3.1 Spaces of Vectors

To a newcomer, matrix calculations involve a lot of numbers. To you, they involve vectors. The columns of Ax and AB are linear combinations of n vectors—the columns of A. This chapter moves from numbers and vectors to a third level of understanding (the highest level). Instead of individual columns, we look at "spaces" of vectors. Without seeing ***vector spaces*** and especially their ***subspaces***, you haven't understood everything about $Ax = b$.

Since this chapter goes a little deeper, it may seem a little harder. That is natural. We

Definition of subspace

If v and w are vectors in the subspace and c is any scalar, then

(i) $v + w$ is in the subspace

(ii) cv is in the subspace.

$\Rightarrow$ all linear combinations stay in the subspace

The column space of A : $C(A)$

Def

The column space consists of the the combinations that are all possible vectors Ax. They fill the column space $C(A)$..

$$Ax = b$$

b is written as the linear combination the colums of A

(II) The column space is very important!!!

To solve $A\mathbb{x} = \mathbb{b}$ is to express $\mathbb{b}$ as a linear combination of the columns.

The system $A\mathbb{x} = \mathbb{b}$ is solvable if and only if $\mathbb{b}$ is in the column space of A

(III) Null space

$N(A) = \mathrm{Ker}(A)$

$$N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \}$$

(III) The difinition of rank

① rank = # pivots

② rank = # independent columns / rows

③ rank = the dimension of the $\left\{ \begin{matrix} \text{column space} \\ \text{row space} \\ \text{nullspace} \end{matrix} \right\}$

(IV) The pivot columns (pivcol)

$$A = \begin{pmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{pmatrix} \xrightarrow[\text{echelon}]{} R = \begin{pmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivcol = (1. 3)

$$EA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} A = R$$

$$E^{-1}R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} R = A$$

$\boxed{\text{The } r \text{ pivot columns of } A \text{ are also the first } r \text{ columns of } E^{-1}}$

$\boxed{\begin{array}{l}\text{The pivot columns are not combinations of earlier columns.} \\ \text{The free columns are combinations of earlier columns.}\end{array}}$

The complete solution to $A\mathbf{x} = \mathbf{b}$

Ex What's the complete solution to the equation below?

$$\underset{A}{\begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{pmatrix}} \underset{\mathbf{x}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} = \underset{\mathbf{b}}{\begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}}$$

$$E(A\ \mathbf{b}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} (A\ \mathbf{b})$$

↑
elimination

$$= \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = (R\ \mathbf{d})$$

① x_p : particular solution

$$A x_p = b \Rightarrow R x_p = d$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}$$

pivot free columns

We set the free variables $x_2 = x_4 = 0$

Then we get the particular solution

$$x_p = \begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix}$$

② x_n : special solution

$$A x_n = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} x_n = 0$$

$$x_n = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} x_4$$

→ Complete Solution

$$x = x_p + x_n = \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} x_4$$

(III) Properties of full column rank matrix A $(r = n)$ $(A \in \mathbb{R}^{m \times n})$

① All columns of A are pivot columns

② There are no free variables

③ The nullspace $N(A)$ contains only the zero vector $x = 0$

④ If $Ax = b$ has a solution then it has only one solution.

⑤ $A^T A$ is invertible.

(IV) Properties of full row rank matrix A $(r = m)$

① All rows have pivots, and R has no zero rows.

② $Ax = b$ has a solution for every

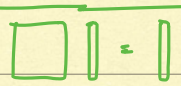
right side b

③ The column space is the whole space $\mathbb{R}^m$

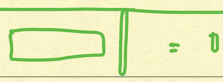
④ There are $n-r = n-m$ special solutions in the nullspace of A.

(iii) The four possibilities for linear equations depending on the rank r

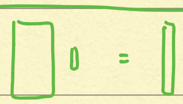
① $r = m = n$ Square and invertible. $Ax = b$ has 1 solution



② $r = m < n$ Short and wide $Ax = b$ has ∞ solutions



③ $r = n < m$ Tall and thin $Ax = b$ has 0 or 1 solution



④ $r < m$, $r < n$ Not full rank $Ax = b$ has 0 or ∞ solutions

(iii) The four types of reduced matrix R

① $r = m = n \longrightarrow R = [\,I\,]$

② $r = m < n \longrightarrow R = [\,I \;\; F\,]$

③ $r = n < m \longrightarrow R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

④ $r < m,\ r < n \longrightarrow R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

(II) Independence

- The definition of the independence

The columns of A are linearly independent when the nullspace $N(A)$ contains only zero vector $x = 0$

(III) independency and rank

$$\boxed{\text{The columns of } A \text{ are independent} \iff r = n \text{ (full column rank)}}$$

(IV) Spanning

A set of vectors spans a space if their linear combinations fill the space.

→ The columns of a matrix span its column space.

Basis

The vectors $v_1 \cdots v_n$ are a basis for $\mathbb{R}^n$ exactly when they are the columns of an n by n invertible matrix.

The dimension

def

The dimension of a space is the number of vectors in every basis

$$\dim(C(A)) = \text{rank } A$$

Four fundamental subspaces.

① $C(A^T)$: row space.

$$\dim(C(A^T)) = r$$

② $C(A)$: column space

$$\dim(C(A)) = r$$

③ $N(A)$: nullspace

$$\dim(N(A)) = n-r$$

④ $N(A^T)$: left nullspace

$$\dim(N(A^T)) = m-r$$

次元定理

The Rank-nullity theorem

$$r + \dim(N(A)) = n$$

Rank nullity