The properties of determinants.

The determinant of an n by n matrix can be found in three ways.

① Pivot formula

… $\det A = \prod_{i=1}^{n} (\text{pivot of } A)_i$

② Big formula

… Add up $n!$ terms

③ Cofactor formula

… Combine n smaller determinants.

$\boxed{\text{CAUTION}}$

The determinant changes sign when two rows / columns are exchanged.

The important properties of determinant for application

① $\det(aA) = a^n \det(A)$

② Cramer's Rule (Determinants give A^{-1} and $A^{-1}b$)

③ When the edges of a box are the rows of A, the volume is $|\det A|$

④ For n special numbers λ, called eigenvalues, the determinants of $A-\lambda I$ is zero

10 rules of determinant

① $\det I = 1$

$$\det\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = 1$$

② Sign reversal

… The determinant changes sign when two rows are exchanged.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\det\begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$\rightarrow$ $\det P = (-1)^n$

(n: the number of row exchange from identity)

(matrix to P)

③ The determinant is a linear function of each row separately.

$$\det\begin{pmatrix} ta & tb \\ c & d \end{pmatrix} = t \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det\begin{pmatrix} a+a' & b+b' \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \det\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$$

$$\rightarrow \quad \det(aA) = a^n \det(A)$$

④ If two rows of A are equal, then $\det A = 0$

From rule ②,

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\det\begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$\Big\downarrow \text{ if } (a, b) = (c, d)$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$$

⑤ Elimination does not change the determinant.

$$\det\begin{pmatrix} a & b \\ c-la & d-lb \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(∵) From rule ③,

$$\det\begin{pmatrix} a & b \\ c-\ell a & d-\ell b \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \det\begin{pmatrix} a & b \\ -\ell a & -\ell b \end{pmatrix}$$

$$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \ell \underbrace{\det\begin{pmatrix} a & b \\ a & b \end{pmatrix}}_{= 0 \text{ (rule ④)}}$$

$$= \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

⑥ A matrix with a row of zeros has $\det A = 0$

$$\det\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} = 0$$

(∵)

$$\det\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \overset{\text{rule ⑤}}{=} \det\begin{pmatrix} c & d \\ c & d \end{pmatrix} \overset{\text{rule ④}}{=} 0$$

⑦ If A is triangular then $\det A = a_{11} a_{22} \cdots a_{nn}$

$$\det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix} = \det\begin{pmatrix} a_{11} & & 0 \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$= a_{11} a_{22} \cdots a_{nn}$$

(∵)

$$\det\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{pmatrix}$$

$\Big\downarrow$ rule ③

$$= a_{11}\cdots a_{nn} \det\begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ 0 & & 1 & a_{n-1n} \\ & & & 1 \end{pmatrix}$$

$\Big\downarrow$ elimination

$$= a_{11}\cdots a_{nn} \det\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$\Big\downarrow$ rule ①

$$= a_{11}\cdots\cdots a_{nn}$$

⑧ If A is singular then $\det A = 0$.
If A is invertible then $\det A \neq 0$

$\because$ Elimination goes from A to U.

If A is singular then U has a zero row

$\downarrow$ rule ⑥

$\det A = 0$

⑨ $\det(AB) = (\det A)(\det B)$

∵ (for n by n matrix A)

$$D(A) = \frac{\det(AB)}{\det(B)}$$

To prove that $D(A) = \det(A)$, we're going to show that $D(A)$ has the same 3 properties as $\det A$.

Property 1 (Determinant of I)

If $A = I$, then

$$\det(A) = 1$$

$$D(A) = \frac{\det(AB)}{\det(B)} = \frac{\det B}{\det B} = 1$$

Property 2 (sign reversal)

If two rows of A are exchanged

↓

the same two rows of AB are exchanged

↓

$$\det(A) \xrightarrow[\text{exchange}]{} -\det(A)$$

$$D(A) \xrightarrow[\text{exchange}]{} -D(A)$$

Property 3 (Linearity)

When row 1 of A is multiplied by t

$\downarrow$

So is row 1 of AB

$\therefore$

$$\det A \to t \det A$$

$$D(A) \longrightarrow t\, D(A)$$

(10) $\det A^T = \det A$

$\because$

A has the usual factorization

$\longrightarrow$ $PA = LU \Leftrightarrow A^T P^T = U^T L^T$

$$\Rightarrow \begin{cases} \det P \det A = \det L \det U \\ \det A^T \det P^T = \det U^T \det L^T \end{cases}$$

$$\Big\downarrow \quad \begin{aligned} &\det L = \det L^T = 1 \\ &\det U = \det U^T \end{aligned}$$

$$\begin{cases} \det P \det A = \det U \\ \det P^T \det A^T = \det L \end{cases}$$

$$\Big\downarrow \quad \begin{aligned} &P^T P = I \\ &\det P^T \det P = 1 \\ &\therefore \begin{cases} \det P^T = \det P = 1 \\ \quad \text{or} \\ \det P^T = \det P = -1 \end{cases} \end{aligned}$$

$$\det A = \det A^T \quad \blacksquare$$

The pivot formula

- Elimination without row exchange

$$A = LU$$

$$\det A = \det L \det U \qquad \left\{ \begin{aligned} L &= \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \end{pmatrix} \\ U &= \begin{pmatrix} d_1 & & \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \end{aligned} \right.$$

$$= \det U$$

$$= \det U$$

$$= \underbrace{d_1 \cdots d_n}$$

- Elimination with row exchange

$$PA = LU$$

$d_1 \cdots d_n$: pivots

$$\det P \det A = \det L \det U$$

$$\underbrace{\det A = \pm (d_1 \cdots d_n)}$$

det A has $n!$ terms!!!

Cramer's Rule

$$A \begin{pmatrix} x_1 & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & & & \\ x_n & 0 & & 1 \end{pmatrix} = \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix} = B_1$$

$$\Big\downarrow \det$$

$$\det A \cdot x_1 = \det B_1$$

$$\therefore x_1 = \frac{\det B_1}{\det A}$$

Using the same idea,

$$A\begin{pmatrix} 1 & & x_1 & & \\ & \ddots \; 1 & \vdots & & \\ & & x_i & & \\ & & \vdots & 1 \; \ddots & \\ & & x_m & & 1\end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ & \cdots & b_i & & \\ & & \vdots & \ddots & \\ a_{m1} & \cdots & b_m & \cdots & a_{mn}\end{pmatrix} = B_i$$

$$\therefore \; x_i = \frac{\det B_i}{\det A}$$