Mistake

$$|3p-2q| < N^{\frac{1}{4}} \Rightarrow (3p-2q)^2 < \sqrt{N} \quad ①$$

Let $A = \frac{3p+2q}{2}$ "not an integer"

$$A^2 - 6N = \frac{1}{4}(9p^2+4q^2+12pq) - 6N$$

$$= \frac{1}{4}(9p^2+4q^2-12pq)$$

$$= \frac{1}{4}(3p-2q)^2 \quad ②$$

for, $p,q \geq 3$ $(3p \neq 2q)$

According to ②, $A^2-6N \overset{>}{\geq} 0 \Rightarrow A \overset{>}{\geq} 6\sqrt{N}$ ③

$$\therefore A-\sqrt{6N} = A-\sqrt{6N} \cdot \frac{A+\sqrt{6N}}{A+\sqrt{6N}} = \frac{A^2-6N}{A+\sqrt{6N}}$$

$$(②③) \quad \leq \frac{(3p-2q)^2}{4(\sqrt{6N}+\sqrt{6N})} = \frac{(3p-2q)^2}{8\sqrt{6N}}$$

$$(①) \quad \leq \frac{\sqrt{N}}{8\sqrt{6N}} = \frac{1}{8\sqrt{6}}$$

$$\Rightarrow A-\sqrt{6N} \leq \frac{1}{8\sqrt{6}} \quad ④$$

$$③④ \Rightarrow \sqrt{6N} < A \leq \sqrt{6N} + \frac{1}{8\sqrt{6}}$$

$$A = \lceil \sqrt{6N} \rceil \qquad A = \frac{\overset{\text{odd}}{3p} + \overset{\text{even}}{2q}}{2}$$

Now $\frac{3p+2q}{2} = A$ (i.e. $\lceil\sqrt{6N}\rceil$), $pq = N$

$$\begin{cases} N = \frac{1}{6}\cdot 3p \cdot 2q, \Leftrightarrow 6N = 3p\cdot 2q \\ A = \frac{3p+2q}{2} \end{cases}$$

Let $3p = A+x$, $2q = A-x$

$$\Rightarrow 6N = A^2 - x^2 \Rightarrow x^2 = A^2 - 6N$$

$$\Rightarrow x = \sqrt{A^2-6N}$$

$$\Rightarrow p = \frac{A+\sqrt{A^2-6N}}{3}, \quad q = \frac{A-\sqrt{A^2-6N}}{2}$$

Correct Answer

$$|3p-2q| < N^{\frac{1}{4}} \Rightarrow (3p-2q)^2 < \sqrt{N} \quad ①$$

Let $A = 3p+2q$ "odd integer"

Observe: $A^2 - 4\cdot 6N = 9p^2 + 4q^2 - 12pq = (3p-2q)^2 \quad ②$

$$② \Rightarrow A^2 - 4\cdot 6N > 0 \Rightarrow A > 2\sqrt{6N} \quad ③$$

Then: $$A - 2\sqrt{6N} = A - 2\sqrt{6N} \cdot \frac{A+2\sqrt{6N}}{A+2\sqrt{6N}} = \frac{A^2 - 4\cdot 6N}{A+2\sqrt{6N}}$$

$$< \frac{(3p-2q)^2}{4\sqrt{6N}} < \frac{1}{4\sqrt{6}} \quad ④$$

$$③④ \Rightarrow 2\sqrt{6N} < A < 2\sqrt{6N} + \frac{1}{4\sqrt{6}}$$

$$\sqrt{6N} = [\sqrt{6N}] + \underbrace{\langle\sqrt{6N}\rangle}_{\in [0,1)}$$

$$\Rightarrow 2[\sqrt{6N}] + 2\langle\sqrt{6N}\rangle < A < 2[\sqrt{6N}] + 2\langle\sqrt{6N}\rangle + \frac{1}{4\sqrt{6}}$$

$$\Rightarrow 2\lfloor\sqrt{6N}\rfloor \leq A \leq 2\lfloor\sqrt{6N}\rfloor + 2$$

$$A = \cancel{2\lfloor\sqrt{6N}\rfloor} \text{ or } 2\lfloor\sqrt{6N}\rfloor + 1 \text{ or } \cancel{2\lfloor\sqrt{6N}\rfloor + 2}$$

$$\begin{cases} N = p\cdot q \\ A = 3p+2q \end{cases} \iff 3N = 3p\cdot q$$

$$\Rightarrow \quad 3N = (A-2q)\cdot q$$

$$\Rightarrow \quad 2q^2 - A\cdot q + 3N = 0$$

$$\Delta = A^2 - 4\cdot 2\cdot 3N = A^2 - 24N$$

$$\therefore \quad q = \frac{A+\sqrt{A^2-24N}}{4} \quad \text{or} \quad \frac{A-\sqrt{A^2-24N}}{4}$$

CHECK whether $A+\sqrt{A^2-24N}$ or $A-\sqrt{A^2-24N}$ is a multiple of 4

Then $p = \frac{1}{3}(A-2q)$