Fourier Series

Consider a continuous function w/ the domain: $t \in [0, T]$. We could also consider periodic function that repeat every T so that $f(t+T) = f(t) \; \forall \, t$.

We attempt to represent this function as a sum over simpler functions. While this approach is quite general & can be used for many basis sets, we are specifically interested in sinusoids

Fourier Series

$$f(t) = f_0 + \sum_{n=1}^{\infty} \left[A_n \cos\left(2\pi \frac{n}{T} t\right) + B_n \sin\left(2\pi \frac{n}{T} t\right) \right]$$

The question is, what values of $\{A_n, B_n\}$ should we choose to accurately represent the functions. ORTHOGONALITY SAVES THE DAY! We note that

$$\int_0^T dt \, \sin\left(2\pi \frac{n}{T} t\right) \sin\left(2\pi \frac{m}{T} t\right) = \int_0^T dt \, \frac{1}{2}\left[\cos\left(2\pi \frac{n-m}{T} t\right) - \cos\left(2\pi \frac{n+m}{T} t\right)\right]$$

$$= \frac{1}{2}\left[\frac{T}{2\pi(n-m)} \sin\left(2\pi \frac{n-m}{T} t\right) - \frac{T}{2\pi(n+m)} \sin\left(2\pi \frac{n+m}{T} t\right)\right]_0^T$$

$$= 0 \quad \text{iff } n \neq \pm m$$

Thus, we obtain:

$$\int_0^T dt \, \sin\left(\frac{2\pi n}{T} t\right) \sin\left(\frac{2\pi m}{T} t\right) = \frac{T}{2}\left[\delta_{nm} + \delta_{n-m}\right]$$

From this, we can crack the expansion and obtain

$$f_0 = \frac{1}{T} \int_0^T dt \, f(t)$$

$$A_n = \frac{2}{T} \int_0^T dt \, f(t) \cos\left(\frac{2\pi n}{T} t\right)$$

$$B_n = \frac{2}{T} \int_0^T dt \, f(t) \sin\left(\frac{2\pi n}{T} t\right)$$

let's try it!

EXAMPLE: square pulse fourier series

Now, we often use: $(C_n e^{+i\phi_n}) e^{-2\pi i f_n t} = A_n \cos(2\pi f_n t) + i B_n \sin(2\pi f_n t)$

$$\Rightarrow C_n = \sqrt{A_n^2 + B_n^2}$$

$$\phi_n = \tan^{-1}\left(\frac{B_n}{A_n}\right)$$

With this notation, we often call $|C_n|^2$ the "power in the signal at that frequency"

Consider basic symmetries. Can we figure out which components will have power in them without numerically calculating it?

QUESTION: why are some fourier coefficients nearly zero for our square-pulse example?

Let's now consider the limit $T \to \infty$. what happens to the frequency spacing in our series? That's right! It becomes smaller & smaller. In the limit of functions defined over the entire real line, we usually define

Fourier Transform $$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-2\pi i k x} f(x)$$

with the associated inverse relation

Inverse Fourier Transform $$f(x) = \int_{-\infty}^{\infty} dk\, e^{2\pi i k x} \tilde{f}(k)$$ } looks like our fourier series

Let's look @ some common behavior of the Fourier transform

→ sharp edges require many fourier components to resolve. (broad-band)

consider

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

so that

$$\tilde{f}(k) = \int_{-\infty}^{\infty} dx\, e^{-2\pi i kx} f(x) = \int_{0}^{\infty} dx\, e^{-2\pi i kx} = \frac{1}{2\pi i k}$$

(ask me about the upper integration bound if you're interested)

we find a relatively fat tail to large fourier conjugate values: $\tilde{f} \propto \frac{1}{k}$

This can be interpreted as a statement about time-frequency uncertainty: smaller time scales correspond to broader frequency scales. You can derrive the Heisenberg uncertainty principle in general terms for any fourier conjugate pair

(position & momentum are fourier conjugates in Quantum Mechanics, so $\Delta x \Delta p \geq 2\pi$)

→ real functions have specific ~~fourier~~ symmetries in their fourier transform

$$f \in \mathbb{R} \Rightarrow f^* = f \Rightarrow \left(\int dk\, e^{2\pi i kx} \tilde{f}(k)\right)^* = \int dk\, e^{2\pi i kx} \tilde{f}(k)$$

$$\int_{-\infty}^{\infty} dk\, e^{-2\pi i kx} \tilde{f}^*(k) = \int dk\, e^{2\pi i kx} \tilde{f}(k)$$

$$-\int_{\infty}^{-\infty} dk\, e^{2\pi i kx} \tilde{f}^*(-k) =$$

$$\Rightarrow \tilde{f}^*(-k) = \tilde{f}(k)$$

we can also see this from counting "degrees of freedom" in some sense

→ linearity: $g = f_1 + f_2 \Rightarrow \tilde{g} = \tilde{f}_1 + \tilde{f}_2$

Can you see why?

→ convolution theorem

convolution theorem

$$h = g * f \equiv \int d\tau\, f(\tau)\, g(t-\tau)$$

$$= \int d\tau \int dk\, e^{2\pi i k \tau}\, \tilde{f}(k) \int dk'\, e^{2\pi i k'(t-\tau)}\, \tilde{g}(k')$$

$$= \int dk'\, e^{2\pi i k' t}\, \tilde{g}(k') \int dk\, e^{2\pi i (k-k')\tau}\, \tilde{f}(k)$$

$$= \int dk'\, e^{2\pi i k' t}\, \tilde{g}(k') \int dk\, \tilde{f}(k) \underbrace{\int d\tau\, e^{2\pi i (k-k')\tau}}_{= \delta(k-k')}$$

(trust me, it is)

$$= \int dk'\, e^{2\pi i k' t} \underbrace{\left[\tilde{g}(k')\, \tilde{f}(k')\right]}$$

convolution in the time domain is multiplication in the frequency domain

→ Parseval's theorem: Can you prove that

$$\int dt\, h(t)\, g(t) = \int dk\, \tilde{h}(k)\, \tilde{g}(k) \; ?$$

→ differentiation is easy in the frequency domain

$$\frac{\partial}{\partial x} f = \frac{\partial}{\partial x} \int dk\, e^{2\pi i k x}\, \tilde{f} = \int dk\, \tilde{f} \frac{\partial}{\partial x} e^{2\pi i k x}$$

$$= \int dk \left[\tilde{f}\, (2\pi i k)\right] e^{2\pi i k}$$

differentiation is "multiplication by k"

Discrete Fourier Transform (DFT)

The DFT is a slightly different operation, but we can think of it as a way to numerically estimate the Fourier Transform. However, there are some complications...

A discrete series typically has 2 main parameters

frequency resolution

sampling rate $\longleftrightarrow$ maximum resolvable frequency

duration $\longleftrightarrow$ frequency spacing

These are related by: $\Delta t = 1/2f_{Nyquist}$ | $f_{Nyquist}$ is the max resolvable frequency (Nyquist Sampling Theorem)

$$T = 1/\Delta f$$

QUESTION: why is there a maximum resolvable frequency? what happens if there are frequencies above that?

EXAMPLE: square-pulse-fft

Window functions:

Consider a discrete sequence as an approximation to a continuous function multiplied by a top-hat window. By the convolution theorem, we know that multiplication in the time-domain is convolution in the frequency domain, so the DFT should look like a smeared-out version of the true function's ~~D~~FT.

sidebands

NOTE: narrow windows have broad frequency support $\Rightarrow$ really smeared

windows with sharp corners have fat tails $\Rightarrow$ side-bands

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EXAMPLE: audacity & filter design

→ low-pass } band-pass filters

high-pass

"knotch" filters