2.3 The Slope and the Tangent Line

Chapter 1 started with straight line graphs. The velocity was constant (at least piecewise). The distance function was linear. Now we are facing polynomials like $x^3 - 2$ or $x^4 - x^2 + 3$, with other functions to come soon. Their graphs are definitely curved. Most functions are not close to linear—except if you focus all your attention near a single point. That is what we will do.

Over a very short range a curve looks straight. Look through a microscope, or zoom in with a computer, and there is no doubt. The graph of distance versus time becomes nearly linear. Its slope is the velocity at that moment. We want to find the line that the graph stays closest to—the "***tangent line***"—before it curves away.

The tangent line is easy to describe. We are at a particular point on the graph of $y = f(x)$. At that point x equals a and y equals $f(a)$ and the slope equals $f'(a)$. ***The tangent line goes through that point*** $x = a$, $y = f(a)$ ***with that slope*** $m = f'(a)$. Figure 2.5 shows the line more clearly than any equation, but we have to turn the geometry into algebra. We need the equation of the line.

EXAMPLE 1 Suppose $y = x^4 - x^2 + 3$. At the point $x = a = 1$, the height is $y = f(a) = 3$. The slope is $dy/dx = 4x^3 - 2x$. At $x = 1$ the slope is $4 - 2 = 2$. That is $f'(a)$:

The numbers $x = 1$, $y = 3$, $dy/dx = 2$ ***determine the tangent line.***

The equation of the tangent line is $y - 3 = 2(x - 1)$, *and this section explains why.*

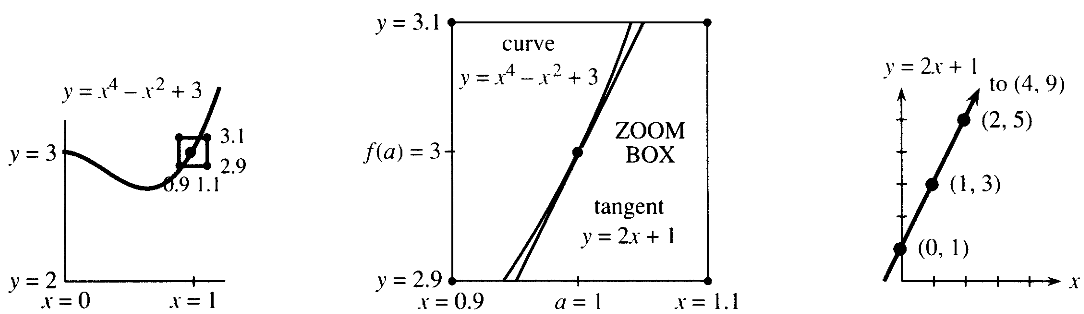

Fig. 2.5 The tangent line has the same slope 2 as the curve (especially after zoom).

THE EQUATION OF A LINE

A straight line is determined by two conditions. We know the line if we know two of its points. (We still have to write down the equation.) Also, if we know ***one point and the slope***, the line is set. That is the situation for the tangent line, which has a known slope at a known point:

1. The equation of a line has the form $y = mx + b$
2. The number m is the slope of the line, because $dy/dx = m$
3. The number b adjusts the line to go through the required point.

I will take those one at a time—first $y = mx + b$, then m, then b.

1. The graph of $y = mx + b$ is not curved. How do we know? For the specific example $y = 2x + 1$, take two points whose coordinates x, y satisfy the equation:

$$x = 0,\ y = 1 \quad \text{and} \quad x = 4,\ y = 9 \quad \text{both satisfy} \quad y = 2x + 1.$$

Those points (0, 1) and (4, 9) lie on the graph. *The point halfway between has* $x = 2$ *and* $y = 5$. That point also satisfies $y = 2x + 1$. ***The halfway point is on the graph.*** If we subdivide again, the midpoint between (0, 1) and (2, 5) is (1, 3). This also has $y = 2x + 1$. The graph contains all halfway points and must be straight.

2. What is the correct slope m for the tangent line? In our example it is $m = f'(a) = 2$. ***The curve and its tangent line have the same slope at the crucial point***: $dy/dx = 2$.

Allow me to say in another way why the line $y = mx + b$ has slope m. At $x = 0$ its height is $y = b$. At $x = 1$ its height is $y = m + b$. The graph has gone *one unit across* (0 to 1) *and m units up* (b to $m + b$). The whole idea is

$$\text{slope} = \frac{\text{distance up}}{\text{distance across}} = \frac{m}{1}. \tag{1}$$

Each unit across means m units up, to $2m + b$ or $3m + b$. A straight line keeps a constant slope, whereas the slope of $y = x^4 - x^2 + 3$ equals 2 only at $x = 1$.

3. Finally we decide on b. The tangent line $y = 2x + b$ must go through $x = 1$, $y = 3$. Therefore $b = 1$. With letters instead of numbers, $y = mx + b$ leads to $f(a) = ma + b$. So we know b:

> **2E** The equation of the tangent line has $b = f(a) - ma$:
>
> $$y = mx + f(a) - ma \qquad \text{or} \qquad y - f(a) = m(x - a). \tag{2}$$

That last form is the best. You see immediately what happens at $x = a$. The factor $x - a$ is zero. Therefore $y = f(a)$ as required. This is the ***point-slope form*** of the equation, and we use it constantly:

$$y - 3 = 2(x - 1) \qquad \text{or} \qquad \frac{y-3}{x-1} = \frac{\text{distance up}}{\text{distance across}} = \text{slope } 2.$$

EXAMPLE 2 The curve $y = x^3 - 2$ goes through $y = 6$ when $x = 2$. At that point $dy/dx = 3x^2 = 12$. The point-slope equation of the tangent line uses 2 and 6 and 12:

$$y - 6 = 12(x - 2), \quad \text{which is also} \quad y = 12x - 18.$$

There is another important line. It is *perpendicular* to the tangent line and *perpendicular* to the curve. This is the ***normal line*** in Figure 2.6. Its new feature is its slope. When the tangent line has slope m, the normal line has slope $-1/m$. (Rule: Slopes of

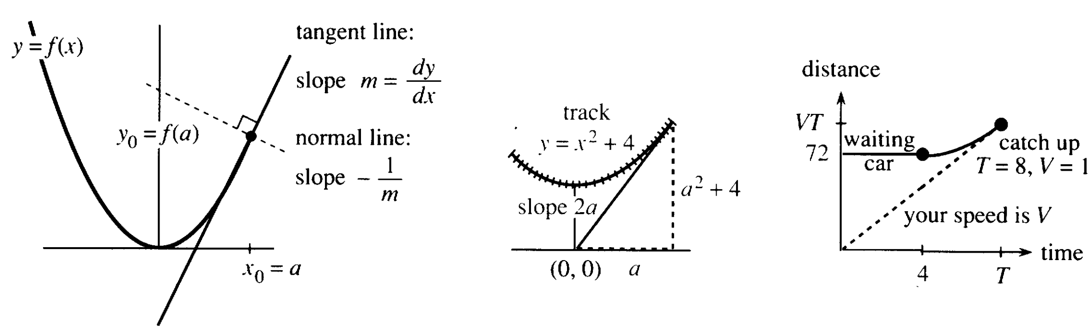

Fig. 2.6 Tangent line $y - y_0 = m(x - x_0)$. Normal line $y - y_0 = -\dfrac{1}{m}(x - x_0)$. Leaving a roller-coaster and catching up to a car.

perpendicular lines multiply to give -1.) Example 2 has $m = 12$, so the normal line has slope $-1/12$:

$$\textit{tangent line}: y - 6 = 12(x - 2) \qquad \textit{normal line}: y - 6 = -\tfrac{1}{12}(x - 2).$$

Light rays travel in the normal direction. So do brush fires—they move perpendicular to the fire line. Use the point-slope form! The tangent is $y = 12x - 18$, the normal is not $y = -\frac{1}{12}x - 18$.

EXAMPLE 3 You are on a roller-coaster whose track follows $y = x^2 + 4$. You see a friend at (0, 0) and want to get there quickly. Where do you step off?

Solution Your path will be the tangent line (at high speed). The problem is ***to choose*** $x = a$ ***so the tangent line passes through*** $x = 0$, $y = 0$. When you step off at $x = a$,

the height is $y = a^2 + 4$ and the slope is $2a$

the equation of the tangent line is $y - (a^2 + 4) = 2a(x - a)$

this line goes through (0, 0) if $-(a^2 + 4) = -2a^2$ or $a = \pm 2$.

The same problem is solved by spacecraft controllers and baseball pitchers. Releasing a ball at the right time to hit a target 60 feet away is an amazing display of calculus. Quarterbacks with a moving target should read Chapter 4 on related rates.

Here is a better example than a roller-coaster. Stopping at a red light wastes gas. It is smarter to slow down early, and then accelerate. When a car is waiting in front of you, the timing needs calculus:

EXAMPLE 4 How much must you slow down when a red light is 72 meters away? In 4 seconds it will be green. The waiting car will accelerate at 3 meters/sec^2. You cannot pass the car.

Strategy Slow down immediately to the speed V at which you will just catch that car. (If you wait and brake later, your speed will have to go below V.) At the catch-up time T, the cars have the same speed and same distance. ***Two conditions***, so the distance functions in Figure 2.6d are tangent.

Solution At time T, the other car's speed is $3(T - 4)$. That shows the delay of 4 seconds. Speeds are equal when $3(T - 4) = V$ or $T = \frac{1}{3}V + 4$. Now require equal distances. Your distance is V times T. The other car's distance is $72 + \frac{1}{2}at^2$:

$$72 + \tfrac{1}{2} \cdot 3(T - 4)^2 = VT \quad \text{becomes} \quad 72 + \tfrac{1}{2} \cdot \tfrac{1}{3}V^2 = V(\tfrac{1}{3}V + 4).$$

The solution is $V = 12$ meters/second. This is 43 km/hr or 27 miles per hour.

Without the other car, you only slow down to $V = 72/4 = 18$ meters/second. As the light turns green, you go through at 65 km/hr or 40 miles per hour. Try it.

THE SECANT LINE CONNECTING TWO POINTS ON A CURVE

Instead of the tangent line through one point, consider the ***secant line through two points***. For the tangent line the points came together. Now spread them apart. The point-slope form of a linear equation is replaced by the ***two-point form***.

The equation of the curve is still $y = f(x)$. The first point remains at $x = a$, $y = f(a)$. The other point is at $x = c$, $y = f(c)$. The secant line goes between them, and we want its equation. This time we don't start with the slope—but m is easy to find.