Diagonizing Matrix.

$$
\left\{
\begin{array}{l}
A \in \mathbb{R}^{n \times n} \\
\text{Eigenvalues} : \lambda_1 \cdots \lambda_n \\
\text{Eigenvectors} : x_1 \cdots\cdots x_n
\end{array}
\right.
$$

$$
A \underbrace{(x_1 \cdots\cdots x_n)}_{= S} = (\lambda_1 x_1 \cdots\cdots \lambda_n x_n)
$$

$$
= \underbrace{(x_1 \cdots\cdots x_n)}_{= S} \underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{= \Lambda}
$$

$$
\Longleftrightarrow \quad A S = S \Lambda
$$

$$
\Longleftrightarrow \quad \underline{A = S \Lambda S^{-1}}
$$

※ Any matrix that has no repeated eigenvalues can be diagonalized.

Application to Differential Equation

$\frac{du}{dt} = \lambda u \implies u(t) = C e^{\lambda t}$

One equation $\quad \frac{du}{dt} = \lambda u \rightarrow u(t) = Ce$

x_i : eigenvectors
λ_i : eigenvalues

n equations $\quad \frac{du}{dt} = Au$

$$u(t) = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$$

$$u(0) = (x_1 \cdots x_n) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

Second order equations.

$$m\ddot{y} + b\dot{y} + ky = 0$$

$\downarrow$ suppose $m = 1$

$$\ddot{y} + b\dot{y} + ky = 0$$

$$\downarrow$$

$$\frac{d}{dt}\begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix}\begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

$$\downarrow$$

Eigenvalues : λ_1, λ_2

Eigenvectors : $x_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

$$\Rightarrow \quad u(t) = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$$

4 Stability (2×2 Matrices)

$$\frac{du}{dt} = Au \quad \Rightarrow \quad u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$\Rightarrow$ stable when $\lambda_1 < 0$ and $\lambda_2 < 0$

$$\Rightarrow \begin{cases} \lambda_1 + \lambda_2 < 0 \\ \lambda_1 \lambda_2 > 0 \end{cases}$$

$$\Leftrightarrow \boxed{\begin{cases} T = Tr(A) < 0 \\ D = \det A > 0 \end{cases}}$$

Symmetric Matrices

① A symmetric matrix has only real eigenvalues.

② The eigenvectors can be chosen

orthonomal

$$\left.\begin{aligned} A &= S \Lambda S^{-1} \\ A^T &= (S^{-1})^T \Lambda S^T \end{aligned}\right.$$

$$\Rightarrow \quad S^{-1} = S^T$$

$$\lambda = a + jb$$

$$A x_i = \lambda x_i$$

$$\Rightarrow \quad A \bar{x}_i = \bar{\lambda} \bar{x}_i$$

transpose

$$\Rightarrow \quad \bar{x}_i^T A = \bar{x}_i^T \bar{\lambda}$$

$\times x_i$

$\times \bar{x}_i$

$$\Rightarrow \quad \left.\begin{aligned} \bar{x}_i^T A x_i &= \bar{x}_i^T \bar{\lambda} x_i \\ \bar{x}_i^T A x_i &= \bar{x}_i^T \lambda x_i \end{aligned}\right. \rightarrow \lambda = \bar{\lambda}$$

Eigenvectors of a real symmetric matrix
(when they correspond to different λ's)

are always perpendicular

∵ Suppose $Ax = \lambda_1 x$, $Ay = \lambda_2 y$ $(\lambda_1 \neq \lambda_2)$

Take dot products.

$$\begin{aligned}(\lambda_1 x_1)^T y &= (Ax)^T y \\ &= x_1^T A y \\ &= x_1^T \lambda_2 y\end{aligned}$$

$$\Leftrightarrow \quad (\lambda_1 - \lambda_2)\, x_1^T y = 0$$

$$\Rightarrow \quad x_1^T y = 0$$

∎

Eigenvalues versus Pivots

$$\boxed{\det A = \prod_{i=1}^{n} d_i = \prod_{i=1}^{n} \lambda_i}$$

(applicable also for non-sym A)

For symmetric matrices,

the pivots and the eigenvalues have the

Same signs.

(4) Positive Definite Matrices

= Symmetric matrices that have positive eigenvalues

= $x^T A x > 0$ for $\forall x \in \mathbb{R}^n$ (Energy - based definition)

When a symmetric matrix has one of these five properties, it has them all.

$$\boxed{\begin{array}{l} \text{(1) All } n \text{ pivots are positive} \\ \text{(2) All } n \text{ upper left determinants are positive} \\ \text{(3) All } n \text{ eigenvalues are positive} \\ \text{(4) } x^T A x \text{ is positive except at } x = 0. \\ \text{(5) } A = R^T R \text{ for a matrix } R \text{ with independent columns.} \end{array}}$$

(5) R: Cholesky factor

$A = LDL^T \Rightarrow R = L\sqrt{D}$

$$A = Q \Lambda Q^T \Rightarrow R = Q\sqrt{\Lambda}$$

Similar Matrices

Definition

Let M be any invertible matrix. Then

$B = M^{-1} A M$ is similar to A

$\boxed{\text{A and B have the same eigenvalues.}}$

$\because$

$$B = M^{-1}AM \iff A = MBM^{-1}$$

$$Ax = \lambda x$$

$$\iff MBM^{-1}x = \lambda x$$

$$\iff B(M^{-1}x) = \lambda (M^{-1}x)$$

Changing variables in differential equation

$$\frac{du}{dt} = Au$$

$$\downarrow \quad v = Mu$$

$$M \frac{dv}{dt} = A M v$$

$$\Leftrightarrow \quad \frac{dv}{dt} = M^{-1} A M v$$

Similar matrices give the same growth or decay.

i.e. A and B have the same eigenvalues.

In going from A to $B = M^{-1}AM$,

some things change and some don't

- Not changed
  - eigenvalues λ_i
  - Trace and determinant $\mathrm{Tr}(A)$. $\det A$
  - Rank r
  - Jordan form

- Changed
  - Eigenvectors u_i
  - Null space $N(A)$
  - Column space $C(A)$

— Row space $C(A^T)$

— Left nullspace $N(A^T)$

— Singular values

SVD (Singular Value Decomposition)

$$A \in \mathbb{R}^{m\times n} \text{ has rank } r$$

$$\begin{cases} u_1 \cdots u_r \in \mathbb{R}^m : \text{evectors of } AA^T \\ v_1 \cdots v_r \in \mathbb{R}^n : \text{evectors of } A^TA \end{cases}$$

singular vectors

singular values : $\sigma_1 \cdots \sigma_r$ ← all positive

$v_1 \cdots v_r \in C(A) \in \mathbb{R}^m$

$u_1 \cdots u_r \in C(A^T) \in \mathbb{R}^n$

$$\begin{cases} Av_1 = \sigma_1 u_1 \\ Av_2 = \sigma_2 u_2 \\ \quad\vdots \end{cases}$$

$A v_r = \sigma_r u_r$

$$\Rightarrow A(v_1 \cdots v_r) = (u_1 \cdots u_r)\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix}$$

$$AV = U\Sigma$$

$$A = U\Sigma V^T$$

$$= \underbrace{u_1\sigma_1 v_1^T + \cdots\cdots + u_r\sigma_r v_r^T}_{\boxed{SVD}}$$

Singular values $\sigma_1 \cdots \sigma_r$

$$A v_1 = \sigma_1 u_1$$

$$\Rightarrow u_1 = \frac{1}{\sigma_1} A v_1 \longrightarrow \sigma_1 = \frac{1}{\|Av_1\|}$$

($\because u_1$ is unit vector)

$$AV = U\Sigma$$

$$A = U\Sigma V^T$$

$$A^TA = V\Sigma U^T U \Sigma V^T$$

$$= V \Sigma \Sigma V^T$$

$$= V \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_r^2 \end{pmatrix} V^T$$

$\Rightarrow$ eigenvalues of A^TA are $\sigma_1^2 \cdots \sigma_r^2$

Ex

$$\boxed{\text{Find the SVD of } A = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}}$$

$$A^TA = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

$$\Rightarrow \quad v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{cases} Av_1 = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} = \sigma_1 u_1 \\ Av_2 = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} = \sigma_2 u_2 \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_1 = 2\sqrt{2} & u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sigma_2 = \sqrt{2} & u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

$A = U\Sigma V^T$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & \\ & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Another way to find the singular values.

$$|A^TA - \lambda I| = (\lambda - 8)(\lambda - 2) \Rightarrow \sigma_1^2 = 8, \quad \sigma_2^2 = 2$$

$$|AA^T - \lambda I| = (\lambda - 8)(\lambda - 2) \Rightarrow \sigma_1^2 = 8 \quad \sigma_2^2 = 2$$

The matrices U and V have the orthonormal bases for all four subspaces.

first r columns of $V \in C(A^T)$

last $n-r$ columns of $V \in N(A)$

first r columns of $U \in C(A)$

last $m-r$ columns of $U \in N(A^T)$

Proof of SVD.

$$\boxed{\sigma_i = \|Av_i\| = \|A^T u_i\|}$$

Start from

$$A^T A v_i = \sigma_i^2 v_i$$

$\times v_i^T$

$$v_i^T A^T A v_i = \sigma_i^2 \|v_i\|^2 = \sigma_i^2$$

$$\Leftrightarrow \quad \|Av_i\|^2 = \sigma_i^2 \rightarrow \|Av_i\| = \sigma_i$$

Start from

$\times u_i^T$

$$AA^T u_i = \sigma_i^2 u_i$$

$$u_i^T A A^T u_i = \sigma_i^2 \|u_i\|^2 = \sigma_i^2$$

$$\Leftrightarrow \quad \|A^T u_i\|^2 = \sigma_i^2 \rightarrow \|A^T u_i\| = \sigma_i$$

$$\boxed{\begin{aligned} Av_i &= \sigma_i u_i \\ A^T u_i &= \sigma_i v_i \end{aligned}}$$

Start from

$$A^T A v_i = \sigma_i^2 v_i$$

$\times A$

$$(AA^T)(Av_i) = \sigma_i^2 (Av_i) \quad \cdots ②$$

$$AA^T u_i = \sigma_i^2 u_i \quad \cdots ①$$

From (P), (Q), we get

$$u_i = \frac{A v_i}{\sigma_i} \Rightarrow A v_i = \sigma_i u_i \quad \blacksquare$$

Start from

$$\times A^T \quad A A^T u_i = \sigma_i^2 u_i$$

$$(A^T A)(A^T u_i) = \sigma_i^2 (A^T u_i) \quad \cdots \text{(P)}$$

$$(A^T A) v_i = \sigma_i^2 v_i \quad \cdots \text{(Q)}$$

From (P), (Q), we get

$$v_i = \frac{A^T u_i}{\sigma_i} \Rightarrow A^T u_i = \sigma_i v_i \quad \blacksquare$$

Table of Eigenvalues and Eigenvectors

How are the properties of a matrix reflected in its eigenvalues and eigenvectors? This question is fundamental throughout Chapter 6. A table that organizes the key facts may be helpful. Here are the special properties of the eigenvalues λ_i and the eigenvectors $\boldsymbol{x}_i$.

Symmetric: $A^{\mathrm{T}} = A$	real λ's	orthogonal $\boldsymbol{x}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Orthogonal: $Q^{\mathrm{T}} = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Skew-symmetric: $A^{\mathrm{T}} = -A$	imaginary λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Complex Hermitian: $\overline{A}^{\mathrm{T}} = A$	real λ's	orthogonal $\overline{\boldsymbol{x}}_i^{\mathrm{T}}\boldsymbol{x}_j = 0$
Positive Definite: $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} > 0$	all $\lambda > 0$	orthogonal since $A^{\mathrm{T}} = A$
Markov: $m_{ij} > 0, \sum_{i=1}^{n} m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $\boldsymbol{x} > 0$
Similar: $B = M^{-1}AM$	$\lambda(B) = \lambda(A)$	$\boldsymbol{x}(B) = M^{-1}\boldsymbol{x}(A)$
Projection: $P = P^2 = P^{\mathrm{T}}$	$\lambda = 1;\ 0$	column space; nullspace
Plane Rotation	$e^{i\theta}$ and $e^{-i\theta}$	$\boldsymbol{x} = (1, i)$ and $(1, -i)$
Reflection: $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$	$\lambda = -1;\ 1, .., 1$	$\boldsymbol{u}$; whole plane $\boldsymbol{u}^{\perp}$
Rank One: $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$	$\lambda = \boldsymbol{v}^{\mathrm{T}}\boldsymbol{u};\ 0, .., 0$	$\boldsymbol{u}$; whole plane $\boldsymbol{v}^{\perp}$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Stable Exponential: $e^{At} \to 0$	all Re $\lambda < 0$	any eigenvectors
Cyclic Permutation: row 1 of I last	$\lambda_k = e^{2\pi ik/n}$	$\boldsymbol{x}_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Tridiagonal: $-1, 2, -1$ on diagonals	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$\boldsymbol{x}_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots\right)$
Diagonalizable: $A = S\Lambda S^{-1}$	diagonal of Λ	columns of S are independent
Symmetric: $A = Q\Lambda Q^{\mathrm{T}}$	diagonal of Λ (real)	columns of Q are orthonormal
Schur: $A = QTQ^{-1}$	diagonal of T	columns of Q *if* $A^{\mathrm{T}}A = AA^{\mathrm{T}}$
Jordan: $J = M^{-1}AM$	diagonal of J	each block gives $\boldsymbol{x} = (0, .., 1, .., ($
Rectangular: $A = U\Sigma V^{\mathrm{T}}$	$\text{rank}(A) = \text{rank}(\Sigma)$	eigenvectors of $A^{\mathrm{T}}A, AA^{\mathrm{T}}$ in V, U