcs at node n
z_a=flow along arc $a \in \mathcal{A}_1 \cup \mathcal{A}_2$
y_a=flow along arc $a \in \mathcal{A}_3$
$\boldsymbol{r_a}$=multiplier for arc $a \in \mathcal{A}$
$\boldsymbol{b_n}$=supply or demand at node $n \in \mathcal{N}$
l_a=lower bound for arc $a \in \mathcal{A}$
u_a=upper bound for arc $a \in \mathcal{A}$.

Then the problem statement follows simply from a material balance at each node.

3.5.2 Problem statement

Given $\{r_a : a \in \mathcal{A}_1\}$, $\{b_n : n \in \mathcal{N}_1\}$ and $\{(l_a, u_a) : a \in \mathcal{A}\}$, the problem is to

$$\text{maximize} \quad \sum_{a \in \mathcal{A}_1 \cap \mathcal{T}} r_a z_a + \underset{\{\boldsymbol{r_a}, \boldsymbol{b_n}\}}{E} \left[\sum_{a \in \mathcal{A}_2 \cap \mathcal{T}} \boldsymbol{r_a} z_a + \sum_{a \in \mathcal{A}_3 \cap \mathcal{T}} \boldsymbol{r_a} y_a \right]$$

$$\text{subject to} \quad \sum_{a \in D_n^+} z_a - \sum_{a \in D_n^-} r_a z_a = b_n, \qquad n \in \mathcal{N}_1,$$

$$\sum_{a \in D_n^+ \cap (\mathcal{A}_1 \cup \mathcal{A}_2)} z_a - \sum_{a \in D_n^- \cap \mathcal{A}_1} r_a z_a - \sum_{a \in D_n^- \cap \mathcal{A}_2} \boldsymbol{r_a} z_a +$$

$$\sum_{a \in D_n^+ \cap \mathcal{A}_3} y_a - \sum_{a \in D_n^- \cap \mathcal{A}_3} \boldsymbol{r_a} y_a = \boldsymbol{b_n}, \quad n \in \mathcal{N}_2,$$

$$l_a \le z_a \le u_a, \quad a \in \mathcal{A}_1 \cup \mathcal{A}_2,$$

$$l_a \le y_a \le u_a, \quad a \in \mathcal{A}_3.$$

3.5.3 Numerical examples

Mulvey and Vladimirou [21] did not provide data for the numerical examples that they discuss [19], so we have created two examples, each with two stages. There are five nodes in each stage: checking, savings, certificate of deposit (CD), cash, and loans, with initial balances of 100, 200, 150, 80, and -80, respectively.

Of course, the yields are specified as random. The smaller problem, using stochastic file `assets.sto.small`, has 100 random realizations, while the larger problem, using `assets.sto.large`, has 37,500 realizations.

3.5.4 Notational reconciliation

Suppose the cardinality of $\mathcal{A}_1 \cup \mathcal{A}_2$ is n_1, and that of $\mathcal{A}_3$ is n_2. Enumerate the arcs so that arcs 1 through n_1 are in $\mathcal{A}_1 \cup \mathcal{A}_2$. Reorder the nodes so that the first N_1 are in the set $\mathcal{N}_1$, and the rest are in $\mathcal{N}_2$. For the set $\Phi \in \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{T}, \mathcal{D}_n^+, \mathcal{D}_n^-\}$, define the diagonal matrix $\Delta_1^\Phi \in \mathbb{R}^{n_1 \times n_1}$ by

$$(\Delta_1^\Phi)_{aa} := \begin{cases} 1 & \text{if } a \in \Phi \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for the set $\Phi \in \{\mathcal{A}_3, \mathcal{T}, \mathcal{D}_n^+, \mathcal{D}_n^-\}$, define the diagonal matrix $\Delta_2^\Phi \in \mathbb{R}^{n_2 \times n_2}$ by

$$(\Delta_2^\Phi)_{aa} := \begin{cases} 1 & \text{if } (n_1 + a) \in \Phi \\ 0 & \text{otherwise.} \end{cases}$$

The notation of (2) requires determinism in all of the first stage coefficients. However, since the only stochastic first stage coefficients are the costs, we can use the expected value. For arc $a \in \mathcal{A}_1 \cup \mathcal{A}_2$ define

$$\bar{r}_a := \ E\ [\boldsymbol{r_a}].$$

Note that for $a \in \mathcal{A}_1$, this is simply r_a. Additionally, define $\boldsymbol{\hat{r}_1} \in \mathbb{R}^{n_1}$ by

$$(\boldsymbol{\hat{r}_1})_a := \begin{cases} r_a & \text{if } a \in \mathcal{A}_1 \\ \boldsymbol{r_a} & \text{if } a \in \mathcal{A}_2. \end{cases}$$

Similarly, let $\boldsymbol{\hat{r}_2} \in \mathbb{R}^{n_2}$ be defined by

$$(\boldsymbol{\hat{r}_2})_a := \boldsymbol{r_{(n_1+a)}}, \quad \forall a \ni (n_1 + a) \in \mathcal{A}_3.$$

To fit the notation of (2), we make the following assignments:

$$x_1 := \left[\begin{array}{c} z_1 \\ \vdots \\ z_{n_1} \\ \hline s_1^1 \\ \vdots \\ s_{n_1}^1 \\ s_1^2 \\ \vdots \\ s_{n_1}^2 \end{array}\right], \quad \bar{r} := \begin{bmatrix} \bar{r}_1 \\ \vdots \\ \bar{r}_{n_1} \end{bmatrix}, \quad c_1 := \left[\begin{array}{c} \bar{r} \\ \hline 0^{2n_1 \times 1} \end{array}\right], \quad b_1 := \left[\begin{array}{c} b_1 \\ \vdots \\ b_{N_1} \\ \hline u_1 \\ \vdots \\ u_{n_1} \\ -l_1 \\ \vdots \\ -l_{n_1} \end{array}\right],$$

$$A_1 := \left[\begin{array}{c|cc} (1^{1\times n_1}\Delta_1^{D_1^+} - \bar{r}^\intercal \Delta_1^{D_1^-}) & & \\ \vdots & \multicolumn{2}{c}{0^{N_1 \times 2n_1}} \\ (1^{1\times n_1}\Delta_1^{D_{N_1}^+} - \bar{r}^\intercal \Delta_1^{D_{N_1}^-}) & & \\ \hline I^{n_1 \times n_1} & I^{n_1 \times n_1} & 0^{n_1 \times n_1} \\ -I^{n_1 \times n_1} & 0^{n_1 \times n_1} & I^{n_1 \times n_1} \end{array}\right],$$

$$
x_2 := \left[\begin{array}{c} y_1 \\ \vdots \\ y_{n_2} \\ \hline s_1^3 \\ \vdots \\ s_{n_2}^3 \\ s_1^4 \\ \vdots \\ s_{n_2}^4 \end{array}\right], \quad \boldsymbol{c_2} := \left[\begin{array}{c} \boldsymbol{\hat{r}_2} \\ \hline 0^{2n_2 \times 1} \end{array}\right], \quad \boldsymbol{b_2} := \left[\begin{array}{c} \boldsymbol{b_{(N_1+1)}} \\ \vdots \\ \boldsymbol{b_{(N_1+N_2)}} \\ \hline u_{(n_1+1)} \\ \vdots \\ u_{(n_1+n_2)} \\ -l_{(n_1+1)} \\ \vdots \\ -l_{(n_1+n_2)} \end{array}\right],
$$

$$
\boldsymbol{T_2} := \left[\begin{array}{c|c} (1^{1\times n_1}\Delta_1^{D^+_{(N_1+1)}} - \boldsymbol{\hat{r}_1}^{\mathsf{T}}\Delta_1^{D^-_{(N_1+1)}})(\Delta_1^{\mathcal{A}_1} + \Delta_1^{\mathcal{A}_2}) & \\ \vdots & 0^{N_2\times 2n_1} \\ (1^{1\times n_1}\Delta_1^{D^+_{(N_1+N_2)}} - \boldsymbol{\hat{r}_1}^{\mathsf{T}}\Delta_1^{D^-_{(N_1+N_2)}})(\Delta_1^{\mathcal{A}_1} + \Delta_1^{\mathcal{A}_2}) & \\ \hline 0^{2n_2\times n_1} & 0^{2n_2\times 2n_1} \end{array}\right],
$$

and

$$
\boldsymbol{A_2} := \left[\begin{array}{c|cc} (1^{1\times n_2}\Delta_2^{D^+_{(N_1+1)}} - \boldsymbol{\hat{r}_2}^{\mathsf{T}}\Delta_2^{D^-_{(N_1+1)}})\Delta_2^{\mathcal{A}_3} & & \\ \vdots & \multicolumn{2}{c}{0^{N_2\times 2n_2}} \\ (1^{1\times n_2}\Delta_2^{D^+_{(N_1+N_2)}} - \boldsymbol{\hat{r}_2}^{\mathsf{T}}\Delta_2^{D^-_{(N_1+N_2)}})\Delta_2^{\mathcal{A}_3} & & \\ \hline I^{n_2\times n_2} & I^{n_2\times n_2} & 0^{n_2\times n_2} \\ -I^{n_2\times n_2} & 0^{n_2\times n_2} & I^{n_2\times n_2} \end{array}\right].
$$

3.6 Cargo network scheduling

Due to Mulvey and Ruszczyński [20]
(Two-stage, mixed integer linear or nonlinear stochastic problem)

`/cargo/4node.cor`, `/4node.tim`, $\begin{cases} \texttt{/4node\_det.sto} \\ \texttt{/4node.sto.16} \\ \texttt{/4node.sto.32} \\ \texttt{/4node.sto.64} \end{cases}$

3.6.1 Description

Mulvey and Ruszczyński [20] provide a two stage network problem for scheduling cargo transportation. The flight schedule is completely determined in stage one, and the amounts of cargo to be shipped are uncertain. The recourse actions are to determine which cargo to place on which flights. Transshipment, getting cargo from node m to node n by more than one flight on more than one route, is allowed. When a transshipment is made, cargo must be unloaded at some intermediate node, so that it may be loaded onto a different route going through the same node. Such nodes are called transshipment nodes. Any undelivered cargo costs a penalty.

The notation is introduced in Table 18. A route $\pi \in \mathcal{P}$ is a finite sequence of nodes $(n_1, n_2, \ldots, n_l)$ to be visited in the course of flying the route.

Table 18: Notation

$\mathcal{N}$=	the set of nodes
$\mathcal{P}$=	the set of routes
$\mathcal{A}$=	the set of aircraft types
$\boldsymbol{b}(m, n)$=	the amount of cargo to be shipped from node m to node n
$c(a)$=	the cost of an hour of flight time for aircraft type a
$h(\pi, a)$=	flight hours required for aircraft type a to complete route π
q=	the unit cargo cost for loading and unloading an aircraft
ρ=	the unit penalty for undelivered cargo
$v(\pi, j)$=	function which returns the jth node in route π
$l(\pi)$=	function which returns the number of nodes in route π
$\sigma(n)$=	the maximum number of landings allowed in node n
$d(a)$=	the maximum payload of an aircraft of type a
	(continued on the next page)

Notation (continued)
$f(m,n)$=the minimum number of flights from node m to node n
$h^{\max}(a)$=the maximum flying hours for aircraft of type a
$h^{\min}(a)$=the minimum flying hours for aircraft of type a
$x(\pi,a)$=the number of aircraft of type a assigned to fly route π
$d(\pi,m,n)$=the amount of cargo delivered directly from m to n on route π
$t(\pi,m,k,n)$=the amount of cargo moving from m to n which is moved to transshipment node k on route π
$s(\pi,k,n)$=the amount of transshipment cargo which is moved from transshipment node k to node n on route π
$y(m,n)$=the amount of cargo moving from m to n which is undelivered
$z(\pi,j)$=the unused capacity of leg j on route π
$U(m,n)$=$\{\pi \in \mathcal{P} : m = v(\pi,j_1), n = v(\pi,j_2), j_1 < j_2\}$
$V_1(n)$=$\{\pi \in \mathcal{P} : n = v(\pi,1)\}$
$V_l(n)$=$\{\pi \in \mathcal{P} : n = v(\pi,l(\pi))\}$
$W(n)$=$\{\pi \in \mathcal{P} : n = v(\pi,j)$ for some $j\}$

The first stage constraints include minimum flight requirements

$$\sum_{a\in\mathcal{A}} \sum_{\pi\in U(m,n)} x(\pi,a) \geq f(m,n), \quad \forall m,n \in \mathcal{N},$$

and maximum landings limits

$$\sum_{a\in\mathcal{A}} \sum_{\pi\in W(n)} x(\pi,a) \leq \sigma(n), \quad \forall n \in \mathcal{N}.$$

Assuming the operation is cyclic, we must end the round in the same state as that in which we began the round. That is,

$$\sum_{\pi\in V_1(n)} x(\pi,a) = \sum_{\pi\in V_l(n)} x(\pi,a), \quad \forall a \in \mathcal{A}, n \in \mathcal{N}.$$

Flying hours are limited by

$$h^{\min}(a) \leq \sum_{\pi\in\mathcal{P}} x(\pi,a)h(\pi,a) \leq h^{\max}(a), \quad \forall a \in \mathcal{A}.$$

For recourse constraints, a cargo material balance yields

$$\sum_{\pi \in \mathcal{P}} \left(d(\pi, m, n) + \sum_{k \in \mathcal{N}} t(\pi, m, k, n) \right) + y(m, n) \geq \boldsymbol{b(m, n)}, \quad \forall m, n \in \mathcal{N}.$$

A balance of all transshipments which go through k and wind up at n gives

$$\sum_{\pi \in \mathcal{P}} \sum_{m \in \mathcal{N}} t(\pi, m, k, n) = \sum_{\pi \in \mathcal{P}} s(\pi, k, n), \quad \forall k, n \in \mathcal{N}.$$

Finally, consider the loading and unloading which must occur throughout the course of a single route. At the initial node, we have

$$\sum_{k \in \mathcal{N}} \left(d(\pi, v(\pi, 1), k) + s(\pi, v(\pi, 1), k) + \sum_{n \in \mathcal{N}} t(\pi, v(\pi, 1), k, n) \right) = \sum_{a \in \mathcal{A}} d(a) x(\pi, a) - z(\pi, 1), \quad \forall \pi \in \mathcal{P}.$$

For the remaining nodes in the route, a payload balance yields

$$\begin{aligned} &\sum_{k \in \mathcal{N}} \left(d(\pi, v(\pi, j), k) + s(\pi, v(\pi, j), k) + \sum_{n \in \mathcal{N}} t(\pi, v(\pi, j), k, n) \right) \\ &\quad - \sum_{k \in \mathcal{N}} \left(d(\pi, k, v(\pi, j)) + s(\pi, k, v(\pi, j)) + \sum_{n \in \mathcal{N}} t(\pi, k, v(\pi, j), n) \right) \\ &\quad = z(\pi, j-1) - z(\pi, j), \quad \forall \pi \in \mathcal{P}, j = 2, \ldots (l(\pi) - 1). \end{aligned}$$

The objective is to minimize the costs and penalties. Mulvey and Ruszczyński [20] provide both a linear objective function

$$\begin{aligned} \text{minimize } Z_1 = &\sum_{\pi \in \mathcal{P}} \sum_{a \in \mathcal{A}} c(a) h(\pi, a) x(\pi, a) + \\ &\underset{\boldsymbol{b(m,n)}}{E} \left\{ q \sum_{\pi \in \mathcal{P}} \sum_{(m,n) \in \pi} \left[d(\pi, m, n) + s(\pi, m, n) + \sum_{k \in \mathcal{N}} t(\pi, m, n, k) \right] \right. \\ &\left. + \rho \sum_{m \in \mathcal{N}} \sum_{n \in \mathcal{N}} y(m, n) \right\}, \end{aligned}$$

and a nonlinear objective function

$$\text{minimize } Z_2 = \sum_{\pi\in\mathcal{P}}\sum_{a\in\mathcal{A}} c(a)h(\pi,a)x(\pi,a)+$$

$$\underset{\boldsymbol{b(m,n)}}{E}\left\{\Phi\left(q\sum_{\pi\in\mathcal{P}}\sum_{(m,n)\in\pi}\left[d(\pi,m,n)+s(\pi,m,n)+\sum_{k\in\mathcal{N}} t(\pi,m,n,k)\right]\right.\right.$$
$$\left.\left.+\rho\sum_{m\in\mathcal{N}}\sum_{n\in\mathcal{N}} y(m,n)\right)\right\},$$

where

$$\Phi(x) = \alpha\exp(\beta x). \tag{42}$$

3.6.2 Problem statement

Given Φ as either the identity function or as in (42), the problem is to

$$\text{minimize } Z = \sum_{\pi\in\mathcal{P}}\sum_{a\in\mathcal{A}} c(a)h(\pi,a)x(\pi,a)+$$

$$\underset{\boldsymbol{b(m,n)}}{E}\left\{\Phi\left(q\sum_{\pi\in\mathcal{P}}\sum_{(m,n)\in\pi}\left[d(\pi,m,n)+s(\pi,m,n)+\sum_{k\in\mathcal{N}} t(\pi,m,n,k)\right]\right.\right.$$
$$\left.\left.+\rho\sum_{m\in\mathcal{N}}\sum_{n\in\mathcal{N}} y(m,n)\right)\right\},$$

subject to

$$\sum_{a\in\mathcal{A}}\sum_{\pi\in U(m,n)} x(\pi,a) \geq f(m,n), \quad \forall m,n\in\mathcal{N},$$

$$\sum_{a\in\mathcal{A}}\sum_{\pi\in W(n)} x(\pi,a) \leq \sigma(n), \quad \forall n\in\mathcal{N},$$

$$\sum_{\pi\in V_1(n)} x(\pi,a) = \sum_{\pi\in V_l(n)} x(\pi,a), \quad \forall a\in\mathcal{A}, n\in\mathcal{N},$$

$$h^{\min}(a) \leq \sum_{\pi\in\mathcal{P}} x(\pi,a)h(\pi,a) \leq h^{\max}(a), \quad \forall a\in\mathcal{A},$$

$$\sum_{\pi\in\mathcal{P}}\left(d(\pi,m,n)+\sum_{k\in\mathcal{N}} t(\pi,m,k,n)\right)+y(m,n) \geq \boldsymbol{b(m,n)}, \quad \forall m,n\in\mathcal{N},$$

$$\sum_{\pi\in\mathcal{P}}\sum_{m\in\mathcal{N}} t(\pi,m,k,n) = \sum_{\pi\in\mathcal{P}} s(\pi,k,n), \quad \forall k,n\in\mathcal{N},$$

$$\sum_{k\in\mathcal{N}}\left(d(\pi,v(\pi,1),k) + s(\pi,v(\pi,1),k) + \sum_{n\in\mathcal{N}} t(\pi,v(\pi,1),k,n)\right) \\ = \sum_{a\in\mathcal{A}} d(a)x(\pi,a) - z(\pi,1), \quad \forall\pi\in\mathcal{P},$$

$$\sum_{k\in\mathcal{N}}\left(d(\pi,v(\pi,j),k) + s(\pi,v(\pi,j),k) + \sum_{n\in\mathcal{N}} t(\pi,v(\pi,j),k,n)\right) \\ - \sum_{k\in\mathcal{N}}\left(d(\pi,k,v(\pi,j)) + s(\pi,k,v(\pi,j)) + \sum_{n\in\mathcal{N}} t(\pi,k,v(\pi,j),n)\right) \\ = z(\pi,j-1) - z(\pi,j), \quad \forall\pi\in\mathcal{P}, j=2,\ldots(l(\pi)-1),$$

$$x(\pi,a), d(\pi,m,n), t(\pi,m,k,n), s(\pi,k,n), y(m,n), z(\pi,j) \geq 0 \\ x(\pi,a)\in\mathbb{Z}.$$

3.6.3 Numerical examples

Mulvey and Ruszczyński [20] did not provide data for the numerical examples that they discuss [19]. Therefore, we have created some examples from a four node network, with node airports A, B, C and E. All flights $\pi\in\mathcal{P}$ have two legs. That is, including the airport of origin, there are three airports in each flight. No direct legs are allowed between A and E, but all other possibilities are allowed. Flights are enumerated according to Table 19. The notation "ABA" means that the flight begins at airport A, flies to airport B, and returns to airport A.

Two types of airplane are considered. Type 0 plane has a maximum payload of 8, maximum flight hours of 480, and costs 5 per flight hour. Type 1 plane has a maximum payload of 6, maximum flight hours of 240, but only costs 4 per flight hour. Both types of airplanes may have flight hours as low as 0. The unit cost, q, for loading and unloading is 1.0, and ρ, the penalty for undelivered cargo is 1300. There are no minimum numbers of flights, and the limit on landings is, for the base problem, 25 for each airport. Flight times for the two plane types are listed in Table 20.

Table 19: Possible flights $\pi \in \mathcal{P}$ for the numerical example

0	ABA	6	BAB	13	ECE	19	CAC
1	ABE	7	BAC	14	ECB	20	CAB
2	ABC	8	BCA	15	ECA	21	CBC
3	ACA	9	BCB	16	EBE	22	CBA
4	ACE	10	BCE	17	EBC	23	CBE
5	ACB	11	BEB	18	EBA	24	CEC
		12	BEC			25	CEB

Table 20: Flight times for numerical example

Airplane Type 0

	A	B	C	E
A	-	5	7	-
B	5	-	4	8
C	7	4	-	5
E	-	8	5	-

Airplane Type 1

	A	B	C	E
A	-	6	8.4	-
B	6	-	4.8	9.6
C	8.4	4.8	-	6
E	-	9.6	6	-

3.6.4 Notational reconciliation

We reconcile the problem in the Numerical examples section to the form of problem (2). Define

$$x_1 := \begin{bmatrix} x(0,0) \\ x(1,0) \\ \vdots \\ x(25,0) \\ x(0,1) \\ x(1,1) \\ \vdots \\ x(25,1) \end{bmatrix}, \qquad \text{and } h_i := \begin{bmatrix} h(0,i) \\ h(1,i) \\ \vdots \\ h(25,i) \end{bmatrix},$$

for $i = 0, 1$, and let $h := \begin{bmatrix} h0 \\ h1 \end{bmatrix}$. We will make use of the incidence matrices $W, V_i \in \mathbb{R}^{5\times 26}, i = 1, 2, 3$, defined by

$$(V_i)_{mn} = \begin{cases} 1 & \text{if node } m \text{ is the } i\text{th node in route } n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$W_{mn} = \begin{cases} 1 & \text{if node } m \text{ is in route } n \\ 0 & \text{otherwise.} \end{cases}$$

We are now ready to define the stage one problem parameters. Let

$$A_1 := \begin{bmatrix} W & W \\ (V_1 - V_3) & 0^{5\times 26} \\ h_1^\mathsf{T} & 0^{1\times 26} \\ 0^{1\times 26} & h_2^\mathsf{T} \\ -h_1^\mathsf{T} & 0^{1\times 26} \\ 0^{1\times 26} & -h_2^\mathsf{T} \end{bmatrix}, \qquad b_1 := \begin{bmatrix} \sigma(1) \\ \vdots \\ \sigma(5) \\ 0^{10\times 1} \\ h^{\max}(0) \\ h^{\max}(1) \\ -h^{\min}(0) \\ -h^{\min}(1) \end{bmatrix},$$

and

$$c_1 := \begin{bmatrix} c(0)h_0 \\ c(1)h_1 \end{bmatrix}.$$

Stage two is a bit more involved. We order the stage two variables into x_2 as shown below. When trying to figure out the ordering rationale for the

d, t and s variables, it will help to look at Table 19.

$$x_2 := \left[\begin{array}{c} d(0,A,B) \\ d(1,A,B) \\ d(2,A,B) \\ \vdots \\ d(25,C,E) \\ d(0,B,A) \\ d(1,B,E) \\ \vdots \\ d(25,E,B) \\ d(1,A,B) \\ d(2,A,E) \\ d(4,A,E) \\ \vdots \\ d(23,C,E) \\ d(25,C,B) \\ \hline t(0,A,B,C) \\ t(0A,B,E) \\ t(1,A,B,C) \\ t(2,A,B,E) \\ t(3,A,C,B) \\ \vdots \end{array}\right. \qquad \left.\begin{array}{c} \vdots \\ t(24,C,E,B) \\ t(25,C,E,A) \\ \hline s(0,B,A) \\ s(1,B,E) \\ \vdots \\ s(25,E,B) \\ \hline y(A,B) \\ y(A,C) \\ y(A,E) \\ y(B,A) \\ \vdots \\ y(E,C) \\ \hline z(0,1) \\ z(1,1) \\ \vdots \\ z(25,1) \\ z(0,2) \\ \vdots \\ z(25,2) \end{array}\right]$$

The y variables follow an ordering we call "ordering $\mathcal{J}$," the alphabetical ordering on all combinations of two nodes. We will make use of the following incidence matrices, defined as

$$U_1 \in \mathbb{R}^{12\times 68}, \qquad (U_1)_{ij} := \begin{cases} 1 & \text{if the } i\text{th pair of } \mathcal{J} \text{ is served by the } j\text{th element} \\ & \text{of } x_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_2 \in \mathbb{R}^{12\times 36}, \qquad (U_2)_{ij} := \begin{cases} 1 & \text{if the } i\text{th pair of } \mathcal{J} \text{ is } (m,n) \text{ in the } j\text{th} \\ & t(\pi,m,k,n), \\ 0 & \text{otherwise,} \end{cases}$$

$$U_3 \in \mathbb{R}^{12\times 36}, \qquad (U_3)_{ij} := \begin{cases} 1 & \text{if the } i\text{th pair in } \mathcal{J} \text{ is } (k,n) \text{ in the } j\text{th} \\ & t(\pi,m,k,n), \\ 0 & \text{otherwise,} \end{cases}$$

$$U_4 \in \mathbb{R}^{12\times 26}, \qquad (U_4)_{ij} := \begin{cases} 1 & \text{if the } i\text{th pair in } \mathcal{J} \text{ is } (k,n) \text{ in the } j\text{th} \\ & s(\pi,k,n), \\ 0 & \text{otherwise,} \end{cases}$$

$$U_5 \in \mathbb{R}^{26\times 36}, \qquad (U_5)_{\pi j} := \begin{cases} 1 & \text{if the } j\text{th } t(p,m,k,n) \text{ has } m = v(\pi,1) \text{ and} \\ & p = \pi, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_6 \in \mathbb{R}^{26\times 16}, \qquad (U_6)_{\pi j} := \begin{cases} 1 & \text{if the } (52+j)\text{th element of } x_2 \text{ is } d(\pi,m,n), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$U_7 \in \mathbb{R}^{26\times 36}, \qquad (U_7)_{\pi j} := \begin{cases} 1 & \text{if the } j\text{th } t(p,m,k,n) \text{ has } k = v(\pi,2) \text{ and } p = \\ & \pi, \\ 0 & \text{otherwise.} \end{cases}$$

We are finished putting the problem into the form (2) if we let

$$A_2 := \left[\begin{array}{ccc|c|c|c|cc} & U_1 & & U_2 & 0^{12\times 26} & I^{12\times 12} & \multicolumn{2}{c}{0^{12\times 52}} \\ & 0^{12\times 68} & & U_3 & -U_4 & 0^{12\times 12} & \multicolumn{2}{c}{0^{12\times 52}} \\ I^{26\times 26} & 0^{26\times 26} & U_6 & U_5 & 0^{26\times 26} & 0^{26\times 12} & I^{26\times 26} & 0^{26\times 26} \\ -I^{26\times 26} & I^{26\times 26} & 0^{26\times 26} & -U_7 & I^{26\times 26} & 0^{26\times 12} & -I^{26\times 26} & I^{26\times 26} \end{array}\right],$$

$$c_2 := \left[\begin{array}{c|c|c|c|c} q(1^{1\times 68}) & q(1^{1\times 36}) & q(1^{1\times 26}) & \rho(a^{1\times 12}) & 0^{1\times 52} \end{array}\right], \qquad \boldsymbol{b_2} := \begin{bmatrix} \boldsymbol{b} \\ 0^{12\times 1} \\ 0^{26\times 1} \\ 0^{26\times 1} \end{bmatrix},$$

and

$$T_2 := \begin{bmatrix} \multicolumn{2}{c}{0^{12\times 52}} \\ \multicolumn{2}{c}{0^{12\times 52}} \\ -d(0)I^{26\times 26} & -d(1)I^{26\times 26} \\ \multicolumn{2}{c}{0^{26\times 52}} \end{bmatrix}.$$

3.7 Telecommunication network planning

Due to Sen, Doverspike and Cosares [22]
(Two stage, mixed integer linear stochastic problem)
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3.7.1 Description

The service of providing private lines to telecommunication customers is one with which most people are not familiar. Such service is used by large corporations between business locations for high speed, private data transmission. Private lines are generally used for a much longer duration than public switched service, and they generally carry more capacity per connection.

A manager of such a network must be constantly looking to the future, deciding where and how much to expand capacity. In this problem formulation, the "how much" is decided beforehand, to some extent, by the imposition of an overall budget. Within the constraints of the budget, expansion is not penalized. The goal is to minimize the unserved requests, while staying within budget.

Such networks are usually very interconnected, so that for any point-to-point demand pair, there is usually more than one route which may service the demand. Each route is made of one or more direct links.

Let n be the number of direct links in the network which might be expanded, and let $x \in \mathbb{Z}^n$ be the vector of expanded capacities in the links, where $\mathbb{Z}$ is the set of integers. Let m be the number of point-to-point pairs to be served by the network, and $\boldsymbol{d} \in \mathbb{Z}^m$ be the random variable of demands for service between the pairs.

The total budget constraint will be denoted by b. Then, the problem is to

$$\begin{array}{ll} \text{minimize} & \underset{\boldsymbol{d}}{E}\ [Q(x, \boldsymbol{d})] \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} x_j \leq b, \\ & \quad x \geq 0, \end{array}$$

where $Q(x, \boldsymbol{d})$ represents the number of unserved requests, subject to network balance constraints.

For point-to-point pair $i = 1, \dots, m$, let $R(i)$ be the set of routes which may be used to satisfy a request for service between the two locations. Additionally, for route $r \in R(i)$, let $a_{ir} \in \mathbb{Z}^n$ be the incidence vector defined by

$$(a_{ir})_j := \begin{cases} 1 & \text{if link } j \in r, \\ 0 & \text{otherwise.} \end{cases}$$

Let $e \in \mathbb{Z}^n$ be the existing capacity in the network.

The recourse variables are s_i, the number of unserved requests, and f_{ir}, the number of connections serving point-to-point pair i over route r. Then, the recourse problem is

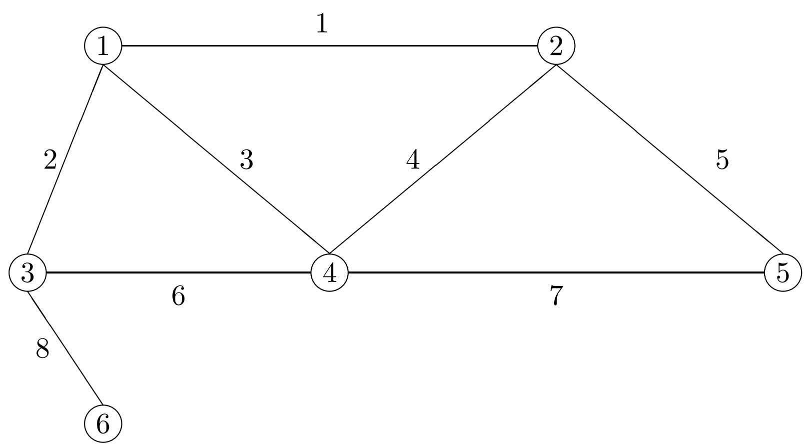
$$\begin{aligned} Q(x, d) := \text{minimize} \quad & \sum_{i=1}^{m} s_i \\ \text{subject to} \quad & \sum_{i=1}^{m} \sum_{r \in R(i)} a_{ir} f_{ir} \leq x + e, \\ & \sum_{r \in R(i)} f_{ir} + s_i = (\boldsymbol{d})_i, \quad \forall i = 1, \dots, m \\ & f_{ir}, s_i \geq 0, \quad \forall i, r \in R(i) \\ & f_{ir} \in \mathbb{Z}, \quad \forall i, r \in R(i). \end{aligned}$$

3.7.2 Problem statement

Given the budget constraint b, and the current condition of the network $\{a_{ir}, e\}$, the problem is to

$$\begin{aligned} \text{minimize} \quad & \underset{\boldsymbol{d}}{E} \left[\sum_{i=1}^{m} s_i \right] \\ \text{subject to} \quad & \sum_{j=1}^{n} (x)_j \leq b, \\ & \sum_{i=1}^{m} \sum_{r \in R(i)} a_{ir} f_{ir} \leq x + e, \\ & \sum_{r \in R(i)} f_{ir} + s_i = (d)_i, \quad \forall i = 1, \dots, m \\ & x, f_{ir}, s_i \geq 0, \quad \forall i, r \in R(i) \\ & x, f_{ir} \in \mathbb{Z}, \quad \forall i, r \in R(i). \end{aligned}$$

Figure 8: Illustration of routing for telephone network example



3.7.3 Numerical example

We have created an example with $2^{15} = 32,768$ random realizations and six nodes. The possible routing is illustrated in Figure 8, and the possible routes connecting each two-node combination are enumerated in Table 21.

The initial capacity of the network, e, is as follows:

route	1	2	3	4	5	6	7	8
capacity	2	2	4	4	2	4	3	1

3.7.4 Notational reconciliation

To put the problem into the notation of (2), let $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}^n$ be slack variables. Then set

$$x_1 := \begin{bmatrix} x \\ z_1 \end{bmatrix} \in \mathbb{R}^{n+1}, \qquad c_1 := 0^{n+1},$$
$$A_1 := 1^{1\times(n+1)}, \qquad b_1 := b \in \mathbb{R},$$

Table 21: Enumeration of all possible routes for telephone network example

node 1 $\rightarrow$ 2		node 1 $\rightarrow$ 3		node 1 $\rightarrow$ 4		node 1 $\rightarrow$ 5		node 1 $\rightarrow$ 6	
0	12	0	13	0	14	0	125	0	136
1	142	1	143	1	124	1	1245	1	1436
2	1452	2	1243	2	1254	2	145	2	12436
3	1342	3	12543	3	134	3	1425	3	125436
4	13452					4	1345		
						5	13425		

node 2 $\rightarrow$ 3		node 2 $\rightarrow$ 4		node 2 $\rightarrow$ 5		node 2 $\rightarrow$ 6		node 3 $\rightarrow$ 4	
0	213	0	24	0	25	0	2136	0	34
1	2143	1	214	1	245	1	21436	1	314
2	243	2	2134	2	2145	2	2436	2	3124
3	2543	3	254	3	21345	3	24136	3	31254
4	25413					4	25436		
5	2413					5	254136		

node 3 $\rightarrow$ 5		node 3 $\rightarrow$ 6		node 4 $\rightarrow$ 5		node 4 $\rightarrow$ 6		node 5 $\rightarrow$ 6	
0	345	0	36	0	45	0	436	0	5436
1	3425			1	425	1	4136	1	54136
2	3145			2	4125	2	42136	2	52436
3	31425			3	43125	3	452136	3	524136
4	34125							4	52136
5	31245							5	542136
6	3125							6	521436

$$x_2 := \begin{bmatrix} f_{11} \\ f_{12} \\ \vdots \\ f_{1R(1)} \\ \vdots \\ f_{mR(m)} \\ s_1 \\ s_2 \\ \vdots \\ s_m \\ z_2 \end{bmatrix}, \qquad c_2 := \begin{bmatrix} 0^{(mR(m))\times 1} \\ 1^{m\times 1} \\ 0^{n\times 1} \end{bmatrix}, \qquad \boldsymbol{b_2} := \left[\begin{array}{c} e \\ \hline \boldsymbol{d} \end{array}\right],$$

$$A_2 := \left[\begin{array}{cccccc|c|c} a_{11} & a_{12} & \cdots & a_{1R(1)} & \cdots & a_{mR(m)} & 0^{n\times m} & I^{n\times n} \\ \hline 1^{1\times R(1)} & 0^{1\times R(2)} & \cdots & 0^{1\times R(m)} & & & & \\ 0^{1\times R(1)} & 1^{1\times R(2)} & & & & & & \\ \vdots & & \ddots & & & & I^{m\times m} & 0^{m\times n} \\ 0^{1\times R(1)} & 0^{1\times R(2)} & \cdots & 1^{1\times R(m)} & & & & \end{array}\right],$$

and

$$T_2 := \left[\begin{array}{cc} -I^{n\times n} & 0 \\ \hline 0^{m\times n} & 0 \end{array}\right].$$

3.8 Bond investment planning

Due to K. Frauendorfer, C. Marohn, M. Schürle [9]
(Multistage, linear stochastic problem)

3.8.1 Description

Frauendorfer, Marohn, and Schürle [9] describe a suite of test problems for multistage stochastic programming, based on bond investments. The test problems are denoted SGPFmYn, where $m \in \{3, 5\}$, and $n \in \{3, 4, 5, 6, 7\}$.

Many business ventures are financed by lending bonds, and many of these ventures also purchase bonds. There is risk in such dealings, as returns on bonds fluctuate, and earnings from the business ventures are uncertain. This risk cannot be modeled by assuming a mean rate of return. Therefore, the scenario is a good one for the application of stochastic programming.

Bonds mature in certain, standard time periods. Suppose we will consider transactions in bonds with standard maturities in the set $\mathcal{D}^S$. Suppose that the longest maturity in $\mathcal{D}^S$ is D months. Then, since the time frame is rolling in such problems, we must include in the model bonds which mature in d months, where $d \in \mathcal{D} = \{1, 2, \ldots, D\}$.

Let $v_t^{d,+}$ be the amount of new borrowing done at time t with maturity $d \in \mathcal{D}^S$, and let $v_t^{d,-}$ be the amount of new lending done in the same circumstances. Then, if v_t^d is the balance of bond transactions at time t with maturity d, we have

$$v_t^d = \begin{cases} v_{t-1}^{d+1} + v_t^{d,+} - v_t^{d,-} & \text{if } d \in \mathcal{D}^S, \\ v_{t-1}^{d+1} & \text{if } d \in \mathcal{D} \backslash \mathcal{D}^S. \end{cases}$$

The total balance of bond transactions at time t is

$$x_t = \sum_{d \in \mathcal{D}} v_t^d.$$

If this quantity is positive, the balance will be used to fund the business venture during the time period t. Rather than writing x_t as a function of historical balances and rates of return, Frauendorfer, Marohn, and Schürle [9] simply express it as the stochastic quantity

$$x_t = x_{t-1} + \boldsymbol{\xi_t},$$

where $\boldsymbol{\xi_t}$ is a random variable.

To limit the sale of bonds, the authors [9] include the constraint

$$\sum_{d \in \mathcal{D}^S} v_t^{d,+} - \sum_{d=1}^{M} v_{t-1}^d \leq \boldsymbol{\xi_t}.$$

Given random rates of return $\boldsymbol{\eta_t^{d,-}}$, $\boldsymbol{\eta_t^{d,+}}$, and $\boldsymbol{\eta_t^x}$, corresponding to the quantities $v_t^{d,-}$, $v_t^{d,+}$, and x_t, respectively, the objective is to maximize the expected return:

$$\mathop{E}_{\boldsymbol{\eta},\boldsymbol{\xi}} \left\{ \sum_{t=0}^{T} \left(\sum_{d \in \mathcal{D}^S} [\boldsymbol{\eta_t^{d,-}} v_t^{d,-} - \boldsymbol{\eta_t^{d,+}} v_t^{d,+}] + \boldsymbol{\eta_t^x} x_t \right) \right\}.$$

The returns at time $t = 0$ are actually deterministic. So the decision variables for time $t = 0$ are the so-called first stage decision variables in the stochastic problem.

3.8.2 Problem statement

We change the problem to a minimization, and separate the first stage variables and constraints from the recourse variables and constraints. We are given all values for the time $t = -1$ decision variables, and the time $t = 0$ values of all η and ξ. The program then is to

$$\begin{aligned} \text{minimize} \quad & \sum_{d \in \mathcal{D}^S} [-\eta_0^{d,-} v_0^{d,-} + \eta_0^{d,+} v_0^{d,+}] - \eta_0^x x_0 + \\ & \qquad \mathop{E}_{\boldsymbol{\eta},\boldsymbol{\xi}} \left\{ \sum_{t=1}^{T} \left(\sum_{d \in \mathcal{D}^S} [-\boldsymbol{\eta_t^{d,-}} v_t^{d,-} + \boldsymbol{\eta_t^{d,+}} v_t^{d,+}] - \boldsymbol{\eta_t^x} x_t \right) \right\} \end{aligned}$$

subject to

$$v_0^d - v_{-1}^{d+1} - v_0^{d,+} + v_0^{d,-} = 0, \quad \forall d \in \mathcal{D}^S, \tag{43}$$

$$v_0^d - v_{-1}^{d+1} = 0, \quad \forall d \in \mathcal{D} \backslash \mathcal{D}^S, \tag{44}$$

$$x_0 - \sum_{d \in \mathcal{D}} v_0^d = 0,$$

$$x_0 - x_{-1} = \xi_0,$$

$$\sum_{d \in \mathcal{D}^S} v_0^{d,+} - \sum_{d=1}^{M} v_{-1}^d \leq \xi_0, \tag{45}$$

$$v_0^{d,+}, v_0^{d,-} \geq 0, \quad \forall d \in \mathcal{D},$$

$$\begin{aligned}
v_t^d - v_{t-1}^{d+1} - v_t^{d,+} + v_t^{d,-} &= 0, \quad \forall d \in \mathcal{D}^S, t = 1, \ldots, T, \\
v_t^d - v_{t-1}^{d+1} &= 0, \quad \forall d \in \mathcal{D} \backslash \mathcal{D}^S, t = 1, \ldots, T, \\
x_t - \sum_{d \in \mathcal{D}} v_t^d &= 0, \quad \forall t = 1, \ldots, T, \\
x_t - x_{t-1} &= \boldsymbol{\xi_t}, \quad \forall t = 1, \ldots, T, \\
\sum_{d \in \mathcal{D}^S} v_t^{d,+} - \sum_{d=1}^{M} v_{t-1}^d &\leq \boldsymbol{\xi_t}, \quad \forall t = 1, \ldots, T, \\
v_t^{d,+}, v_t^{d,-} &\geq 0, \quad \forall d \in \mathcal{D}, t = 1, \ldots, T.
\end{aligned} \tag{46}$$

3.8.3 Numerical examples

A total of ten numerical examples in SMPS format [4] are available from Birge's POSTS web site [12]. Since the only coefficients to be specified in this model are stochastic, specifying any one problem here would require duplicating the stochastic file from the set of SMPS files. Therefore, we refer the reader to the publicly available SMPS files [12].

3.8.4 Notational reconciliation

We may rearrange the equations represented by (43) and (44) so that they are in ascending order, by d. Then we have D constraints, each with right-hand sides v_{-1}^{d+1}, and left hand sides depending on whether $d \in \mathcal{D}^S$ or not. We replace all v_t^d with the term $(vp_t^d - vm_t^d)$, with the added constraints that $vp_t^d, vm_t^d \geq 0$.

Let $\{d1, d2, \ldots, dN\} = \{d \in \mathcal{D}^S\}$. We define the matrix $\Delta^S \in \mathbb{R}^{D \times N}$ by

$$\Delta^S := \left[\begin{array}{cccc} e^{d1} & e^{d2} & \cdots & e^{dN} \end{array} \right],$$

where $e^i \in \mathbb{R}^D$ is the ith unit vector.

Let s_t be the slack variable associated with constraint (45) or (46). Assign the new notation

$$b_1 := \begin{bmatrix} (vp_{-1}^2 - vm_{-1}^2) \\ (vp_{-1}^3 - vm_{-1}^3) \\ \vdots \\ (vp_{-1}^D - vm_{-1}^D) \\ 0 \\ 0 \\ \xi_0 + x_{-1} \\ \xi_0 + \sum_{d=1}^{M}(vp_{-1}^d - vm_{-1}^d) \end{bmatrix}, \qquad c_1 := \begin{bmatrix} -\eta_0^x \\ 0 \\ \eta_0^{d1,+} \\ \vdots \\ \eta_0^{dN,+} \\ -\eta_0^{d1,-} \\ \vdots \\ -\eta_0^{dN,-} \\ 0^{2D\times 0} \end{bmatrix},$$

and

$$x_1 := \begin{bmatrix} x_0 & s_0 & v_0^{d1,+} & \cdots & v_0^{dN,+} & v_0^{d1,-} & \cdots & v_0^{dN,-} \\ & & & vp_0^1 & \cdots & vp_0^D & vm_0^1 & \cdots & vm_0^D \end{bmatrix}^{\mathsf{T}}.$$

Also, let

$$A_1 := \begin{bmatrix} 0^{D\times 1} & 0^{D\times 1} & -\Delta^S & \Delta^S & I^{D\times D} & -I^{D\times D} \\ 1 & 0 & 0^{1\times N} & 0^{1\times N} & -1^{1\times D} & 1^{1\times D} \\ 1 & 0 & 0^{1\times N} & 0^{1\times N} & 0^{1\times D} & 0^{1\times D} \\ 0 & 1 & 1^{1\times N} & 0^{1\times N} & 0^{1\times D} & 0^{1\times D} \end{bmatrix}.$$

Analogous assignments are made for $t = 2, 3, \ldots, T$, except that $\boldsymbol{c_t}$ is made stochastic for these times, because $\boldsymbol{\eta}$ is stochastic. Also,

$$\boldsymbol{b_t} := \begin{bmatrix} 0^{D\times 1} \\ 0 \\ 0 \\ \boldsymbol{\xi_{t-1}} \\ \boldsymbol{\xi_{t-1}} \end{bmatrix},$$

and

$$T_t := \begin{bmatrix} 0^{D\times 1} & 0^{D\times 1} & 0^{D\times N} & 0^{D\times N} & -I^{D\times D} & I^{D\times D} \\ 0 & 0 & 0^{1\times N} & 0^{1\times N} & 0^{1\times D} & 0^{1\times D} \\ -1 & 0 & 0^{1\times N} & 0^{1\times N} & 0^{1\times D} & 0^{1\times D} \\ 0 & 0 & 0^{1\times N} & 0^{1\times N} & -W_1 & W_1 \end{bmatrix},$$

where

$$W_1 := \begin{bmatrix} 1^{1\times M} & 0^{1\times(D-M)} \end{bmatrix}.$$

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