Standard form

$$\boxed{\begin{array}{ll} \text{min.} & f_0(x) \\ \text{s.t.} & f_i(x) \le 0 \\ & h_i(x) = 0 \end{array}}$$

Equivalent problems

… We call two problems equivalent if from a solution of one, a solution of the other is readily found, and vice versa.

Scaling

$$\boxed{\text{Standard form}}$$

$\Updownarrow$ equivalent

$$\boxed{\begin{array}{ll} \text{min.} & \alpha_0 f(x) \\ \text{s.t.} & \alpha_i f_i(x) \le 0 \\ & \beta_i h_i(x) = 0 \end{array} \quad \begin{pmatrix} \alpha_i > 0 \\ \beta_i \neq 0 \end{pmatrix}}$$

(iii) Change of variables

standard form

$\cdot\ x = \phi(z)$

$\cdot$ image of ϕ covers the problem domain D, i.e. $\phi(\mathrm{dom}\,\phi) \supseteq D$

equivalent

$$
\begin{aligned}
\text{min.}\quad & f_0(\phi(z)) \\
\text{s.t.}\quad & f_i(\phi(z)) \leq 0 \\
& h_i(\phi(z)) \leq 0
\end{aligned}
$$

(iv) Transformation of objective and constraint functions

standard form

ψ_0 : monotone increasing

$\psi_1 \cdots \psi_m$: $\psi_i(u) \leq 0$ if $u \leq 0$

$\psi_{m+1} \cdots \psi_{m+p}$: $\psi_i(u) = 0$ if $u = 0$

equivalent

$$\boxed{\begin{aligned} \min.\quad & \psi_0(f_0(x)) \\ \text{s.t.}\quad & \psi_i(f_i(x)) \le 0 \\ & \psi_{m+i}(h_i(x)) = 0 \end{aligned}}$$

Ex

$$\min.\quad \|Ax-b\|_2$$

$$\downarrow \quad \psi_0 = u^2 \quad (u \ge 0)$$

$$\min.\quad \|Ax-b\|_2^2.$$

⑪ Slack variables

$$\boxed{\text{Standard form}}$$

equivalent

$$f_i(x) \le 0 \xrightarrow[\text{(u⸴ᵻ3 u)}]{\text{slack}} f_i(x) + s_i(x) = 0$$

$$\boxed{\begin{aligned} \min.\quad & f_0(x) \\ \text{s.t.}\quad & f_i(x) + s_i(x) = 0 \\ & s_i(x) \ge 0 \\ & h_i(x) = 0 \end{aligned}}$$

Eliminating equality constraints

$\boxed{\text{Standard form}}$

equivalent

$\phi : \mathbb{R}^k \to \mathbb{R}^n$ is such that

$h_i(\phi(z)) = 0$ for some $z \in \mathbb{R}^k$

$$\boxed{\begin{aligned} \text{min. } & f_0(\phi(z)) \\ \text{s.t. } & f_i(\phi(z)) \le 0 \end{aligned}}$$

Eliminating linear equality constraints.

$$\boxed{\begin{aligned} \text{min. } & f_0(x) \\ \text{s.t. } & f_i(x) \le 0 \\ & Ax = b \end{aligned}}$$

equivalent

- let x_0 denote any solution of $Ax = b$.
- $F \in \mathbb{R}^{n \times k}$, $C(F^T) = N(A)$
  row space null space

$\Rightarrow$ general solution of $Ax = b$ is given by $Fz + x_0$

$$\boxed{\begin{aligned} &\text{min.} && f_0(Fz + x_0) \\ &\text{s.t.} && f_i(Fz + x_0) \le 0 \end{aligned}}$$

④ Introducing equality constraints

$$\boxed{\begin{aligned} &\text{min.} && f_0(A_0 x + b_0) \\ &\text{s.t.} && f_i(A_i x + b_i) \\ &&& h_i(x) = 0 \\ & x \in \mathbb{R}^n, && A_i \in \mathbb{R}^{k_i \times n}, \quad f_i : \mathbb{R}^{k_i} \to \mathbb{R} \end{aligned}}$$

$$\Big\downarrow \; y_i = A_i x + b_i$$

equivalent

$$\boxed{\begin{aligned} &\text{min.} && f_0(y_0) \\ &\text{s.t.} && f_i(y_i) \le 0 \\ &&& y_i = A_i x + b_i \\ &&& h_i(x) = 0 \end{aligned}}$$

Optimizing over some variables

$$\inf_{x,y} f(x,y) = \inf_x \left(\inf_y f(x,y) \right)$$

Epigraph problem form

$\boxed{\text{Standard form}}$

$\downarrow$

$$\boxed{\begin{aligned} \text{min.} \quad & t \\ \text{s.t.} \quad & f_0(x) - t \le 0 \\ & f_i(x) \le 0 \\ & h_i(x) = 0 \end{aligned}}$$

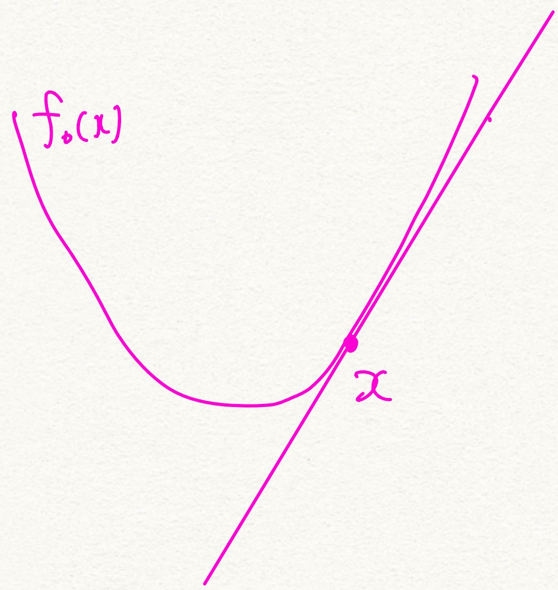
Convex optimization problem in standard form

$$\begin{aligned} \text{min.} \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \le 0 \\ & a_i^T x = b_i \end{aligned}$$

- $f_0 \cdots f_m$ are convex
- equality constraint functions are affine.

④ An optimality criterion for differentiable f_0

If f_0 is convex and differentiable, then



$$f_0(y) \geq f_0(x) + \nabla f_0(x)^T (y - x)$$

$$(x, y \in \mathrm{dom} f_0)$$

Let X denote the feasible set, i.e.

$$X = \{ x \mid f_i(x) \leq 0, \quad h_i(x) = 0 \}$$

Then x is optimal iff $x \in X$ and

$$f_0(y) - f_0(x) \geq \nabla f_0(x)^T (y - x) \geq 0$$

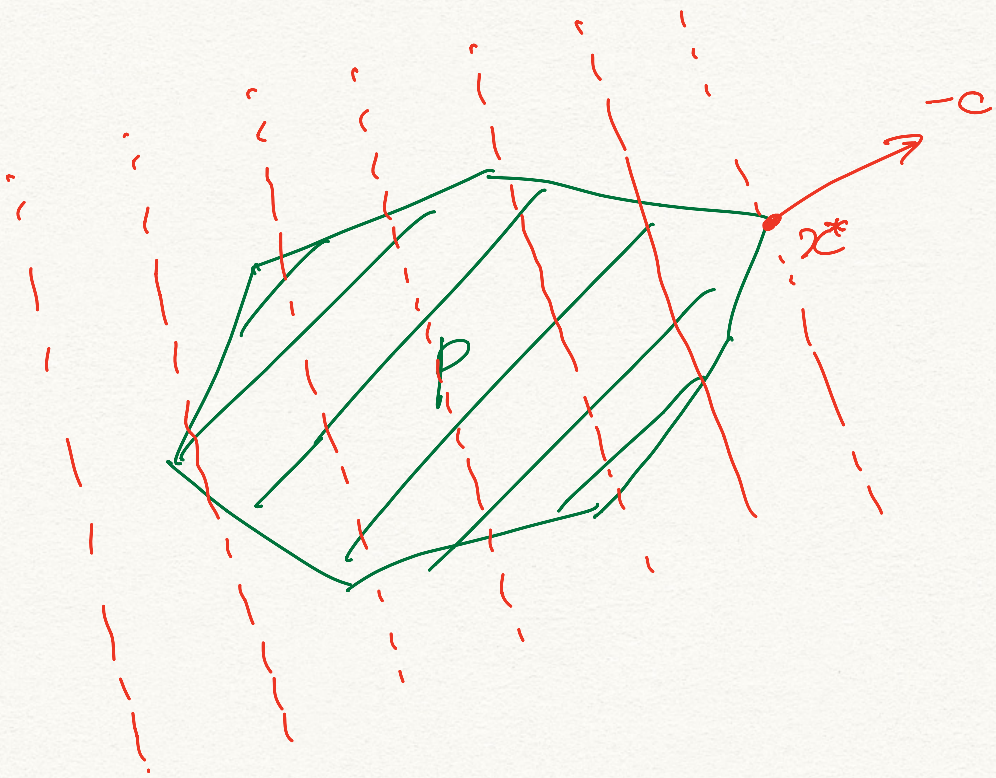
for all $y \in X$

LP (Linear Program)

$$\boxed{\begin{array}{ll} \text{min.} & c^T x + d \\ \text{s.t.} & Gx \preceq h \\ & Ax = b \end{array}}$$

where $G \in \mathbb{R}^{m \times n}$, $A \in \mathbb{R}^{p \times n}$.

It is common to omit the constant d.



① Standard form LP

$$\boxed{\begin{array}{ll} \text{min.} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}}$$

② Inequality form LP

$$\boxed{\begin{array}{ll} \text{min.} & c^T x \\ \text{s.t.} & Ax \leq b \end{array}}$$

Linear-fractional programming

$$\boxed{\begin{aligned} \text{min.} \quad & \frac{c^T x + d}{e^T x + f} \\ \text{s.t.} \quad & Gx \preceq h \\ & Ax = b \end{aligned}}$$

$(\mathrm{dom}\, f_0 = \{ x \mid e^T x + f > 0 \})$

$$\downarrow \qquad y = \frac{x}{e^T x + f}, \qquad z = \frac{1}{e^T x + f}$$

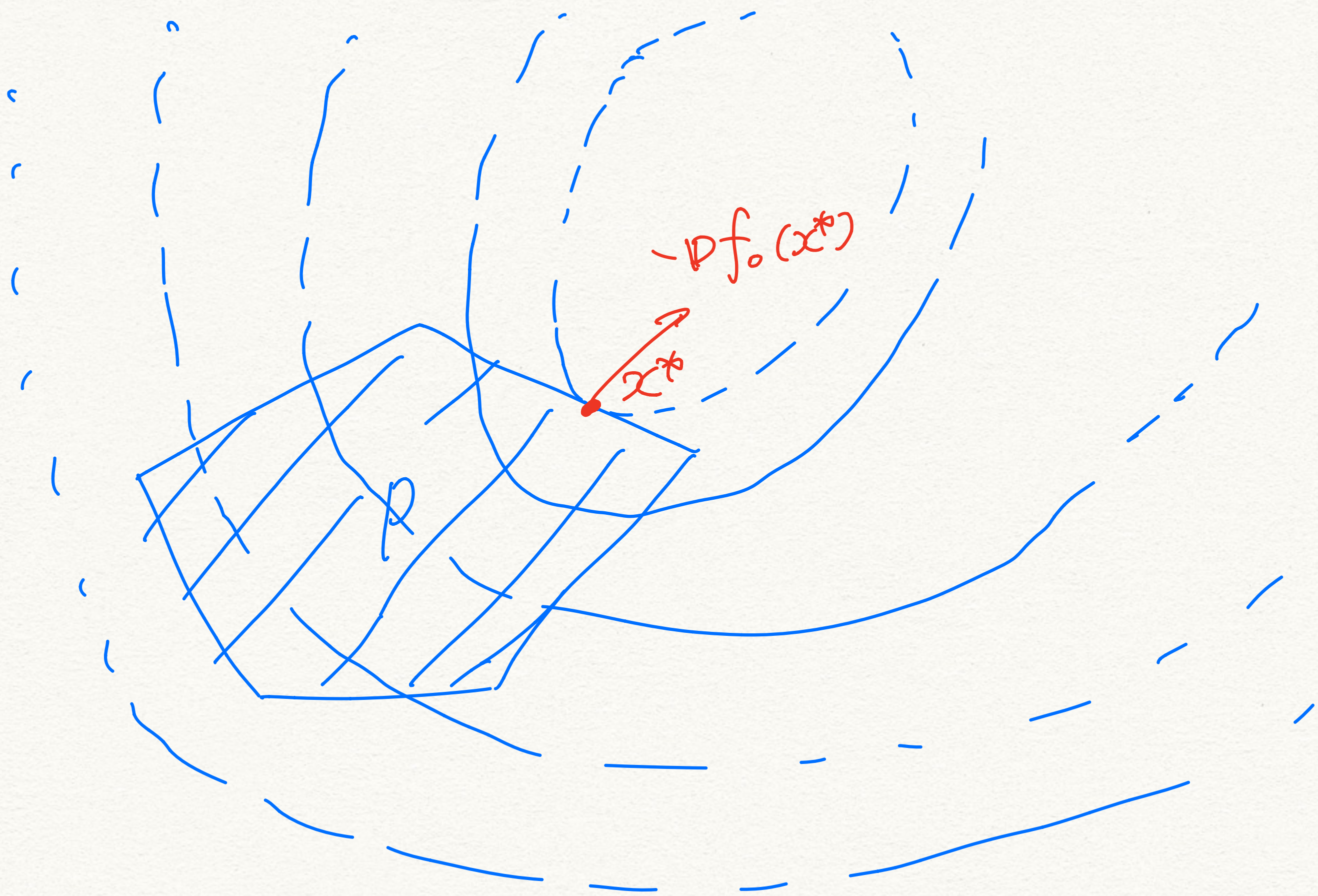
standard form

$$\boxed{\begin{aligned} \text{min.} \quad & c^T y + dz \\ \text{s.t.} \quad & Gy - hz \preceq 0 \\ & Ay - bz = 0 \\ & e^T y + fz = 1 \\ & z \geq 0 \end{aligned}}$$

11 QP (Quadratic Program)

$$\boxed{\begin{aligned} \text{min.} \quad & \frac{1}{2}x^T P x + q^T x + r \\ \text{s.t.} \quad & Gx \preceq h \\ & Ax = b \end{aligned}}$$

where $P \in S_+^n$, $G \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{p\times n}$



(iii) QCQP (Quadratically constrained Quadratic Program)

If the objective and inequality functions are quadratic,

$$
\boxed{
\begin{aligned}
\text{min.} \quad & \frac{1}{2} x^T P_0 x + q_0^T x + r_0 \\
\text{s.t.} \quad & \frac{1}{2} x^T P_i x + q_i^T x + r_i \leq 0 \\
& Ax = b
\end{aligned}
}
$$

(iv) SOCP (Second-Order Cone Program)

$$
\boxed{
\begin{aligned}
\text{min.} \quad & f^T x \\
\text{s.t.} \quad & \|A_i x + b_i\|_2 \leq c_i^T x + d_i \\
& Fx = g
\end{aligned}
}
$$

where $x \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n_i \times n}$, $F \in \mathbb{R}^{p \times n}$

$$
\text{QCQP} \xleftarrow{c_i = 0} \text{SOCP} \xrightarrow{A_i = 0} \text{LP}
$$

(iii) Monomials and Posynomials

- Monomial

$$f(x) = c\, x_1^{a_1} x_2^{a_2} \cdots\cdots x_n^{a_n}$$

$$(\mathrm{dom} f = \mathbb{R}_{++}^n,\ c > 0)$$

Closed under multiplication and division

- Posynomial

$$f(x) = \sum_{k=1}^{K} c_k\, x_1^{a_{1k}} x_2^{a_{2k}} \cdots\cdots x_n^{a_{nk}}$$

$$(c_k > 0)$$

Closed under addition, multiplication and nonnegative scaling.

(iv) GP (Geometric Program)

Optimization of the form

$$\boxed{\begin{aligned} \min.\ & f_0(x) \\ \text{s.t.}\ & f_i(x) \leq 1 \\ & h_i(x) = 1 \end{aligned}}$$

where $f_0 \dots f_m$ are posynomials and

$h_1 \dots h_p$ are monomials

is called GP (with domain $D = \mathbb{R}^n_{++}$)

Converting GP to convex

Consider the posynomial

$$f(x) = \sum_{k=1}^{K} c\, x_1^{a_{1k}} \cdots x_n^{a_{nk}}$$

we use the variables defined as

$$y_i = \log x_i, \qquad x_i = e^{y_i}$$

then

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} \cdots x_n^{a_{nk}}$$

$$= \sum_{k=1}^{K} c_k e^{a_{1k} y_1} \cdots e^{a_{nk} y_n}$$

$$= \sum_{k=1}^{K} c_k e^{a_k^T y}$$

$$= \sum_{k=1}^{K} e^{a_k^T y + b_k} \qquad (e^{b_k} = c_k)$$

So the GP can be expressed as

$$
\boxed{
\begin{aligned}
\min.\quad & \sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}} \\
\text{s.t.}\quad & \sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \le 1 \\
& e^{g_i^T y + h_i} = 1
\end{aligned}
}
$$

$$\Big\downarrow \text{ take the logarithm}$$

$$
\boxed{
\begin{aligned}
\min.\quad & \log\left(\sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}}\right) \\
\text{s.t.}\quad & \log\left(\sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}}\right) \le 0 \\
& g_i^T y + h_i = 0
\end{aligned}
}
$$

This is convex !!!