Notes about inverse matrix

① The inverse exists if and only if elimination produces n pivots.

② If there is a nonzero vector x such that $Ax = 0$, then A cannot have an inverse.

③ A diagonal matrix has an inverse provided no diagonal entries are zero

If $A = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{pmatrix}$

④ Inverse of an elimination matrix E

$$E \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrices used in elimination

① The matrix P for row exchange

Permutation matrix : $P_{23} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

② The elimination matrix E

$$E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ex

$$\underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}}_{b}$$

⇒ Augmented matrix : $B = \begin{pmatrix} A & b \end{pmatrix} = \begin{pmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{pmatrix}$

$$E_{21} B = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B = \underbrace{\begin{pmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{pmatrix}}_{B'}$$

subtracting (row 1 × 2) from row 2 of B

$$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 4 & -2 & 2 \end{pmatrix}$$

$$E_{31}B' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} B' = \begin{pmatrix} 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{pmatrix}$$

$$E_{32}B'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} B'' = \overset{``B"}{\begin{pmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{pmatrix}}$$

$$\underbrace{\qquad\qquad}_{= U} \qquad \blacksquare$$

LU decomposition example

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$E_{21}A = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A = \overset{``A'"}{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 1 & 2 \end{pmatrix}}$$

$$E_{32}A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix} A' = \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{pmatrix}}_{= U}$$

$$\Rightarrow \quad E_{32} E_{21} A = U$$

$$\Leftrightarrow \quad A = (E_{32} E_{21})^{-1} U$$

$$= E_{21}^{-1} E_{32}^{-1} U$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix} U$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix}}_{=L} U = LU$$

$\blacksquare$

The facts that is very important in practice.

When a row of A starts with zeros, so does that row of L

When a column of A starts with zeros, so does that column of U

Split U into (diag)(upper)

$$\begin{pmatrix} d_1 & & \\ \end{pmatrix} \begin{pmatrix} 1 & u_{12}/d_1 & \cdots\cdots & u_{1n}/d_1 \\ \end{pmatrix}$$

$$U = \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & d_n \end{pmatrix} \begin{pmatrix} 1 & u_{23}/d_2 & \cdots & u_{2n}/d_2 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \end{pmatrix}$$

$$\rightarrow \boxed{A = LDU}$$

(One square system) = (Two triangular systems)

(iii) The cost of elimination

- Column 1 elimination $\cdots$ $n(n-1) \rightarrow n^2$

$$\begin{pmatrix} a_{11} & * & \cdots & * \\ a_{21} & * & \ddots & \vdots \\ \vdots & \vdots & \ddots & \\ a_{n1} & * & \cdots & * \end{pmatrix} \xrightarrow[n(n-1)\ \text{subtraction}]{n(n-1)\ \text{multiplication}} \begin{pmatrix} a_{11} & * & \cdots & * \\ 0 & * & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$$

- Column i elimination $\cdots$ $(n-i+1)^2$.

$$\Downarrow$$

$$n^2 + (n-1)^2 + \cdots + 2^2 + 1^2 = \frac{1}{3} n \left(n + \frac{1}{2}\right)(n+1)$$

Elimination on A requires $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions.

(iii) Inverse of permutation matrix P

$$P^{-1} = P^{T}$$