Definition of orthogonality

Orthogonal subspaces :

if $v^T w = 0$ for $\forall v \in V, \forall w \in W$

then V and W are orthogonal.

Fundamental Theorem of linear algebra

$$C(A^T) \perp N(A)$$

row space **nullspace.**

$\boxed{\text{proof 1}}$

$$C(A^T) = \{ A^T y \in \mathbb{R}^n \mid y \in \mathbb{R}^m \}$$

$$N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

Let x be any vector in $N(A)$.

$\rightarrow \quad x \in N(A)$

$$\Rightarrow \quad Ax = \begin{pmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{pmatrix} x = 0$$

$$\Rightarrow \begin{cases} (\text{row } 1) \cdot x = 0 \\ \vdots \\ (\text{row } m) \cdot x = 0 \end{cases} \longrightarrow C(A^T) \perp N(A)$$

$\boxed{\text{proof 2}}$

Let $x \in N(A)$, $A^T y \in C(A^T)$.

$$x^T (A^T y) = (Ax)^T y = 0^T y = 0$$

Ex

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{pmatrix} \rightarrow R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{cases} x = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in N(A) \\ A^T y = \begin{pmatrix} 1 & 5 \\ 3 & 2 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 5y \\ 3x + 2y \\ 4x + 7y \end{pmatrix} \end{cases}$$

$$\Rightarrow x^T (A^T y) = 0$$

Another fundamental theorem of linear algebra

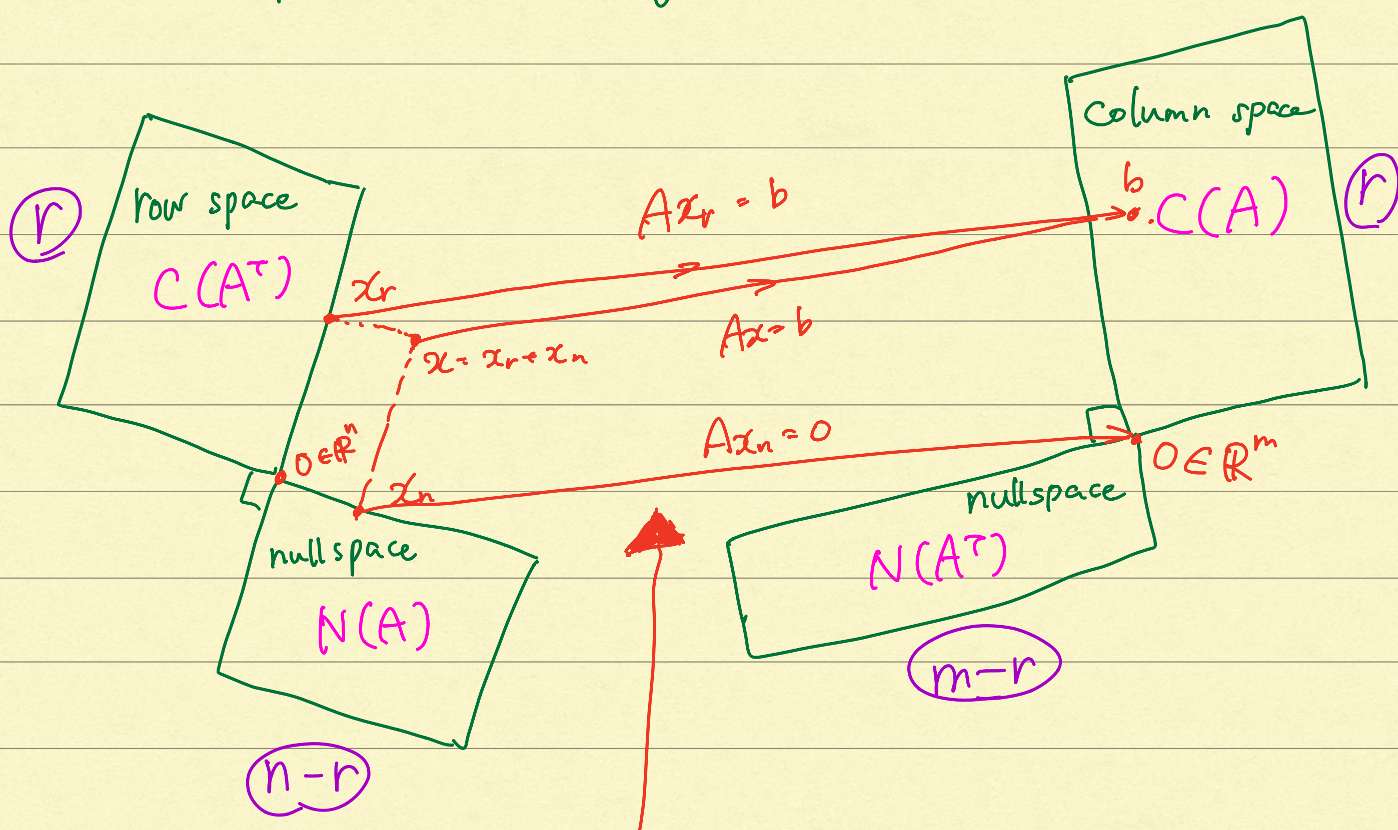
$$C(A) \perp N(A^T)$$

column space nullspace.

$$\begin{cases} Ay \in C(A) \\ x \in N(A^T) \end{cases}$$

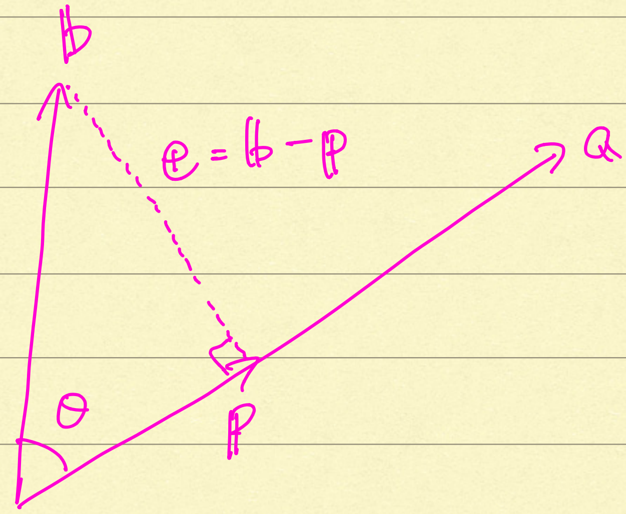
$$x^T(Ay) = (A^T x)^T y = 0^T y = 0$$

The pairs of orthogonal subspaces



$\boxed{\text{Every vector in row space moves to the column space by multiplying } A\text{ !!}}$

Projection onto a line.



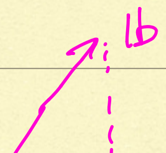
$$p = \hat{x}\,a$$

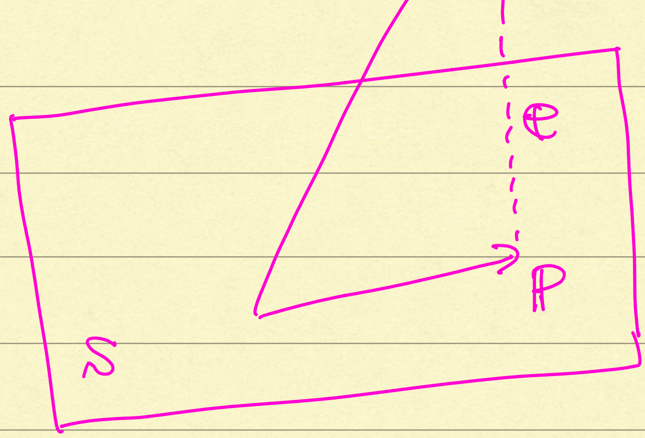
$$e^T a = 0$$

$$\Leftrightarrow \quad (b - \hat{x}a)^T a = 0$$

$$\Leftrightarrow \quad \hat{x} = \frac{b^T a}{a^T a}$$

Projection onto a subspace





$$\mathbf{p} = A\hat{x}$$

$$A = (a_1 \cdots\cdots a_n)$$

$a_1 \cdots\cdots a_n$ span the subspace S

$$\downarrow$$

$$\left\{\begin{array}{l} a_1^T(\mathbf{b} - A\hat{x}) = 0 \\ \vdots \\ a_n^T(\mathbf{b} - A\hat{x}) = 0 \end{array}\right. \iff A^T(\mathbf{b} - A\hat{x}) = 0$$

$$\Downarrow$$

$$\hat{x} = (A^TA)^{-1}A^T\mathbf{b}$$

$$\mathbf{p} = A\hat{x} = A(A^TA)^{-1}A^T\mathbf{b}$$

$$\downarrow \quad \mathbf{p} = P\mathbf{b}$$

$P = A(A^TA)^{-1}A^T$

projection matrix

Summary

$\boxed{n = 1}$

$\hat{x} = \dfrac{a^Tb}{a^Ta}$

$p = a\dfrac{a^Tb}{a^Ta}$

$P = \dfrac{aa^T}{a^Ta}$

$\boxed{n > 1}$

$\hat{x} = (A^TA)^{-1}A^Tb$

$p = A(A^TA)^{-1}A^Tb$

$P = A(A^TA)^{-1}A^T$

$p = A\hat{x}$: projected vector

$p = Pb$: P is projection matrix

A^TA is invertible if and only if A has linearly independent columns

Procedure

① A^TA has the same nullspace as A

② When the columns of A are linearly independent, its nullspace contains only the zero vector. Then A^TA, with the same nullspace, is invertible.

proof of ①

(i)

$$Ax = 0 \quad \times A^T$$

$$\Rightarrow A^TAx = 0$$

$$Ax = 0 \Rightarrow A^TAx$$

(ii)

$$A^TAx = 0 \quad \times x^T$$

$$\Rightarrow x^TA^TAx = 0$$

$$\Rightarrow (Ax)^TAx = 0$$

$$\Rightarrow \|Ax\|^2 = 0$$

$$\Rightarrow Ax = 0$$

$$A^TAx = 0 \Rightarrow Ax = 0$$

$$\therefore \quad A^T A x = 0 \iff Ax = 0$$

Orthonomal matrix Q

$$\underbrace{Q = (q_1 \cdots q_n)}_{} \quad (q_j \in \mathbb{R}^m)$$

$$\boxed{\begin{array}{l} q_1 \cdots q_n \text{ are orthonomal} \\ \Rightarrow \quad q_i^T q_j = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases} \end{array}}$$

Properties

$$Q^T Q = I$$

$$\|Qx\| = \|x\| \quad \text{for every vector } x$$

$$(Qx)^T (Qy) = x^T Q^T Q y = x^T y$$

Examples

$$\text{Rotation : } Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\text{Permutation : } P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Reflection : $Q = I - 2uu^T$

(u is any unit vector)

Projection Using Orthonormal Bases.

Suppose the basis vectors are actually orthonormal.

$$a_1, \cdots, a_n \rightarrow q_1, \cdots, q_n$$

$$A^T A \rightarrow Q^T Q = I$$

$$\boxed{\begin{aligned} \hat{x} &= (Q^T Q)^{-1} Q^T b = Q^T b \\ p &= Q \hat{x} = Q Q^T b \\ P &= Q Q^T b \end{aligned}}$$

The Gram - Schmidt Process

a, b, c : linearly independent

$\downarrow$

A, B, C : orthogonal

$\downarrow$

$$q_1, q_2, q_3 : \text{orthonomal}$$

Step 1

$$A = a$$

Step 2

$$B = b - \frac{A^T b}{A^T A} A \quad \left(\Rightarrow A^T B = 0 \right)$$

Step 3

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B \quad \left(\Rightarrow \begin{array}{l} A^T C = 0 \\ B^T C = 0 \end{array} \right)$$

Step 4

$$q_1 = \frac{A}{\|A\|} \quad q_2 = \frac{B}{\|B\|} \quad q_3 = \frac{C}{\|C\|}$$

QR Factorization

$$A = \begin{pmatrix} a & b & c \end{pmatrix} \rightarrow Q = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}$$

$$A = QR$$

$$= \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & \end{pmatrix}$$

→ Least squares

$$(A^TA)\hat{x} = A^Tb$$

$$\Leftrightarrow \quad R^TR\hat{x} = R^TQ^Tb$$

$$\Leftrightarrow \quad R\hat{x} = Q^Tb$$

$$\Leftrightarrow \quad \hat{x} = R^{-1}Q^Tb$$

easy!!