Linear Transformation

$$T(cv + dw) = cT(v) + dT(w)$$

Affine transformation

$$T(v) = \underline{Av + u_0} \quad : \text{linear-plus-shift}$$

{ Straight lines stay straight

Equally spaced points go to equally spaced points.

$$u = c_1 v_1 + \cdots + c_n v_n$$

$$\downarrow$$

$$T(u) = c_1 T(v_1) + \cdots + c_n T(v_n)$$

range, kernel

range : corresponds to column space

kernel : corresponds to nullspace

Matrices for derivative and integral

$\mathbb{R}^4$ $\qquad$ $\mathbb{R}^3$

$$\begin{array}{ccc} V & \xrightarrow{\text{derivative}} & W \\ & \xleftarrow[\text{integral}]{} & \\ \text{cubics} & & \text{quadratics} \end{array}$$

$\left\{\begin{array}{l} \text{the basis for } V \text{ is } 1,\ x,\ x^2,\ x^3 \\ \text{the basis for } W \text{ is } 1,\ x,\ x^2 \end{array}\right.$

- derivative $V \in \mathbb{R}^4 \to W \in \mathbb{R}^3 : A \in \mathbb{R}^{3\times 4}$

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 3d \end{pmatrix}$$

$$a + bx + cx^2 + dx^3 \to b + 2cx + 3dx^2$$

- integral $W \in \mathbb{R}^3 \to V \in \mathbb{R}^4 : A \in \mathbb{R}^{4\times 3}$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}}_{A^{-1}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ \frac{1}{2}b \\ \frac{1}{3}c \end{pmatrix}$$

$$a + bx + cx^2 \to ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3$$

$$AA^{-1} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

$$A^{-1}A = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in \mathbb{R}^{4\times 4}$$

⑩ Key rule for constructing a matrix

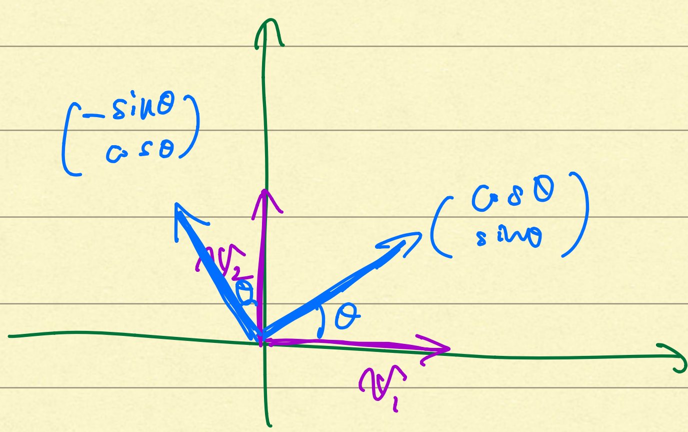
The j th column of A is found by applying T to the j th basis vector v_j

$$T(v_j) = \text{combination of basis vectors of } W$$

$$= a_{1j} w_1 + \cdots + a_{mj} w_m$$

⑪ T: rotates every vector by the angle θ

standard vectors : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



$$\begin{cases} \vec{v}_1 = a_{11}\begin{pmatrix}1\\0\end{pmatrix} + a_{21}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a_{11}\\a_{21}\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \\ \vec{v}_2 = a_{12}\begin{pmatrix}1\\0\end{pmatrix} + a_{22}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a_{12}\\a_{22}\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \end{cases}$$

$$A = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$