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Remedial for Natural Science(Mathematics)

Calculus I (□□□ □□□□ □□□□□□□)

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Chapter 1: Solving Equations and Inequalities

1.1 Equations involving exponents and radicals

Equations are equality of expressions. There are different types of equations that depend on the variable(s) considered. When the exponent has other than one then the variable in use is said to be an equation involving exponents.

Example: $2^2 \times 2^n = 2^{n+2}, 2^x = 16, x^3 = 8$ etc.

Rule : For a>0, $a^x = a^y$ if and only if $x = y$.

Solve the following equations:

A. $8^x = 2^{2x+1}$ B.$9^{x-3} = 27^{3x}$ C. $9^{2x}27^{1-x} = 81^{2x+1}$ D.$16^{3x+4} = 2^{3x}64^{-4x+1}$

1.2 System of linear equations in two variables

Definition:

Any equation that can be reduced to the form $ax + b = 0$, where $a, b \in \mathbb{R}$, and $a \neq 0$, is called a linear equation in one variable.

Example: a) $3x + 4 = 5$ b) $-2x - 6 = 7$ c) $\frac{2}{3}x - \frac{3}{2}$ d) $\sqrt{3}\,x - \pi = 0$

An equation of the type $cx + dy = e$, where c, d and e are arbitrary constants and c ≠ 0, d ≠ 0, is called a linear equation in two variables. An equation in two variables of the form $cx + dy = e$ can be reduced to the form $y = ax + b$.

Definition:

A set of two or more linear equations is called a system of linear equations. Systems of two linear equations in two variables are equations that can be represented as

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

Where, $a_1, b_1, a_2, b_2 \neq 0$.

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the system has infinite solutions. In this case, every order that satisfies one of the component equations also satisfies the second. Such system is said to be dependent.

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If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the system has no solution. This means the two component equations do not have common solution. In this case the system is said to inconsistent.

If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ then the system has one solution. This means there is only one ordered pair that satisfies both equations. In this case the system is called independent.

Example: The following are examples of systems of linear equations in two variables.

a) $\begin{cases} 3x + 2y = 5 \\ 5x + 4y = 10 \end{cases}$

Definition:

A solution to a system of linear equations in two variables means the set of ordered pairs (x, y) that satisfy both equations.

Solution to a system of linear equations in two variables

These include the graphical method, substitution method and elimination method.

i) Graphical Method

In this method, we need to draw graphs of the two equations on the same system.

If the lines intersect there is one solution, if the lines are parallel, the system has no solution,

If the lines coincide then there are infinite solutions to the system, every point (ordered pair) on the line satisfies both equations.

ii) Substitution Method

Steps to solve the system using this method:

- Take one of the linear equations from the system and write one of the variables in terms of the other.
- Substitute your result into the other equation and solve for the second equation
- Substitute this result into one of the equations and solve for the first equation

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Example: Solve the following using substitution methods

a) $\begin{cases}2x-3y=5\\5x+3y=9\end{cases}$ b) $\begin{cases}2x-4y=5\\-6x+12y=-15\end{cases}$ c) $\begin{cases}2x-y=1\\3x-2y=-4\end{cases}$

iii) Elimination Method

To solve a system of two linear equations by the elimination method elimination method elimination method, you follow the following steps.

- Select one of the variables and make the coefficients of the selected variable equal but opposite in sign in the two equations.
- Add the two equations to eliminate the selected variable and solve for the resulting variable.
- Substitute this result again into one of the equations and solve for the remaining variable.

Example: Solve the following equations using elimination methods

a) $\begin{cases}2x-3y=5\\5x+3y=9\end{cases}$ b) $\begin{cases}7x+5y=11\\-3x+3y=-3\end{cases}$ c) $\begin{cases}2x-4y=8\\x-2y=4\end{cases}$

Word problems leading to a system of linear equations

Example: 1. Simon has twin younger brothers. The sum of the ages of the three brothers is 48 and the difference between his age and the age of one of his younger brothers is 3. How old is Simon?

Solution: Let x be the age of Simon and y be the age of each of his younger brothers. The sum of the ages of the three brothers is 48. Then the equation becomes:

$\begin{cases}\mathrm{x+y+y=48}\\x-y=3\end{cases}$, and $\begin{cases}\mathrm{x+2y=48}\\x-y=3\end{cases}\Leftrightarrow\begin{cases}\mathrm{x+2y=48}\\-x+y=-3\end{cases}$, multiplying the second by -1. Then adding the two $3y=45, y=15$. Hence, $x=48-2(15)=18$. Thus, age of Simon is 18 years.

2. The sum of two numbers is 64. Twice the larger number plus five times the smaller number is 20. Find the two numbers.

3. In a two-digit number, the sum of the digits is 14. Twice the tens digit exceeds the units digit by one. Find the numbers.

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1.3 Equations involving absolute values

Definition The absolute value of a number x, denoted by $|x|$, is defined as follows.

$$|x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$$

Note: 1. For any real number x, $|x| = |-x|$.

2. For any real number x, $|x|$ is always non-negative.

Example: Evaluate each of the following.

a) $|2-5|$ b) $|-3-4|$ c) $|8-3|$ d) $|2-(-5)|$

Solution: a) $-(-3) = 3$ b) $-(-7) = 7$ c) 5 d) 7

Note: For any non-negative number a, $|x| = a$, means $x = a$ or $x = -a$.

Example: Solve the following

a) $|2x-3| = 5$ b) $|-2x+3| = 4$ c) $|x+1| = |x-1|$ d) $|3x+2| = |2x-1|$

Solution: a) $|2x-3| = 5$, $2x-3 = 5$ or $2x-3 = -5$

$$2x = 8\ or\ 2x = -2, x = 4\ or\ x = -1$$

b) Answer: $x = \frac{-1}{2}, or\ \frac{7}{2}$

c)$|x+1| = |x-1| \Leftrightarrow x+1 = x-1\ or\ x+1 = -(x-1)$, 1=-1(False) and $x+1 = -x+1, x = 0$, hence the solution is $x = 0$.

d) Answer: $x = -3\ or\ \frac{-1}{5}$

Properties of absolute value

For any real numbers

1. $x \leq |x|$ 2. $|xy| = |x||y|$ 3. $\sqrt{x^2} = |x|$ 4. $|x+y| \leq |x| + |y|$ (Triangle inequality) 5. If $y \neq 0$, $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ 6. $-|x| \leq x \leq |x|$

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1.4 Inequalities involving absolute values

Theorem: Solutions of the equation $|x| = a$.

For any real number a, the equation |x|= a has

i) two solutions $x = a$ and $x = -a$, if $a > 0$;
ii) one solution, $x = 0$, if $a = 0$; and
iii) no solution, if $a < 0$.

Theorem: Solution of $|x| < a$ and $|x| \leq a$

For any real number a > 0,

i) the solution of the inequality $|x| < a$ is $-a < x < a$.
ii) the solution of the inequality $|x| \leq a \; is -a \leq x \leq a$.

Example: Solve each of the following absolute value inequalities:

a) $|2x - 5| < 3$ b) $|3 - 5x| \leq 1$

Note: In | x| < a, if a < 0 the inequality |x| < a has no solution.

Theorem: Solution of x > a and x ≥ a

For any real number a, if a > 0, then

i) the solution of the inequality $|x| > a \; is \; x < -a \; or \; x > a$.
ii) the solution of the inequality $|x| \geq a \; is \; x \leq -a \; or \; x \geq a$.

Example: Solve each of the following inequalities:

a) $|5 + 2x| > 6$ b) $\left|\frac{3}{5} - 2x\right| \geq 1$ c) $|3 - x| > -2$

Solution: a) Answer: The solution set is $\left\{x: x < \frac{-11}{2} \; or \; x > \frac{1}{2}\right\} = \left(-\infty, \frac{-11}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

b) Answer: The solution set is $\left\{x: x \leq \frac{-1}{5} \; or \; x \geq \frac{4}{5}\right\} = \left(-\infty, \frac{-1}{5}\right] \cup \left[\frac{4}{5}, \infty\right)$
c) Answer: The solution set is the set of real numbers.

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1.5 System of linear inequalities in two variables

When two or more linear equations involve the same variables, they are called a system of linear equations. An ordered pair that satisfies all the linear equations of a system is called a solution of the system.

In a system of equations, $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$, if " $=$ " is replaced by " $>$ "," $<$ " " $\leq$ " or " $\geq$ ", the system becomes a system of linear inequalities.

Example: Find the solution of the following system of inequalities graphically:

a) $\begin{cases} y \geq -3x + 2 \\ y < x - 2 \end{cases}$ b) $\begin{cases} x + y < 3 \\ y < x - 2 \\ y \geq 0 \end{cases}$ c) $\begin{cases} x + y > 0 \\ y - x \leq 1 \\ x \leq 2 \end{cases}$

Solution: Draw the graphs and determine the region which satisfies the inequalities.

a) Answer: $\{(x, y): -3x + 2 \leq y < x - 2 \; and \; 1 < x < \infty\}$
b) Answer: the set of (x, y) such that $x \in [0, 3)$ and $y \in [0, 3 - x)$
c) The solution is the triangular region with vertices $(\frac{-1}{2}, \frac{1}{2})$, $(2,3)$ and $(2, -2)$ except those points on the line $y + x = 0$.

1.6 Quadratic equations and inequalities

Recall that for real numbers a and b, any equation that can be reduced to the form $ax + b = 0$, where $a \neq 0$ is called a linear equation.

Following the same analogy, for real numbers a, b and c, any equation that can be reduced to the for $a\,x^2 + bx + c = 0$, where $a \neq 0$ is called a quadratic equation.

Quadratics equations can be solved by the method of factorization, the method of completing the square, and the general formula.

Expressions are combinations of various terms that are represented as a product of variables or numbers and variables.

Example: $x^2 + 2x, 2x^2 + 4x + 2$, etc. are expressions. x^2 and $2x$ are the terms in $x^2 + 2x$ and $2x^2$, $4x$, and 2 are the terms in $2\,x^2 + 4x + 2$.

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Factorizing the difference of two squares

$$x^2 - a^2 = (x + a)(x - a)$$

Example: Factorize the following

a) $x^2 - 9$ b) $x^2 - 36$ c) $x^2 - 6$ d) $4x^2 - 32$

Factorizing trinomials

To factorize a trinomial $ax^2 + bx + c$ by grouping terms, if you are able to find two numbers p and q such that p + q = b and pq = ac.

Example: Factorize $x^2 + 5x + 6$.

Solution: Two numbers whose sum is 5 and product 6 are 2 and 3. So, in the expression, we write 2 x + 3x instead of 5 x:

$x^2 + 5x + 6 = x^2 + (2x + 3x) + 6$ because 2 x + 3x = 5x.

$= (x^2 + 2x) + (3x + 6)$ (grouping into two parts)

$=x(x + 2) + 3(x + 2)$

$= (x + 2)(x + 3)$ … (factorizing each part), because $x + 2$ is a common factor.

Example: Factorize

a) $3x^2 - 14x - 5$ b) $x^2 + 4x + 4$ c) $2x^2 + 10x + 12$

1. Solving quadratic equations using the method of factorization

$$ax^2 + bx + c = P(x)$$

In order to solve a quadratic equation by factorization, go through the following steps:

i) Clear all fractions and square roots (if any).
ii) Write the equation in the form $P(x) = 0$
iii) Factorize the left hand side into a product of two linear factors.
iv) Use the zero-product rule to solve the resulting equation

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Zero-product rule: If a and b are two numbers or expressions $ab = 0$ then and if either $a = 0$ *or* $b = 0$ or both $a = 0$ and $b = 0$.

Example: Solve each of the following quadratic equations

a) $4x^2 - 16 = 0$ b) $x^2 + 9x + 8 = 0$ c) $2x^2 - 6x + 7 = 3$

2. Solving quadratic equations by completing the square

In general, go through the following steps in order to solve a quadratic equation by the method of completing the square:

$$ax^2 + bx + c = 0$$

i) Write the given quadratic equation in the standard form.
ii) Make the coefficient of x^2 unity, if it is not.
iii) Shift the constant term to R.H.S.(Right Hand Side)
iv) Add $\left(\frac{1}{2} coefficient\ of\ x\right)^2$ on both sides
v) Express L.H.S.(Left Hand Side) as the perfect square of a suitable binomial expression and simplify the R.H.S.
vi) Take square root of both the sides.
vii) Obtain the values of x by shifting the constant term from L.H.S. to R.H.S.

Note: The number we need to add (or subtract) to construct a perfect square is determined by using the following product formulas:

$$x^2 + 2ax + a^2 = (x + a)^2$$

$$x^2 - 2ax + a^2 = (x - a)^2$$

Example: Solve each of the following quadratic equations by using the method of completing the square.

a) $x^2 - 6x + 10 = 0$ b) $x^2 - 12x + 20 = 0$ c) $2x^2 - x - 6 = 0$ d) $2x^2 + 3x - 2 = 0$

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3. Solving quadratic equations using the quadratic formula

The quadratic equation $x^2 + bx + c = 0$, for $a \neq 0$, by applying the method of completing the square $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ called quadratic formula.

Note: The quadratic equation $x^2 + bx + c = 0$, for $a \neq 0$, has a solution, then the solution is determined by $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

i) if $b^2 - 4ac > 0$, then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ represents two numbers $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$

ii) if $b^2 - 4ac = 0$,then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-b}{2a}$, is the only one solution

iii) if $b^2 - 4ac < 0$,then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ has no solution

The expression $b^2 - 4ac$ is called discriminant.

Example: Solve each of the following quadratic equations by using the formula.

a) $x^2 + 8x + 15 = 0$ b) $3x^2 - 12x + 2 = 0$ c) $x^2 + +3x - 2 = 0$

Theorem: Viete's theorem

If the roots of $ax^2 + bx + c = 0, a \neq 0$ are $r_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$= and $r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$,

then $r_1 + r_1 = \frac{-b}{a}$ and $r_1 \times r_2 = \frac{c}{a}$

Example

1. If the sum of the roots of the equation $3x^2 + kx + 1 = 0$ is 7, then what is the value of k?
2. If one of the roots of the equation $x^2 - 4x + k = 0$ exceeds the other by 2, then find the roots and determine the value of k.
3. Determine the value of k so that the equation $x^2 + kx + k - 1 = 0$ has exactly one real root.
4. The area of a rectangle is 21 square centimeters. If one side exceeds the other by 4 cm, then find dimensions of the rectangle.

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Quadratic Inequalities

Definition: An inequality that can be reduced to any one of the following forms:

$$ax^2 + bx + c \leq 0 \; or \; ax^2 + bx + c < 0,$$

$$ax^2 + bx + c \geq 0 \; or \; ax^2 + bx + c > 0$$

where a, b and c are constants and a ≠ 0, is called a quadratic inequality.

i) Solving Quadratic Inequalities Using Product Properties

Product properties:

1. $mn > 0$, if and only if i
   i. $m > 0$ and $n > 0$ or
   ii. $m < 0$ and $n < 0$
2. $mn < 0$, if and only if
   i. $m > 0$ and $n < 0$ or
   ii. $m < 0$ and $n > 0$

Example: Solve each of the following inequalities:

a. $(x + 1)(x - 3) > 0$ b. $3x^2 - 2x \geq 0$ c. $-2x^2 + 9x + 5 < 0$ d. $x^2 - x - 2 \leq 0$

ii) Solving Quadratic Inequalities Using the Sign Chart Method

The "sign chart" method allows to find the sign in each interval.

Step 1. Factorize

Step 2. Draw a sign chart, noting the sign of each factor and hence the whole expression

Step 3. Read the solution from the last line of the sign chart

Example: Solve each of the following inequalities using the sign chart method:

a. $6 + x - x^2 \leq 0$ b. $2x^2 + 3x - 2 \geq 0$

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Note: If $ax^2 + bx + c$ not factorizable then either $ax^2 + bx + c > 0$ for all values of x or $ax^2 + bx + c < 0$ for all values of x. As a result, the solution set of $ax^2 + bx + c > 0$ or $ax^2 + bx + c \geq 0$ is either $(-\infty, \infty)$ or $\{\}$.

iii) Solving Quadratic Inequalities Graphically

In order to use graphs to solve quadratic inequalities, it is necessary to understand the nature of quadratic functions and their graphs.

i. If $a > 0$, then the graph of the quadratic function $f(x) = ax^2 + bx + c$ is an upward parabola.
ii. If $a < 0$, then the graph of the quadratic function $f(x) = ax^2 + bx + c$ is a downward parabola.

The graph of a quadratic function has both its ends going upward or downward depending on whether a is positive or negative.

$$f(x) = ax^2 + bx + c$$

i. crosses the x-axis twice, if $b^2 - 4ac > 0$
ii. touches the x-axis at a point, if $b^2 - 4ac = 0$
iii. does not touch the x-axis at all, if $b^2 - 4ac < 0$

Example: Solve each of the following quadratic inequalities, graphically:

a. $x^2 - 3x + 2 < 0$ b. $x^2 + 4x + 5 > 0$ c. $-x^2 + 2x + 3 < 0$

Note: If the quadratic equation $ax^2 + bx + c = 0, a \neq 0$ has discriminant $b^2 - 4ac < 0$, then the equation has no real roots. Moreover,

i. the solution set of $ax^2 + bx + c \geq 0$ is the set of all real numbers if $a > 0$ and is empty set if $a < 0$.
ii. the solution set of $ax^2 + bx + c \leq 0$ is the set of all real numbers if $a < 0$ and is empty set if $a > 0$.

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1.7 Simplification of rational expressions and solving rational equations

Definition: A rational expression is the quotient $\frac{P(x)}{Q(x)}$ of two polynomials $P(x)$ and $Q\ (x)$, where $Q(x) \neq 0$. P(x) is called the numerator and Q (x) is called the denominator.

Example: a. $\frac{x-2}{2x^2-2x+3}$ $\quad$ $b. \frac{x^3+3x+4}{4}$ $\quad$ $c. \frac{1}{x^4-2}$

Note: The domain of rational function is $\{\ x:\ x \text{ is a real number and } Q(x) \neq 0\}$.

Steps to find the domain of a rational expression:

Step 1: Set the denominator of the expression equal to zero and solve.

step 2: The domain is the set of all real numbers except those values found in step 1.

Definition: We say that a rational expression is reduced to lowest terms (or in its lowest terms or in simplest form), if the numerator and denominator do not have any common factor other than 1.

To simplify a rational expression*:*

1. Find the domain.
2. Factorize the numerator and denominator completely.
3. Divide the numerator and denominator by any common f actor (i.e. cancel like terms).

Example: a. $\frac{6x^2+23x+20}{2x^2+5x-12}$

Solution: $\frac{6x^2+23x+20}{2x^2+5x-12} = \frac{(3x+4)(2x+5)}{(2x-3)(x+4)}$, For $x \neq \frac{3}{2}$ and $x \neq -4$.

b. $\frac{x^3+3x^2}{x+3}$

Solution: $\frac{x^3+3x^2}{x+3} = \frac{x^2(x+3)}{x+3} = x^2$ For all $\mathbb{R}$.

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Decomposition of Rational Expressions into Partial Fractions

Definition: In a rational expression $\frac{P(x)}{Q(x)}$, if the degree of $P(x)$ is less than that of $Q\ (x)$, then $\frac{P(x)}{Q(x)}$ is called a proper rational expression. Otherwise it is called improper.

Note: $ax^2 + bx + c$ is not reducible in real numbers, if

Theorem: Linear and quadratic factor theorem

For a polynomial with real coefficients, there always exists a complete factorization involving only linear and/or quadratic factors (raised to some power of natural number k ≥ 1), with real coefficients, where the linear and quadratic factors are not reducible relative to real numbers.

	Factor in the Denominator	Corresponding term in the Partial Fraction
1	$ax + b$	$\frac{A}{ax+b}$, where A is constant
2	$(ax\ +\ b)^k$	$\frac{A_1}{(ax + b)} + \frac{A_1}{(ax\ +\ b)^2} + \cdots + \frac{A_1}{(ax\ +\ b)^k}$ Where $A_1, A_{2,} \ldots, A_k$
3	$(ax^2 + bx + c)$ With $b^2 - 4ac <$ 0	$\frac{Ax+B}{ax^2+bx+c}$, where A is constant
4	$(ax^2 + bx + c)^{\boldsymbol{k}}$ With $b^2 - 4ac <$ 0	$\frac{A_1x + B_1}{(ax^2 + bx + c)} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$ Where $A_1, A_{2,} \ldots, A_k\ and\ B_1, B_2, \ldots, B_k$ are constants

Example: a. $\frac{5x+7}{x^2+2x-3}$ b. $\frac{6x^2-14x-27}{(x+2)(x-3)^2}$ c. $\frac{5x^2-8x+5}{(x-2)(x^2-x+1)}$

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Solving rational equations

Definition: A rational equation is an equation that can be reduced to the form $\frac{P(x)}{Q(x)} = 0$ where $P(x)$ *and* $Q(x)$ are polynomials and $Q\ (x)\ \neq\ 0$.

To solve rational equations, you follow the following steps:

step 1: Factorize all the denominators and determine their LCM.

step 2: Restrict the values of the variable that make the LCM equal to 0.

step 3: Multiply both sides of the rational equation by the LCM and simplify.

step 4: Solve the resulting equation.

step 5: Check the answers against the restricted values in step 2. Any such value must be excluded from the solution.

Example: a. $\frac{3}{x+2} - \frac{1}{x} = \frac{1}{5x}$

Solution: $\frac{3}{x+2} - \frac{1}{x} = \frac{1}{5x} \Rightarrow \frac{3}{x+2}\left(\frac{(x+2)x}{1}\right) - \frac{1}{x}\left(\frac{(x+2)x}{1}\right) = \frac{1}{5x}\left(\frac{(x+2)x}{1}\right)$

$\Rightarrow 3x - x - 2 = \frac{1}{5}x + \frac{2}{5}$

$\Rightarrow x = \frac{4}{3}$

b. $\frac{3a-5}{a^2+4a+3} + \frac{2a+2}{a+3} = \frac{a-1}{a+1}$

Rational Functions and Their Graphs

Definition: A rational function is a function of the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ *and* $Q(x)$ are polynomials and $Q\ (x)\ \neq\ 0$.

Example: a. $f(x) = \frac{x+1}{2x^3+x^2+x}$ b. $g(x) = \frac{2x^2+2x-1}{1}$

$f(x)$ *and* $g(x)$ are example of rational function.

$\frac{2x^2+2x-1}{1}$ is the same as with $2x^2 + 2x - 1$ so any polynomial functions are rational function.

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Note:

- A rational function is said to be in lowest terms, if $p\ (x)\ and\ q\ (x)$ have no common factor other than 1.
- The domain of a rational function f is the set of all real numbers except the values of x that make the denominator $q\ (x)$ zero.

These two behaviors of $f\ near\ x\ =\ 0$ are denoted as follows.

- $f\ (x)\ \to\ -\infty\ as\ x\ \to\ 0^-$
- $f\ (x)\ \to\ \infty\ as\ x\ \to\ 0^+$

In this case, the line $x\ =\ 0\ (the\ y-axis)$ is called a vertical asymptote of the graph of f.

In addition, we have:

- $f\ (x)\ \to\ 0\ as\ x\ \to -\ \infty$
- $f\ (x)\ \to\ 0\ as\ x\ \to\ \infty$

Here, the line $y\ =\ 0\ (the\ x-axis)$ is called a horizontal asymptote of the graph of f.

Definition:

- The line $x\ =\ a$ is called a vertical asymptote of the graph of f

  if $f\ (x)\ \to\ \infty\ or\ f\ (x)\ \to -\ \infty\ as\ x\ \to\ a$, either from the left or from the right.
- The line $y\ =\ b$ is called a horizontal asymptote of the graph of f if $f\ (x) \to\ b\ as$

  $x\ \to\ \infty\ or\ as\ x\ \to\ -\infty$.

Rules for asymptotes and holes

Note: Let $f(x) = \frac{P(x)}{Q(x)} = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}$, be a rational function, where n is the largest exponent in the numerator and m is the largest exponent in the denominator.

- The graph will have a vertical asymptote at $\mathrm{x\ =\ a}$ if $\mathrm{q\ (a)\ =\ 0}$ and $\mathrm{p\ (a)\ \neq\ 0}$. In case $p(a)\ =\ q(a)\ =\ 0$, the function has either a hole at $x\ =\ a$ or requires further simplification to decide.
- If $n\ <\ m$, then the x-axis is the horizontal asymptote.
- If $n\ =\ m$, then the line $y = \frac{a}{b}$ is a horizontal asymptote.
- If n = m + 1, the graph has an oblique asymptote and we can find it by long division.

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- If $n > m + 1$, the graph has neither an oblique nor a horizontal asymptote.

Example: Give the vertical and horizontal asymptotes, if they have:

a. $f(x) = \frac{1}{x+2}$ b. $g(x) = \frac{x-2}{x^2-4}$ c. $h(x) = \frac{x^2-1}{x^2+3x+2}$

The zeros of a rational function

Definition: Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function. An element a in the domain of f is called a zero of f, if and only if $p\ (a)\ =\ 0$.

Example: Find the zeros of the following rational functions:

a. $f(x) = \frac{x^2+3x+2}{x^2-2x-3}$ b. $g(x) = \frac{x^2-6x+9}{x^2-9}$

Graphs of Rational Functions

Steps to sketch the graph of a rational function:

Step 1: Reduce the rational function to lowest terms and check for any open holes in the graph.

Step 2: Find x-intercept(s) by setting the numerator equal to zero.

Step 3: Find the y-intercept (if there is one) by setting x = 0 in the function.

Step 4: Find all its asymptotes (if any).

Step 5: Determine the parity (i.e. whether it is even or odd or neither).

Step 6: Use the x-intercepts and vertical asymptote(s) to divide the x-axis into intervals. Choose a test point in each interval to determine if the function is positive or negative there. This will tell you whether the graph approaches the vertical asymptote in an upward or downward direction.

Step 7: Sketch the graph! Except for breaks at the vertical asymptotes or cusps, the graph should be a nice smooth curve with no sharp corners.

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To draw the graph of $f(x) = \frac{P(x)}{Q(x)}$ Horizontal asymptote

We need to find	Criteria
Domain	$R\backslash \{x: q\ (x)\ =\ 0\}$
x – intercept	$Zero\ of\ f$
y - intercept	$x\ =\ 0\ and\ 0\ \in\ domain\ of\ f$
Vertical asymptote	P $(x)\ \neq\ 0\ and\ q\ (x)\ =\ 0$
Horizontal asymptote	Degree of $p\ (x)\ \leq\ Degree\ of\ q\ (x)$
Oblique asymptote	Degree of $p\ (x)$ = Degree $q\ (x)\ +\ 1$
Parity	f is odd or even or neither

Example: a. $g(x) = \frac{1}{x^2}$ b. $g(x) = \frac{x+1}{(x-2)(x+3)^2}$ c. $g(x) = \frac{x^2+5x+6}{x+1}$

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Chapter Two: Relations and Functions

2.1. Definition and Examples of relations:

Definition:

Let A and B be non-empty sets. A relation R from A to B is any subset of $A \times B$.

In other words, R is a relation from A to B if and only if $R \subseteq (A \times B)$.

Example: Let A = { 1, 2, 3, 4} and B = { 1, 3, 5}

$R_1 = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$ is a relation from A to B

Because $R_1 \subseteq (A \times B)$.

Is R_1 a relation from B to A ? Justify.

Activity:

Let $A = \{1, 2, 4, 6, 7\}$ and $B = \{5, 12, 7, 9, 8, 3\}$

Let R_1 and R_2 be relations given by: $R_1 = \{(x, y) \mid x \in A, y \in B, x > y\}$ and

$$R_2 = \{(x, y) \mid x \in A, y \in B, x = \tfrac{1}{2}y\}$$

Then List all the elements of R_1 and R_2 ?

Domain and Range of a relation

Definition

Let R be a relation from a set A to a set B. Then:-

Domain of $R = \{ x : (x, y)$ belongs to R for some $y\}$

Range of $R = \{ y : (x, y)$ belongs to R for some $x\}$

Activity:

Let $A = \{x: 1 \leq x < 10\}$ and $B = \{2, 4, 6, 8\}$. If R is a relation from A to B given by $R = \{(x, y): x + y = 12\}$, then find the domain and the range of R.

Graphs of Relations involving Inequalities

To sketch graphs of relations involving inequalities, do the following:

1. Draw the graph of a line(s) in the relation on the xy −coordinate system.
2. If the relating inequality is $\leq$ or $\geq$, use a solid line; if it is $<$ or $>$, use a broken line.
3. Then take arbitrary ordered pairs represented by points, one from one side and th

Other from another side of the line(s), and determine which of the pairs satisfy the relation.

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4. The region that contains points representing the ordered pair satisfying the relation will be the graph of the relation.

Note: A graph of a relation when the relating phrase is an inequality is a region on the coordinate system.

Example: Sketch the graph of the relation

$R = \{(x, y): y \geq x + 2$ and $y > -x, x \in \mathbb{R}$ and $y \in \mathbb{R}\}$.

Solution:

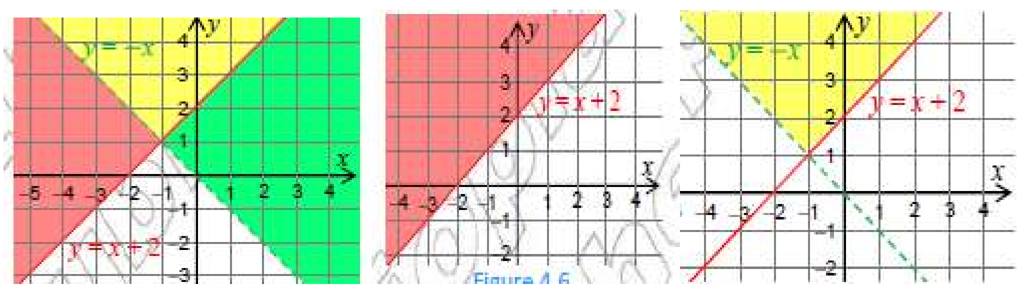
First sketch the graph of the relation

$R = \{(x, y): y \geq x + 2, x \in \mathbb{R}$ and $y \in \mathbb{R}\}$.Next, on the same diagram, sketch the graph of

$R = \{(x, y): y > -x, x \in \mathbb{R}$ and $y \in \mathbb{R}\}$

The two shaded regions have some overlap. The intersection of the two regions is the graph of the relation. So, taking only the common region, we obtain the graph of the relation

$R = \{(x, y): y \geq x + 2$ and $y > -x, x \in \mathbb{R}$ and $y \in \mathbb{R}\}$ as shown in figure below.



2.2. Definition and examples of functions

Definition: A function is a relation such that no two ordered pairs have the same first-coordinates and different second-coordinates, which means that for any given x in the domain of f, there is a unique pair (x, y) belonging to the function f.

Example 1: Consider the relation $R = \{(1, 2), (7, 8), (4, 3), (7, 6)\}$ Since 7 is paired with both 8 and 6 the relation R is not a function

Example 2: Let $A = \{1,2,3,4\}$ and $B = \{1,6,8,11,15\}$. Which of the following are functions from A to B?

a) f defined by $f(1)= 1, f(2)= 6, f(3)= 8, f(4)= 8$

b) f *defined by* $f(1)= 1, f(2)= 6, f(3)= 15$

c) f *defined by* $f(1)= 6, f(2)= 6, f(3)= 6, f(4)= 6$

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Solution:

a) f is a function because to each element of A there corresponds exactly one element of B .

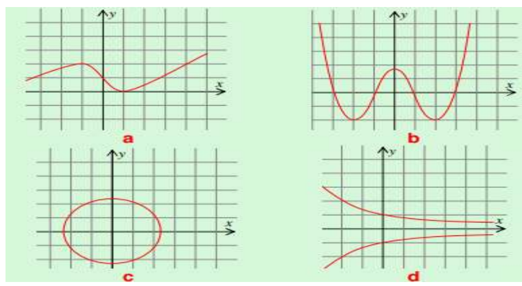
b) f is not a function because there is no element of B which correspond to 4($\in$ A).

c) f is a function because to each element of A there corresponds exactly one element of B.

<u>**Vertical line test**</u>

A set of points in the Cartesian plane is the graph of a function, if and only if no vertical line intersects the set more than once.

Example: For the following graph use the vertical line test to determine it is a function or not?



Solution : (a) and (b) represent functions since no vertical line intersect the graph more than once, but (c) and (d) are not functions.

Domain and range of a function

Definition: Let f be a function from a set A to a set B. Then

i) Domain of f $= \{: \ (x\,, y)$ belongs to f for some y }

ii) Range of f $= \{\, y : \ (x, y)$ belongs to R for some $x\}$

Example 1: Consider the relation $R = \{(1, 2), (7, 8), (4, 3), (7, 6)\}$ Since 7 is paired with both 8 and 6 the relation R is not a function.

Example 2: Let $R = \{(1, 2), (7, 8), (4, 3)\}$. This relation is a function because no first-coordinate is paired (mapped) with more than one element of the - second coordinate.

Example 3: Determine whether the following equations determine y as a function of , if so, find the domain of the function. $a)\ y = -3x - 5 \quad b)\ y = \frac{2x}{3x-5} \qquad c)\ y^2 = x$

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Solution:3 a) For each x there is a unique y . Therefore, y = $-3x + 5$ is a function. Its domain is the set of all real numbers.

b) Looking at the equation $y = \frac{2x}{3x-5}$carefully, we can see that each x-value uniquely determines a y-value (one x-value cannot produce two different y-values). Therefore, $y = \frac{2x}{3x-5}$is a function?

The domain consists of all real numbers except $\frac{5}{3}$. Thus, Domain f $=\{x : x \neq \frac{5}{3}\}$.

c) For the equation $y^2 = x$, if we choose x = 9 we get $y^2 = 9$, which gives $y = \pm 3$. In other words, there are two y – values associated with $x = 9$. Therefore $y^2 = x$ is not a function?

EXERCISES

1. Determine whether each of the following relations is a function or not, and give reasons for those that are not functions.

   a) $R = \{(-1, 2), (1, 3), (-1, 3)\}$

   b) $R = \{(1, 1), (1, 3), (-1, 3), (2, 1)\}$

   c) $R = \{(x, y)$: x is the area of triangle y $\}$

   $d)$ $R = \{(x, y): y = x^2 + 3\}$

   $e)$ $R = \{(x, y): y < x\}$

2. Find the domain and the range of each of the following functions:

   a) $f(x) = 3$ b) $f(x) = 1 - 3x$ c) $f(x) = \sqrt{x+4}$ d) $\frac{1}{2x}$

2.3. Classification of functions

Definition: A function $f : A \to B$ is said to be

i) Odd, if and only if, for any $x \in A, f(-x) = -f(x)$.

ii) Even, if and only if, for any $x \in A,$ $f(-x) = f(x)$. The evenness or oddness of a function is called its parity.

Example: a. $f(x) = x^3$ is odd, since $f(-x) = (-x)^3 = -x^3 = -f(x)$.

b. $f(x) = x^2$ is even since $f(-x) = (-x)^2 = x^2 = f(x)$.

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c. $f\ (x) = x + 1$ is neither even nor odd since $f\ (-x) = -x + 1 \neq -(x + 1) = -f\ (x)$ and $f\ (-x) = -x + 1 \neq x + 1 = f\ (x)$.

One-to-One Functions

Definition: A function f: A → B is said to be one-to-one (an injection), if and only if, each element of the range is paired with exactly one element of the domain, i.e.,

$f\ (x_1) = f\ (x_2) \Rightarrow x_1 = x_2$ for any $x_1$and $x_2 \in A$.

Note: This is the same as saying $x_1 \neq x_2 \Rightarrow f\ (x_1) \neq f\ (x_2)$.

Example 1 : Show that $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f\ (x) = 2x$ is one-to-one.

Solution: Let $x_1, x_2 \in \mathbb{R}$ be any two elements such that $f\ (x_1) = f\ (x_2)$.

Then, $2x_1 = 2x_2 \Rightarrow \frac{1}{2}(2x_1) = \frac{1}{2}(x_2) \Rightarrow x_1 = x_2$.

Thus, f is one-to-one.

Example 2: Show that $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f\ (x) = x^2$ is not one-to-one.

Solution: Take $x_1 = 2$ and $x_2 = -2$.

Obviously, $x_1 \neq x_2$i.e$2 \neq -2$ But $f(x_1) = f(2) = 2^2 = 4 = (-2)^2 = f(-2) = f(x_2)$. This means there are numbers $x_1, x_2 \in \mathbb{R}$ for which $x_1 \neq x_2 \Rightarrow f\ (x_1) \neq f\ (x_2)$ does not hold.

Thus, f is not one-to-one.

The horizontal line test:

A function f : A → B is one-to-one, if and only if any horizontal line crosses its graph at most once.

Example 3 : Using the horizontal line test, show that f : ℝ→ℝ given by $f\ (x) = 2x$ is one-to-one.

Solution: It is clear that any horizontal line crosses the graph of $y = 2x$ at most once. Hence, $f\ (x) = 2x$ is a one-to- one function.(observe the figure below)

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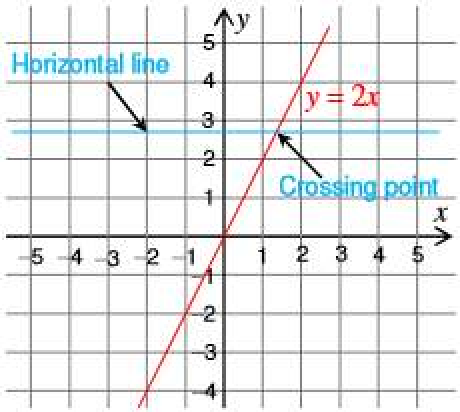

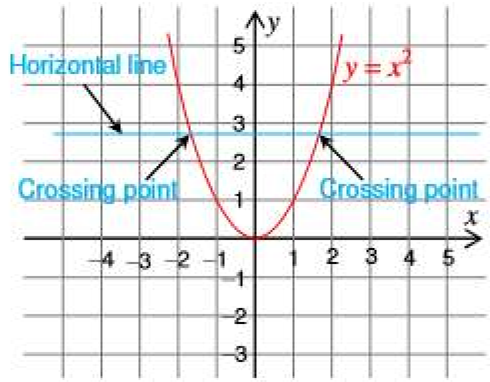

Example 4: Using the horizontal line test, show that $f: \mathbb{R} \to \mathbb{R}$ given by $f\ (x)\ =\ x^2$ is not one-to-one. **Solution** : A horizontal line crosses the graph of $y\ =\ x^2$ at two points (Figure 2). Thus, f is not one- to- one.

EXERCISES

1. Which of the following functions are one-to-one?

a) $f\ =\ \{(-2,2),(-1,3),(0,1),(4,1),(5,6)\}$

b) $f\ =\ \{\ (x,y) :\ y$ is a brother $of\ x\ \}$

c) $g: (0,\infty) \to \mathbb{R}$, given by $g\ (x)\ =\ 5\ log\ x.$

d) $h: \mathbb{R} \to \mathbb{R}$, given by $h(x) = |x-1|$

Onto Functions

Definition: A function $f : A\ \to\ B$ is onto (a surjection), if and only if, Range of $f\ =\ B.$

Example1: Let f be defined by the Venn diagram in Figure below.

Range of $f\ =\ B$. Therefore, f is onto.

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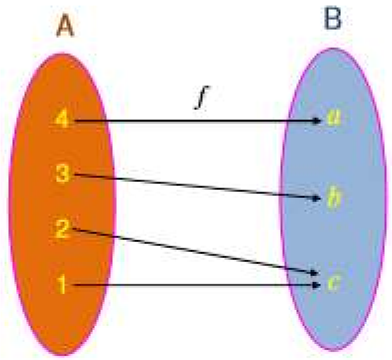

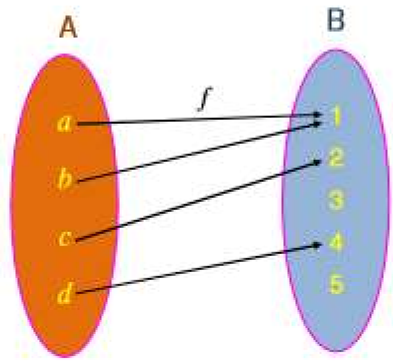

Example 2: In Figure 2 above, Range of $f = \{1, 2, 4\} \Rightarrow$ Range of $f \neq B$

Thus, f is not onto.

One- to –one correspondence

Definition: A function : $f: A \to B$ is a one-to-one correspondence (a bijection), if and

Only if, f is one-to-one and onto.

Example : Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 5x - 7$. Show that f is a one-to- one correspondence.

Solution : Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2) \Rightarrow 5x_1 - 7 = 5x_2 - 7 \Rightarrow 5x_1 - 7 + 7 = 5x_2 - 7 + 7 \Rightarrow 5x_1 = 5x_2 \Rightarrow x_1 = x_2$

So, f is one-to-one.

Let $y \in \mathbb{R}$. Is there $x \in \mathbb{R}$ such that $y = f(x)$?

If there is, it can be found by solving $y = f(x) = 5x - 7 \Rightarrow \mathrm{y} + 7 = 5\mathrm{x} \Rightarrow \mathrm{x} = \frac{y+7}{5} \in \mathrm{R}$

Then $f(x) = f\left(\frac{y+7}{5}\right) \Rightarrow 5\left(\frac{y+7}{5}\right) - 7 = y$.

So f is onto.

Therefore, f is a one-to-one correspondence?

Example3: Let $f : \mathbb{R} \to \mathbb{R}$be given by $f(x) = 3^x$. Check whether or not f is a oneto-one correspondence.

Solution: For any $x_1, x_2 \in \mathbb{R}$,

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$$f(x_1) = f(x_2) \Rightarrow 3^{x_1} = 3^{x_2} \Rightarrow \frac{3^{x_1}}{3^{x_2}} = 1 \Rightarrow 3^{x_1 - x_2} = 1 = 3^0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$$

Thus, f is one-to-one. But, f is not onto, because negative numbers cannot be images. For instance, take $y = -4$.

Since $3^x > 0$, for $every\ x \in \mathbb{R}$, it is not possible to have $x \in \mathbb{R}$, for which $3^x = -4$

Thus, f is not onto

Therefore, f is not a one-to-one correspondence.

EXERCISES

1.Which of the following functions are onto?

a .$f: \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 3x + 5$

b .$g: [0, \infty) \to R$, defined by $g(x) = 3 - \sqrt{x}$

2.Show whether each of the following functions is a one-to-one correspondence or not.

a . $f: \mathbb{R} \to \mathbb{R}, f(x) = \frac{3x+1}{5}$. b. $g: [0, \infty) \to [0, \infty), g(x) = \sqrt{x}$

INVERSE FUNCTIONS

Definition: If f is a function from A to B ($f: A \longrightarrow B$), then the inverse of f, denoted by f^{-1} and defined by $f^{-1} = \{(b, a): (a, b) \in R\}$, is a function from B to A.

Note: $f^{-1}$is a function, if and only if f is one-to-one.

Notation: If the inverse of f is , then g is denoted by f^{-1} . In this case f is called invertible.

Example 1: Find the inverse of each of the following functions.

a. $f(x) = 4x - 3$. b $f(x) = \frac{x}{x-1}\ x \neq 1$

Solution

a. $f = \{(x, y): y = 4x - 3\}$ and

$$f^{-1} = \{(x, y): x = 4y - 3\} = \left\{(x\ y): \frac{x+3}{4} = y\right\} \Rightarrow f^{-1}(x) = \frac{x+3}{4}$$

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b. $f = \{(x,y): y = \frac{x}{x-1}, x \neq 1\}$

$f^{-1} = \left\{(x,y): x = \frac{y}{y-1}, x \neq 1\right\} = \{(x,y): x(y-1) = y, x \neq 1\} =$

$\{(x,y): y(x-1) = x, x \neq 1\} = \{(x,y): y = \frac{x}{x-1}, x \neq 1\}$.

Therefore $f^{-1}(x) = \frac{x}{x-1}$.

EXERCISES

1. Determine the inverse of each of the following functions. Is the inverse a function?

   a. $f\,(x) \;=\; 3\,log\,2x$ b. $h\,(x) \;= -\,5x \;+\; 13$

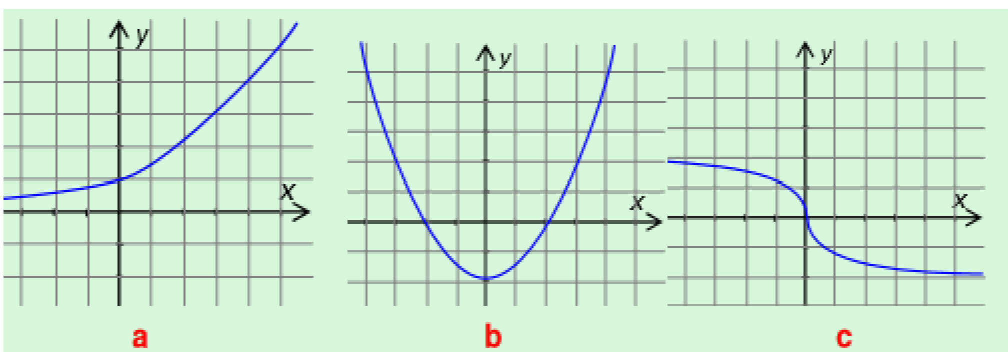
2. Are the following functions inverses of each other (in their respective domain)?

   a. $f(x) = 3x + 2; g(x) = \frac{x-2}{3}$ b. $f(x) = \sqrt{x};\; g(x) = x^2$

3. Which of the following functions are invertible? If they are not, can you restrict the domain to make them invertible?

   a. $f(x) = x^3$ b. $g(x) = 4 - x^2$

4. Which of the following functions are invertible?



2.4. Operations on functions and composition of functions

Definition: (Sum, Difference, Product and Quotient of two functions)

Let $f\,(x)$ and $g(x)$ be two functions. We define the following four functions:

1. $(f + g)(x) = f(x) + g(x)$…The sum of the two functions
2. $(f - g)(x) = \;f\,(x) - g(x)$…The difference of the two functions
3. $(f.g)(x) = f(x)g(x)$…The product of the two functions
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ ……The quotient of the two functions (provided $g(x) \neq 0$)

The domain of these functions is defined as:

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i. $dom(f+g) = dom(f) \cap dom(g)$

ii. $dom(f-g) = dom(f) \cap dom(g)$

iii. $dom(f.g) = dom(f) \cap dom(g)$

iv. $dom\left(\frac{f}{g}\right) = dom(f) \cap dom(g) \setminus \{x: g(x) = 0\}$

Example: Let $f\ (x) = \sqrt[4]{x+1}$ and $g(x) = \sqrt{9-x^2}$.Find formulas for $f+g$, $f+g$, $f+g$ and $\frac{f}{g}$ and give their domains.

Solution:

First find the domains of f and g: $dom(f) = [-1, \infty)$ and $dom(g) = [-3,3]$.

$$f+g = \sqrt[4]{x+1} + \sqrt{9-x^2} \qquad dom(f+g) = [-1,3]$$

$$f-g = \sqrt[4]{x+1} - \sqrt{9-x^2} \qquad dom(f-g) = [-1,3]$$

$$f.g = \sqrt[4]{x+1}.\sqrt{9-x^2} \qquad dom(f.g) = [-1,3]$$

$$\frac{f}{g} = \frac{\sqrt[4]{x+1}}{\sqrt{9-x^2}} \qquad dom\left(\frac{f}{g}\right) = [-1,3)$$

Exercise: Let $f\ (x) = 3x^2 + 2$ and $g(x) = 5x - 4$. Find each of the following and its domain

a. $(f+g)(x)$
b. $(f+g)(x)$
c. $(f+g)(x)$
d. $(f+g)(x)$
e. $(f+g)(x)$
f. $\left(\frac{f}{g}\right)(x)$

Definition: (Composition of functions)

Given two functions $f\ (x)$ and $g(x)$, the composition of the two functions is denoted by $f \circ g$and is defined by:

$$(f \circ g)(x) = f\ (g(x))$$

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$(f \circ g)(x)$is read as "f composed with g of x". The domain of $f \circ g$ consists of those x’sin the domain of g whose range values are in the domain of f, i.e. those x’s for which $g(x)$ is in the domain of f.

Exercise:

1. Let $f(x) = \frac{6x}{x^2-9}$ and $g(x) = \sqrt{3x}$. Find $(f \circ g)(12)$ and $(g \circ f)(x)$ and its domain.

   __**Ans**__. The domain of $f \circ g$ is $[0,3) \cup (3, \infty)$

2. Given $f(x) = 5x^2 - 3x + 2$ and $g(x) = 4x + 3$, find
   a. $(f \circ g)(-2)$
   b. $(g \circ f)(2)$
   c. $(f \circ g)(x)$
   d. $(g \circ f)(x)$

3. Let $f(x) = \frac{x}{x+1}$ and$f(x) = \frac{2}{x-2}$. find
   a. $(fog)(x)$ and its domain
   b. $(gof)(x)$ and its domain

Ans: $\mathrm{Dom}(fog) = \{x: x \neq \pm 1\}$ and $\mathrm{Dom}(gof) = \{x: x \neq -1\}$

Definition: (Equality of functions)

Two functions are said to be equal if and only if the following two conditions hold:

i. The functions have the same domain;
ii. Their functional values are equal at each element of the domain

Example: let $f(x) = \frac{x^2-25}{x-5}$ and $g(x) = x + 5$. The function f and g are not equal because $Dom(f) \neq Dom(g)$.

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2.5. Types of Functions

Polynomial Functions

Definition: Let n be a non-negative integer and let $a_n, a_{n-1}, \ldots, a_1, a_0$ be real numbers

With $a_n \neq 0$. Then the function

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

is called a **polynomial function** in variable x of degree n.

Note that: In the definition of a polynomial function

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

i. $a_n, a_{n-1}, \ldots, a_1, a_0$ are called the **coefficients** of the polynomial function (or simply the polynomial).

ii. The number a_n is called the **leading coefficient** of the polynomial function and $a_n x^n$ is the **leading term**.

iii. The number a_0 is called the **constant term** of the polynomial.

iv. The number n (the exponent of the highest power of x), is the **degree** of the polynomial.

v. Each individual expression $a_n x^n$, $a_{n-1}x^{n-1}, \ldots, a_1 x, a_0$ making up the polynomial is called a **terms** of the polynomial.

vi. The domain of a polynomial function is $\mathbb{R}$.

vii. Constant function, linear function and quadratic function are all special cases of a wider class of functions called polynomial functions.

Exercise: For the polynomial function $p(x) = \frac{x^2-2x^5+8}{4} + \frac{7}{8}x - x^3$

a. What is its degree?
b. Find a_n, a_{n-1}, a_{n-2} and a_2
c. What is the constant term?
d. What is the coefficient of x?

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2.5.1. Operations on Polynomial Functions

Let $f(x)$ and $g(x)$ be polynomial functions.

$$(f + g)(x) = f(x) + g(x), for\ all\ x \in \mathbb{R}$$

$$(f - g)(x) = f(x) - g(x), for\ all\ x \in \mathbb{R}.$$

$$(f \cdot g)(x) = f(x) \cdot g(x), for\ all\ x \in \mathbb{R}.$$

$$(f \div g)(x) = f(x) \div g(x) = \frac{f(x)}{g(x)},\ provided\ that\ g(x) \neq 0,\ for\ all\ x \in \mathbb{R}.$$

Exercise: Let $f(x) = 4x3 - 3x + 5$ and $g(x) = 2x - 3$. Find

i. $f + g$

ii. $f - g$

iii. $f . g$

iv. Divide f by g

Theorems on polynomial functions (Division, remainder, factor and location theorems)

Theorem: (Polynomial division theorem)

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)\,q(x) + r(x)$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $f(x)$ divides exactly into $d(x)$.

Note: In division theorem $f(x)$ is called **dividend**,$d(x)$ is called **divisor**, $q(x)$ is called **quotient** and $r(x)$ is called **remainder**. That is $f(x) = d(x)\,q(x) + r(x)$ can be expressed as: $\boldsymbol{Dividend} = (\boldsymbol{divisor})\,(\boldsymbol{quotient}) + \boldsymbol{remainder}$.

Theorem: (Remainder theorem)

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let c be any real number. If $f(x)$ is divided by the linear polynomial $(x - c)$, then the remainder is $f(c)$.

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Theorem: (Factor theorem)

Let $f(x)$ be a polynomial of degree greater than or equal to one, and let c be any real number, then

i. $x - c$ is a factor of $f(x)$, if $f(c) = 0$, and

ii. $f(c) = 0$, if $x - c$ is a factor of $f(x)$.

Definition: For a polynomial function $f(x)$ and a real number c, if $f(c) = 0$, then c is called the **zero (or the root)** of a polynomial function $f(x)$.

Note that:

1. If $x - c$ is a factor of $f(x)$, then c is a zero of $f(x)$.
2. A polynomial function cannot have more zeros than its degree.
3. If $r_1, r_2, \dots, r_n$ are the roots of $p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$, then it can be factorized as $p(x) = a_n(x - r_1)(x - r_2) \dots (x - r_n)$.

Definition: If $(x - c)^k$ is a factor of $f(x)$, but $(x - c)^{k+1}$ is not, then c is said to be a zero of multiplicity k of $f(x)$.

Theorem: (Location theorem)

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a)$ and $f(b)$ have opposite signs (*i.e.* $f(a)f(b) < 0$), then there is at least one zero of f between a and b.

Theorem: (Rational root test)

If the rational number $\frac{p}{q}$ in its lowest terms, is a zero of the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

with integer coefficients, then p must be a factor of a_0 and q must be a factor of a_n.

Example: Find all the rational zeros of the polynomial

a. $g(x) = \frac{1}{2}x^4 - 2x^3 - \frac{1}{2}x^2 + 2x$

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b. $g(x) = 2x^3 + 9x^2 + 7x - 6$

a. **Solution**: Let $h(x) = 2g(x)$. Thus $h(x)$ will have the same zeros, but has integer coefficients.

$$h(x) = x^4 - 4x^3 - x^2 + 4x$$

x is a factor, so $h(x) = x(x^3 - 4x^2 - x + 4) = xk(x)$

$k(x)$ has a constant term of 4 and leading coefficient of 1.

The possible rational zeros are

$$\pm 1, \quad \pm 2, \quad \pm 4$$

Using the **remainder theorem**, $k(1) = 0, k(-1) = 0$ and $k(4) = 0$

So, by the **factor theorem** $k(x) = (x - 1)(x + 1)(x - 4)$

Hence, $h(x) = x\,k(x) = x(x - 1)(x + 1)(x - 4)$ and

$$g(x) = \frac{1}{2}h(x) = \frac{1}{2}x(x - 1)(x + 1)(x - 4)$$

Therefore, the zeros of $g(x)$ are 0, ± 1 *and* 4.

b. Possible values of p are factors of -6. These are $\pm 1, \pm 2, \pm 3$ and ± 6.

Possible values of q are factors of 2. These are $\pm 1, \pm 2$.

The possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$

Of these 12 possible rational zeros, at most 3 can be the zeros of g.

Check that $f(-3) = 0, f(-2) = 0$ and $f\left(\frac{1}{2}\right) = 0$

Using the factor theorem, we can factorize $g(x)$ as:

$2x^3 + 9x^2 + 7x - 6 = (x + 3)(x + 2)(2x - 1)$.

So, $g(x) = 0$ at $x = -3, x = -2$ and at $x = \frac{1}{2}$

Therefore $-3, -2$ and $\frac{1}{2}$ are the only (rational) zeros of g

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Exercise:

1. Using the location theorem, show that the polynomial $f\ (x) = x^5 - 2x^2 - 1$ has a zero between $x = 1$ and $x = 2$.
2. Find the remainder by dividing $f(x)$ by $d(x)$ in each of the following pairs of polynomials
   a. $f\ (x) = x^3 - x^2 + 8x - 1;\ d\ (x) = x + 2$
   b. $f\ (x) = x^4 + x^2 + 2x + 5;\ d\ (x) = x - 1$
3. In each of the following pairs of polynomials, write $f\ (x) = d\ (x)\ q\ (x) + r\ (x)$ form.
   a. $f\ (x) = x^3 - 2x^2 + x + 5;\ d\ (x) = x^2 + 1$
   b. $f\ (x) = x^4 + x^2 - 2\ ;\ d\ (x) = x^2 - x + 3$
4. Show that the polynomial $f\ (x) = x^5 - 2x^2 - 1$ has a zero or not between $x = 1$ and $x = 2$.

   Graphs of polynomial functions

Note:

1. If the graph of a function f crosses the x-axis at $(x_1, 0)$, thenx_1 is the **x-intercept** of the graph. If the graph of a $f$crosses the y-axis at $(0, y_1)$, theny_1 is the **y-intercept** of the graph of f.
2. The graph of a polynomial function **has no jumps, gaps, holes and sharp corners**.
3. The graph of a polynomial function is **a smooth and continuous curve** which means there is no break anywhere on the graph.
4. For every value of x in the domain of a polynomial function $p(x)$, there is exactly one value y where $y = p\ (x)$.
5. The graph of a polynomial function of degree n crosses (meets) the x-axis at most $n$times. So (as stated previously), every polynomial function of degree n has at most n zeros (roots).
6. In general, the behavior of the graph of a polynomial function as x decreases without bound to the left or as x increases without bound to the right can be **determined by its degree (even or odd) and by its leading coefficient**.

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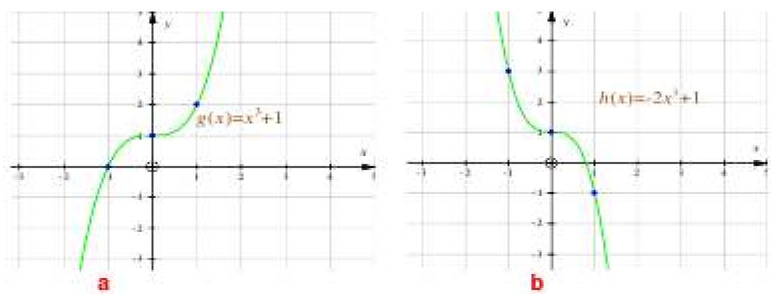
7. The graph of the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ eventually rises or falls.
8. Note that the graph of a polynomial function of degree n has at most n – 1 turning points.
9. If the degree of a polynomial function f is odd, then the range of f is the set of all real number.

To understand properties of polynomial functions, we shall see the graphs of the following higher degree polynomial functions.

Example: By sketching the graphs of $g(x) = x^3 + 1$ and $h(x) = -2x^3 + 1$, observe their behaviors and generalize for odd n when $|x|$ is large.

Solution: The sketches of the graphs of g and h are as follows.



In figure-a, the coefficient of the leading term is 1 which is positive. As a result when x becomes large in absolute value and x negative, $g(x)$ is negative but large in absolute value (The graph moves down). When x takes large positive values, $g(x)$ becomes large positive.

In figure-b, the coefficient of the leading term is – 2 which is negative. As a result, when x becomes large in absolute value for x negative, $h(x)$ becomes large positive. When x takes large positive values, $h(x)$ becomes negative but large in absolute value.

Example: By sketching the graphs of $g(x) = 2x^4$ and $h(x) = -x^4 + 2x^2 + 1$, observe their behavior and generalize for even n when $|x|$ is large.

Solution: The sketches of the graphs of g and h are as follows.

1. Note that the graph of a polynomial function of degree n has at most n – 1 turning points.
2. If the degree of a polynomial function f is odd, then the range of f is the set of all real number.

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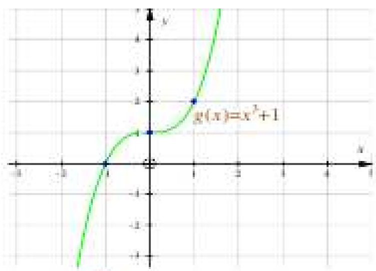
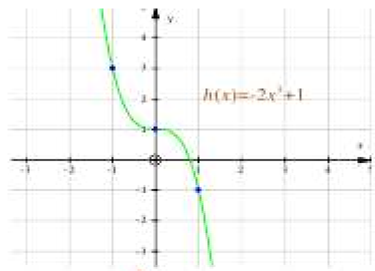
3. To understand properties of polynomial functions, we shall see the graphs of the following higher degree polynomial functions.
4. **Example**: By sketching the graphs of $g(x) = x^3 + 1$ and $h(x) = -2x^3 + 1$, observe their behaviors and generalize for odd n when $|x|$ is large.
5. **Solution**: The sketches of the graphs of g and h are as follows.



a

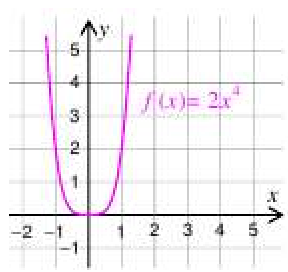


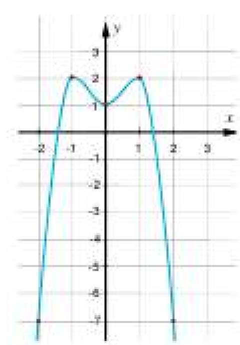
b

In figure-a, the coefficient of the leading term is 1 which is positive. As a result when x becomes large in absolute value and x negative, $g(x)$ is negative but large in absolute value (The graph moves down). When x takes large positive values, $g(x)$ becomes large positive.

In figure-b, the coefficient of the leading term is – 2 which is negative. As a result, when x becomes large in absolute value for x negative, $h(x)$ becomes large positive. When x takes large positive values, $h(x)$ becomes negative but large in absolute value.

Example: By sketching the graphs of $g(x) = 2x^4$ and $h(x) = -x^4 + 2x^2 + 1$, observe their behavior and generalize for even n when $|x|$ is large.
Solution: The sketches of the graphs of g and h are as follows.





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From figure-a, when | x | takes large values, $g(x)$ becomes large positive and the graph opens upward. On the other hand, from figure-b, when | x | takes large values, $h(x)$ becomes negative but large in absolute value and the graph opens downward.

When n is even, the graph of f opens upward for$a_n > 0$ and opens downward for$a_n < 0$.

A point of f that is either a maximum point or minimum point on its domain is called **extreme point** of f.

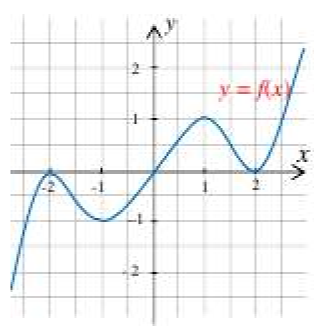
Example: Consider the polynomials

a. $f(x) = x(x-2)^2(x+2)^4$ b. $f(x) = 3x^4 + 4x^3$

Solution:

a. The function f has zeros at $x = 0$ with multiplicity 1, a zero of multiplicity 2 at $x = 2$ and a zero of multiplicity 4 at $x = -2$,
As shown in the figure below, it has a local maximum at $x = -2$ and does not change sign at $x = -2$. Also, f has a relative (local) minimum at $x = 2$ and does not change sign here. Both $x = -2$ and $x = 2$ are zeros of even multiplicity.

On the other hand, $x = 0$ is a zero of odd multiplicity, $f(x)$changes sign at $x = 0$, and does not have a turning point at $x = 0$.



b. $f(x) = 3x^4 + 4x^3$ can be expressed as $f(x) = x^3(3x+4)$
The degree of f is even and the leading coefficient is positive. Hence, the graph rises up as | x | becomes large.

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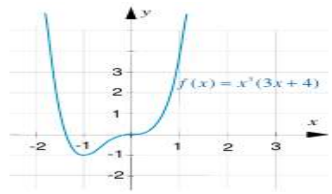

The function has a zero at $x = -\frac{4}{3}$ and changes sign at point$\left(-\frac{4}{3}, 0\right)$. The graph of f has a local minimum at a point $(-1, -1)$.

Also f has a zero at $x = 0$ and changes sign here. So, $x = 0$ is of odd multiplicity. There is no local minimum or maximum at $(0, 0)$.

The above observations can be generalized as follows:

1. If c is a zero of odd multiplicity of a function f, then the graph of the function crosses the x-axis at $x = c$ and does not have a relative extreme at $x = c$.
2. If c is a zero of even multiplicity, then the graph of the function touches (but does not cross) the x-axis at $x = c$ and has a local extremum at $x = c$.

Note that for $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, a_n \neq 0$

If $a_n > 0$and n is odd, $p(x)$ becomes large positive as x takes large positive values and $p(x)$ becomes negative but large in absolute value as the absolute value of x becomes large for x negative.

Discuss the cases where:

i. $a_n > 0$ and n is even
ii. $a_n < 0$ and n is even
iii. $a_n < 0$ and n is odd
iv. $a_n > 0$ and n is odd

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2.5.2. Rational Function and their graph

Definition: A rational function is a function of the form$\mathrm{R(x)} = \frac{\mathrm{P(x)}}{\mathrm{Q(x)}}$, where the nominator $P(x)$ and denominator $Q(x)$ are polynomials.

Note:

i. The domain of a rational function consists of all real number x except those for which makes the denominator is zero.

ii. The zeros of a rational function are the values of x makes the value of the numerator is zero.

iii. The line $x = a$ is a vertical asymptote of the rational function if the functional value of the denominator at $x = a$ is zero,but the functional value at $x = a$ is nonzero.

iv. The line $y = b$ is a horizontal asymptote of the function f if $f(x)$ approaches b as x approaches $\pm\infty$.

v. If the degree of the numerator is equal to the degree of the denominator, then there is a horizontal asymptote at $y = \frac{\text{the leading coefficent of numerator}}{\text{the leading coefficent of denominator}}$.

vi. If the degree of the numerator is less than from the degree of the denominator, then there horizontal asymptote x −axis.

vii. If the degree of the numerator is greater than from the degree of the denominator, then there is not a horizontal asymptote, but has an oblique one. The equation is found by doing long division and the quotient is the equation of oblique asymptote ignoring the remainder.

Example:

1. The following functions are rational

   a. $f(x) = \frac{1+x}{x+2}$

   b. $f(x) = \frac{x^2-2x-3}{x+4}$

   c. $f(x) = \frac{x^2+2x-3}{x^3+2x^2-3x-5}$

2. The following functions are not rational

   a. $f(x) = \frac{1+\sqrt{x}}{x+4}$

   b. $f(x) = \frac{x^2-2x-3}{x^{\frac{1}{3}}+4}$

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c. $f(x) = \sqrt{\frac{x^2-2x-3}{x+4}}$

3. Find the domain, range, vertical asymptote, horizontal asymptote , hole and oblique asymptote of the following function if they have

a. $f(x) = \frac{1+x}{x+2}$

Solution: The given rational function has

- ✓ Domain is all real number except $x = -2$
- ✓ Rang is
- ✓ Vertical asymptote denoted by VA is $x = -2$
- ✓ Horizontal asymptote denoted by HA is $y = 1$
- ✓ The function has no oblique asymptote
- ✓ The function has no hole

Exercise: Find the domain, range, vertical asymptote, horizontal asymptote, hole and oblique asymptote of the following function if they have

$$f(x) = \frac{x^2-2x-3}{x+4}$$

Graph of rational functions

Let $\mathrm{R(x)} = \frac{\mathrm{P(x)}}{\mathrm{Q(x)}}$ is a rational functions, then the graph will be hold the following

1. Simplify f, if possible and find its domain and range.
2. Find and plot the x-intercept and y-intercept(if any)
3. Find the zeros of the numerator (if any) by setting the numerator equal to zero. Then plot the corresponding -intercepts.
4. Find the corresponding vertical asymptote using dashed vertical lines and plot the corresponding holes using open circles (if any).
5. Find and sketch any other asymptote of the graph using dashed lines.
6. Plot at least one point between and one point beyond each -intercept and vertical asymptote.
7. Use smooth curves to complete the graph of the function

Example: Sketch the graph of the following rational function

a. $f(x) = \frac{4x+1}{2x+1}$

b. $f(x) = \frac{4\mathrm{x}^2+\mathrm{x}}{2\mathrm{x}^2+\mathrm{x}}$

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c. $f(x) = \sqrt{\frac{x^2-2x-3}{x+4}}$

4. Find the domain, range, vertical asymptote, horizontal asymptote , hole and oblique asymptote of the following function if they have

b. $f(x) = \frac{1+x}{x+2}$

Solution: The given rational function has

- ✓ Domain is all real number except $x = -2$
- ✓ Rang is
- ✓ Vertical asymptote denoted by VA is $x = -2$
- ✓ Horizontal asymptote denoted by HA is $y = 1$
- ✓ The function has no oblique asymptote
- ✓ The function has no hole

Exercise: Find the domain, range, vertical asymptote, horizontal asymptote , hole and oblique asymptote of the following function if they have

$$f(x) = \frac{x^2-2x-3}{x+4}$$

Graph of rational functions

Let $R(x) = \frac{\mathrm{P(x)}}{\mathrm{Q(x)}}$ is a rational functions, then the graph will be hold the following

1. Simplify $R(x) = \frac{\mathrm{P(x)}}{\mathrm{Q(x)}}$, if possible and find its domain and range.
2. Find and plot the x −intercept and y −intercept(if any)
3. Find the zeros of the numerator (if any) by setting the numerator equal to zero. Then plot the corresponding -intercepts.
4. Find the corresponding vertical asymptote using dashed vertical lines and plot the corresponding holes using open circles (if any).
5. Find and sketch any other asymptote of the graph using dashed lines.
6. Plot at least one point between and one point beyond each -intercept and vertical asymptote.
7. Use smooth curves to complete the graph of the function

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Example: Sketch the graph of the following rational function

a. $f(x) = \frac{4x+1}{2x+1}$

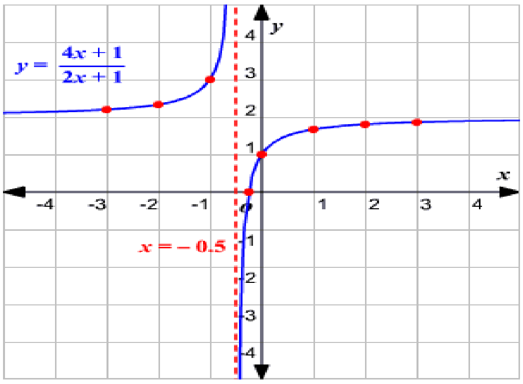
b. $f(x) = \frac{4x^2+x}{2x^2+x}$

c. $f(x) = \frac{1}{x}$

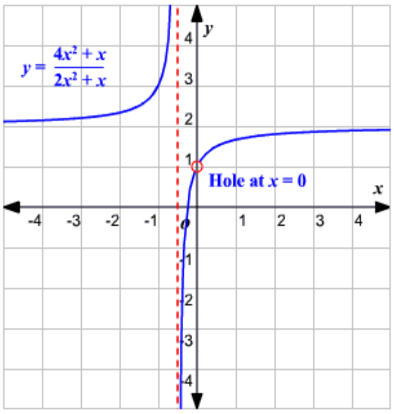
Solution:

a. The given rational function is $f(x) = \frac{4x+1}{2x+1}$ and we have

1. Domain is all real number except $x = -0.5$
2. x-intercept with $y = 0$ is $(1, 0)$ and y-intercept with $x = 0$ is $(0, -\frac{1}{4})$
3. VA is $x = -0.5$, and HA is $y = 2$
4. The graph has no hole and let as plot those intercept and asymptote on the $xy -$ Plane as follow



b. The graph of the functions $f(x) = \frac{4x^2+x}{2x^2+x}$ by listing all guidelines is looks like



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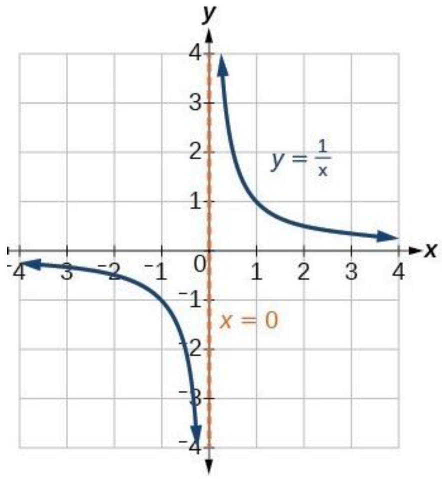
c. The graph of the function $f(x) = \frac{1}{x}$ by listing all above guide lines is looks like



2.5.3. Exponential Function and their graph

Definition: Exponential function is a function $f: \mathbb{R} \to (0, \infty)$ defined by $f(x) = a^x$, where ''x" is variable in any real number and "a" is a constant which is called the base of the function with, $a > 0$, and $a \neq 1$.

Example: Determine whether the following functions are exponential or not

a. $f(x) = 2^x$

b. $f(x) = \left(\frac{1}{2}\right)^x$

c. $f(x) = x^x$

d. $f(x) = x^2$

Solution:.

a. The function $f(x) = 2^x$ is exponential function with basis 2 and exponent .

b. The function $f(x) = \left(\frac{1}{2}\right)^x$ is exponential function with base $\frac{1}{2}$ and exponent x.

c. The function $f(x) = x^x$ is not exponential function, since the base is not arbitrary constant

d. The function $f(x) = x^2$ is not exponential function, since the base is not arbitrary constant and the exponent also not variable.

Rules of Exponential Function

Some important exponential rules are given below:

If $a > 0$, and $b > 0$, then the following hold true for all the real numbers x and y:

✓ $a^x a^y = a^{x+y}$ $\frac{a^x}{a^y} = a^{x-y}$ $(a^x)^y = a^{xy}$

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✓ $a^x b^x = (ab)^x$ $\frac{a^x}{b^x} = (\frac{a}{b})^x$ $a^0 = 1$ $a^{-x} = \frac{1}{a^x}$

Example: Solve the value of x for the following problem

a. $\left(\frac{1}{4}\right)^x = 64$

b. $4^{x^2} = 2^x$

c. $16^x 2^x = 4^6$

d. $2^{2x} + 2^{x+2} - 12 = 0$

Graph of Exponential functions

Let $f(x) = a^x$ is an Exponential functions, then the graph will be hold the following

1. The domain is the set of all real numbers:
2. The range is the set of positive real numbers:
3. There are no x-intercepts
4. The y-intercept is $(0, 1)$
5. The x-axis (the line y $=$ 0) is a horizontal asymptote, but has no vertical asymptote.
6. An exponential function is increasing when a $>$ 1 and decreasing when $0 < a < 1$.
7. An exponential function is continuous and one to one,
8. The exponential function is invertible.
9. The graph of the function is smooth.

Example: Sketch the graph of the following exponential functions

a. $f(x) = 2^x + 1$

b. $f(x) = \left(\frac{1}{2}\right)^x$

c. $f(x) = 2^{x+3}$

d. $f(x) = 2^x - 3$

Solution:

a. The graph of the function $f(x) = 2^x + 1$ will be draw with the following necessary things

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b. The domain is the set of all real numbers:

c. The range is the set of positive real numbers:

d. There are no x-intercepts

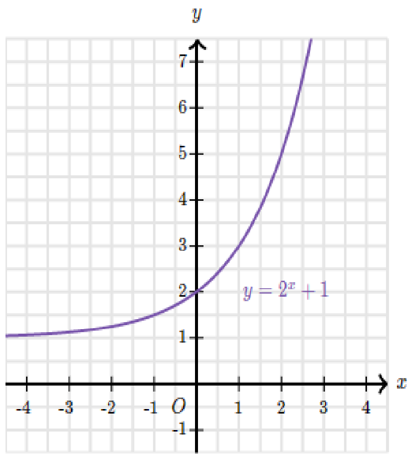
e. The y-intercept is $(0, 2)$

f. The x −axis is a horizontal asymptote, but has no vertical asymptote.

g. The function is increasing since $\mathrm{a} = 2 > 1$

h. The function is continuous and one to one,

i. The graph of the function $f(x) = 2^x + 1$ is smooth and the it is given below



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b. The graph of the function $f(x) = \left(\frac{1}{2}\right)^x$ will be draw with the following necessary things

i. The domain is the set of all real numbers:

ii. The range is the set of positive real numbers:

iii. There are no x −intercepts

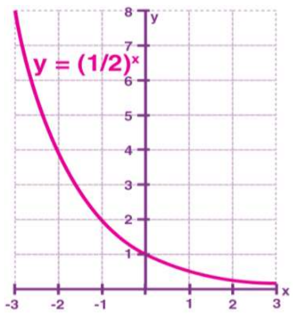
iv. The y-intercept is $(0, 1)$.

v. The x −axis is a horizontal asymptote, but has no vertical asymptote.

vi. The function is decreasing since $a = \frac{1}{2} < 1$.

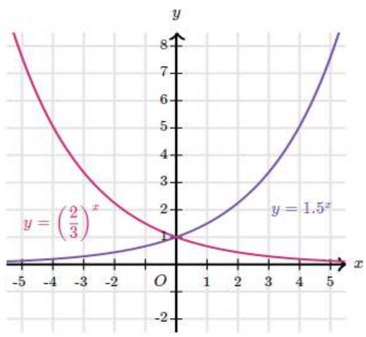
vii. The function is continuous and one to one.

viii. The graph of the function $f(x) = \left(\frac{1}{2}\right)^x$ is smooth and the it is given below



Example 2: Draw the graph of the function $f(x) = \left(\frac{2}{3}\right)^x$ and $f(x) = \left(\frac{3}{2}\right)^x$ on the same xy −plane and observe their difference.

Solution: By determining all necessary guide line of the functions their graphs is given below:



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2.5.4. Logarithmic Functions, Equation involving logarithms and Graphs of logarithmic functions

Logarithms:

Logarithms can be thought of as “the inverse” of exponents.

For example, we know that the following exponential equation is true : $3^2 = 9$.

In this case, the base is 3 and the exponent is 2. We write this equation in logarithm Form (with identical meaning) as $\log_3 9 = 2$

We read this as "the logarithm of 9 to the base 3 is 2".

Example: Write an equivalent exponential statement for the following:

(a) $\log_{12} 144 = 2$ (b) $\log_4 \left(\frac{1}{64}\right) = -3$ (c) $\log_{10} \sqrt{10} = \frac{1}{2}$

Solution

(a) $\log_{12} 144 = 2$ can be written in exponential form as $12^2 = 144$

(b) $\log_4 \left(\frac{1}{64}\right) = -3$ ……………………………………$4^{-3} = \frac{1}{64}$

The following are laws of logarithms:

Let $a, x,$ and y be positive real numbers and $\neq 1$, then

(a) $\log_a(xy) = \log_a(x) + \log_a(y)$

(b) $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

(c) $\log_a x^k = k\log_a(x)$ …for $k \in \mathbb{R}$

(d) $\log_a a = 1$

(e) $\log_a 1 = 0$

(f) $\log_{a^k}(x) = \frac{1}{k}\log_a(x)$ …for $k \in \mathbb{R} \setminus \{0\}$

(g) $\log_a x = \frac{\log_c x}{\log_c a}$ …for positive real numbers c and $c \neq 1$

(h) $(a)^{\log_a x} = x$

Example: Simplify the following:

(a) $\log_2(64 \times 1024)$ (b) $\log_2\left(\frac{32}{256}\right)$ (c) $\log_{\left(\frac{1}{3}\right)} 81$

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Equation involving logarithms

Properties or laws of logarithms play a major role in solving logarithmic equations.

Example: Find the domain and Solve each of the following for , then check that your solutions are valid.

(a) $\log_2(\mathrm{x}-3)=5$

(b) $\log_3(x+1)-\log_3(x+3)=1$

(c) $\log_{10}(x+3)+\log_{10}(x)=1$

(d) $\log_5(4x-7)=\log_5(x+5)$

(e) $(3)^{(1-\log_2(1-x)}=\frac{3}{2}$

(f) $\log_2(3-x)+\log_2(1-x)=3$

(g) $2\log_9 x+9\log_x 3=10$

Graphs of logarithmic functions

Study the graphs of $f(x)=\log_2 x$ and $g(x)=\log_{\left(\frac{1}{2}\right)} x$ to answer the following questions:

1. What are the domains of f and g ?
2. For which values of x is $f(x)=\log_2 x$ negative? Positive?
3. For which values of x is $g(x)=\log_{\left(\frac{1}{2}\right)} x$ negative? Positive?
4. What is the range of f and g ?
5. What is the range of f and g ?
6. Does $f(x)=\log_2 x$ increase as x increase? What about $g(x)=\log_{\left(\frac{1}{2}\right)} x$?
7. Do the graphs cross the y-axis?
8. What is the asymptote of the graphs?

In general, the graph of $f(x)=\log_b x$ for $b>1$ and $0<b<1$ looks like the given below.

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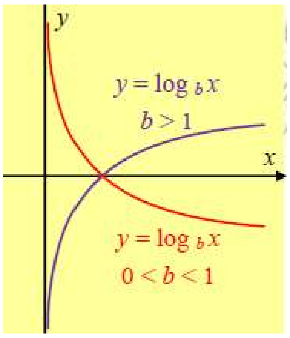

Basic properties:

(I) The graph of $f(x) = \log_b x$, for $b > 1$ has the following properties.

1. The domain is the set of all positive real numbers.
2. The range is the set of all real numbers.
3. The graph includes the point $(1, 0)$ i.e. the x-intercept of the graph is 1.
4. The function increases, as x increases.
5. The y-axis is a vertical asymptote of the graph.
6. The values of the function are negative for $0 < x < 1$ and

   They are positive for $x > 1$.

(II) The graph of $f(x) = \log_b x$, for $0 < b < 1$ has the following properties.

1. The domain is the set of all positive real numbers.
2. The range is the set of all real numbers.
3. The graph has its x-intercept at $(1, 0)$ i.e. its x-intercept is 1.
4. The function decreases as x increases.
5. The y-axis is an asymptote of the graph.
6. The values of the function are positive when $0 < x < 1$ and

   They are negative for $x > 1$.

Power function:

A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)That is given as:

$f(x) = kx^n$ where k is constant of variation and n is the power.

Example: Circumference of a circle(C) $= 2\pi r$ where $k = 2\pi$ and $n = 1$.

Area of the circle (A) $= \pi r^2$ where $k = \pi$ and $n = 2$.

Example: The following are parental functions which are power functions

$f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = \frac{1}{x}$ and $f(x) = \sqrt{x}$

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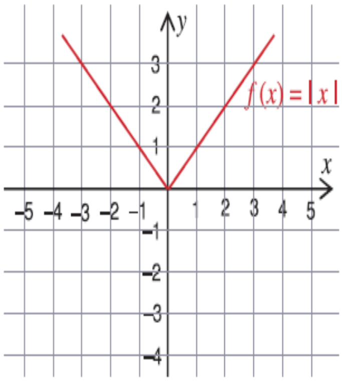
Then find the value of k and n according to the definition and also sketch their graphs.

Absolute value function:

An absolute value function is a function in algebra where the variable is inside the absolute value bars. This function is also known as the modulus function and it is defined as: $f(x) = |x| = \begin{cases} x \ldots\ldots for\ x \geq 0 \\ -x \ldots\ldots for\ x < 0 \end{cases}$

Note: The graph of $f(x) = |x|$ looks like the following:



Where: Domain of an absolute value function $= \mathbb{R}$

Range of an absolute value function $= [\,0, \infty\,)$

Properties of Absolute value:

1. Non-negativity… … … … … … $|\,x\,| \geq 0$
2. Positive-definiteness… … … $|\,x\,| = 0 \Leftrightarrow x = 0$
3. Multiplicativeness……….. $|\,xy| = |x||y|$
4. Subadditivity… … … … … … $|\,x + y| \leq |\,x\,| + |\,y\,|$

Additional properties of Absolute value:

1. $|\,x\,| = a \Leftrightarrow x = \pm a$
2. $|\,x\,| \leq a \Leftrightarrow -a \leq x \leq a$
3. $|\,x\,| \geq a \Leftrightarrow x \geq a \ \ or \ \ x \leq -a$
4. $|\,x\,| = |-x\,|$
5. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ for $y \neq 0$.
6. $|\,x - y| \leq |\,x - z\,| + |z - y|$.

<u>Activity</u>:

(a) Sketch the graph of $f(x) = |\,2x + 5\,|$

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(b) Solve the equation $|x + 5| = 2x + 3$

(c) Solve the inequality $|4x - 3| \leq 2x + 3$

(d) Solve the inequality $|4x - 3| \leq 2x + 3$ and $|x + 2| \geq 3$

Sign or Signum Function

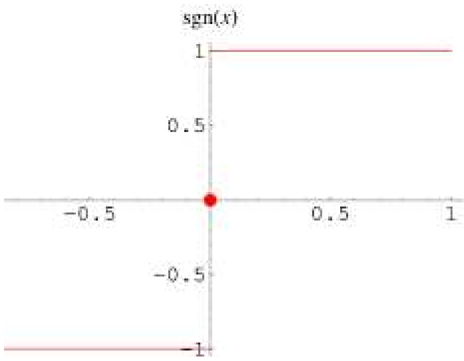
Definition: The signum function, read as signum *x*, is written as sgn *x* and is defined by:

$$f(x) = sgn\ x = \begin{cases} -1 \ldots \ldots for\ x < 0 \\ 0 \ldots \ldots for\ x = 0 \\ 1 \ldots \ldots for\ x > 0 \end{cases}$$

Note: The domain of Signum function is $\mathbb{R}$ and its range is $\{-1,0,1\}$.

Example: $(-4) = -1$, $f(0) = 0$,$f(8) = 1$

Note: The following is the graph of Sgn function.



The Greatest integer function

The greatest integer function or Stair function denoted by $f(x) = [\ x\ ]$ and defined as:

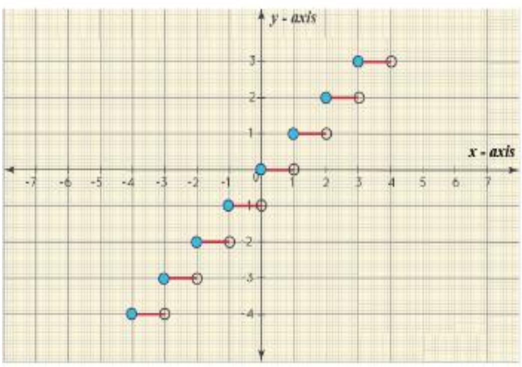
$f(x) = [\ x\ ]$ =The greatest integer that is less than or equal to x.

Note: Domain of the greatest integer function is $\mathbb{R}$, and its range is $\mathbb{Z}$.

Example:

$f(3.6) = 3$, $f(3) = 3$, $f(-4.2) = -5$, $f(-4) = -4$.

Note: The following is the graph of greatest integer function.



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2.5.5. Trigonometric Functions (Sine, Cosine, and Tangent) and their graphs

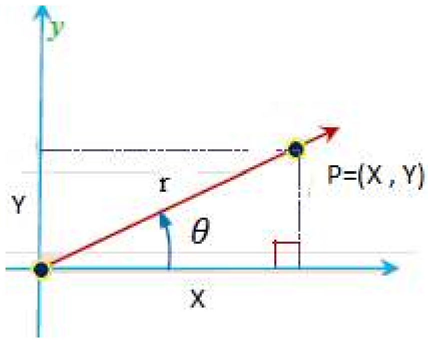
Definition: Trigonometric Function for any angle θ

A right triangle can be formed from an initial side x and a terminal side r, where r is the radius and hypotenuse of the right triangle. (See figure below) The Pythagorean Theorem tells us that $x^2 + y^2 = r^2$, therefore $r = \sqrt{x^2 + y^2}$. the six trigonometric functions of the angle defined as follows.

$$\mathbf{sin}(\theta) = \frac{opp}{hyp} = \frac{y}{r}$$

$$\mathbf{cos}(\theta) = \frac{adj}{hyp} = \frac{x}{r}$$

$$\mathbf{tan}(\theta) = \frac{opp}{adj} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}, for\ \theta \neq n\pi + \frac{\pi}{2}, n\ is\ any\ integer$$



THE UNIT CIRCLE

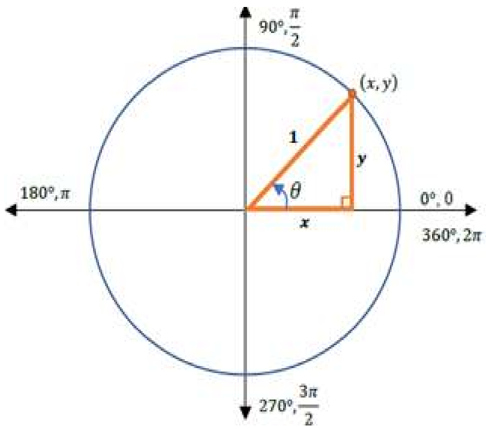
The circle with Centre at (0, 0) and radius 1 unit is called the unit circle.

From the unit circle, you can define the following six trigonometric function

$$\mathbf{sin}\,\theta = y$$

$$\mathbf{cos}\,\theta = x$$

$$\mathbf{tan}\,\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$



BASIC TRIGONOMETRIC IDENTITIES

1. **Pythogrous identities**

$$sin^2\theta + cos^2\theta = 1$$

$$1 + tan^2\theta = sec^2\theta$$

$$1 + cot^2\theta = csc^2\theta$$

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2. **Negative angle identities**

$$sin(-\theta) = -sin\theta cos(-\theta) = cos\theta tan(-\theta) = -tan\theta$$
$$csc(-\theta) = -csc\theta sec(-\theta) = sec\theta cot(-\theta) = -cot\theta$$

3. **Sumand diference identities**

$$sin(\alpha + \beta) = sin\alpha cos\beta + cos\alpha sin\beta$$
$$sin(\alpha - \beta) = sin\alpha cos\beta - cos\alpha sin\beta$$
$$cos(\alpha + \beta) = cos\alpha cos\beta - sin\alpha sin\beta$$
$$cos(\alpha - \beta) = cos\alpha cos\beta + sin\alpha sin\beta$$
$$tan(\alpha + \beta) = \frac{tan\,\alpha + tan\beta}{1 - tan\alpha\, tan\,\beta}$$
$$tan(\alpha - \beta) = \frac{tan\,\alpha - tan\beta}{1 + tan\alpha\, tan\,\beta}$$

4. **Co-function Identities**

$$cos(90° - \theta) = sin\theta$$
$$sin(90° - \theta) = cos\theta$$
$$tan(90° - \theta) = cot\theta$$

Solving trigonometric equations:

Example:

$$2sin^2x + 3cosx - 3 = 0$$

Solution

$$sin^2x + 3cosx - 3 = 0$$
$$2(1 - cos^2x) + 3cosx - 3 = 0$$
$$2cos^2x - 3cosx + 1 = 0$$

Factorize

$$(2cosx - 1)(cosx - 1) = 0$$
$$2cosx - 1 = 0 \; and \;\; cosx - 1 = 0$$

The solution in the interval $[0,2\pi]$,

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$2cosx - 1 = 0 \rightarrow cosx = \frac{1}{2} \rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$ and $cosx - 1 = 0 \rightarrow \cos x = 1 \rightarrow x = 0$

The general solution is $x = \frac{\pi}{3} + 2\pi n,\ \frac{5\pi}{3} + 2\pi n, 0 + 2\pi n$

GRAPH OF SINE, COSINE AND TANGENT FUNCTION

A function that repeats its values at regular intervals is called a periodic function. Hence sine, cosine and tangent function are periodic functions.

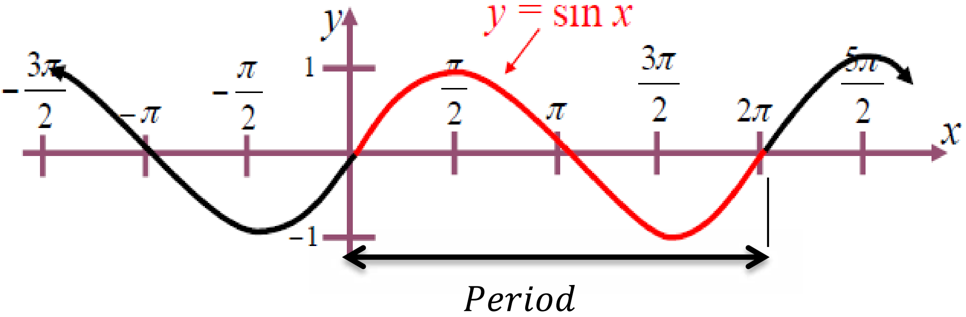
The sine and cosine functions are repeats after every $360^0 or\ 2\pi$radians and the tangent function is repeats after $180^0\ or\ \pi$. Therefore, $360^0 or\ 2\pi$ are period of sine and cosine function and $180^0\ or\ \pi$ is the period of tangent function.

Amplitude: is the maximum or minimum vertical distance between the graph and the x-axis. Amplitude is always positive.

Graph of the sine function: periodic

To determine the graph of $y = \sin x$, we construct a table of values for $y = \sin x$.

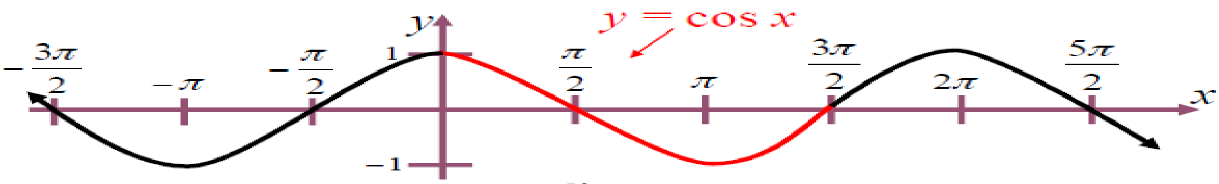
x	$\frac{-3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
$y = sinx$	1	0	-1	0	1	0	-1	0	1

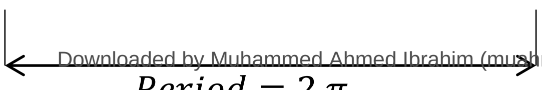


Graph of the cosine function: periodic

To determine the graph of $y = cos\ x$, we construct a table of values for $y = cos\ x$.

x	$\frac{-3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
$y = cosx$	0	-1	0	1	0	-1	0	1	0





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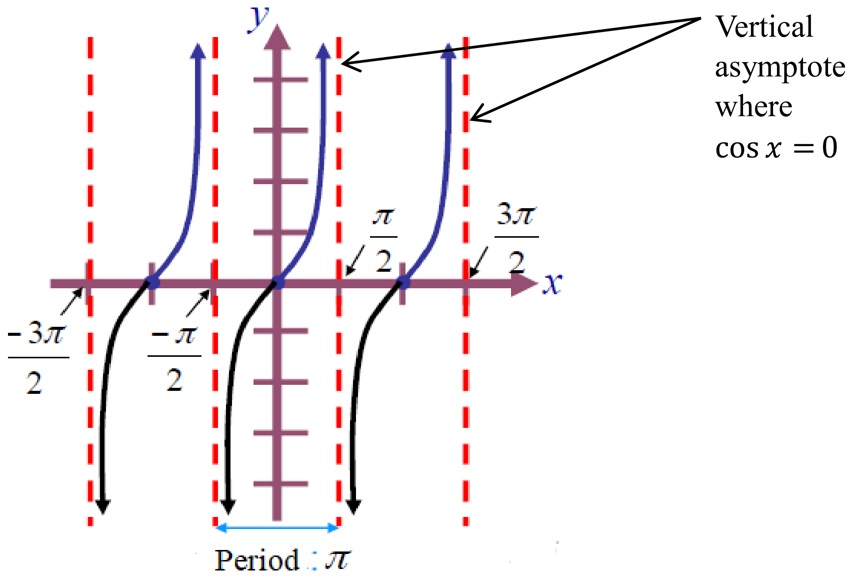
Graph of Tangent Function: Periodic

To determine the graph of $y = tanx$, we construct a table of values for $y = \tan x$

x	$\frac{-3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$y = \tan x$	-	0	-	0	-	0	-



Charactrstics of sine, cosine and tangent function

function	Period	Domain	Range	Symmetry	Even/odd
$y = \sin x$	2π	$\mathbb{R}$ or $(-\infty, \infty)$	$[-1, -1]$	orgin	Odd b/c $\forall\, x \in domain, \sin(-x) = -\sin(x)$
$y = \cos x$	2π	$\mathbb{R}$ $or(-\infty, \infty)$	$\mathbb{R}$ or $(-\infty, \infty)$	y- axis	Even b/c $\forall\, x \in domain, \cos(-x) = \cos(x)$
$y = \tan x$	π	$\mathbb{R}/\left\{x \neq \frac{\pi}{2} \pm n\pi, n \in \mathbb{Z}\right\}$	$\mathbb{R}$ or $(-\infty, \infty)$	orgin	Odd b/c $\forall\, x \in domain, \tan(-x) = -tan(x)$

Review questions for chapter two

1. Which of the following relation is not a function?

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A. $R = \{(x, y): y = x + 1\}$

B. $R = \{(x, y): y\ is\ a\ father\ of\ x\}$

C. $R = \{(x, y): x\ is\ a\ father\ of\ y\}$

D. $R = \{(x, y): y = x^2 + 1\}$

2. Find the domain and range of the following relation.

   A. $R = \{(x, y): x^2 - y = 1\}$

   B. $R = \left\{(x, y): y = \frac{x}{\sqrt{4-x}}\right\}$

3. Let $f(x) = ax + b$ and $g(x) = \frac{1}{x+1}$. If $f(g(0)) = 3$ and $g(f(0)) = \frac{1}{2}$, then find the value(s) of a and b?

4. If $x - \frac{1}{2}$ and $x + \frac{1}{4}$ are factors of $P(x) = x^3 + ax^2 + bx + c$ and when $P(x)$ is divided by -1 , then the remainder is $\frac{5}{12}$ then find the value(s) of a,b and c ?

5. Find the remainder when $P(x) = 3x^6 + 5x^2 + x + 2$ divided by $x^2 - 3x + 2$.

6. Find the zeros of $R(x) = \frac{x^3-1}{3x^2+2x-5}$.

7. Sketch the graph of $R(x) = \frac{x^2+1}{x+1}$

8. Solve $\left(2^{\sqrt{2x+1}}\right)^2 - 8\left(2^{\sqrt{2x+1}}\right) + 15 = 0$

9. If $\log 2 = 0.3.10$ and $\log 3 = 0.4771$, then find $\log \sqrt[3]{54} =$?

10. State the domain and solve for x ?

    (a) $\log_3 4x = \log_6 x$

    (b) $3\log(3x + 5) = 2\log x^3$

    (c) $(2.5)^{1-(\log_3 x)^2} = (0.16)^{2+\log_{\sqrt{3}} x}$

11. Let $f(x) = \log_2(3 - 2x)$ for $< \frac{3}{2}$, then find $f^{-1}(x)$?

12. Sketch the graph of the following functions.

    (a) $f(x) = 1 + 3^x$

    (b) $f(x) = 1 + \log_3 x$

13. If $\sin(x + y) = \frac{3}{4}$ and $\sin(x - y) = \frac{1}{2}$, then show that $\frac{\tan x}{\tan y} = 5$.

14. Find the value of x for the following equations.

    (a) $2\sin\left(x - \frac{\pi}{4}\right) = cos\left(x + \frac{\pi}{4}\right)$

    (b) $\tan(x - \theta) = \frac{3}{2}$ and $\tan(\theta) = 2$

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Chapter 3: Geometry and Measurement

Introduction

Why do you study Geometry?

- ❖ Geometry teaches you how to think clearly. Of all the subjects taught at high school level, Geometry is one of the lessons that gives the best training in correct and accurate methods of thinking.

  The study of Geometry has a practical value. If someone wants to be an artist, a designer, a carpenter, a tinsmith, a lawyer or a dentist, the facts and skills learned in Geometry are of great value.
- ❖ *Abraham Lincoln* borrowed a Geometry text and learned the proofs of most of the plane Geometry theorems so that he could make better arguments in court.
- ❖ *Leonardo da Vinci* obtained the "Mona Lisa" smile by tilting the lips so that the ends lie on a circle which touches the outer corners of the eyes. The outline of the top of the head is the arc of another circle exactly twice as large as the first. In the same artist's "Last Supper", the visible part of Christ conforms to the sides of an equilateral triangle.

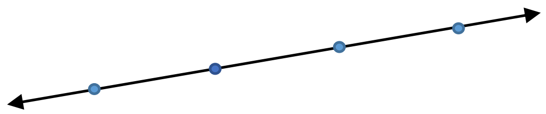
Plane Geometry (sometimes called Euclidean Geometry) is a branch of Geometry dealing with the properties of flat surfaces and plane figures, such as triangles, quadrilaterals or circles.

Theorem on triangles, special quadrilaterals, circles, Regular polygons, congruency and similarity, area of triangle and parallelogram, surface area and volume of solid figures (Prism, Cylinder, Cone and Sphere) and Frustum of Pyramids and cones are the major topics that are going to be covered in this unit.

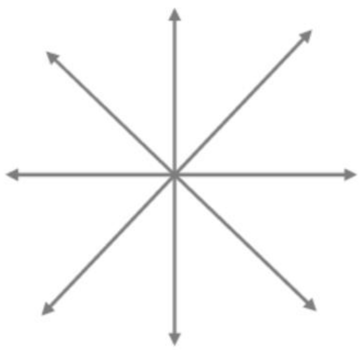
3.1. Theorems on Triangles

In previous grades, you have learnt that a triangle is a polygon with three sides and is the simplest type of polygon.

Three or more points that lie on one line are called **collinear points**. Three or more lines that pass through one point are called **concurrent lines**.



Collinear points



Concurrent lines

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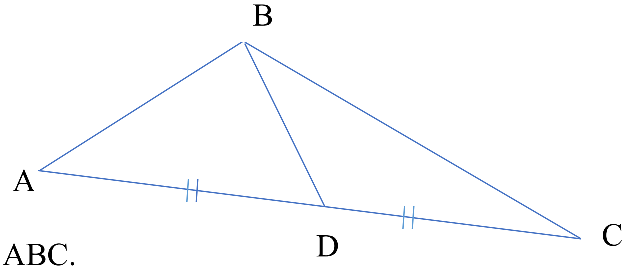
Theorems about collinear points and concurrent lines are called **incidence theorems**. Some such theorems are stated below. Recall that a line that divides an angle into two congruent angles is called an **angle bisector** of the angle.

A line that divides a line segment into two congruent line segments is called a **bisector** of the line segment. When a bisector of a line segment forms right angle with the line segment, then it is called the **perpendicular bisector** of the line segment.

Median of a triangle

A **median** of a triangle is a line segment drawn from any vertex to the mid-point of the opposite side.



Here $\overline{BD}$ is a median of triangle ABC.

Theorem 3.1

The medians of a triangle are concurrent at a point $\frac{2}{3}$ of the distance from each vertex to the mid-point of the opposite side.

Thus, $OE = \frac{1}{3}AE,\ AO = \frac{2}{3}AE.$

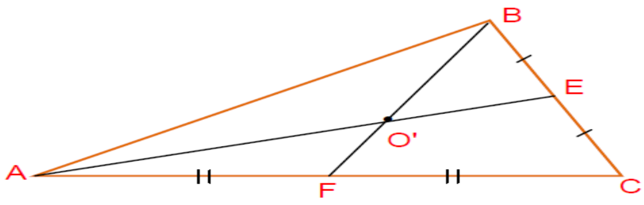


Figure 3.2

Example 1. In Figure 3.3, $\overline{AN}, \overline{CM}$ and $\overline{BL}$ are medians of ΔABC. If AN = 12 cm, OM = 5 cm and BO = 6 cm, find BL, ON and OL.

Solution:

By Theorem 3.1, $BO = \frac{2}{3}BL$ and $AO = \frac{2}{3}AN$. Substituting $6 = \frac{2}{3}BL$ and $AO = \frac{2}{3} \times 12$. So BL = 9 cm and AO = 8 cm. Since $BL = BO + OL$, $OL = BL - BO = 9 - 6 = 3$ cm.

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Now, $AN = AO + ON$ gives $ON = AN - AO = 12 - 8 = 4$ cm. $BL = 9$ cm, $OL = 3$ cm and $ON =$ 4 cm

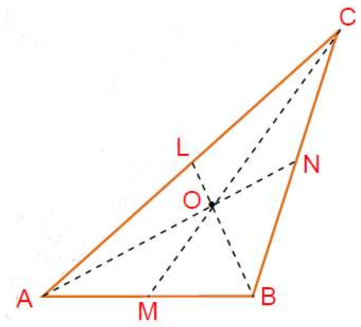

Figure 3.3

Note: The point of intersection of the medians of a triangle is called the **centroid** of the triangle.

Altitude of a triangle

The **altitude** of a triangle is a line segment drawn from a vertex, perpendicular to the opposite side, or to the opposite side produced. The **altitudes** through B and A for the triangles are shown below in Figure 3.4.

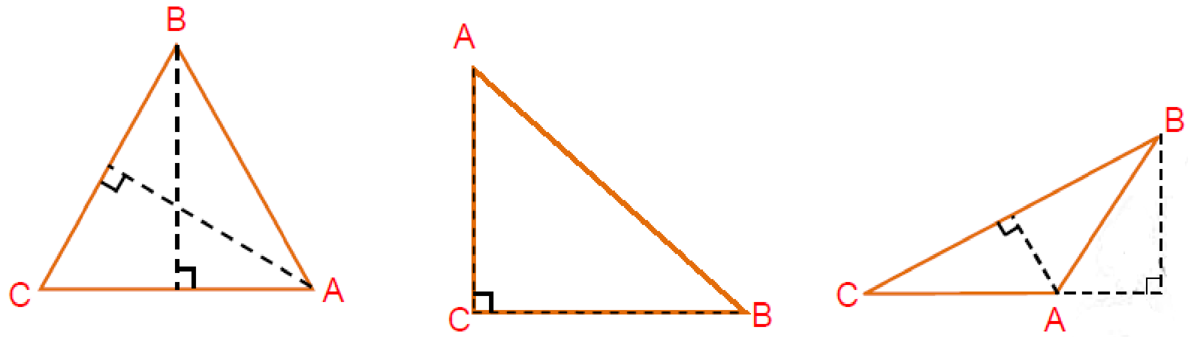

Figure 3.4

Theorem 3.2

The perpendicular bisectors of the sides of any triangle are concurrent at a point which is equidistant from the vertices of the triangle.

Let ΔABC be given and construct perpendicular bisectors on any two of the sides. The perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ are shown in **Figure 3.5.** These perpendicular bisectors intersect at a point O; they cannot be parallel. (Why?)

Using a ruler, find the lengths AO, BO and CO. Observe that the intersection point O is equidistant from each vertex of the triangle.

Note that the perpendicular bisector of the remaining side BC must pass through the point O. Therefore, the point of intersection of the three perpendicular bisectors is equidistant from the three vertices of ΔABC.

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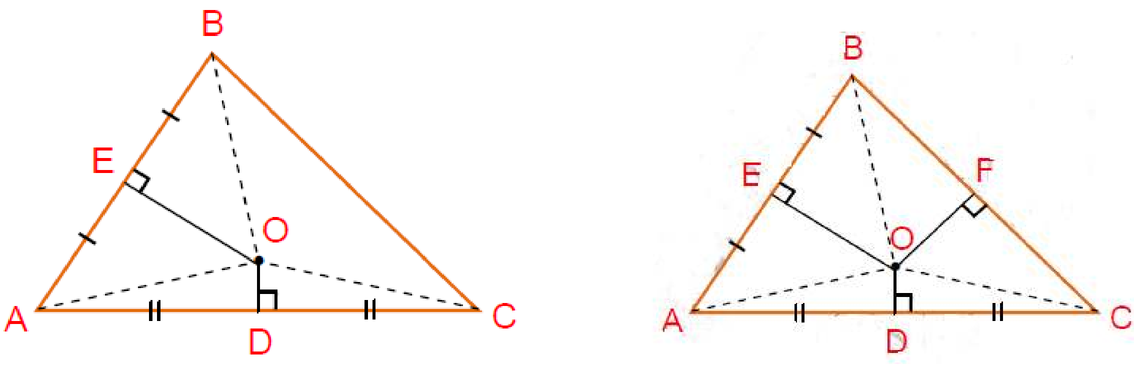

Figure 3.5

Note: The point of intersection of the perpendicular bisectors of a triangle is called circumcentre of the triangle.

Theorem 3.3

The altitudes of a triangle are concurrent.

Angle bisector of a triangle

Theorem 3.4

The angle bisectors of any triangle are concurrent at a point which is equidistant from the sides of the triangle.

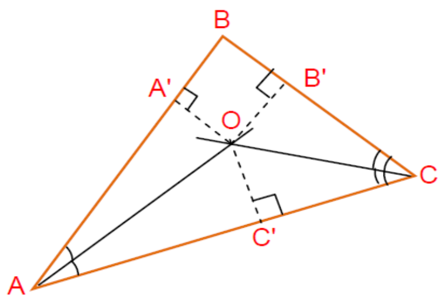

Figure 3.6

Note: The point of intersection of the bisectors of the angles of a triangle is called the incentre of the triangle.

Example 2. In a right angle triangle ΔABC, <C is a right angle, AB = 8 cm and CA = 6 cm. Find the length of $\overline{CO}$ where O is the point of intersection of the perpendicular bisectors of ΔABC.

Solution: The perpendicular bisector of CA is parallel to CB.

Hence, O is on AB.

Therefore, AO = 4. (By Theorem 3.2, $AO = BO$).

By Theorem 3.2, O is equidistant from A, B and C

Therefore, $CO = AO$ = 4 cm.

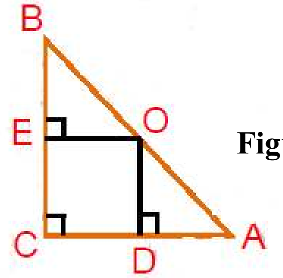

Figure 3.7

Altitude theorem

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The altitude theorem is stated here for a right angled triangle. It relates the length of the altitude to the hypotenuse of a right angled triangle, to the lengths of the segments of the hypotenuse.

Theorem 3.5 (Altitude theorem)

In a right angled triangle ABC with altitude $\overline{CD}$ to the hypotenuse $\overline{AB}$, $\frac{AD}{DC} = \frac{CD}{DB}$.

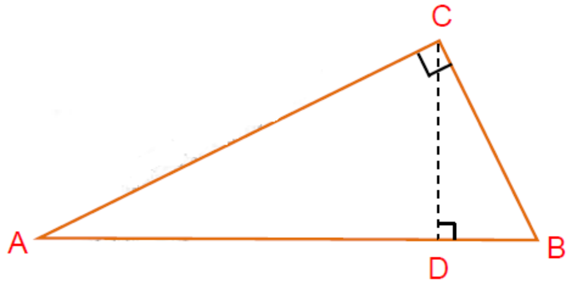

Figure 3.8

Note: The square of the length of the altitu … of the lengths of the segments of the hypotenuse. That is, $(CD)^2 = (AD)\,(DB)$ or $(AD)\,(DB) = (CD)\,(DC)$.

Example 3. In ΔABC, $\overline{CD}$ is the altitude to the hypotenuse $\overline{AB}$, AD= 9 cm and BD = 4 cm. How long is the altitude $\overline{CD}$?

Solution: Let h = CD. From the Altitude Theorem, $(CD)_2 = (AD)\,(BD)$

Substituting, $h^2 = 9 \times 4 = 36\ cm^2$. So, h = 6 cm. The length of the altitude is 6 cm.

Theorem 3.6 (Menelaus' theorem)

If points D, E and F on the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively of ABC (or their extensions) are collinear, then $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$. Conversely, if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$, then the points D, E and F are collinear.

Note: 1 For a line segment AB, we use the convention: $AB = -BA$.

2 If F is in $\overline{AB}$, then $\frac{AF}{FB} = r > 0$.

3.2. Special Quadrilaterals

In this section, we consider the following special quadrilaterals: trapezium, parallelogram, rectangle, rhombus and square.

Keep in mind the mathematical definitions of each of the above quadrilaterals.

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Trapezium

Definition 3.1

A trapezium is a quadrilateral where only two of the sides are parallel.

In Figure 3.9, the quadrilateral *ABCD* is a trapezium. The sides *AD* and *BC* are nonparalle sides of the trapezium *ABCD*.

Note that if the sides *AD* and *BC* of trapezium *ABCD* are congruent, then the trapezium is called an isosceles trapezium.

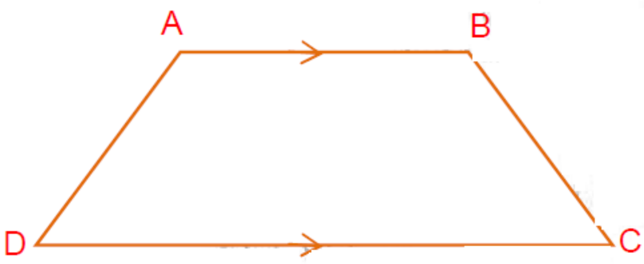

Figure 3.9

Parallelogram

Definition 3.2

A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.

In Figure 3.10, the quadrilateral *ABCD* is a parallelogram.

$AB \mathbin{//} DC$ and $AD \mathbin{//} BC$.

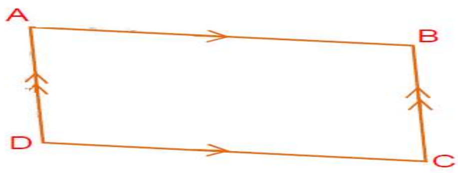

Figure 3.10

Properties of a parallelogram and tests for a quadrilateral to be a parallelogram are stated in the following theorem:

Theorem 3.6

a. The opposite sides of a parallelogram are congruent.

b. The opposite angles of a parallelogram are congruent.

c. The diagonals of a parallelogram bisect each other.

d. If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

e. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

f. If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

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Rectangle

Definition 3.3

A rectangle is a parallelogram in which one of its angles is a right angle.

In Figure 3.11, the parallelogram *ABCD* is a rectangle whose angle *D* is a right angle. What is the measure of each of the other angles of the rectangle *ABCD*?

Some properties of a rectangle

1. A rectangle has all properties of a parallelogram.
2. Each interior angle of a rectangle is a right angle.
3. The diagonals of a rectangle are congruent

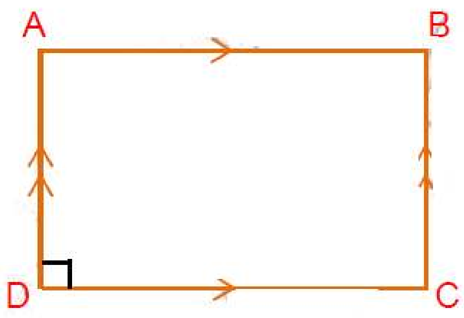


Figure 3.11

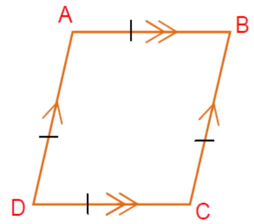
Rhombus

Definition 3.4

A rhombus is a parallelogram which has two congruent adjacent sides.

Some properties of a rhombus

1. A rhombus has all the properties of a parallelogram.
2. A rhombus is an equilateral quadrilateral.
3. The diagonals of a rhombus are perpendicular to each other.
4. The diagonals of a rhombus bisect its angles.



Figure

Square

Definition 3.5

A square is a rectangle which has congruent adjacent sides.

In Figure 3.13, the rectangle *ABCD* is a square.

Some properties of a square

1. A square has the properties of a rectangle.
2. A square has all the properties of a rhombus.

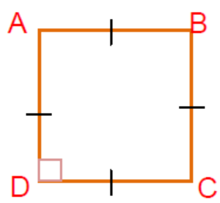


Figure 3.13

Theorem 3.7

If the diagonals of a quadrilateral are congruent and are perpendicular bisectors of each other, then the quadrilateral is a square.

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Proof:-

Given $\overline{AC}, \overline{BD}, \overline{AC}$ and $\overline{BD}$ are perpendicular bisectors of each other.

To prove: *ABCD* is a square.

Let O be the point of intersection of *AC* and *BD*.

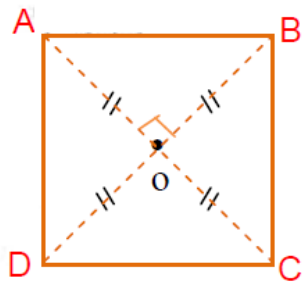

Figure 3.14

3.3. Circles

In this section, you are going to study circles and the lines and angles associated with them. Of all simple geometric figures, a circle is perhaps the most appealing. Have you ever considered how useful a circle is? Without circles there would be no watches, wagons, automobiles, steamships, electricity or many other modern conveniences. Recall that a circle is a plane figure, all points of which are equidistant from a given point called the center of the circle. As you recall from Grade 9, in Figure 3.15, PQ is a chord of the circle with center *O*. *AB* is a chord (diameter) $\widehat{AXC}$ is an arc of the circle.

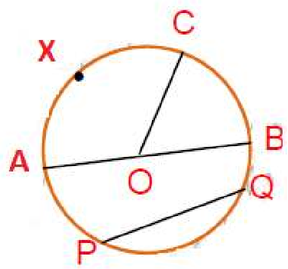

Figure 3.15

If *A* and *C* are not end-points of a diameter, $\widehat{AXC}$ is a minor arc. <*BOC* is a central angle. $\widehat{AXC}$ or arc *AXC* is said to subtend <*AOC* or <*AOC* intercepts arc *AXC*.

3.3.1. Angles and Arcs Determined by Lines Intersecting Inside and, On a Circle,

We now extend the discussion to angles whose vertices do not necessarily lie at the center of the circle.

In a circle, an inscribed angle is an angle whose vertex

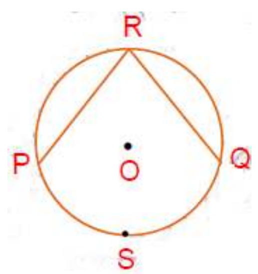

Figure 3.16

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lies on the circle and whose sides are chords of the circle.

In Figure 3.16, angle PRQ is inscribed in the circle.

We also say that ÐPRQ is inscribed in the arc PRQ and

ÐPRQ is subtended by arc PSQ (or $\widehat{PSQ}$).

Measure of a central angle: Note that the measure of a central angle is the measure of the arc it intercepts.

So, m (<POQ) = m ($\widehat{PXQ}$).

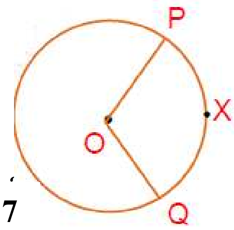

Figure 3.17

Theorem 3.8

The measure of an angle inscribed in a circle is half the measure of the

Proof:-

Given Circle O with <B an inscribed angle intercepting $\widehat{AC}$.

To prove $m\ (< ABC\ )\ =\ ½\ m(\widehat{AXC})$ where X is a point as shown in Figure 3.18.

To prove Theorem 3.8, we consider three cases.

Case 1: Suppose that one side of ΔABC is a diameter of the circle with center O.

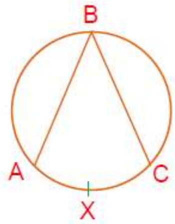

Figure 3.18

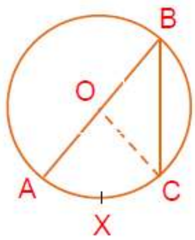

Figure 3.19

Therefore, m (<ABC) = ½m $(\widehat{AXC})$

Case 2: Suppose that A and C are on opposite sides of the diameter through B, as shown in Figure 3.20.

Figure 3.20

Therefore, m (<ABC) = ½ $m(\widehat{AXC})$

Case 3: Suppose that A and C are on the same side of the diameter through B as shown in Figure 3.21.

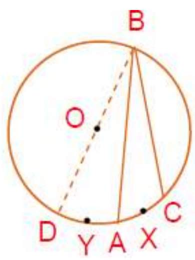

Figure 3.21

Therefore, m (<ABC) =1/2 $m(\widehat{AXC})$ in all cases and the theorem holds.

Example 1. In Figure 3.22, m(<PXQ) = 110^o. Find the measure of <PRQ.

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Solution: By theorem 3.8, we have m (<PRQ) $=\frac{1}{2}$ m (PXQ) = $\frac{1}{2}(110^o)$ = 55^o

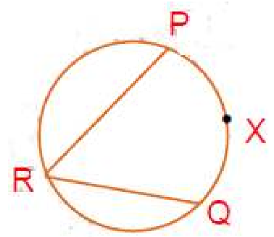

Figure 3.22

Corollary 3.1

An angle inscribed in a semi-circle is a right angle.

Proof:-

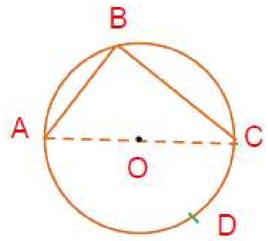

Figure 3.23

In Figure 3.23, ΔABC is inscribed in semi-circle ABC.

Đ ABC is subtended by arc ADC, which is a semi-circle.

The measure of arc ADC is 180^o or p radians.

By theorem 3.8, m (Đ ABC) =1/2 m ($\widehat{ADC}$)

$=\frac{1}{2}(180^o) = 90^o \; or \frac{\pi}{2}$ radians.

Corollary 3.2

An angle inscribed in an arc less than a semi-circle is obtuse.

Proof:-

$$m(\quad ABC) = \frac{1}{2}\, m\,(\widehat{ADC})$$

But m ($\widehat{ABC}$) < length of a semi-circle

$$m\,(\widehat{ABC}) < 180^o$$

Therefore, ($\widehat{ADC}$) > 180^o

$$m(\quad ABC) = \frac{1}{2} m\,(\widehat{ADC}) > \frac{1}{2}(180^o)$$

$$m(\quad ABC) > 90^o. \text{ So, } \quad ABC \text{ is an obtuse angle.}$$

Corollary 3.3

An angle inscribed in an arc greater than a semi-circle is acute.

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Theorem 3.9

Two parallel lines intercept congruent arcs on the same circle.

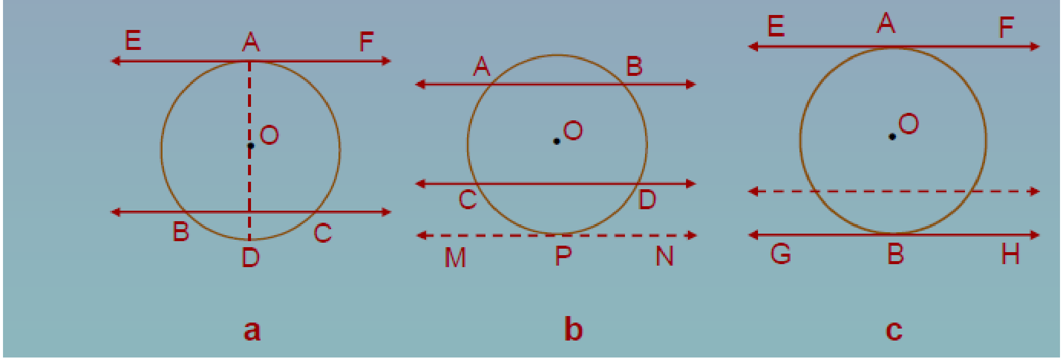

Figure 3.25

Proof:-

To prove this fact, you have to consider the following three possible cases:

a. When one of the parallel lines, $\overline{EF}$ is a tangent line and the other $\overline{BC}$ is a secant line as shown in Figure 3.25a.

b. When both parallel lines $\overline{AB}$ and $\overline{CD}$ are secants as shown in Figure 3.25 b.

c. When both parallel lines $\overline{EF}$ and $\overline{GH}$ are tangents as shown in Figure 3.25 c.

Case a:

Given A circle with center O, $\overline{EF}$ and $\overline{BC}$ are two parallel lines such that $\overline{EF}$ is a tangent to the circle at A and $\overline{BC}$ is a secant.

To prove: $\widehat{AB} \equiv \widehat{AC}$

	Statement		Reason
1	Draw diameter $\overline{AD}$	1	Construction.
2	$\overline{AD}$ $\overline{EF}$ and $\overline{AD}$ $\overline{BC}$	2	A tangent is perpendicular to the diameter drawn to the point of tangency and also $\overline{EF} // \overline{BC}$ is given.
3	$\widehat{BD}$ $\widehat{CD}$	3	Any perpendicular from the centre of a circle to a chord bisects the chord and the arc subtended by it.
4	$\widehat{AB}$ $\widehat{AC}$	4	$\widehat{ABD}$ $\widehat{ACD}$ (semicircles) and step 3.

Other case are left to the readers.

Theorem 3.10

An angle formed by a tangent and a chord drawn from the point of tangency is measured by half the arc it intercepts.

Theorem 3.11

The measure of an angle formed by two chords intersecting inside a circle is half the sum of the measures of the arc subtending the angle and its vertically opposite angle.

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Example 2. In Figure 3.26, m (<MRQ) = 30^o, and m (<MQR) = 40^o. Write down the measure of all the other angles in the two triangles, DPSM and DQMR. What do you notice about the two triangles?

Solution: m (<QMR) = $180^o - (30^o + 40^o) = 110^o$ (why?)

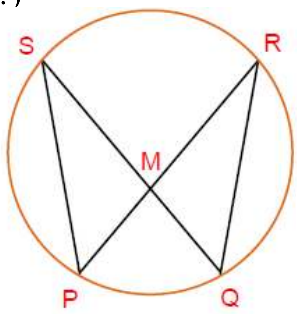


Figure 3.26

m (<RQS) =1/2 m ($\widehat{RS}$)

Therefore, 40^o =1/2m ($\widehat{RS}$).

$\therefore m$ ($\widehat{RS}$) = 80^o.

$$m(\quad PRQ) = \frac{1}{2} m(\widehat{PQ})$$

$$\text{Hence, } 30^\circ = \frac{1}{2} m(\widehat{PQ})$$

$$m(\widehat{PQ}) = 60^\circ$$

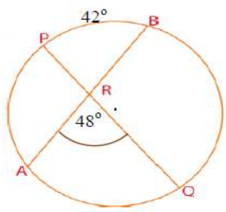
$$m(\quad PSQ) = \frac{1}{2} m(\widehat{PQ}) = \frac{1}{2}(60^\circ) = 30^\circ$$

$$m(\quad RPS) = \frac{1}{2} m(\widehat{RS}) = \frac{1}{2}(80^\circ) = 40^\circ$$

The two triangles are similar by AA similarity.

Example 3. An angle formed by two chords intersecting within a circle is 48^o, and one of the intercepted arcs measures 42^o. Find the measures of the other intercepted

Solution: Consider Figure 3.27.



Figure

$$m(\quad PRB) = \frac{1}{2} m(\widehat{PB}) + \frac{1}{2} m(\widehat{AQ})$$

$$48^\circ = \frac{1}{2}(42^\circ) + \frac{1}{2}(\widehat{AQ})$$

$$\Rightarrow 96^\circ = 42^\circ + m(\widehat{AQ})$$

$$54^\circ = m(\widehat{AQ})$$

3.3.2. Angles and Arcs Determined by Lines Intersecting Outside a Circle

What happens if two secant lines intersect outside a circle? In Figure 3.28, $\overline{AB}$ and $\overline{XY}$ intersect at P outside the circle. They intercept arcs BY and AX. Draw the chord AC parallel to $\overline{XY}$. Can you see that the measure of <XPA is half the difference between the measures of arcs $\widehat{BY}$ and $\widehat{AX}$? Can you prove it?

This is stated in theorem 3.12.

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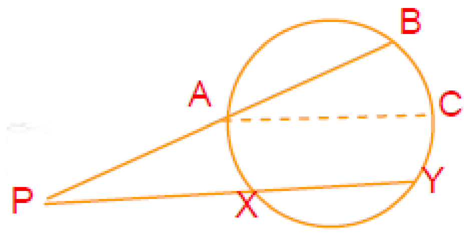

Theorem 3.12

Figure 3.28

The measure of the angle formed by the lines of two chords intersecting outside a circle is half the difference of the measure of the arcs they intercept.

The product property, $(PA)(PB) = (PX)(PY)$ is also true when two chords intersect outside a circle.

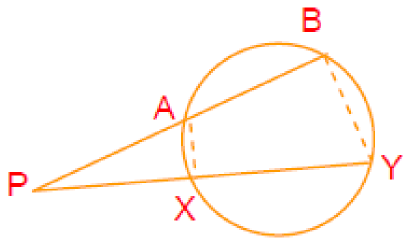

Figure 3.29

Theorem 3.13

The measure of an angle formed by a tangent and a secant drawn to a circle from a point outside the circle is equal to one-half the difference of the measures of the intercepted arcs.

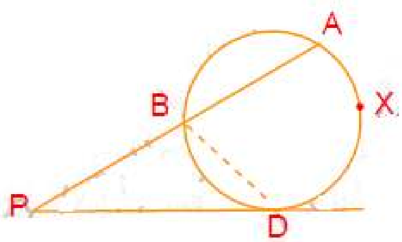

Figure 3.30

Theorem 3.14

If a secant and a tangent are drawn from a point outside a circle, then the square of the length of the tangent is equal to the product of the lengths of line segments given by

$(PA)^2 = (PB)(PC)$.

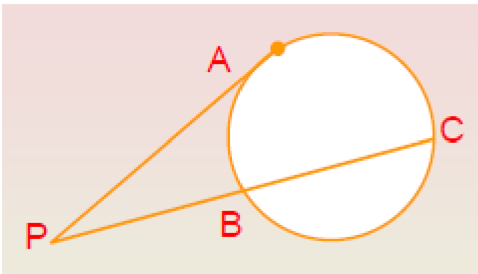

Figure 3.31

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3.4. Regular Polygons

A polygon whose vertices are on a circle is said to be inscribed in the circle. The circle is circumscribed about the polygon. In Figure 3.32a, the polygon *ABCDE* is inscribed in the circle or the circle is circumscribed about the polygon.

Definition 3.6

> A polygon is a simple closed curve, formed by the union of three or more line segments, no two of which in succession are collinear. The line segments are called the sides of the polygon and the end points of the sides are called the vertices.

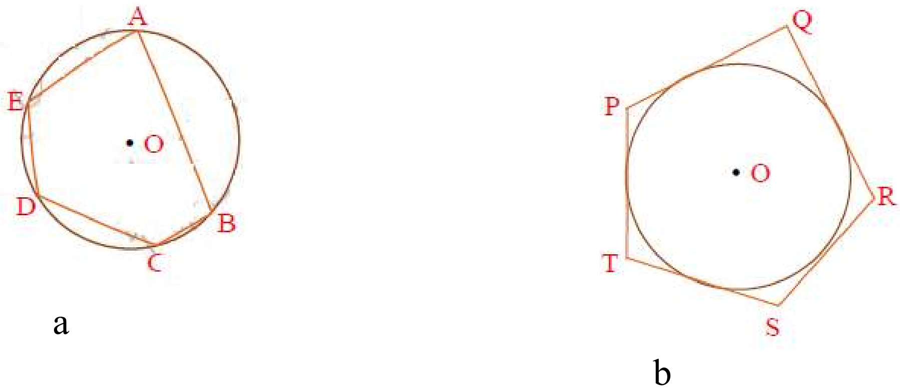

Figure 3.32

A polygon whose sides are tangent to a circ ircumscribed about the circle. In Figure 3.32b, the pentagon *PQRST* is circumscribed about the circle. The circle is inscribed in the pentagon.

Perimeter, Area of a Regular Polygon

Theorem 3.15

> Formulae for the length of side s, apothem a, perimeter P and area A of a regular polygon with n sides and radius r are
>
> **1** $s = 2r\sin\dfrac{180^o}{n}$ **3** $P = 2nr\sin\dfrac{180^o}{n}$
>
> **2** $a = r\cos\dfrac{180^o}{n}$ **4** $A = \dfrac{1}{2}aP = \dfrac{1}{2}nr^2\sin\dfrac{360^o}{n}$

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3.5. Congruency and Similarity

3.5.1. Congruency of Triangles

Triangles that have the same size and shape are called *congruent triangles*. That is, the six parts of the triangles (three sides and three angles) are correspondingly congruent. If two triangles, ΔABC and ΔDEF are congruent like those given below, then we denote this as

$$\Delta ABC \cong \Delta DEF.$$

The notation "□" means "*is congruent to*".

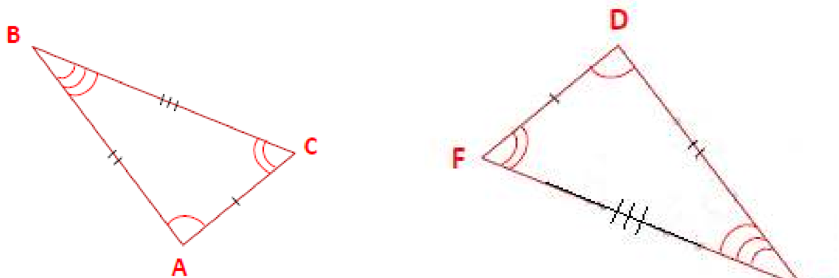
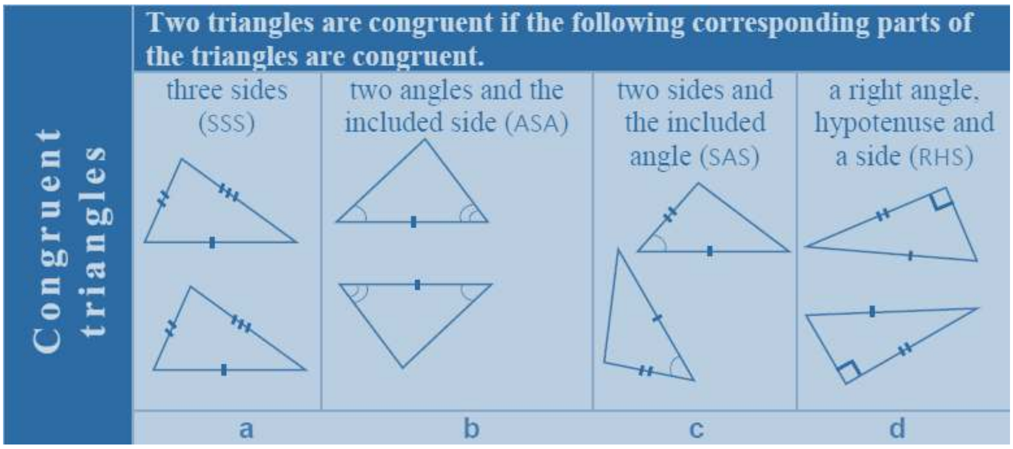


Figure 3.33

Parts of congruent triangles that "match" are called corresponding parts. For example, in the triangles above, □B corresponds to □E and $\overline{AC}$ corresponds to $\overline{DF}$. Two triangles are congruent when all of the corresponding parts are congruent. However, you do not need to know all of the six corresponding parts to conclude that the triangles are congruent. Each of the following Theorems states that three corresponding parts determine the congruence of two triangles.



3.5.2. Similarity of Triangles

Two polygons of the same number of sides are similar, if their corresponding angles are congruent and their corresponding sides have the same ratio.

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If quadrilateral ABCD is similar to quadrilateral WXYZ, we write ABCD ~ WXYZ. (The symbol ~ means "*is similar to*"). Corresponding angles of similar polygons are congruent. You can use a protractor to make sure the angles have the same measure.

$$\angle A \cong \angle W \qquad \angle B \cong \angle X$$
$$\angle C \cong \angle Y \qquad \angle D \cong \angle Z$$

A special relationship also exists between the corresponding sides of the polygons. Compare the ratios of lengths of the corresponding sides:

$$\frac{AB}{WX} = \frac{BC}{XY} = \frac{CD}{YZ} = \frac{DA}{ZW}$$

3.5.3. Theorems on Similarity of Triangles

Two triangles are said to be similar, if

a. their corresponding sides are proportional (have equal ratio), and

b. their corresponding angles are congruent.

That is, $\Delta ABC \sim \Delta DEF$ if and only if

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} \text{ and}$$
$$\angle A \cong \angle D,\ \angle B \cong \angle E,\ \angle C \cong \angle F$$

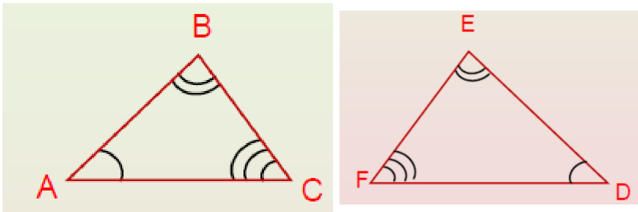

Figure 3.34

The following theorems on similarity of triangles will check whether or not two triangles are similar.

Theorem 3.16 (SSS similarity theorem)

If the three sides of one triangle are proportional to the three corresponding sides of another triangle, then the two triangles are similar.

Theorem 3.17 (SAS similarity theorem)

Two triangles are similar, if two pairs of corresponding sides of the two triangles are proportional and if the included angles between these sides are congruent.

Theorem 3.18 (AA similarity theorem)

If two angles of one triangle are congruent to two corresponding angles of another triangle, then the two triangles are similar.

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Theorem 3.19

If the ratio of the lengths of the corresponding sides of two similar triangles is k, then

i. the ratio of their perimeters is k

ii. the ratio of their areas is k^2.

Theorem 3.20

If the ratio of the lengths of any two corresponding sides of two similar polygons is k, then

i. the ratio of their perimeters is k.

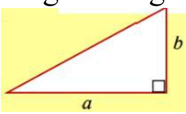
ii. the ratio of their areas is k^2.

3.6. Area of Triangle and Parallelogram

Areas of triangles

The area A of a right angle triangle with perpendicular sides of length a and b is given by

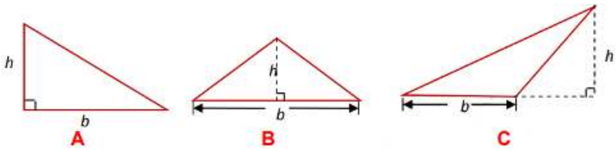
$$A = \frac{1}{2}ab$$



The area A of any triangle with base b and the corresponding height h is given by

$$A = \frac{1}{2}ah$$

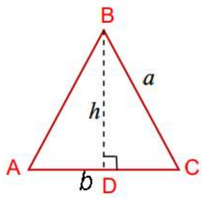
The base and corresponding height of a triangle may appear in any one of the following forms.



The area A of any triangle with sides a and b units long and angle C (∠C) included between these sides is

$$A = \frac{1}{2}ab\,sin\,(\angle C)$$

Proof: - Let Δ ABC be given such that BC = a and AC = b.



Case i: Let ∠C be an acute angle. Consider the height h drawn from B to AC. It meets AC at D (see Figure). Now,

$area\ of\ \Delta ABC = \frac{1}{2}ab$ (1) Since Δ BCD is right-angled with hypotenuse $sin(\angle C) = \frac{h}{a}\ implies\ h = a\sin(\angle C)$

Replacing h by a $sin\ (\angle C)$ in 1 we obtain Area of $\Delta ABC\ = \frac{1}{2}\ ab\ sin(\angle C)$ as required.

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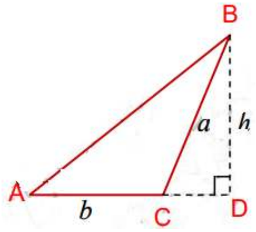
Case ii: Let ∠C be an obtuse angle. Draw the height from B to the extended base AC. It meets the extended base AC at D. Now,



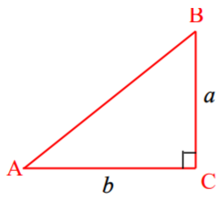
$$\text{Area of } \Delta ABC = Area\ of\ \Delta ABD - Area\ of\ \Delta BDC$$

$$= \frac{1}{2}\ AD.h - \frac{1}{2}\ CD.h = \frac{1}{2}\ h(AD - CD)$$

$$= \frac{\mathbf{1}}{\mathbf{2}} h.AC = \frac{\mathbf{1}}{\mathbf{2}}\ hb \qquad (2)$$

In the right-angled triangle BCD, $(180^{\circ} - C) = \frac{h}{a}$; Therefore, $h = a\ sin\ (180^{\circ} - C)$; Since $sin\ (180^{\circ} - C) = sin\ C$, we have $h = a\ sin\ (\angle\ C)$. Hence, replacing h by a $sin\ (\angle C)$ in 2 we obtain; $Area\ of\ \Delta ABC = \frac{1}{2}\ ab\ sin\ (\angle C)$ as required.

Case iii: Let ∠C be a right angle.



$A = \frac{1}{2}\ ab = \frac{1}{2} ab\ (sin\ 90\ )$ Since $(sin\ 90^{o} = 1)$

$= \frac{1}{2} ab\ sin\ (\ \angle\ C)$ (as required)

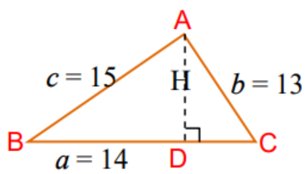
This completes the proof.

Now we state another formula called Heron's formula, which is often used to find the area of a triangle when its three sides are given.

Theorem 3.21 (Heron's formula)

The area A of a triangle with sides a, b and c units long and semi- perimeter $S = \frac{1}{2}(a + b + c)$ is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$

Example 1: Given ΔABC. If AB = 15 units, BC = 14 units and AC = 13 units, find

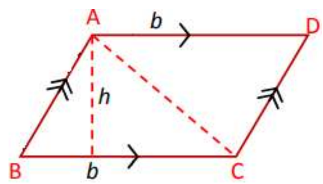


a. The area of ΔABC.

b. The length of the altitude from the vertex A.

c. The measure of the angle

Area of parallelograms



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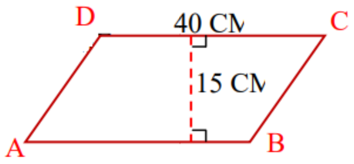
Theorem: The area A of a parallelogram with base b and perpendicular height h is

$$A = bh$$

Example 2: If one pair of opposite sides of a parallelogram have length 40 cm and the distance between them is 15cm. Find the area of parallelogram.



3.7. Surface Areas and Volumes of solid figures (prism, cylinder, cone and sphere)

Geometrical figures that have three dimensions (length, width and height) are called **solid figures**. For example, cubes, prisms, cylinders, cones and pyramids are three dimensional solid figures.

For the cylinder D, the region R is called its **lower base** or simply base and R' is its **upper base**. The line ℓ is called its **directrix** and the perpendicular distance between E_1 and E_2 is the altitude of D. If ℓ is perpendicular to E_1, then D is called a **right cylinder**, otherwise it is an **oblique cylinder**. If R is a circular region, then D is called a **circular cylinder**.

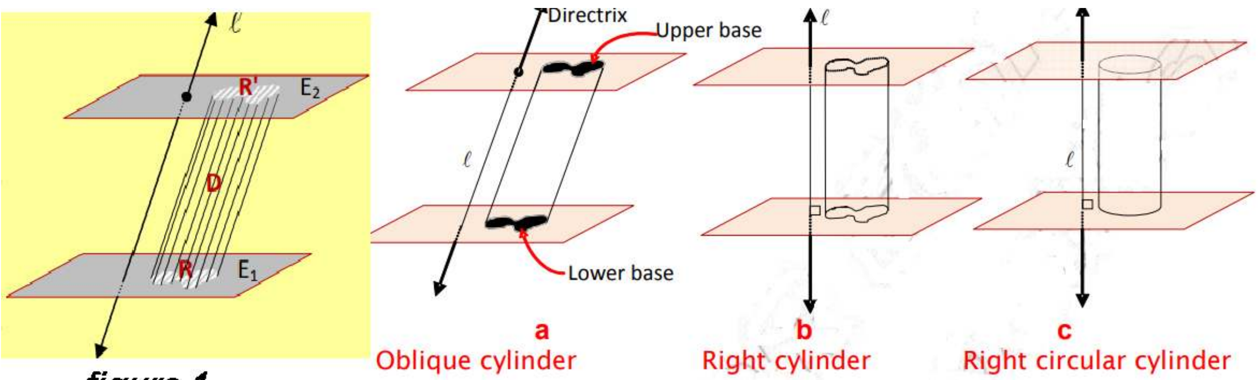
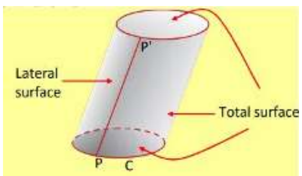


figure 1

Let C be the bounding curve of the base region R. The union of all the elements $\overline{PP'}$ for which P belongs to C is called the **lateral surface** of the cylinder. The **total surface** is the union of the lateral surface and the bases of the cylinder.



Definition 3.7

In the above figure 1:-

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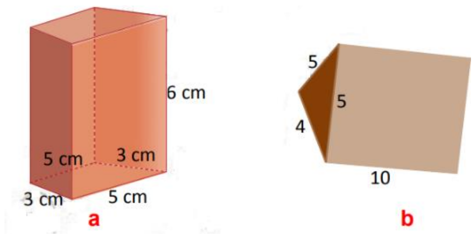
- If R is a polygonal region, then D is called a **prism**.
- If R is parallelogram region, then D is a **parallelepiped**.
- If R is a triangular region, then D is a **triangular prism**.
- If R is a square region, then D is a **square prism**.
- A **cube** is a square right prism whose altitude is equal to the length of the edge of the base.

If we denote the lateral surface area of a prism by A_L, the area of the base by A_B, altitude h and the total surface area by A_T, then

$A_L = Ph$; Where P is the perimeter of the base and h is the height of the prism.

$$A_T = 2A_B + A_L$$

Example 1: Find the lateral surface area of each of the following right prisms.

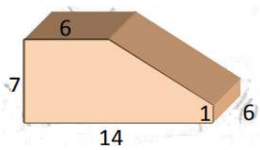


The measurement of space completely enclosed by the bounding surface of a solid is called its Volume.

Volume of any Prism

The volume (V) of any **prism** equals the product of its base area (A_B) and altitude (h). That is, $V = A_B h$.

Example 3: Find the total surface area and volume of the following prism.



Volume of a right circular cylinder

The volume (V) of a **circular cylinder** is equal to the product of the base area (A_B) and its altitude (h). That is,

$$V = A_B h$$

$$V = r^2 h,$$

where r is the radius of the base.

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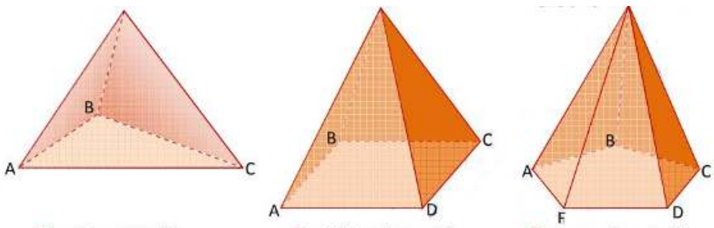
Example 4: Find the volume of the cylinder whose base circumference is 12 cm and whose lateral area is 288 cm^2.

Pyramids, Cones and Spheres

Definition 3.8

A **pyramid** is a solid figure formed when each vertex of a polygon is joined to the same point not in the plane of the polygon.



The **altitude** of a pyramid is the length of the perpendicular from the vertex to the plane containing the base. The **slant height** of a regular pyramid is the altitude of any of its lateral faces.

Definition 3.9

The solid figure formed by joining all points of a circle to a point not on the plane of the circle is called a **cone**.

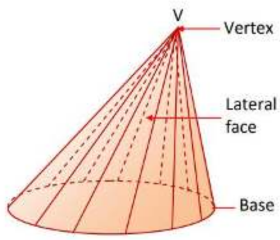


figure 2

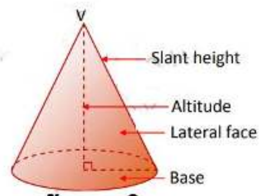


figure 3

The figure shown in Figure 2 represents a cone. Note that the curved surface is the **lateral surface** of the cone. A right circular cone (see Figure 3) is a cone with the foot of its altitude at the centre of the base. A line segment from the vertex of a cone to any point on the boundary of the base (circle) is called the **slant height**.

Surface Area

The lateral surface area of a regular pyramid is equal to half the product of its slant height and the perimeter of the base. That is,

$$A_L = \frac{1}{2} P\ell,$$

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Where A_L denotes the lateral surface area;

P denotes the perimeter of the base;

ℓ denotes the slant height.

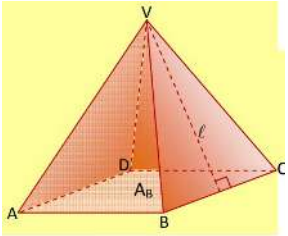


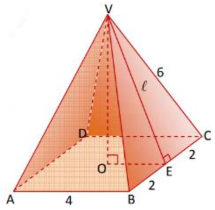
figure 4

The total surface area (A_T) of a pyramid is given by

$$A_T = A_B + A_L = A_B + \frac{1}{2}P\ell$$

Figure 4 where A_B is area of the base.

Example 1: A regular pyramid has a square base whose side is 4 cm long. The lateral edges are 6 cm each.

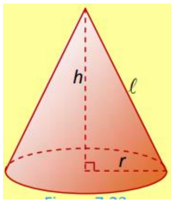


a) What is its slant height?
b) What is the lateral surface area?
c) What is the total surface area?

The lateral surface area of a right circular cone is equal to half the product of its slant height and the circumference of the base. That is,

$$A_L = \frac{1}{2}P\ell = \frac{1}{2}(2r)\ell = rl;$$

$$l = \sqrt{h^2 + r^2}$$

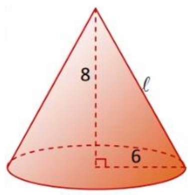


Where A_L denotes the lateral surface area, ℓ represents the slant height, r stands for the base radius, and h for the altitude. The total surface area (A_T) is equal to the sum of the area of the base and the lateral surface area. That is,

$$A_T = A_B + A_L = r\ell + r^2 = r(\ell + r)$$

Example 2: The altitude of a right circular cone is 8 cm. If the radius of the base is 6 cm, then find its:

a) slant height
b) lateral surface area
c) total surface area.



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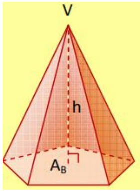
Volume of any Pyramid

The volume of any pyramid is equal to one third the product its base area and its altitude.

That is,$V = \frac{1}{3}A_B h$,



where V denotes the volume, A_B the area of the base and h the altitude.

Example 3: Find the volume of the pyramid given in the Example 1 above.

Volume of a Circular Cone

The volume of a circular cone is equal to one-third of the product of its base area and its altitude. That is, $= \frac{1}{3}A_B h = \frac{1}{3}r^2 h$,

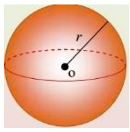
where V denotes the volume, r the radius of the base and h the altitude.

Example 4: Find the volume of the right circular cone given in Example 2 above.

Surface area and volume of a sphere

Definition 3.10

A sphere is a closed surface, all points of which are equidistant from a point called the center.

The surface area (A) and the volume (V) of a sphere of radius r are  given by

$$A = 4r^2 \text{ and } V = \frac{4}{3}r^3$$

Example 6: Find the surface area and volume of a spherical gas balloon with a diameter of 10 m

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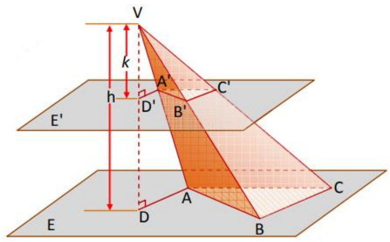
3.8. Frustums of Pyramids and Cones

Definition 3.11

If a pyramid or a cone is cut by a plane parallel to the base, the intersection of the plane and the pyramid (or the cone) is called **a horizontal cross-section** of the pyramid (or the cone).

Let us now examine the relationship between the base and the cross-section. Let ΔABC be the base of the pyramid lying in the plane E. Let h be the altitude of the pyramid and let $\Delta A'\ B'\ C'$ be the cross-section at a distance k units from the vertex.

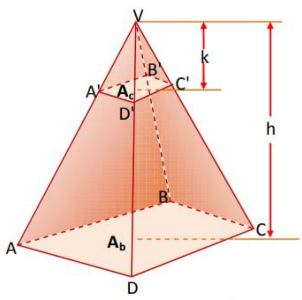
1. Let D and D' be the points at which the perpendicular from V to E meet E and $E\ '$, respectively. We have, $\Delta VA'D' \sim \Delta VAD$. This follows from the fact that if a plane intersects each of two parallel planes, it intersects them in two parallel lines, and an application of the AA similarity theorem.
   Hence, $\frac{VA'}{VA} = \frac{VD'}{VD} = \frac{k}{h}$
2. Similarly, $\Delta\ VD'B' \sim \Delta VDB$ and hence
   $$\frac{VB'}{VB} = \frac{VD'}{VD} = \frac{k}{h}$$
   Then, from 1 and 2 and the SAS similarity theorem, we get,



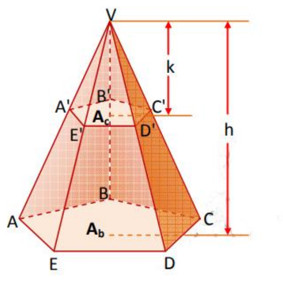
3. $\Delta\ VA'B' \sim \Delta\ VAB$. Therefore, $\frac{A'B'}{AB} = \frac{VA'}{VA} = \frac{k}{h}$
   By an argument similar to that leading to (3), we have
4. $\frac{B'C'}{BC} = \frac{k}{h}$ and $\frac{A'C'}{AC} = \frac{k}{h}$ Hence, by the SSS similarity theorem $\Delta ABC \sim \Delta A'B'C'$

Theorem 3.22

In any pyramid, the ratio of the area of a cross-section to the area of the base is $\frac{k^2}{h^2}$ where h is the altitude of the pyramid and k is the distance from the vertex to the plane of the cross-section.



$$\frac{A_c}{A_b} = \frac{\text{area}(A'B'C'D')}{\text{area}(ABCD)} = \frac{k^2}{h^2}$$



$$\frac{\text{area}(A'B'C'D'E')}{\text{area}(ABCD)} = \frac{k^2}{h^2}$$

However, when a pyramid is cut by a plane parallel to the base, the part of the pyramid between the vertex and the horizontal cross-section is again a pyramid whereas the other part is not a pyramid.

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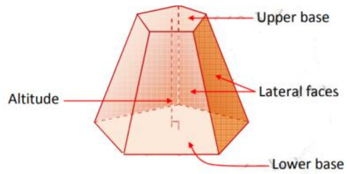
Example 1: The area of the base of a pyramid is 90 cm^2 . The altitude of the pyramid is 12 cm. What is the area of a horizontal cross-section 4 cm from the vertex?

Frustum of a Pyramid

Definition 3.12

A f**rustum of a pyramid** is a part of the pyramid included between the base and a plane parallel to the base.The base of the pyramid and the cross-section made by the plane parallel to it are called **the bases of the frustum**. The other faces are called **lateral faces**. The total surface of a frustum is the sum of the lateral surface and the bases. The altitude of a frustum of a pyramid is the perpendicular distance between the bases.



Example 2: The lower base of the frustum of a regular pyramid is a square 4 cm long, the upper base is 3 cm long. If the slant height is 6 cm, find its lateral surface area

Note:

i. The lateral faces of a frustum of a pyramid are trapeziums.
ii. The lateral faces of a frustum of a regular pyramid are congruent isosceles trapeziums.
iii. The slant height of a frustum of a regular pyramid is the altitude of any one of the lateral faces.
iv. The lateral surface area of a frustum of a pyramid is the sum of the areas of the lateral faces.

Example 3: The lower base of the frustum of a regular pyramid is a square of side s units long. The upper base is s' units long. If the slant height of the frustum is ℓ, then find the lateral surface area.

Frustum of a Cone

Definition 3.12

A **frustum of a cone** is a part of the cone included between the base and a horizontal cross-section made by a plane parallel to the base.

For a frustum of a cone, the bases are the base of the cone and the cross-section parallel to the base. The **lateral surface** is the curved surface that makes up the frustum. The altitude is the perpendicular distance between the bases.

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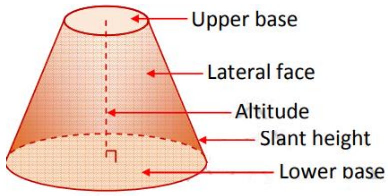
The **slant height** of a frustum of a right circular cone is that part of the slant height of the cone which is included between the bases.

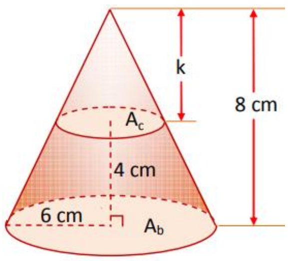


Theorem: The lateral surface area (A_L) of a frustum of a regular pyramid is equal to half the product of the slant height (ℓ) and the sum of the perimeter (P) of the lower base and the perimeter (P') of the upper base. That is,

$$A_L = \frac{1}{2}\,\ell\,(P + P')$$

The lateral surface (curved surface) of a frustum of a circular cone is a trapezium whose parallel sides (bases) have lengths equal to the circumference of the bases of the frustum and whose height is equal to the height of the frustum.

Example 4: A frustum of height 4 cm is formed from a right circular cone of height 8 cm and base radius 6 cm as shown in Figure. Calculate the lateral surface area of the frustum.

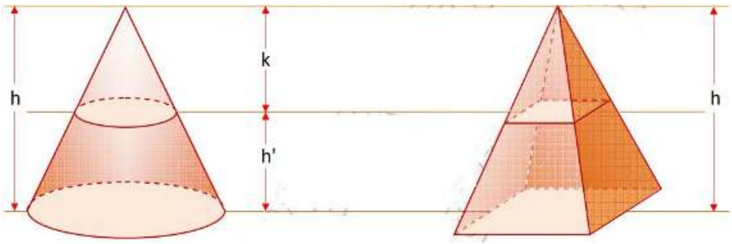


Theorem 3.23

For a frustum of a right circular cone with altitude h and slant height ℓ, if the circumferences of the bases are c and c', then the lateral surface area of the frustum is given by

$$A_L = \frac{1}{2}\,\ell\,(c + c') = \frac{1}{2} l(2r + 2\,r') = \ell\,(r + r')$$

Note that the upper and lower bases of the frustum of a pyramid are similar polygons and that of a cone are similar circles.



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Let h = the height (altitude)of the complete cone or pyramid.

k = the height of the smaller cone or pyramid.

A = the base area of the bigger cone or pyramid (lower base of the frustum)

A' = the base area of the completing cone or pyramid (upper base of the frustum)

h' = h – k = the height of the frustum of the cone or pyramid.

V = the volume of the bigger cone or pyramid.

V ' = the volume of the smaller cone or pyramid (upper part).

V_f = the volume of the frustum

$V = \frac{1}{3} Ah$ and $' = \frac{1}{3} A' k$, consequently the volume (V_f) of the frustum of the pyramid is

$V_f = V - V' = \frac{1}{3} Ah - \frac{1}{3} A'k = \frac{1}{3} (Ah - A'k)$

Using this notion, we shall give the formula for finding the volume of a frustum of a cone or pyramid as follows

$$V_f = \frac{h'}{3}(A + A' + \sqrt{AA'})$$

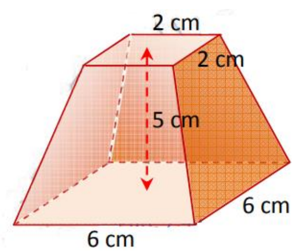
Where A is the lower base area, A' the upper base area and h' is the height of a frustum of a cone or pyramid.

From this, we can give the formula for finding the volume of a frustum of a cone in terms of r and r' as follows:

$$V_f = \frac{h'}{3}(r^2 + (r')^2 + rr')$$

where r is the radius of the bigger (the lower base of the frustum) cone and r' is the radius of the smaller cone (upper base of the frustum).

Example 7: A frustum of a regular square pyramid has height 5 cm. The upper base is of side 2 cm and the lower base is of side 6 cm. Find the volume of the frustum.



Unit summary

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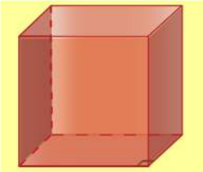
Prism

$$A_L = Ph$$

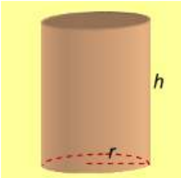


$$A_T = 2A_b + A_L$$

$$V = A_b h$$

Right Circular Cylinder

$$A_L = 2rh$$

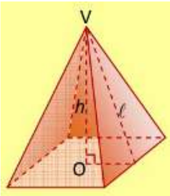


$$A_T = 2r^2 + 2rh$$
$$= 2r(r+h)$$

$$V = r^2 h$$

Regular pyramid

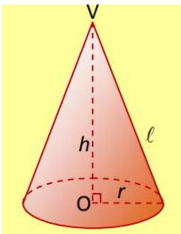
$$A_L = \frac{1}{2}Pl$$



$$A_T = 2A_b + \frac{1}{2}Pl$$

$$V = \frac{1}{3}A_b h$$

Right circular cone



$$A_L = rl$$

$$A_T = r^2 + rl = r(r+l)$$

$$V = \frac{1}{3} r^2 h$$

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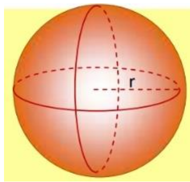
Sphere

$A = 4r^2$

$V = \frac{4}{3}r^3$

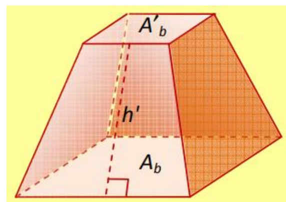


Frustum of a pyramid

$A_L \; = \; \frac{1}{2}l(P + P')$

$A_T \; = \; \frac{1}{2}l(P + P') + A_b + A'_b$

$V \; = \frac{1}{3}\; h'(A_b + A'_b + \sqrt{A_b A'_b})$



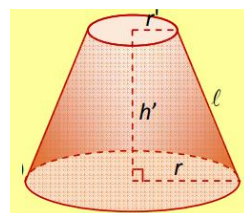
Frustum of a cone

$A_L \; = \; \frac{1}{2}l(2r + 2r') = l(r + r')$

$A_T \; = \; \frac{1}{2}l(2r + 2r') + r^2 + (r')^2$

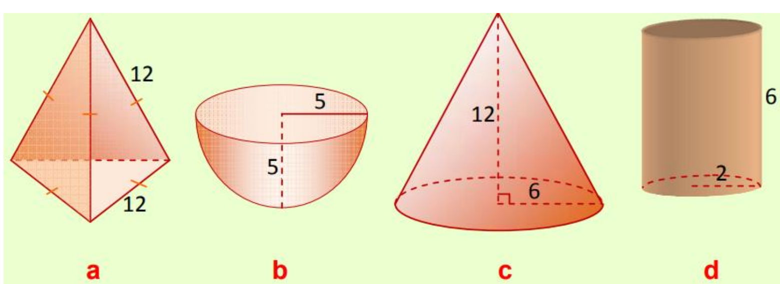
$= l(r + r') + (r^2 + (r')^2)$

$V \; = \frac{1}{3}\; h'(r^2 + (r')^2 + rr')$



<u>Review Exercises</u>

1. Find the lateral surface area and volume of each of the following figures.



2. A lateral edge of a right prism is 6 cm and the perimeter of its base is 36 cm. Find the area of its lateral surface.

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3. The height of a circular cylinder is equal to the radius of its base. Find its total surface area and its volume, giving your answer in terms of its radius r.
4. What is the volume of a stone in an Egyptian pyramid with a square base of side 100 m and a slant height of $50\sqrt{2}$ m for each of the triangular faces?
5. Find the total surface area of a regular hexagonal pyramid, given that an edge of the base is 8 cm and the altitude is 12 cm.
6. Find the area of the lateral surface of a right circular cone whose altitude is 8 cm and base radius 6 cm.
7. Find the total surface area of a right circular cone whose altitude is h and base radius is r. (Give the answer in terms of r and h)
8. When a lump of stone is submerged in a rectangular water tank whose base is 25 cm by 50 cm, the level of the water rises by 1 cm. What is the volume of the stone?

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Chapter 4: Coordinate Geometry

4.1. Distance between Two Points

Definition: The distance d between any two points $P\ (x_1, y_1)$ and Q (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Examples:

1. Find the distance between the given points.

a. $A\ (1, -5)\ and\ B\ (7, 3)$

b. $E(\ 2, 1) and\ F\ (\ -6\ , 3\ )$

c. $P\ (\sqrt{2}, \sqrt{3})$ and Q $(2\sqrt{2}, 2\sqrt{3}))$

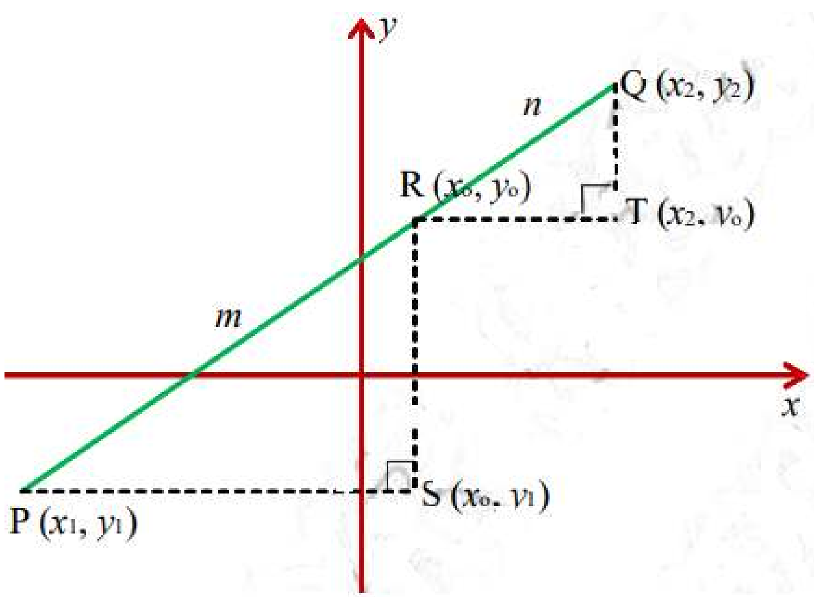
d. $G\ (a, -b)$ and $H\ (-a, b)$

2. Let $A\ (3, -7)$ and $B\ (-1, 4)$ be two adjacent vertices of a square. Calculate the area of the square.

4. $P\ (3, 5)$ and $Q\ (1, -3)$ are two opposite vertices of a square. Find its area.

DIVISION OF A LINE SEGMENT

Given a line segment PQ with end point coordinates $P\ (x_1, y_1)$ and Q (x_2, y_2), let us find the coordinates of the point $R\ (x_0, y_0)$ dividing the line segment PQ internally in the ratio m: n, i.e., $\frac{PR}{RQ} = \frac{m}{n}$, where m and n are given positive real numbers. See the fig below.



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Then, The point $R(x_0, y_0)$ dividing the line segment PQ internally in the ratio m: n is given by $R(x_0, y_0) = (\frac{nx_1+mx_2}{n+m}, \frac{ny_1+my_2}{n+m})$.

Examples

1. Find the coordinates of the point R that divides the line segment with end- points $A(6, 2)$ and $B(1, -4)$ in the ratio 2:3.

2. A line segment has end-points (–2, –3) and (7, 12) and it is divided into three equal parts. Find the coordinates of the points that trisect the segment.

The mid-point formula

A point that divides a line segment into two equal parts is the mid-point of the segment.

Definition: The mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) is given by $M(x_0, y_0) = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$.

Examples

1.Find the coordinates of the mid-point of the line segments joining the points:

a. $A\ (1, 4)$ and $B\ (-2, 2)$ b. $E(1 + \sqrt{2}, 2)$ and $F(2 - \sqrt{2}, 8)$

2. The mid-point of a line segment is $M\ (-3, 2)$. One end-point of the segment is $P\ (1, -3)$. Find the coordinate of the other end-point.

4.2. Equation of A Line

Gradient (slope) of a Line

Definition: If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are points on a line with $x_1 \neq x_2$, then the gradient(slope) of the line, denoted by m, is given by $m = \frac{y_2-y_1}{x_2-x_1}$

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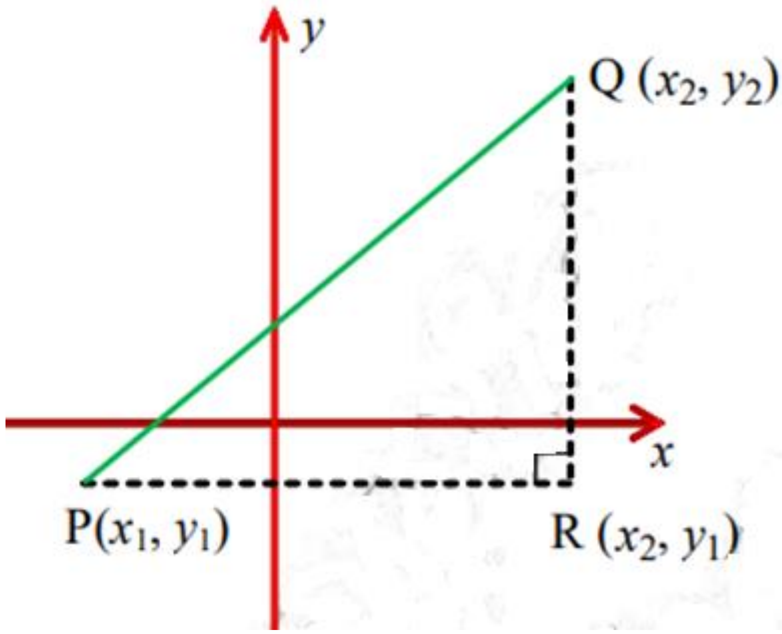


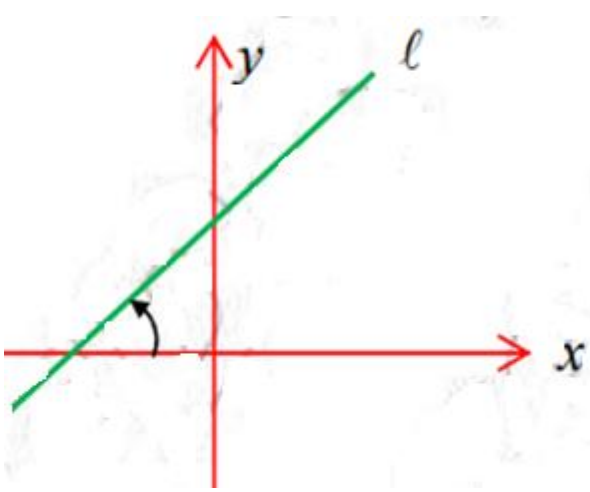
Examples:

Find the gradients/slopes of the lines passing through the following points:

a. $A\ (4\ ,3)$ and $B\ (8, 11)$ b. $C\ (\sqrt{2}\ , -9)$ and $D\ (2\sqrt{2}\ , -7)$

Slope of a Line in Terms of Angle of Inclination

The angle measured from the positive x-axis to a line, in anticlockwise direction, is called the inclination of the line or the angle of inclination of the line. This angle is always less than 180^0 .



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Examples:

1.Find the slope of a line, if its inclination is:

a. 60^0 b. 135^0

2. Find the angle of inclination of the line containing the points

a. A(3, –3) and B(–1, 1) b. C(0, 5) and D(4, 5).

Next we will see the different forms of equations of a line one by one as follow.

The point-slope form of equation of a line

Let $A(x_1, y_1)$ be a given point and $B(x, y)$ be any other point on the line with slope m. Then the **point-slope form** of the equation of a line is given by

$y - y_1 = m\,(x - x_1)$ or $y = y_1 + m\,(x - x_1)$.

Example

Find the equation of the straight line with slope $\frac{-3}{2}$ and which passes through the point $(-3, 2)$.

The slope-intercept form of equation of a line

If the line passes through the point (0, b) with slope , then the **slope-intercept form** of the equation of a line is given by $y = mx + b$.

Example

Find the equation of the line with slope $\frac{-2}{3}$ and y-intercept 3.

The two-point form of equation of a line

Consider a straight line which passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$. If $R\,(x, y)$ is any point on the line other than $P\ or\ Q$, then the **two-point form** of the equation of a line is given by $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$.

Example

Find the equation of the line passing through the points P (–1, 5) and Q (3, 13).

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The general equation of a line

The general form of equation of a line is given by $Ax + By + C = 0$, where $A \neq 0$ and $B \neq 0$.

Example. Find the slope and y-intercept of the line whose general equation is $3x - 6y - 4 = 0$.

PARALLEL AND PERPENDICULAR LINES

Theorem: If two non-vertical lines ℓ_1 and ℓ_2 are parallel to each other, then they have the same slope.

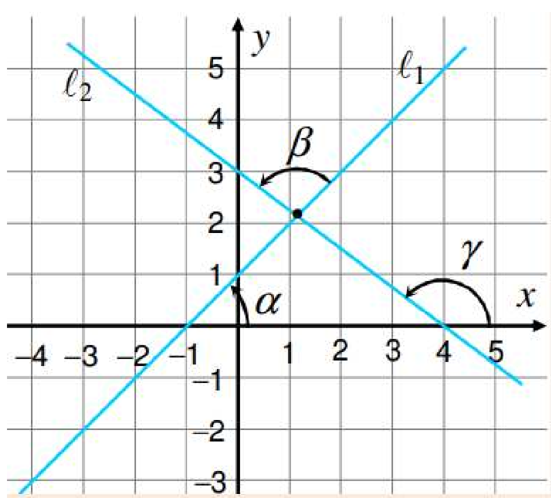
Theorem: Two non-vertical lines having slopes m_1 and m_2 are perpendicular, if and only if $m_1 \cdot m_2 = -1$.

Example: Find the equation of the line passing through the point P(2, 5) and:

a. parallel to the line passing through the points A (3, 1) and B (−1, 3)
b. parallel to the line ℓ: x + y = 2
c. perpendicular to the line joining the points A (−1, 2) and B (4, −2)
d. perpendicular to the line ℓ: y = x + 1.

Angle Between Two Lines

Definition: The angle between two intersecting lines ℓ1 and ℓ2 is defined to be the angle β measured counter-clockwise from ℓ1 to ℓ2.



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Example: Find the tangent of the angle between the given lines.

a. $\ell 1: y = -3x + 2$; $\ell 2: y = -x$ b. $\ell 1: 3x - y - 2 = 0$; $\ell 2: 4x - y - 6 = 0$

Distance between a Point and a Line

The distance d from the point $P(h,k)$ to the line $l: Ax + By + C = 0$ is $d = \frac{|Ah+Bk+C|}{\sqrt{A^2+B^2}}$

Examples:

1. Find the distance from each point to the given line.

   a. $P(-3,2)$; $5x + 4y - 3 = 0$ b. $P(4,0)$; $2x - 3y - 2 = 0$

2. Find the distance between the pairs of parallel lines whose equations are given below:

   a. $2x - 3y + 2 = 0$ and $2x - 3y + 6 = 0$ b. $4y = 3x - 1$ and $8y = 6x - 7$

4.3. Conic-Sections

Definition: A **locus** is a system of points, lines or curves on a plane which satisfy one or more given conditions.

Examples:

The following are examples of loci (plural of locus).

1. The set $\{(x, y) \in \mathbb{R}^2 : y = 3x + 5\}$ is a line in the coordinate plane.

2. The set of all points on the x-axis which are at a distance of 3 units from the origin is $\{(-3,0),(3,0)\}$.

There are four basic conic sections and we will investigate one by one as follows.

Circles

Definition: A **circle** is the locus of a point that moves in a plane with a fixed distance from a fixed point. The **fixed distance** is called the **radius** of the circle and the **fixed point** is called the **center** of the circle.

From the above definition, for any point $P(x,y)$ on a circle with centre $C(h,k)$ and radius r, $PC = r$ and by the distance formula we have,

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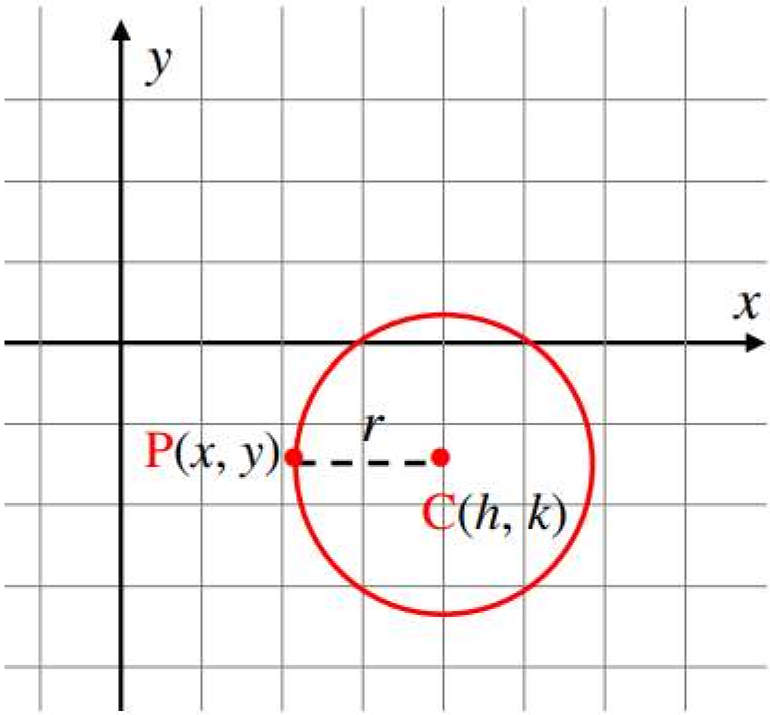
$$\sqrt{(x-h)^2+(y-k)^2}=r$$

Thus, the **standard form of the equation of a circle**, with center $C(h,k)$ and radius r is given by $(x-h)^2+(y-k)^2=r^2$



If the centre of a circle is at the origin (i.e. h = 0, k = 0), then the above equation becomes,

$x^2+y^2=r^2$.

Examples:

1.Write down the standard form of the equation of a circle with the given centre and radius.

a. $C\,(0,0),\ \ r\ =\ 8$ b . $C\,(2,-7),\ \ r\ =\ 9$

2. Write the standard form of the equation of the circle with centre at $C\,(2,3)$ and that passes through the point $P\,(7,-3)$.

3. Give the centre and radius of the circle,

a. $(x-5)^2+(y+7)^2=64$ b . $x^2+y^2+6x-8y=0$

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NOTES:

1. If the perpendicular distance from the centre of a circle to a line is less than the radius of the circle, then the line intersects the circle at two points. Such a line is called a **secant line** to the circle.

2. If the perpendicular distance from the centre of a circle to a line is equal to the radius of the circle, then the line intersects the circle at only one point. Such a line is called a **tangent line** to the circle and the point of intersection is called the **point of tangency**.

3. If the perpendicular distance from the centre of a circle to a line is greater than the radius of the circle, then the line does not intersect the circle.

Examples:

1. Find the intersection of the circle with equation $(x-1)^2+(y+1)^2 = 25$ with each of the following lines.

   a. $4x - 3y - 7 = 0$ b. $x = 4$

2. For the circle $(x+1)^2+(y-1)^2 = 13$, show that $y = \frac{3}{2}x - 4$ is a tangent line.

3. Give the equation of the line tangent to the circle with equation $(x+1)^2+(y-1)^2 = 13$ at the point $P(-3, 4)$.

Parabolas

Definition: A **parabola** is the locus of points on a plane that have the same distance from a given point and a given line. The point is called the **focus** and the line is called the **directrix** of the parabola.

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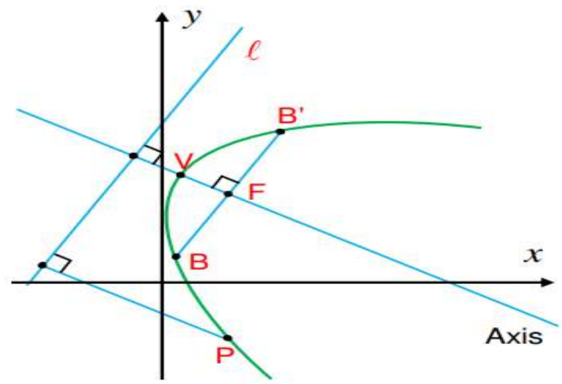
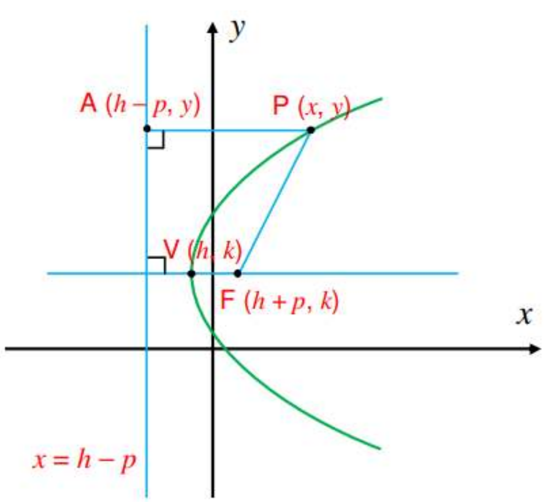


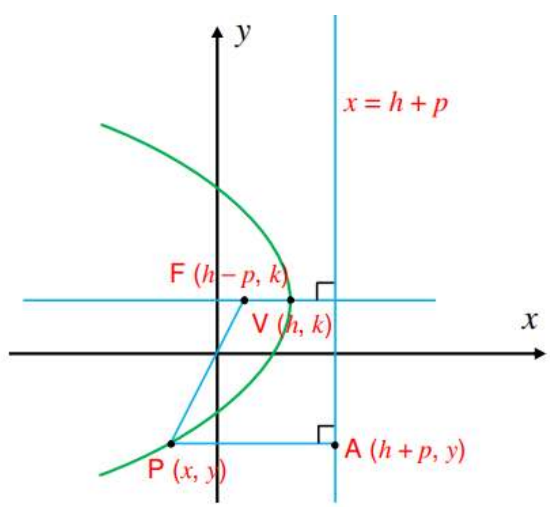
Here are some terminologies for parabolas.

- F is the **focus** of the parabola.
- The line ℓ is the **directrix** of the parabola.
- The line which passes through the focus F and is perpendicular to the directrix ℓ is called the **axis** of the parabola.
- The point V on the parabola which lies on the axis of the parabola is called the **vertex** of the parabola.
- The chord BB' through the focus and perpendicular to the axis is called **the latus rectum** of the parabola.
- The distance p = VF from the vertex to the focus is called the **focal length** of the parabola.

Equation of a parabola whose axis is parallel to the x-axis



a



b

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Let $V(h, k)$ be the vertex of the parabola. The axis of the parabola is the line $y = k$. If the focus of the parabola is to the right of the vertex of the parabola, then the focus is $F(h + p, k)$ and the equation of the directrix is $x = h - p$. Let $P(x, y)$ be a point on the parabola. Then the distance from P to F is equal to the distance from P to the directrix. That is, PF= PA , where $A(h - p, y)$.

This implies $\sqrt{(x-(h+p))^2+(y-k)^2} = \sqrt{(x-(h-p))^2+(y-y)^2}$

Thus, the **standard form of equation of a parabola** is $(\boldsymbol{y}-\boldsymbol{k})^2 = \boldsymbol{4p}(\boldsymbol{x}-\boldsymbol{h})$ with vertex $V(h, k)$, focal length p; the focus F is to the right of the vertex and its axis is parallel to the x-axis.

NOTE:

The equation $(\boldsymbol{y}-\boldsymbol{k})^2 = \pm\boldsymbol{4p}(\boldsymbol{x}-\boldsymbol{h})$ represents a parabola with:

- Vertex $V(h, k)$
- Focus $(h \pm p, k)$
- directrix : $x = h \pm p$
- axis of symmetry $y = k$
- If the sign in front of p is positive, then the parabola opens to the right.
- If the sign in front of p is negative, then the parabola opens to the left.

Examples:

1.Find the equation of the directrix, the focus of the parabola, the length of the latus rectum and draw the graph of the parabola $y^2 = 4x$

2. Find the equation of the directrix and the focus of each parabola and draw the graph of each of the following parabolas.

a. $4y^2 = -12x$ b. $(y-2)^2 = 6(x-1)$ c. $y^2 - 6y + 8x + 25 = 0$

3. Find the equation of the parabola with vertex $V(-1, 4)$ and focus $F(5, 4)$.

Equation of a parabola whose axis is parallel to the y-axis

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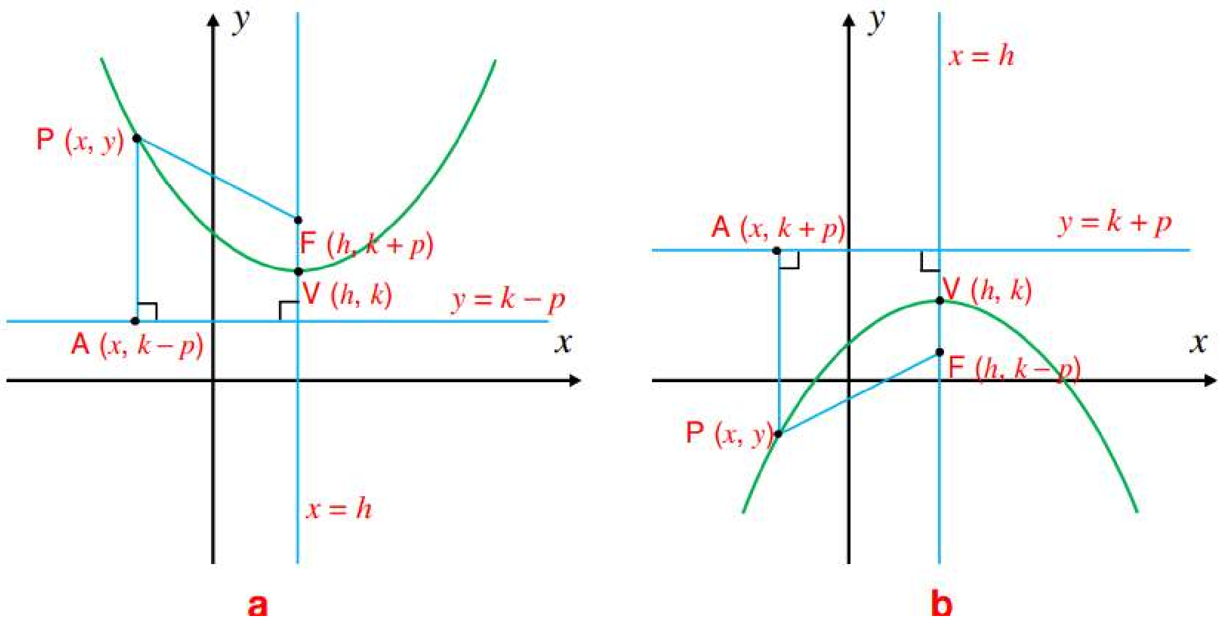

Let $V(h, k\ )$ be the vertex of the parabola. The axis of the parabola is the line $x\ = h$. If the focus of the parabola is above the vertex of the parabola, then the focus is $F(h\ , k + p)$ and the equation of the directrix is $y = k - p$. Let $P(x\ , y)$ be a point on the parabola. Then the distance from P to F is equal to the distance from P to the directrix. That is, PF= PA , where $A(x, k -\ p)$.

This implies $\sqrt{(x-h)^2 + (y-(k+p))^2} = \sqrt{(x-x)^2 + (y-(k-p))^2}$

Thus, the **standard form of equation of a parabola** is $\boldsymbol{(x-h)^2 = 4p(y-k)}$ with vertex $V(h\ , k\ )$ and whose axis is parallel to the *y*-axis.

NOTE:

The equation $\boldsymbol{(x-h)^2 = \pm 4p(y-k)}$ represents a parabola with:

- ❖ Vertex $V\ (h, k)$
- ❖ Focus $(h\ , k \pm p\ )$
- ❖ directrix : $y\ = k \mp\ p$
- ❖ axis of symmetry $x\ = h$
- ❖ If the sign in front of p is positive, then the parabola opens upward.
- ❖ If the sign in front of p is negative, then the parabola opens downward.

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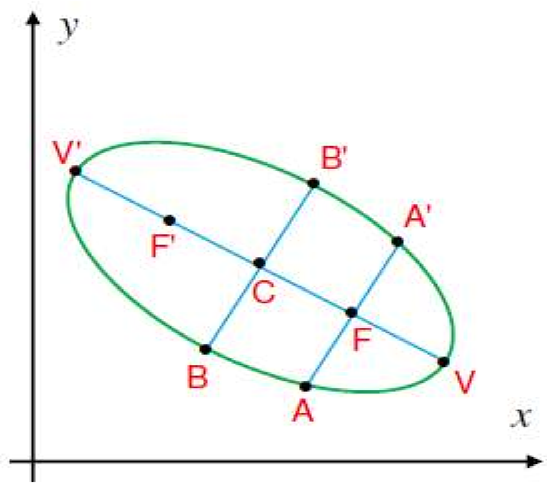
Examples:

Find the vertex, focus and directrix of the following parabolas; sketch the graphs of the parabolas in *b* and *c*.

a. $x^2 = 16y$ b. $-2x^2 = 8y$ c. $(x-2)^2 = 8(y+1)$ d. $x^2 + 12y - 2x - 11 = 0$

Ellipses

Definition: An **ellipse** is the locus of all points in the plane such that the sum of the distances from two given fixed points in the plane, called the **foci**, is constant.



Here are some terminologies for ellipses.

- F and F' are **foci**.
- V, V', B and B' are called **vertices** of the ellipse.
- V'V is called the **major axis** and BB ' is called the **minor axis**.
- C, which is the intersection point of the major and minor axes is called the **centre** of the ellipse.
- CV and CV' are called **semi-major axes** and CB and CB' are called **semiminor axes.**
- Chord AA' which is perpendicular to the major axis at F is called the **latus rectum** of the ellipse.
- The distance from the centre to a focus is denoted by c.
- The length of the **semi-major** axis is denoted by *a* and the length of the **semi-minor axis** is denoted by *b*.

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- The eccentricity of an ellipse, usually denoted by e, is the ratio of the distance between the two foci to the length of the major axis, that is,

$e = \frac{\text{distance between the two foci}}{\text{lengh of the major axis}} = \frac{c}{a}$ which is a number between 0 and 1.

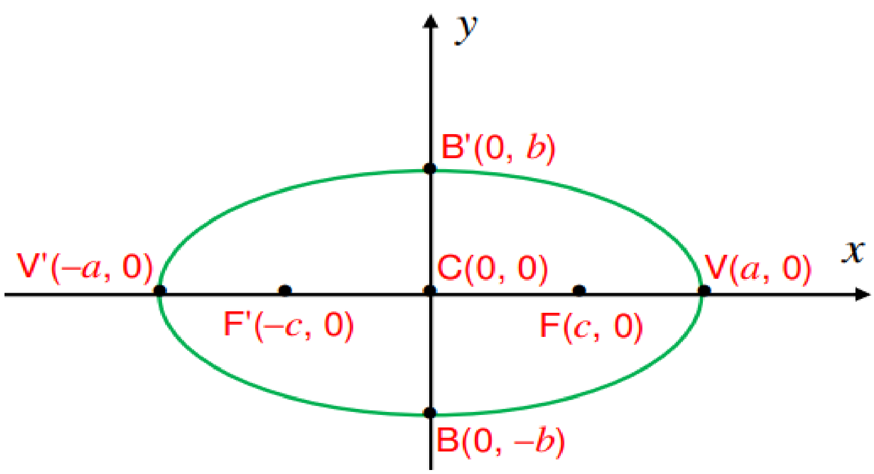
Note that V'F' = VF and VF + VF' = VV' = 2a, according to the definition. If P is any point on the ellipse, we have, PF + PF' = 2a. Since B is on the ellipse, we also have that BF + BF' = 2a. But BF = BF'. This implies BF = a. By using Pythagoras Theorem for right angled triangle ΔBCF , we get,

$$CB^2 + CF^2 = BF^2$$

But CB = b, CF = c and BF = a. Therefore a, b and c have the relation, $c^2 + b^2 = a^2$

Equation of an ellipse whose centre is at the origin

Case I: When the major axis of the ellipse is parallel to the x-axis as shown below.



From the discussion so far, we have, PF' + PF = 2a.

This implies $\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$

After simplifying this equation, the **standard form of an equation of an ellipse** whose major axis is horizontal and centre is at (0, 0) is given by $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$

Equation of an ellipse whose centre is C (h, k) different from the origin

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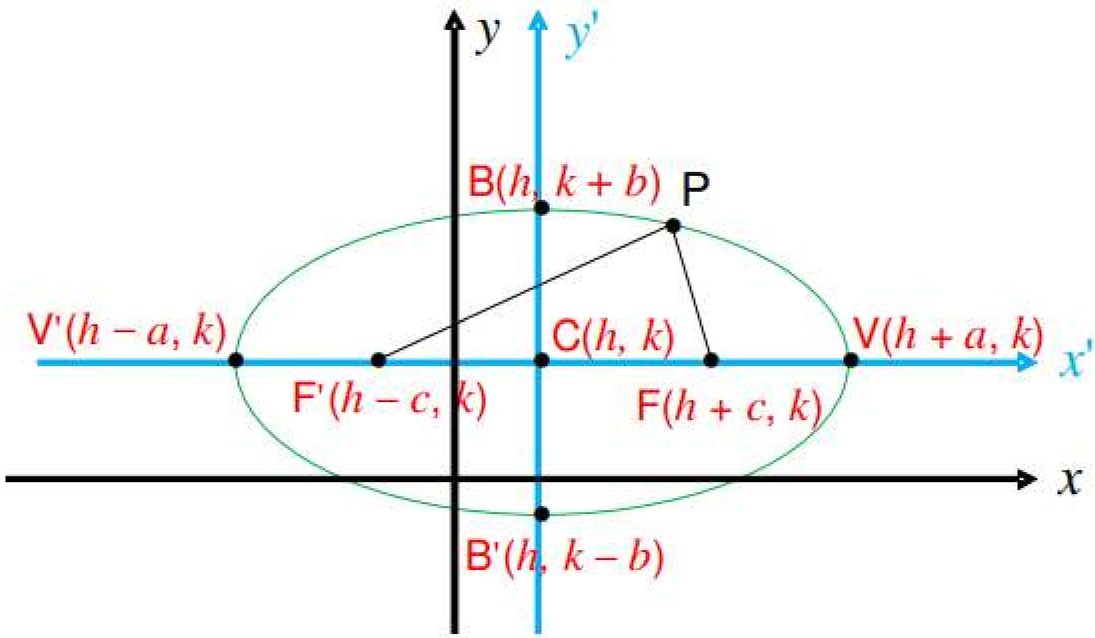
Consider the following figure.



Then, $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=\mathbf{1}$ is the standard equation of an ellipse with centre at C(h, k) and major axis parallel to the x-axis.

Case II: When the major axis is vertical, the standard equation of the ellipse is given by:

$\frac{y^2}{a^2}+\frac{x^2}{b^2}=\mathbf{1}$ when C (0 , 0) and $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=\mathbf{1}$ when C(h, k).

Examples:

1. Find the centre, the foci, the vertices and the eccentricity of each ellipse whose equation is given. Also sketch the graph of each ellipse.

a. $\frac{(x-3)^2}{25}+\frac{(y-4)^2}{16}=1$ b. $\frac{(y+2)^2}{25}+\frac{(x-1)^2}{4}=1$ c. $\frac{(y-2)^2}{25}+\frac{(x-3)^2}{5}=1$ d. $\frac{(x-2)^2}{9}+\frac{(y-1)^2}{1}=1$

2. Find the equation of the ellipse with

a. centre at (1, 4) and vertices at (10, 4) and (1, 2)

b. foci at (–1, 0), (1, 0) and the length of the major axis 6 units.

c. vertex at (6, 0), focus at (–1, 0) and centre at (0, 0).

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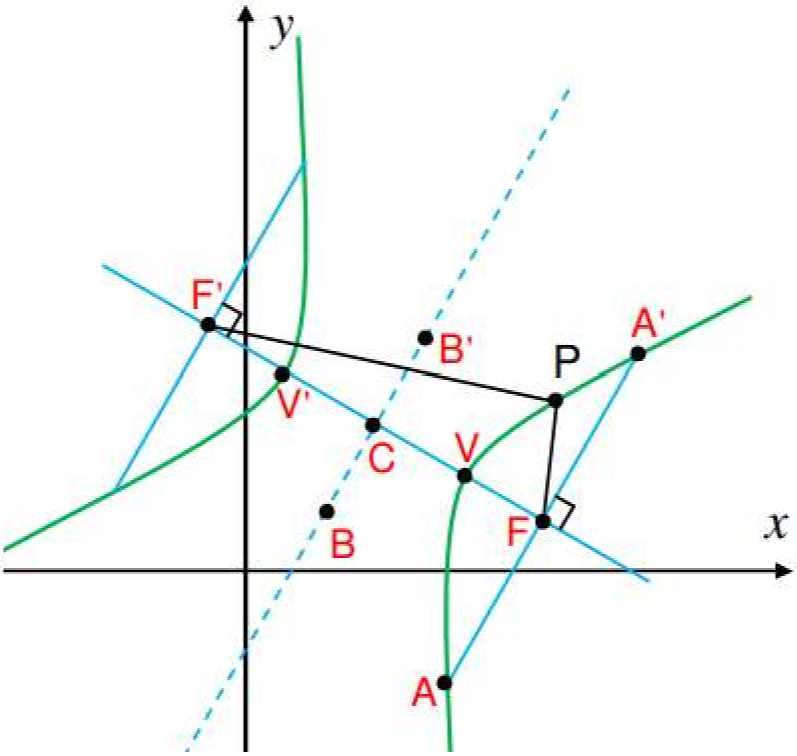
Hyperbolas

Definition: A **hyperbola** is defined as the locus of points in the plane such that the difference between the distances from two fixed points is a constant. The fixed points are called **foci**. The point midway between the foci is called the **centre** of the hyperbola.



Here are some terminologies for hyperbolas.

- F and F' are the **foci** of the hyperbola.
- C is the **centre** of the hyperbola.
- The points V and V' on each branch of the hyperbola nearest to the centre are called **vertices**.
- V'V is called the **transverse axis** of the hyperbola and CV = CV' is denoted by a and CF = CF' is denoted by c.
- Denote $c^2 - a^2$ by b^2 so that $b = \sqrt{c^2 - a^2}$
- The segment of symmetry perpendicular to the transverse axis at the centre, which has length 2b, is called the **conjugate axis**.
- The end points B and B' of the **conjugate axis** of the hyperbola are called co-vertices.

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- The **eccentricity** of the hyperbola, usually denoted by **e**, is the ratio of the distance between the two foci to the length of the transverse axis, that is,

  $e = \frac{\text{distance between the two foci}}{\text{lengh of the transverse axis}} = \frac{c}{a}$ which is a number greater than 1.

- The chords with end points on the hyperbola passing through the foci and perpendicular to FF' are called the **latus-rectums**.

Equation of a hyperbola with center at the origin and whose transverse axis is horizontal

Consider a hyperbola with foci F'(–c, 0), F(c, 0) and centre C (0, 0).

Then, a point P(x, y) is on the hyperbola, if and only if

$$\sqrt{(x-c)^2+y^2} - \sqrt{(x-c)^2+y^2} = \pm 2a$$

Thus, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the **standard form of equation of a hyperbola** with centre at C(0, 0) and transverse axis horizontal.

Remark: The **standard equation of a hyperbola** with centre at C(h, k) and transverse axis parallel to the x-axis is defined by $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.

Similarly, when the transverse axis is vertical, the standard equation of the hyperbola is given by:

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ when C (0 , 0) and

$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ when C(h, k).

The following table gives all possible standard forms of equations of hyperbolas.

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Equation	Centre	Transverse axis	Asymptotes
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	(0, 0)	horizontal	$y = \pm\frac{b}{a}x$
$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$	(h, k)	horizontal	$y-k=\left(\pm\frac{b}{a}(x-h)\right)$
$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$	(0, 0)	vertical	$y = \pm\frac{a}{b}x$
$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$	(h, k)	vertical	$y-k=\left(\pm\frac{a}{b}(x-h)\right)$

Examples:

1. Find the equation of a hyperbola, if the foci are F (2, 5) and F'(−4, 5) and the transverse axis is 4 units long. Draw the graph of the hyperbola.

2. Find the equation of the hyperbola with vertices (1, 2) and (1, – 2), and b = 2.

3. Sketch the hyperbola with equation $16y^2 - 9x^2 = 144$.

4. Find the centre, foci, vertices, eccentricity and the equations of the asymptotes of each hyperbola given below. Also sketch their graph.

a. $\frac{(x+3)^2}{9}-\frac{(y+6)^2}{36}=1$ b. $\frac{(y-3)^2}{25}-\frac{(x-2)^2}{25}=1$ c. $9x^2-16y^2=144$

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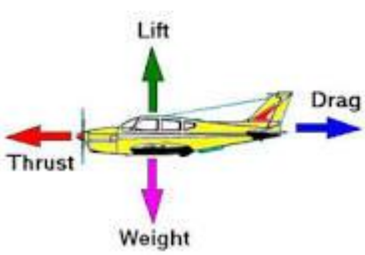
Chapter 5: Vectors and Transformation of the Plane

5.1. Introduction to vectors and scalars

We observe that there are some quantities and processes in our world that depend on the direction in which they occur, and there are some quantities that do not depend on direction. For example, the volume of an object, the three-dimensional space that an object occupies, does not depend on direction. If we have a 10 cubic centimeter block of iron and we move it up and down and then left and right, we still have a 10 cubic centimeter block of iron. On the other hand, the location, of an object does depend on direction. If we move the 10 cubic centimeter block 5 km to the north, the resulting location is very different than if we moved it 5 km to the east.

Definition: A physical quantity which depends on both magnitude and direction is called a vector quantity. A physical quantity which depend only on magnitude is called a scalar quantity.

Examples:

Vector quantities	Scalar Quantities
Displacement	**Length**
Force	**Area**
Velocity	**Volume**
Acceleration	**Mass**
Momentum	**Density**
Drag	**Pressure**
Lift	**Temperature**
Trust	**time**
Weight	**Energy**
	**Work** **Power**
(Forces acting on a flying airplane)	

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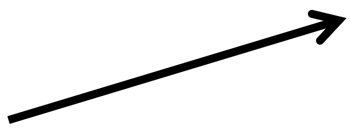
5.2. Representation of vectors

Geometrically, a vector is represented by using a directed line segment or an arrow. An arrow contains a head and a tail. The head of the arrow points the direction of the vector.

- The starting point of the vector is called its tail (or) the initial point of the vector.
- The ending point of the vector is called its head (or) the terminal point of the vector.



Notation: A vector is denoted by a lower case letter with an arrow over it. For example $\vec{a}$, $\vec{u}$, and $\vec{w}$ denote vectors. A vector in a plane (or space) with initial point A and terminal point B is also denoted by $\overrightarrow{AB}$.

Components of a Vector: Any vector $\overrightarrow{AB}$ in space with initial point A(a_1, a_2, a_3) and terminal point B(b_1, b_2, b_3) has components $b_1 - a_1$, $b_2 - a_2$ and $b_3 - a_3$. Then we write vector $\overrightarrow{AB}$ in terms the ordered triples of numbers, $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$.

Equality of Vectors: Two vectors $\vec{a}$ =(a_1, a_2, a_3) and $\vec{b} = (b_1, b_2, b_3)$ are said to be equal written $\vec{a} = \vec{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.

Examples: 1. If the point Q(2,-1,3) is the terminal point of vector $\overrightarrow{RQ} = (0,4,-1)$, then Find the coordinates of R.

Solution: Let R has coordinates R(x, y, z). Then $\overrightarrow{RQ} = (2 - x, -1 - y, 3 - z)$, Tthis implies that $= (2 - x, -1 - y, 3 - z) = (0,4,-1)$, Equating the corresponding components we get

$2 - x = 0 \Rightarrow$ x=2,

$-1 - y = 4 \Rightarrow y = -5$ and

$3 - z = -1 \Rightarrow z = 4$.

Therefore R has coordinates $R(2, -5,4)$.

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Definition: (Zero Vectors)

A vector where each of its components equals to zero is called a zero vector.

Remark: The direction of a zero vector is indeterminate (undefined)

Definition: (Position Vector)

Let P(x, y, z) be a point in space and O is the origin of the coordinate system. The vector $\vec{r} = \overrightarrow{OP}$ is called the position vector of the point P relative to the origin. i.e. if P(x, y, z) is a point in space, then its position vector is $\vec{r} = (x, y, z)$.

Definition: (length /magnitude/norm of a vector)

Let $\vec{a}$ =(a_1, a_2, a_3) be a vector in space. Then the length or magnitude of vector $\vec{a}$ denoted by $\|\vec{a}\|$, is defined by

$$\|\vec{a}\| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2}.$$

Remarks:

a) The magnitude of a vector is a non-negative number. i.e for any vector $\vec{a}$, $\|\vec{a}\| \geq 0$.
b) A vector whose magnitude is 1 is called a unit vector.
c) The vector $\mathbf{i} = (1,0,0), \mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$ are called the standard unit vectors. A vector $\vec{a}$ =(a_1, a_2, a_3) can be represented in terms of the special unit vectors **i, j** and **k** as $\vec{\mathrm{a}} = \mathrm{a}_1\mathbf{i} + \mathrm{a}_2\mathbf{j} + \mathrm{a}_3\mathbf{k}$

5.3. Operations on vectors

Definition: (Addition of Vectors)

For any two vectors $\vec{a} = (a_1, a_2, a_3)$and $\vec{b} = (b_1, b_2, b_3)$ in space the sum $\vec{a} + \vec{b}$ of the two vector is a vector given by

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

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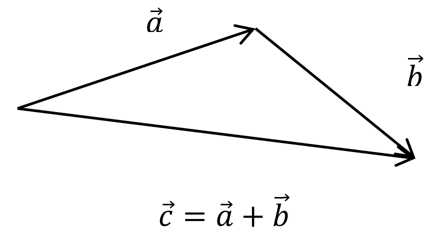
Remark:

To find the sum $\vec{a}+\vec{b}$ of two vectors $\vec{a}$ and $\vec{b}$, arrange the vectors in a head-to-tail fashion and connect the tail of $\vec{a}$ to the head of $\vec{b}$ we get the vector $\vec{c}=\vec{a}+\vec{b}$.



Definition: (Scalar Multiplication)

For any vector $\vec{a}=(a_1,a_2,a_3)$ and a scalar (number) m, the scalar multiple of $\vec{a}$, denoted by m$\vec{a}$, is a vector given by $m\vec{a}=(ma_1,ma_2,ma_3)$.

Remarks:

a) For any scalar m, $\|m\vec{a}\| = |m|\ \|m\vec{a}\|$
b) If $m>0$, $\vec{a}$ and $m\vec{a}$ have the same direction. If $m<0$, $\vec{a}$ and $m\vec{a}$ have opposite directions. If m=0, then $m\vec{a}=\vec{0}$.
c) For any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$

Properties of addition and scalar multiplication

For vectors $\vec{a},\vec{b}$, $\vec{c}$ and scalars *m, n*, the operations addition and scalar multiplication on vectors obey the following:

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2. $\vec{a}+\vec{O}=\vec{a}$
3. $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$
4. $m(n\vec{a})=(mn)\vec{a}$
5. $m(\vec{a}+\vec{b})=m\vec{b}+m\vec{a}$

Definition: (The dot / scalar product of Vectors)

The dot/scalar product $\vec{a}.\vec{b}$ of two vectors $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)$ is a scalar given by

$$\boldsymbol{\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3}$$

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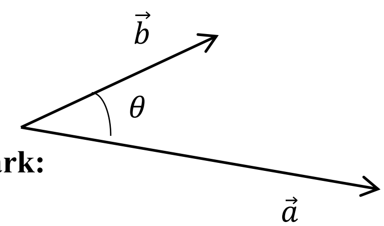
Definition: (Angle between two vectors)

For any two non-zero vectors $\vec{a}$ and $\vec{b}$, the angle between the two vectors is defined to be the angle θ measured from $\vec{a}$ to $\vec{b}$ when they have the same initial point and $0 \leq \theta \leq 180^0$.



Remark:

1. If θ is the angle between two non-zero vectors $\vec{a}$ and $\vec{b}$, then the dot product of the vectors can also defined to be $\vec{a}.\vec{b} = \| \vec{a}\| \| \vec{b}\| \, cos(\theta)$.
2. From the definition of the dot product, the angle θ between vectors $\vec{a}$ and $\vec{b}$ can be obtained from the formula $cos(\theta) = \frac{\vec{a}.\vec{b}}{\| \vec{a}\| \| \vec{b}\|}$.

Properties of the dot product

For vectors $\vec{a}, \vec{b}$, $\vec{c}$ and scalars *m, n* the dot product of vectors have the following properties:

a. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$
b. $\vec{a}.(\vec{b} + \vec{c}) = (\vec{a}.\vec{b}) + (\vec{a}.\vec{c})$
c. $m(n\vec{a}) = (mn)\vec{a}$
d. $m(\vec{a}.\vec{b}) = (m\vec{a}).\vec{b} = \vec{a}.(m\vec{b})$
e. $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$
f. $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{k} = \mathbf{i}.\mathbf{k} = 0$

Exercise:

1. Given the vectors $\vec{u} = (1, -3, 4)$, $\vec{v} = (2, 3, -2)$, and $\vec{w} = (0, -1, 1)$, find:
   a. $-\vec{u} + 3\vec{w}$
   b. $-3w + 2\vec{v}$
   c. $\vec{u}.\vec{v}$

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d. $\vec{u}.\overrightarrow{(v - \vec{w})}$

e. The angle θ between $\vec{u}$ and $\vec{v}$

2. Given the vectors $\vec{a} = (0, -3,4)$ and $\vec{b} = (6\ ,0, -8)$, then:

a. Find $\|\vec{a} + \vec{b}\|$.

b. Find a unit vector in the direction of $\|\vec{a} - \vec{b}\|$.

c. Find the angle between $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

<u>Transformation of the plane</u>

When a point P is given a new position, the point P is said to have a transformation.

There are two categories of transformation (motion)

i. Non-Rigid motion
ii. Rigid motion

i. Non-Rigid motion- A transformation that does not preserve distance

$$\overline{AB} \neq \overline{A'B'}$$

ii. Rigid motion- A transformation that preserves distance

- Segment moved that into congruent segment
- Angle into congruent angle
- Triangle into congruent triangle
- Preserve distance $\overline{AB} = \overline{A'B'}$

iii. Identity transformation : A transformation the image of every point is itself
Identity transformation is a rigid motion.

There are three types of rigid motions (transformations) these are

i. Translations
ii. Reflections
iii. Rotations.

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Definition: If every point of a figure is moved along the same direction through the same distance, then the transformation is called a translation or parallel movement.

If point P is translated to point P', then the vector $\overline{PP'}$ is said to be the translation vector. If $u = (h, k)$ is a translation vector, then the image of the point $P(x, y)$ under the translation will be the point $P'(x + h, y + k)$.

Example: Let T be a translation that takes the origin to $(1, 2)$. Determine the translation vector and find the images of the following points.

a. (2,–1) b. (–3, 5) c. (1, 2)

Solution: $T((0,0)) = (1,2) \Rightarrow u = (1,2)$ is the translation vector. $\Rightarrow x \to x + 1$ and $y \to y + 2$. Thus

a. $T((2,-1)) = (2 + 1, -1 + 2) = (3,1)$
b. $T((-3,5)) = (-3 + 1, 5 + 2) = (-2,7)$
c. $T((1,2)) = (1 + 1, 2 + 2) = (2,4)$.

Example: Let the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ be translated by the vector $u = (h, k)$. Show that $|\overline{PQ}| = |\overline{P\,Q}|$.

Solution: Clearly $P' = (x_1 + h, y_1 + k)$ and $Q' = (x_2 + h, y_2 + k)$.

Then, $|\overline{P'Q'}| = \sqrt{(x_2 + h - x_1 - h)^2 + (y_2 + k - y_1 - k)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \overline{PQ}$.

The above example shows that a translation is a rigid motion. You can state a translation formula in terms of coordinates as follows:

1. If (h, k) is a the translation vector, then
   a. the origin is translated to (h, k) i.e., $(0,0) \to (h, k)$
   b. the point $P(x, y)$ is translated to $P'(x + h, y + k)$.

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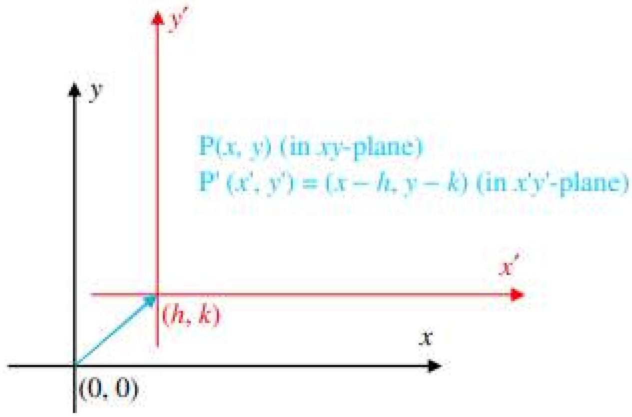

2. If the translation vector is $\overline{AB}$ where $A = (a, b)$ and $B = (c, d)$, then
   a. the origin is translated to $(c - a, d - b)$, and
   b. the point $P(x, y)$ is translated to $(x + c - a, y + d - b)$.

Example: If a translation T takes the origin to $P'(1, 2)$, then $T(x, y) = (x + 1, y + 2)$ and $T(-2, 3) = (-2 + 1, 3 + 2) = (-1, 5)$.

Example: If a translation T takes the point $(-1, 3)$ to the point $(4, 2)$, then find the images of the following lines under the translation T.

a. $\ell : y = 2x - 3$ b. $\ell : 5y + x = 1$

Solution: The translation vector is $(h, k) = (4 - (-1), 2 - 3) = (5, -1)$. Thus, the point $P(x, y)$ is translated to the point $P'(x + 5, y - 1)$. A translation maps lines onto parallel lines. Let ℓ' be the image of ℓ under T. Then,

a. $\ell' : y - (-1) = 2(x - 5) - 3 \Rightarrow \ell : y = 2x - 14$
b. $\ell' : 5(y + 1) + (x - 5) = 1 \Rightarrow \ell' : 5y + x = 1 \Rightarrow \ell' = \ell$.

Example: Determine the equation of the curve $2x^2 + 3y^2 - 8x + 6y = 7$, when the origin is translated to the point $A(2, -1)$.

Solution: The translation vector is $(h, k) = (2, -1)$. Thus, the point $P(x, y)$ is translated to the point $P'(x + 2, y - 1)$. Substituting $x - 2$ and $y + 1$ in the equation, you obtain $2(x - 2)^2 + 3(y + 1)^2 - 8(x - 2) + 6(y + 1) = 7$.

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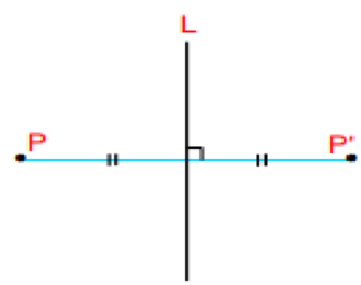
Expanding and simplifying, the equation of the curve becomes $2x^2 + 3y^2 - 16x + 12y + 26 = 0$.

Reflections

Definition: Let L be a fixed line in the plane. A reflection M about a line L is a transformation of the plane onto itself which carries each point P of the plane into the point P' of the plane such that L is the perpendicular bisector of PP'.

The line L is said to be the line of reflection or the axis of reflection.



Note: Every point on the axis of reflection is its own image.

Notation: The reflection of point P about the line L, is denoted by $M(P)$. i.e., $P' = M(P)$.

Reflection has the following properties:

1. A reflection about a line L has the property that, if for two points P and Q in the plane, $P = Q$, then $M(P) = M(Q)$. Hence, reflection is a function from the set of points in the plane into the set of points in the plane.
2. A reflection about a line L maps distinct points to distinct points, i.e., if $P \neq Q$, then $M(P) \neq M(Q)$. Equivalently, it has the property that if, for two points P, Q in the plane, $M(P) = M(Q)$, then $P = Q$. Thus, reflection is a one-to-one mapping.
3. For every point P' in the plane, there exists a point P such that $M(P) = P'$. If the point P' is on L, then there exists $P = P'$ such that $M(P) = P'$. Thus, reflection is an onto mapping.

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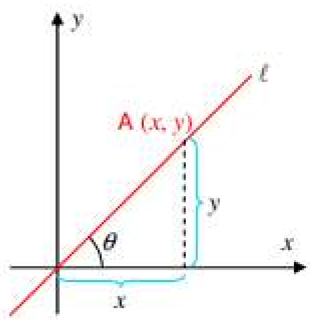
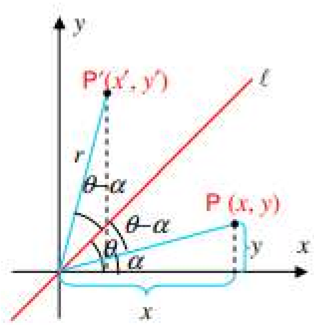
Theorem: A reflection M is a rigid motion. That is, if $P' = M(P)$ and $Q' = M(Q)$, then

$PQ = P'Q'$.

Reflection in the line $y = mx$, where $m = tan\,\theta$

Let ℓ be a line passing through the origin and making an angle with the positive x-axis. Then, the slope of ℓ is given by $m = \tan\theta$ and its equation $y = mx$.





You will now find the image of a point $P(\,x, y)$ when it is reflected about this line.

Let $P'(x', y')$ be the image of $P(\,x, y)$. The coordinates of p are:

$x = r\cos\alpha$ and $y = r\sin\alpha$.

The coordinates of P' are: $x' = r\,cos\,(2\theta - \alpha\,)$ and $' = r\,sin\,(2\theta - \alpha\,)$.

Expanding $cos(2\,\theta - \alpha\,)$ and $ssin\,(2\theta - \alpha\,)$. Now, use the following trigonometric identities that you will learn in

1. $Sine$ of the sum and the difference
   - ✓ $sin\,(x + y) = sin\,x\,cos\,y + cos\,x\,sin\,y$
   - ✓ $sin\,(x - y) = sin\,x\,cos\,y - cos\,xsin\,y$
2. $Cosine$ of the sum and difference
   - ✓ $cos\,(x + y) = cos\,x\,cos\,y - sin\,x\,sin\,y$
   - ✓ $cos\,(x - y) = cos\,x\,cos\,y + sin\,x\,sin\,y$

Using these trigonometric identities, you obtain:

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$x' = r\,[cos\,2\theta\;\; cos\alpha\; + sin\,2\theta\, sin\,\alpha] = \;(r\cos\alpha)\;\, cos\,2\theta + (r\sin\alpha)\;\, sin\,2\theta = xcos\,2\theta +$ $ysin\,2\theta$,

$y' = r\,[sin\,2\theta\;\; cos\alpha - sin\,\alpha\, sin\,2\theta] = \;(r\cos\alpha)\;\, sin\,2\theta - (r\sin\alpha)\cos 2\theta = xsin\,2\theta -$ $ycos\,2\theta$.

Thus, the coordinates of $P'(\,x', y')$, the image of the point $P(\,x, y)$ when reflected about the line $y = \;mx$ is:

$$x' \;=\; x\,cos\,2\theta \;+\; y\,sin\,2\theta$$

$$y' = \;x\,sin\,2\,\theta - \;y\,cos\,2\theta$$

Where θ is the angle of inclination of the line ℓ: $y \;=\; mx$.

Based on the value of θ, you will have the following four special cases:

1. When $\theta \;=\; 0$, you will have reflection in the x-axis. Thus, (x, y) is mapped to $(x, -y)$.
2. When $\theta = \frac{\pi}{4}$, you will have reflection about the line $y \;=\; x$ and hence (x, y) is mapped to (y, x).
3. When $\theta = \frac{\pi}{2}$, you will have reflection in the y – axis and (x, y) is mapped to $(-x, y)$.
4. When $\theta = \frac{3\pi}{2}$, you will have reflection about the line $y \;= -x$ and (x, y) is mapped to $(-y, -x)$.

Example: Find the images of the points $(3, 2)$, $(0, 1)$ and $(-5, 7)$ when reflected about the line $y \;=\; mx$, where $m \;=\; tan\,\theta$ and $\theta \;=\; \frac{\pi}{4}$.

Solution: This is actually a reflection about the line $y \;=\; x$. Thus, the images of $(3, 2)$, $(0, 1)$ and $(-5, 7)$ are (2, 3), (1, 0) and (7, –5), respectively.

Example: Find the images of the points $P\;(3, 2), Q\;(0, 1)$ and $R\;(-5, 7)$ when reflected about the line $= \frac{x}{\sqrt{3}}\,x$.

Solution: Since $\theta \;= \frac{1}{\sqrt{3}}$, you have $\theta = \frac{\pi}{6}$. Thus, if $P'(x', y')$ is the image of P, then

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$$x' = x\,cos2\theta + y\sin 2\theta = 3\cos\,(\frac{\pi}{3}) + 2\sin(\frac{\pi}{3}) = 3\times\frac{1}{2} + 2\times\frac{\sqrt{3}}{2} = \frac{3+2\sqrt{3}}{2}$$

$$y' = x\,sin2\theta - y\cos\,2\theta = 3\sin\left(\frac{\pi}{3}\right) - 2\cos\left(\frac{\pi}{3}\right) = 3\times\frac{\sqrt{3}}{2} + 2\times\frac{1}{2} = \frac{3\sqrt{3}}{2} - 1$$

Hence, the image of $P(3,2)$ is $P'(\frac{3+2\sqrt{3}}{2}, \frac{3\sqrt{3}}{2} - 1)$. Similarly, you can show that the image of $Q(0,1)$ and $R(-5,7)$ are $Q'(\frac{\sqrt{3}}{2}, \frac{-1}{2})$ and $R'(\frac{-5+7\sqrt{3}}{2}, \frac{-5\sqrt{3}-7}{2})$, respectively.

Note:

1. If a line ℓ' is perpendicular to the axis of reflection L, then L' is its own image.
2. If the centre of a circle C is on the line of reflection L, then the image of C is itself.
3. If the centre O of a circle C has image O' when reflected about a line L, then the image circle has centre O' and radius the same as C.
4. If ℓ' is a line parallel to the line of reflection L, to find the image of L' when reflected about L, we follow the following steps.

   Step a: Choose any point P on ℓ'

   Step b: Find the image of P, $M(P) = P'$

   Step c: Find the equation of ℓ', which is the line passing through P' with slope equal to the slope of ℓ.

<u>Reflection in the line $y = mx + b$</u>

Let $\ell : y = mx + b$ be the line of reflection, where $m \in \mathbb{R}\backslash\{0\}$. Let $P(x, y)$ be a point in the plane, not on ℓ. Let $P'(x', y')$ be the image of $P(x, y)$ when reflected about the line ℓ.

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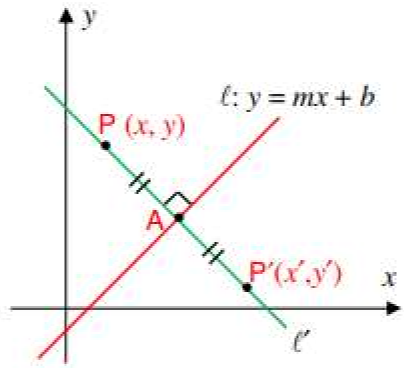

Let ℓ' be the line passing through the points $P\ (x, y)$ and $P'\ (x', y')$.Then, ℓ' is perpendicular to ℓ, since ℓ is perpendicular to $\overline{PP'}$. Since the slope of ℓ is m, the slope of ℓ' is . Thus, one can determine the equation of the line ℓ'. If A is the point of intersection of ℓ and ℓ', taking A as the mid-point of $\overline{PP'}$, we can find the coordinates of P'. Thus, to find the image of a point $P(x, y)$ when reflected about a line ℓ, we follow the following four steps.

Step 1: Find the slope of the line ℓ, say m.

Step 2: Find the equation of the line ℓ', which passes through the point $P(x, y)$ and has slope $-\frac{1}{m}$.

Step 3: Find the point of intersection A of ℓ and ℓ' which serves as the midpoint of $\overline{PP'}$.

Step 4: Using A as the mid-point of $\overline{PP'}$, find the coordinates of P'.

Example: Find images of the following lines and circles after reflection in the line $y = 2x - 3$.

a. $2y + x = 1$ b. $y = 2x + 1$ c. $y = 3x + 4$ d. $x^2 + y^2 - 4x - 2y + 4 = 0$ $e. x^2 + y^2 - 2x - 23 + 8 = 0$.

Solution: a. The image of $\ell: 2y + x = 1$ is itself. Explain!

b. $\ell: y = 2x + 1$ is parallel to the reflecting axis.

Hence $\ell' : y = 2x + b$. We need to determine b.

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Let (a, b) be any point on ℓ, say $(0, 1)$, so that its image lies on ℓ'. By the above reflecting procedure, $M((0,1)) = (a', b') \Rightarrow \frac{b'-1}{2} = -\frac{1}{2} \Rightarrow a' = -2b' + 2$

Also, the midpoint of $(0, 1)$ and (a', b') which is $(\frac{a'}{2}, \frac{b'+1}{2})$ lies on the reflecting axis $\Rightarrow \frac{b'+1}{2} = 2\left(\frac{a'}{2}\right) \Rightarrow a' = \frac{b'}{2} + \frac{7}{2}$. But $' = -2b' + 2 \Rightarrow 2b' + 2 = \frac{b'}{2} + \frac{7}{2}$.

$\Rightarrow b' = -\frac{3}{5} \Rightarrow a' = \frac{16}{5} \Rightarrow (\frac{16}{5}, -\frac{3}{5})$ lies on ℓ'

$\Rightarrow -\frac{3}{5} = 2\left(\frac{16}{5}\right) + b, b = -7 \Rightarrow \ell': y = 2x - 7.$

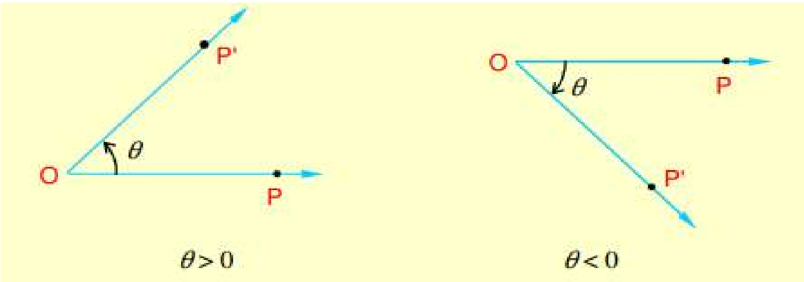
$c - d$ exercise.

Rotation

Definition: A rotation R about a point O through an angle θ is a transformation of the plane onto itself which carries every point P of the plane into the point P' of the plane such that $OP = OP'$ and $m(\angle POP') = \theta$. O is called the centre of rotation and θ is called the angle of rotation.

Note:

i. The rotation is in the counter clockwise direction, if $\theta > 0$ and in the clockwise direction if $\theta < 0$.



ii. Rotation is a rigid motion.

Example: Find the image of point $A\ (1, 0)$ when it is rotated through 30° about the origin.

Solution: Let the image of $A(1,0)$ be $A'(a, b)$ as shown in the following figure.

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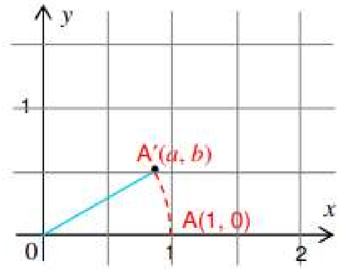


But from trigonometry, $(a, b) = (r\,cos\,\theta, r\,sin\,\theta)$ where $r = 1$ and $\theta = 30^{\circ}$ in this example. Therefore, the image of $A\ (1, 0)$ is $A'(\frac{\sqrt{3}}{2}, \frac{1}{2})$.

Notation:

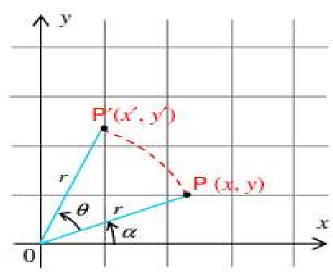
If R is rotation through an angle θ, then the image of $P\ (x, y)$ is denoted by $R_{\theta}(x, y)$. In the above example, $R_{30^{\circ}}(1,0) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

At this level, we derive a formula for a rotation R about $O(0, 0)$ through an angle θ.

Theorem: Let R be a rotation through angle θ about the origin. If $R_{\theta}(x, y) = (x', y')$, then

$x' = xcos\theta - y\sin\theta$

$y' = x\,sin\,\theta + ycos\theta.$



From trigonometry, we have,

$(x, y) = (r\cos\alpha, r\sin\alpha)$ and $(x', y') = (r\cos(\alpha + \theta)), r\sin(\alpha + \theta))$

$$\Rightarrow r\cos(\alpha + \theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta = x\cos\theta - y\sin\theta$$

$$r\sin(\alpha + \theta) = r\sin\alpha\cos\theta + r\cos\alpha\sin\theta = y\cos\theta + x\sin\theta$$

Therefore $R_{\theta}\ (x, y) = (x\,cos\,\theta - y\,sin\,\theta, y\,cos\,\theta + x\,sin\,\theta).$

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Note: Let R be a counter-clockwise rotation through an angle θ about the origin. Then

i. $\theta = \frac{\pi}{2} \Rightarrow R(x, y) = (-y, x)$

ii. $\theta = \pi \Rightarrow R(x, y) = (-x, -y)$

iii. $\theta = \frac{3\pi}{2} \Rightarrow R(x, y) = (y, -x)$

iv. $\theta = 2n\pi \;\; for\; n \in Z \Rightarrow R$ is the identity transformation.

v. Every circle with center at the center of rotation is fixed.

Example: Using the formula, find the images of the following points in rotation about the origin through the indicated angle.

a. $(4, 0)\; ;\; 60^{\circ}$ b. $(1, 1); -\frac{\pi}{6}$ c. $(1, 2);\; 450^{\circ}$

Note:

As in the case of translation and reflection, to find the image of a circle under a given rotation we follow the following steps:

1. Find the center and radius of the given circle
2. Find the image of the center of the circle under the given rotation.
3. Equation of the image circle will be an equation of the circle centered at the image of the center of the given circle with radius the same as the radius of the given circle.

Example: Find the image of the circle $(x - 3)^2 + (y + 5)^2 = 1$ when it is rotated through $\frac{5\pi}{3}$ about $(4, -3)$.

Note:

One can also obtain the image of a line under a given rotation as follows:

- ✓ Choose two points on the line.
- ✓ Find the images of the two points under the given rotation.

Thus, the image line will be the line passing through the two image points.

Example: Find the equation of the line $\ell : 3x - 2y = 1$ after it has been rotated -135° about$(-2,3)$.

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Chapter 6: Matrix and Determinant

6.1 Definition of matrix and Types of matrix

Definition 6.1.1: A rectangular arrangement of mn numbers (real) in to m horizontal rows and n vertical columns enclosed by a pair of brackets [ ], such as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m1} & \dots & a_{mn} \end{bmatrix}$$

Is called an $\boldsymbol{m \times n}$ (read "m by n") **matrix** or a matrix of **order** $\boldsymbol{m \times n}$.

<u>Note</u>: 1. The **order** of a matrix is the number of rows and columns it has. When we say a matrix is a 3 by 4 matrix, we are saying that it has 3 rows and 4 columns.

2. An $m \times n$ matrix contains $\boldsymbol{mn}$ elements (entries) abbreviated as $A = (a_{ij})_{m \times n}$

Example 6.1.1: Let $\boldsymbol{A} = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 3 & 6 \end{bmatrix}$.

Solution: Since A has 2 rows and 3 columns, we say A has order 2×3, where the number of rows is specified first. The element 6 is in the position a_{23} (read a two three) because it is in row 2 and column 3.

Exercise 6.1.1: What is the value of $\boldsymbol{a_{23}}$ and $\boldsymbol{a_{32}}$ in? $\boldsymbol{A} = \begin{bmatrix} -1 & 4 & 7 \\ 2 & 3 & 1 \\ 5 & 7 & 8 \end{bmatrix}$ **?**

6.1.1. Types of matrices:

Certain types of matrices, which play important roles in matrix theory.

1. **Column matrix:** a matrix which have only one column which is called a column vector.

   Example $A = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$ is a 3×1 column matrix

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2. **Row matrix:** : A matrix that has exactly one row is called a row matrix

   Example $\boldsymbol{A} = \begin{bmatrix} 5 & 2 & -1 & 4 \end{bmatrix}$ is a 1×4 row matrix

3. **Zero or Null Matrix**: A matrix whose entries are all 0 is called a **zero** or **null matrix**. It is usually denoted by $0_{\boldsymbol{m}\times\boldsymbol{n}}$ or more simply by **0**. For example, $0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a 2×4 zero matrix

4. **Square Matrix**: An $\boldsymbol{m}\times\boldsymbol{n}$ matrix is said to be a **square matrix** of **order n** if **m = n**. That is, if it has the same number of columns as rows.

   Example $\begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix}$ is a square matrices of 2

5. **Diagonal Matrix:** A square matrix is said to be diagonal if each of the entries not falling on the main diagonal is zero.

Example: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is a diagonal matrix.

6. **Identity Matrix or Unit Matrix**: A square matrix is said to be identity matrix or unit matrix if all its main diagonal entries are 1's and all other entries are 0's

   Example: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7. **Scalar matrix:** A diagonal matrix whose all the diagonal elements are equal is called

   a scalar matrix.

Example, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a scalar matrix.

8. **Lower triangular matrix:** a square matrix whose entries above the main diagonal are all zero

   Example $\begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 4 & 0 & 5 \end{bmatrix}$ is lower triangular matrix

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9. **Upper triangular matrix:** a square matrix whose entries below the main diagonal are all zero

   Example $\begin{bmatrix} 2 & 4 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$ is the upper triangular matrix

6.1.2. Algebra of matrices

1. Addition of matrices:

Let A and B be two matrices of the same order. Then the addition of A and B, denoted by A + B, is the matrix obtained by adding corresponding entries of A and B. Thus, if

$$\boldsymbol{A} = (\boldsymbol{a_{ij}})_{\boldsymbol{m\times n}} \text{ and } \boldsymbol{B} = (\boldsymbol{b_{ij}})_{\boldsymbol{m\times n}}, \text{ then } \boldsymbol{A} + \boldsymbol{B} = (\boldsymbol{a_{ij}} + \boldsymbol{b_{ij}})_{\boldsymbol{m\times n}}.$$

And $A - B = (a_{ij} - b_{ij})_{m\times n}$

Example Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 5 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 4 & 2 \\ 3 & 6 & 1 \end{bmatrix}$

Find, if possible. **a)** A + B **b)** $A - B$

Properties of Addition of Matrices

1. Matrix addition is commutative. That is, if **A** and **B** are two matrices of the same order, then **A + B = B + A**.
2. Matrix addition is associative. That is, if **A, B** and **C** are three matrices of the same order, then **(A + B) + C = A + (B + C).**
3. Existence of additive identity. That is, if 0 is the zero matrix of the same order as that of the matrix **A**, then **A + 0 = A = 0 + A**.
4. Existence of additive inverse. That is, if A is any matrix, then **A + (-A) = 0 = (-A) + A**

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2. Multiplication of a Matrix by a Scalar

Let **A** be b an $m \times n$ matrix and k be a real number (called **a scalar**). Then the multiplication of **A** by k, denoted by k **A**, is the $m \times n$ matrix obtained by multiplying each entry of **A** by k. This operation is called **scalar multiplication**.

Example**:** If $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$. Find 2A + 3B.

$$2A + 3B = \begin{bmatrix} 0 & 4 & 6 \\ 4 & 2 & 8 \end{bmatrix} + \begin{bmatrix} 21 & 18 & 9 \\ 3 & 12 & 15 \end{bmatrix} = \begin{bmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{bmatrix}$$

3. Matrix multiplication: Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$ matrices such that the number of column of A equals the number of rows of matrix B then the product $AB = C = (c_{ik})_{m\times p}$

Where $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + \ldots + a_{in} b_{nk} = \sum_{j=1}^{n} a_{ij} b_{jk} \quad i = 1,2,\ldots,m \text{ and } k = 1,2,\ldots,p$

Thus, the product **AB** is the $m \times p$ matrix, where each entry c_{ik} of **AB** is obtained by multiplying corresponding entries of the i^{th} row of **A** by those of the k^{th} column of **B** and then finding the sum of the results.

Find the product **AB** if $A = \begin{bmatrix} 1 & -4 \\ 5 & 3 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 & 1 & 6 \\ 2 & 7 & 3 & 8 \end{bmatrix}$

Solution: Since the number of columns of **A** is equal to the number of rows of B, the product **AB=C** is defined. Since **A** is 3×2 and B is 2×4, the product AB will be 3×4

$$AB = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}$$

The entry c_{11} is obtained by summing the products of each entry in row 1

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of A by the corresponding entry in column 1 of B, that is.

$c_{11} = (1)(-2) + (-4)(2) = -10$. Similarly, for C_{21} we use the entries in

row 2 of A and those in column 1 of B, that is $C_{21} = (5)(-2) + (3)(2) = -4$.

Also, $C_{12} = (1)(4) + (-4)(7) = -24$

$C_{13} = (1)(1) + (-4)(3) = -11$

$C_{14} = (1)(6) + (-4)(8) = -26$

$C_{22} = (5)(4) + (3)(7) = 41$

$C_{23} = (5)(1) + (3)(3) = 14$

$C_{24} = (5)(6) + (3)(8) = 54$

$C_{31} = (0)(-2) + (2)(2) = 4$

$C_{32} = (0)(4) + (2)(7) = 14$

$C_{33} = (0)(1) + (2)(3) = 6$

$C_{34} = (0)(6) + (2)(8) = 16$

$$\text{Thus } \boldsymbol{AB} = \begin{bmatrix} -10 & -24 & -11 & -26 \\ -4 & 41 & 14 & 54 \\ 4 & 14 & 6 & 16 \end{bmatrix}$$

Note: 1. $AB \neq BA$

2. $AB = 0$ Does not necessarily imply A = 0 or B = 0.

3. AB = AC does not necessarily imply B = C.

6.1.3. Transpose of a matrix

Definition : Let **A** be an $\boldsymbol{m \times n}$ matrix. The transpose of **A**, denoted by $\boldsymbol{A^t}$, is the $\boldsymbol{n \times m}$ matrix obtained from **A** by interchanging the rows and columns of **A**. Thus the first row of **A** is the first column of $\mathbf{A^t}$, the second row of **A** is the second column of $\mathbf{A^t}$ and so on.

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Example : If $A = \begin{pmatrix} 2 & -4 & 6 \\ 3 & 1 & 4 \end{pmatrix}$, then $A^t = \begin{pmatrix} 2 & 3 \\ -4 & 1 \\ 6 & 4 \end{pmatrix}$

Properties:

a) If **A** and **B** have the same order, $(A \pm B)^t = A^t \pm B^t$

b) For a scalar k, $(kA)^t = kA^t$

c) If A is $m \times n$ and **B** is $n \times p$, then $(AB)^t = B^t A^t$.

d) $(A^t)^t = A$

Definition: A square matrix A is called a symmetric matrix if $A^t = A$.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & -5 \\ 3 & -5 & 6 \end{bmatrix}$ *is a symmetric matrix*

6.2 Determinants and its properties

Definition: (Determinant of order 1): Let $A = (a)$ be a square matrix of order 1. Then determinant of A is defined as the number a itself. That is, $|A| = a$.

Example $|3| = 3$, $|-5| = -5$ and $|0| = 0$

Definition : (Determinant of order 2): Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix, then

$$|A| = ad - bc.$$

Determinants of 3×3 matrices

To define the determinant of a 3 ×3 matrix, it is useful to first define the concepts of minor and cofactor. Let $A = (aij)3 \times 3$. Then the matrix Aij is a 2×2 matrix which is found by crossing out the i^{th} row and the j^{th} column of A

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Definition let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ Then $Mij = \left| Aij \right|$ is called the minor of the element a_{ij} and $C_{ij} = (-1)^{i+j} \left| Aij \right|$ is called the cofactor of the elementaij.

Example Find the minors and cofactors of the entries $a22$, $a33$ and $a12$ of the matrix $\begin{pmatrix} -3 & 4 & -7 \\ 1 & 2 & 0 \\ -4 & 8 & 11 \end{pmatrix}$

Definition :(Determinant of order n): If A is a square matrix of order n (n >2), then its determinant may be calculated by multiplying the entries of any row (or column) by their cofactors and summing the resulting products. That is,

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots.. + a_{in}C_{in} \quad \textbf{Or} \quad \det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots.. + a_{nj}C_{nj}$$

Remark: It is a fact that determinant of a matrix is unique and does not depend on the row or column chosen for its evaluation.

Example 4.1.2: Find the determinant of the following matrix A first by expanding along The $1st$ row and then expanding along the 2nd column, where $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ -3 & 2 & 5 \end{pmatrix}$

Solution: Choose a given row or column. Let us arbitrarily select the first row. Then

Along row 1:$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 2(\ 1)2 \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 1 & 4 \\ -3 & 5 \end{vmatrix} + 0(-1^4 \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix} = 2[(1 \times 5 - (2 \times 4)] + (-1)(1 \times 5 \ - (4 \times -3) + (0)(1 \times 2) - (-3 \times 1) = 2(-3) - 17 + 0 = -23$

Check ***Along Column 2***

Properties of determinants

The following properties hold. All the matrices considered are square matrices:

1. If B is found by interchanging two rows (columns) of A, then $|B|$ = - $|A|$.
2. If B is found by multiplying one row (one column) of A by a scalar r, then $|B| = r|A|$.
3. If B is a matrix obtained by adding a multiple of a row (column) of A to another Row (column) of A, then $|B| = |A|$.

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4. If A has a row (or a column) of zeros, then the determinant of A is zero.
5. If A has two identical rows (or columns), then the determinant of A is zero.
6. $|A| = |A^t|$

6.3. Inverse of a square matrix

Definition: A square matrix A is said to be invertible or non-singular, if and only if there is a square matrix B such that $AB = BA = I$, where I is the identity matrix that has the same order as A.

Note:

- Only a square matrix can have an inverse.
- The inverse of a square matrix A, whenever it exists is denoted by A^{-1}.
- A and A^{-1} have the same order.
- A matrix that does not have an inverse is called singular.

Example

a. Show that $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$ are inverse of each other.

b. Given $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, find A^{-1} (if it exists).

Solution

a. $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$

Thus, they are inverses of each other.

b. Suppose $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $AA^{-1} = I_2$.

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} a+c & b+d \\ 2a+3c & 2b+3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\Rightarrow \begin{cases} a+c=1 \\ 2a+3c=0 \end{cases}$ and $\begin{cases} b+d=0 \\ 2b+3d=1 \end{cases}$

Solving these gives you, $a = 3, b = -1, c = -2$ and $d = 1$.

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Hence $A^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

In the above example, you have seen how to find the inverse of invertible matrices. Sometimes, this method is time consuming. There is another method of finding inverses of invertible matrices, using the adjoint.

Definiton: The adjoint of a square matrix $A = (a_{ij})$ is defined as the transpose of the matrix $C = (c_{ij})$ where c_{ij} are the cofactors of the elements a_{ij}. Adjoint of A is denoted by $adj\ A$, i.e $adjA = (c_{ij})^T$.

Example: Find $adj\ A$ if $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 4 & 0 & 0 \end{pmatrix}$.

Solution: $c_{11} = (-1)^{1+1}\begin{vmatrix} 3 & -1 \\ 0 & 0 \end{vmatrix} = 0,\ c_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} = -4,$

$c_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = -12,\ c_{21} = (-1)^{2+1}\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0,$

$c_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = -4,\ c_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 4 & 0 \end{vmatrix} = 0,$

$c_{31} = (-1)^{3+1}\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} = -3,\ c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3,$

$c_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3.$

Then matrix $C = \begin{pmatrix} 0 & -4 & -12 \\ 0 & -4 & 0 \\ -3 & 3 & 3 \end{pmatrix}$ and, adj $A = C^T = \begin{pmatrix} 0 & 0 & -3 \\ -4 & -4 & 3 \\ -12 & 0 & 3 \end{pmatrix}$.

Theorem

A square matrix A is invertible, if and only if $|A| \neq 0$. If A is invertible, then $A^{-1} = \frac{1}{|A|}$ adj

Example: Find the inverse of $A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 1 \\ -4 & 5 & 2 \end{pmatrix}$

Solution first find adj A

$C_{11=(-1)^{1+1}}\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = -1;\quad C_{12=(-1)^{1+2}}\begin{vmatrix} 0 & 1 \\ -4 & 2 \end{vmatrix} = -4;\quad C_{13=+}\begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix} = 8;$

$C_{21=+}\begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} = 19;\ C_{22=+}\begin{vmatrix} 1 & 3 \\ -4 & 2 \end{vmatrix} = 14;\ C_{23=-}\begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = 3;$

$C_{31=+}\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} = -8;\ C_{32=-}\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = -1;\ C_{33=+}\begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = 2$

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Thus, adj A= $\begin{pmatrix} -1 & 19 & -8 \\ -4 & 14 & -1 \\ 8 & 3 & 2 \end{pmatrix}$ Next, find$|A|$.

$|A| = a_{11}\, c_{11} + a_{12}\, c_{12} + a_{13}\, c_{13} = (-1)(-1) + (-2)(-4) + (3)(8) = 31$. Since $|A| \neq 0$,, then A is invertible and

$$A^{-1} = \frac{1}{|A|} \text{adj (A)} = \frac{1}{31} = \begin{pmatrix} -1 & 19 & -8 \\ -4 & 14 & -1 \\ 8 & 3 & 2 \end{pmatrix} == \begin{pmatrix} \frac{-1}{31} & \frac{19}{31} & \frac{-8}{31} \\ \frac{-4}{31} & \frac{14}{31} & \frac{-1}{31} \\ \frac{8}{31} & \frac{3}{31} & \frac{2}{31} \end{pmatrix}$$

Example: Show that $\begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$ is not invertible.

Solution: $\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix}$ = (1)(-6)-(3)(-2)= 0. Thus, the inverse does not exist.

Theorem

If A and B are two invertible matrices of the same order, then$(AB)^{-1} = B^{-1}A^{-1}$.

Proof:

If A and B are invertible matrices of the same order, then $|A| \neq 0$ and $|B| \neq 0$.

$$|AB| = \big||A|\big|\ |B| \neq 0$$

Hence, AB is invertible with inverse $(AB)^{-1}$. On the other hand, $(AB)(B^{-1}A^{-1})A^{-1} = A(BB^{-1})A^{-1} = \mathrm{A}(I)A^{-1} = AA = I$ and similarly $B^{-1}A^{-1} = I$.

Therefore $B^{-1}A^{-1}$ is an inverse of AB and inverse of a matrix is unique.

Example 5 verify that $(AB)^{-1} = B^{-1}A^{-1}$, for the following matrices:

$A = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 2 \\ 3 & 1 \end{pmatrix}$

Solution $|A| = 2$ and $|B| = -9$. To find adj(A), interchange the diagonal elements and take the negatives of the non-diagonal elements. Thus,

$adjA = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$ and $adjB = \begin{pmatrix} 1 & -2 \\ -3 & -3 \end{pmatrix}$

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It follows that, $A^{-1} = \frac{1}{|A|}$ adj $(A) = \frac{1}{2}\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{pmatrix}$, while

$$B^{-1} = \frac{1}{|B|}\, adj\ (B) = -\frac{1}{9}\begin{pmatrix} 1 & -2 \\ -3 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Thus gives us $B^{-1}A^{-1}, = \begin{pmatrix} -\frac{1}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{pmatrix} = \begin{pmatrix} -\frac{13}{18} & \frac{5}{9} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

On the other hand $AB = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -6 & 10 \\ -6 & 13 \end{pmatrix}$, so that

$|AB| = -18$ and $adj(AB) = \begin{pmatrix} 13 & -10 \\ 6 & -6 \end{pmatrix}$

$$(AB)^{-1} = -\frac{1}{18}\begin{pmatrix} 13 & -10 \\ 6 & -6 \end{pmatrix} = \begin{pmatrix} -\frac{13}{18} & \frac{5}{9} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

Therefore , $(AB)^{-1} = B^{-1}A^{-1}$.

6.4.System of linear equation with two or three variables

Definition: an eqution $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $a_1, a_2, \ldots, a_n, b$ are constants and $x_1, x_2, \ldots, x_n$ are variables is called a linear equation. If $b = 0$, then the linear equation is said to be homogenous.

A linear system with m eqution in n unkowns(variables) $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \qquad (*)$$

The system of equations $(*)$ is equivalent to $AX = B$, where

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$$A = (a_{ij})_{mxn}, X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{pmatrix}.$$

Matrix A is called the coefficient matrix of the system and the matrix

$$(A/B) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$ is called the augmented matrix of the system.

Example which of the folowing are systems of linear equations?

a. $\begin{cases} 5x - 23y = 0 \\ x + 14y = 12 \end{cases}$ b. $\begin{cases} 5x^2 - 23y = 6 \\ x + 14y = 12 \end{cases}$ c. $\begin{cases} 5x - 23y + z = 6 \\ x + 14y - 4z = 18 \end{cases}$

solution a and b are systems of linear equations but b is not a linear equation because the first equation in the system is not linear in x.

Example give the agumented matrix of the following systes of equations

a. $\begin{cases} 2x + 5y = 1 \\ 3x - 8y = 4 \end{cases}$ b. $\begin{cases} 2x - y + z = 3 \\ 3x - 2y + 8z = -24 \\ x + 3y + 4y = -2 \end{cases}$

solution a. $\begin{pmatrix} 2 & 5 & 1 \\ 3 & -8 & 4 \end{pmatrix}$ b. $\begin{pmatrix} 2 & -1 & 1 & 3 \\ 3 & -2 & 8 & -24 \\ 1 & 3 & 4 & -2 \end{pmatrix}$

6.4.1.Elementary operation on matrices

Two system of linear eqution are equivalent , if and only if they have exactly the same solution.

To change system of linear equations in to equivalent system, we use any of the following three elementary(also called Gaussian) operations.

Swapping : Interchange two equations of the system

Rescaling: Multiply an equation of the system by a non-zero constant.

Pivoting: Add a constant multiple of one another equation of the system.

Elementary row operation

Swapping: Interchange two rows of a matrix.

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Rescaling: Multiplying a row of a matrix by a non-zero constant.

Pivoting: Add a constant multiple of one row of the matrix onto another row.

Elementary column operation

Swapping : Interchange two columns of a matrix.

Rescaling: Multiplying a column of a matrix by a non-zero constant.

Pivoting: Add a constant multiple of one column of the matrix onto another row.

Definition: Two matrices are said to be row(column) equivalent, if and only if one is obtained from the other by performing the other by any of the elementary operations.

Example solve the system of equtions given below by using the agumented matrix

$$\begin{cases} x - 2y + z = 7 \\ 3x + y - z = 2 \\ 2x + 3y + 2z = 7 \end{cases}$$

Solution write the agumented matrix $\begin{pmatrix} 1 & -2 & 1 & 7 \\ 3 & 1 & -1 & 2 \\ 2 & 3 & 2 & 7 \end{pmatrix}$

$R_2 \to R_2 - 3R_1$ $\quad \begin{pmatrix} 1 & -2 & 1 & 7 \\ 0 & 7 & -4 & -19 \\ 2 & 3 & 2 & 7 \end{pmatrix}$

$R_3 \to R_3 - R_2$ $\quad \begin{pmatrix} 1 & -2 & 1 & 7 \\ 0 & 7 & -4 & -19 \\ 0 & 0 & 4 & 12 \end{pmatrix}$

It corresponds to the system of equation

$$\begin{cases} x - 2y + z = 7 \\ 7y - 4z = -19 \\ \quad 4z = 12 \end{cases}$$

By back substitution $z = 3, y = -1, x = 2$

The solution set is$\{2, -1,3\}$.

Definition: A matrix is said to be in Row Echelon Form if

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1. A zero frow (if ther is) comec at the bottom.

2. The first nonzero element in each nonzero row is 1.

3. The number of zeros preceding the first nonzero element in each nonzero row except the first row is greater than the number of such zeros in the preceding row.

Example: Which of the following matrix is in echelon form ?

$$A = \begin{pmatrix} 1 & -2 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 3 & 0 \\ 3 & 3 & 6 \end{pmatrix}$$

Solution: A is in echelon form but B is not in echelon form.

A system of linear equations has either

1. no solution, or
2. exactly one solution, or
3. Infinitely many solutions.

We say that a linear system is **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Solving a linear system

This is the process of finding the solutions of a linear system. We first see the technique of elimination (Gaussian elimination method) and Cramer's rule.

6.4.2.Gaussian Elimination Method

The Gaussian elimination method is a standard method for solving linear systems. It applies to any system, no matter whether $\boldsymbol{m < n}$, $\boldsymbol{m = n}$ or $\boldsymbol{m > n}$ (where $\boldsymbol{m}$ and $\boldsymbol{n}$ are number of equations and variables respectively). We know that equivalent linear systems have the same solutions. Thus the basic strategy in this method is to replace a given system with an equivalent system, which is easier to solve.

The basic operations that are used to produce an equivalent system of linear equations are the following:

1. Replace one equation by the sum of itself and a multiple of another equation.
2. Interchange two equations
3. Multiply all the terms in an equation by a non zero constant.

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Example: Solve the system

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ x_2 - 2x_3 &= 4 \\ -2x_1 - 3x_2 - 3x_3 &= 5 \end{aligned}$$

Solution: We perform the elimination procedure with and without matrix notation of the system. For each step we put the resulting system and its augmented matrix side by side for comparison:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ x_2 - 2x_3 &= 4 \\ -2x_1 - 3x_2 - 3x_3 &= 5 \end{aligned} \qquad \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & 4 \\ -2 & -3 & -3 & 5 \end{bmatrix}$$

We keep x_1 in the first equation and eliminate it from the other equations. For this replace the third equation by the sum of itself and two times equation 1.

$$\begin{array}{ll} 2.[eq.1]: & 2x_1 + 6x_2 - 2x_3 = 4 \\ \underline{+[eq.3]}: & \underline{-2x_1 - 3x_2 - 3x_3 = 5} \\ [New\,eq.3] & 3x_2 - 5x_3 = 9 \end{array}$$

We write the new equation in place of the original third equation:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ x_2 - 2x_3 &= 4 \\ 3x_2 - 5x_3 &= 9 \end{aligned} \qquad \xrightarrow{R_3 \to R_3 + 2R_1} \qquad \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 3 & -5 & 9 \end{bmatrix}$$

Next use the x_2 in equation 2 to eliminate $3x_2$ in equation 3.

$$\begin{array}{ll} -3.[eq.2]: & -3x_2 + 6x_3 = -12 \\ \underline{+[eq.3]}: & \underline{3x_2 - 5x_3 = 5} \\ [New\,eq.3] & x_3 = -3 \end{array}$$

The resulting equivalent system is:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ x_2 - 2x_3 &= 4 \\ x_3 &= -3 \end{aligned} \qquad \xrightarrow{R_3 \to R_3 - 3R_2} \qquad \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Now we eliminate the $-2x_3$ term form equation 2. For this we use x_3 in equation 3.

$$\begin{array}{ll} 2.[eq.3]: & 2x_3 = -6 \\ \underline{+[eq.2]}: & \underline{x_2 - 2x_3 = 4} \\ [New\,eq.3] & x_3 \quad = -3 \end{array}$$

From this we get

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$$\begin{aligned} x_1 + 3x_2 - x_3 &= 2 \\ x_2 \quad &= -2 \\ x_3 &= -3 \end{aligned} \qquad \xrightarrow{R_2 \to R_2 + 2R_3} \qquad \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Again by using the x_3 term in equation 3, we eliminate the $-x_3$ term in equation 1.

$$\begin{array}{ll} 1.[eq.3]: & x_3 = -3 \\ +[eq.1]: & x_1 + 3x_2 - x_3 = 2 \\ \hline [New\ eq.1] & x_1 + 3x_2 \quad = -1 \end{array}$$

Thus we get the system

$$\begin{aligned} x_1 + 3x_2 &= -1 \\ x_2 &= -2 \\ x_3 &= -3 \end{aligned} \qquad \xrightarrow{R_1 \to R_1 + R_3} \qquad \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Finally, we eliminate the $3x_2$ term in equation 1. We use the x_2 term in equation 2 to eliminate the $3x_2$ term above it.

$$\begin{array}{ll} -3.[eq.2]: & -3x_2 = 6 \\ +[eq.1]: & x_1 + 3x_2 = 2 \\ \hline [New\ eq.1] & x_1 \quad = 5 \end{array}$$

So we have an equivalent system (to the original system) that is easier to solve.

$$\begin{aligned} x_1 \quad &= 5 \\ x_2 \quad &= -2 \\ x_3 &= -3 \end{aligned} \qquad \xrightarrow{R_1 \to R_1 - 3R_2} \qquad \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Thus the system has only one solution, namely (5, -2, -3) or $\mathbf{x}_1 = 5, \mathbf{x}_2 = -2, \mathbf{x}_3 = -3$. To verify that (5, -2, -3) is a solution, substitute these values in to the left side of the original system, and compute:

5 + 3(-2) - (-3) = 5 – 6 + 3 = 2

-2 - 2(-3) = -2 + 6 = 4

-2(5) - 3(-2) - 3(-3) = -10 + 6 + 9 = 5

It is a solution, as it satisfies all the equation in the given system (3).

In Gaussian elimination method we either transform the augmented matrix to an echelon matrix or a reduced echelon matrix. That is we either find an echelon form or the reduced echelon form

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of the augmented matrix of the system. An echelon form of the augmented matrix enables us to answer the following two fundamental questions about solutions of a linear system. These are:

1. Is the system consistent; that is, does at least one solution exists?
2. If a solution exists, is it the only one; that is, is the solution unique?

Example: Determine if the following system is consistent. If so how many solutions does it have?

$$\begin{aligned} \boldsymbol{x}_1 - \boldsymbol{x}_2 + \boldsymbol{x}_3 &= 3 \\ \boldsymbol{x}_1 + 5\boldsymbol{x}_2 - 5\boldsymbol{x}_3 &= 2 \\ 2\boldsymbol{x}_1 + \boldsymbol{x}_2 - \boldsymbol{x}_3 &= 1 \end{aligned}$$

Solution: The augmented matrix is $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 1 & 5 & -5 & 2 \\ 2 & 1 & -1 & 1 \end{bmatrix}$

Let us perform a finite sequence of elementary row operations on the augmented matrix.

$$[A|B] = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 1 & 5 & -5 & 2 \\ 2 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to -R_1 + R_2} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 6 & -6 & -1 \\ 2 & 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 \to -2R_1 + R_3}$$

$$\begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 6 & -6 & -1 \\ 0 & 3 & -3 & -5 \end{bmatrix} \xrightarrow{R_3 \to \frac{-1}{2}R_2 + R_3} \begin{bmatrix} 1 & -1 & 1 & 3 \\ 0 & 6 & -6 & -1 \\ 0 & 0 & 0 & -\frac{9}{2} \end{bmatrix}$$

The corresponding linear system of the last matrix is

$$\begin{aligned} \boldsymbol{x}_1 - \boldsymbol{x}_2 + \boldsymbol{x}_3 &= 3 \\ 6\boldsymbol{x}_2 - 6\boldsymbol{x}_3 &= -1 \qquad (*) \\ 0 &= \frac{-9}{2} \end{aligned}$$

But the last equation $0.\boldsymbol{x}_1 + 0.\boldsymbol{x}_2 + 0.\boldsymbol{x}_3 = \frac{-9}{2}$ is never true. That is there are no values $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ that satisfy the new system (*). Since (*) and the original linear system have the same solution set, the original system is inconsistent (has no solution).

Example: Use Gaussian elimination to solve the linear system

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$$\begin{aligned} 2\mathbf{x} - \mathbf{y} + \mathbf{z} &= 2 \\ -2\mathbf{x} + \mathbf{y} + \mathbf{z} &= 4 \\ 6\mathbf{x} - 3\mathbf{y} - 2\mathbf{z} &= -9 \end{aligned}$$

Solution: The augmented matrix of the given system is

$$[\mathbf{A}|\mathbf{B}] = \begin{bmatrix} 2 & -1 & 1 & 2 \\ -2 & 1 & 1 & 4 \\ 6 & -3 & -2 & -9 \end{bmatrix}$$

Let us find an echelon form of the augmented matrix first. From this we can determine whether the system is consistent or not . If it is consistent we go a head to obtain the reduced echelon form of **[A|B] ,** which enable us to describe explicitly all the solutions.

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ -2 & 1 & 1 & 4 \\ 6 & -3 & -2 & -9 \end{bmatrix} \xrightarrow{R_2 \to R_1 + R_2} \begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 2 & 6 \\ 6 & -3 & -2 & -9 \end{bmatrix} \xrightarrow{R_3 \to -3R_1 + R_3}$$

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & -5 & -15 \end{bmatrix} \xrightarrow{R_3 \to \frac{5}{2}R_2 + R_3} \begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_2}$$

$$\begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to -R_2 + R_1} \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1}$$

$$\begin{bmatrix} 1 & \frac{-1}{2} & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The associated linear system to the reduced echelon form of **[A|B]** is

$$\begin{aligned} \mathbf{x} - \tfrac{1}{2}\mathbf{y} &= \tfrac{-1}{2} \\ \mathbf{z} &= 3 \\ 0 &= 0 \end{aligned}$$

The third equation is $0\mathbf{x} + 0\mathbf{y} + 0\mathbf{z} = 0$. It is not an inconsistency, it is always true what ever values we take for x, y, z. The system is consistent

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and if we assign any value λ for y in the first equation, we get $\mathbf{x} = \frac{-1}{2} + \frac{1}{2}\lambda$.

From the second we have z = 3. Thus $\mathbf{x} = \frac{-1}{2} + \frac{1}{2}\lambda$, $y = \lambda$ and $z = 3$ is the solution of the given system, where λ is any real number. There are an infinite number of solutions, for example,

$$x = \tfrac{-1}{2},\ y = 0,\ z = 3$$

$$x = 0,\ y = 1,\ z = 3 \quad \text{and so on.}$$

In vector form the general solution of the given system is of the form $(\frac{-1}{2} + \frac{1}{2}\lambda, \lambda, 3)$ where $\lambda \in \Re$. What dose this represents in $\Re^3$?

Remark: A system of linear equation $AX = B$ is **consistent** if and only if the ranks of the coefficient matrix and the augmented matrix are equal.

6.4.3.Cramer's rule for solving system of linear equations

Suppose we have to solve a system of n linear equations in n unknowns **Ax** = **b.** Let $\mathbf{A_i(b)}$ be the matrix obtained from **A** by replacing column **i** by the vector **b** and $\mathbf{A^k}$ be the **k**-th column vector of matrix **A**.

$$A_i(b) = [\mathbf{A}^1\ \ \mathbf{A}^2\ \ \ldots\ \underset{\substack{\uparrow \\ \text{column } i}}{\mathbf{b}}\ \ldots\ \mathbf{A}^n]$$

Now let $\mathbf{e_1}, \mathbf{e_2}, \ldots \mathbf{e_n}$ be columns of the n × n identity matrix **I** and $\mathbf{I_i(x)}$ be the matrix obtained from **I** by replacing column **i** by **x.**

If **Ax** = **b** then by using matrix multiplication we have

$$\mathbf{AI_i(x) = A[e_1\ \ldots\ x\ \ldots\ e_n] = [Ae_1\ \ldots\ Ax\ \ldots\ Ae_n]}$$
$$= [\mathbf{A}^1\ \ldots\ \mathbf{b}\ \ldots\ \mathbf{A}^n] = \mathbf{A_i(b)}$$

By the multiplicative property of determinants, (det**A**)(det$\mathbf{I_i(x)}$) = det $\mathbf{A_i(b)}$

The second determinant on the left is x_i. (Make a cofactor expansion along the ith row.) Hence (det **A**). x_i = det $\mathbf{A_i(b)}$. Therefore if det$\mathbf{A} \neq \mathbf{0}$ then we have $\mathbf{x_i} = \dfrac{\det \mathbf{A_i(b)}}{\det \mathbf{A}}$.

This method for finding the solutions of n linear equations in n unknowns is known as Cramer's Rule.

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Example: Solve the following system of linear equations by Cramer's Rule.

$$\begin{aligned} 2\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 &= 6 \\ \mathbf{x}_1 + 4\mathbf{x}_2 - 2\mathbf{x}_3 &= -4 \\ 3\mathbf{x}_1 \quad + \mathbf{x}_3 &= 7 \end{aligned}$$

Solution: Matrix form of the given system is $\mathbf{Ax} = \mathbf{b}$

where $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 4 & -2 \\ 3 & 0 & 1 \end{pmatrix}$ $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}$ **and** $\mathbf{b} = \begin{pmatrix} 6 \\ -4 \\ 7 \end{pmatrix}$

By Cramer's Rule, $\mathbf{x_i} = \dfrac{\mathbf{det\,A_i(b)}}{\mathbf{det\,A}}$ $(\mathbf{i} = 1, 2, 3)$

$$\det A = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 4 & -2 \\ 3 & 0 & 1 \end{vmatrix} = 3$$

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{\begin{vmatrix} 6 & -1 & 1 \\ -4 & 4 & -2 \\ 7 & 0 & 1 \end{vmatrix}}{2} = \frac{6}{3} = 2 \quad , \; x_2 = \frac{\det A_2(b)}{\det A} = \frac{\begin{vmatrix} 2 & 6 & 1 \\ 1 & -4 & -2 \\ 3 & 7 & 1 \end{vmatrix}}{2} = \frac{-3}{3} = -1 \quad \text{and}$$

$$x_3 = \frac{\det A_3(b)}{\det A} = \frac{\begin{vmatrix} 2 & -1 & 6 \\ 1 & 4 & -4 \\ 3 & 0 & 7 \end{vmatrix}}{2} = \frac{3}{3} = 1.$$

Example: Solve the following system of linear equations by Cramer's Rule.

$$\begin{aligned} 2x_1 + x_2 \quad &= 7 \\ -3x_1 + 2x_3 &= -8 \\ x_2 + 2x_3 &= -3 \end{aligned}$$

Solution: Matrix form of the given system is $\mathbf{Ax} = \mathbf{b}$

where $A = \begin{pmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ *and* $b = \begin{pmatrix} 7 \\ -8 \\ -3 \end{pmatrix}$

By Cramer's Rule, $\mathbf{x_i} = \dfrac{\mathbf{det\,A_i(b)}}{\mathbf{det\,A}}$ $(\mathbf{i} = 1, 2, 3)$

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$$\det A = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4$$

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{\begin{vmatrix} 7 & 1 & 0 \\ -8 & 0 & 1 \\ -3 & 1 & 2 \end{vmatrix}}{4} = \frac{6}{4} = \frac{3}{2} \quad , \quad x_2 = \frac{\det A_2(b)}{\det A} = \frac{\begin{vmatrix} 2 & 7 & 0 \\ -3 & -8 & 1 \\ 0 & -3 & 2 \end{vmatrix}}{4} = \frac{16}{4} = 4 \quad \text{and}$$

$$x_3 = \frac{\det A_3(b)}{\det A} = \frac{\begin{vmatrix} 2 & 1 & 7 \\ -3 & 0 & -8 \\ 0 & 1 & -3 \end{vmatrix}}{4} = \frac{-14}{4} = \frac{-7}{2}.$$

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Chapter 7: Limits and Continuity

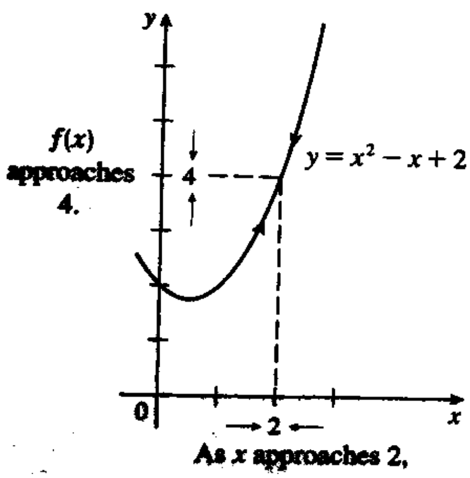
7.1. Limit of Functions

The basic idea underlying the concept of the limit of a function $\boldsymbol{f}$ at a point $'\boldsymbol{a}'$ is to study the behavior of $\boldsymbol{f}$ at points close to $'\boldsymbol{a}'$ but not necessary equal to $'\boldsymbol{a}'$.

Example

Let us investigate the behavior of the function $f(x) = x^2 - x + 2$ for values of x near to 2.

The following table gives values of $f(x)$ for values of x close to 2 but not equal to 2.



x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

From the above table and the graph of $\boldsymbol{f}$, we see that when x is close to 2 (on either side of 2), $f(x)$ is close to 4.

In this case, we say that

"the limit of the function $\mathrm{f(x) = x^2 - x + 2}$ as x approaches 2 is equal to 4".

Symbolically.

$$\lim_{x\to 2}(x^2 - x + 2) = 4$$

Intuitive Definition: If $f\ (x)$ gets closer and closer to a single real number L as x gets closer and closer to a(but not necessary equal to a) , then we say that the limit of $f\ (x)$ as x approaches a is L.

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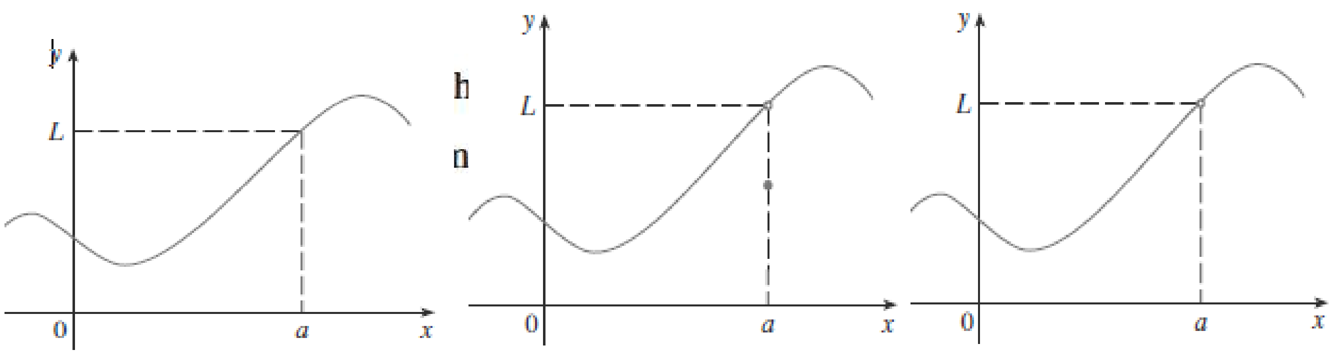
Symbolically, this is written as

$$\lim_{x \to a} f(x) = L$$

Note: the phrase "*but not necessary equal to a*" in the definition of limit means that in finding the limit of $f(x)$as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a from both sides.

Observe the following figure. In part (c), $f(a)$ is not defined and in part (b), $f(a) \neq L$. But in



(a) (b) (c)

each case, regardless of what happens at, it is true that $\lim_{x \to a} f(x) = L$

Basic limit theorems

Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

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exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

7. $\lim_{x \to a} c = c$

8. $\lim_{x \to a} x = a$

9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer
(If n is even, we assume that $a > 0$.)

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

$\left[\text{If } n \text{ is even, we assume that } \lim_{x \to a} f(x) > 0.\right]$

Example

Evaluate the following limits and justify each step.

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(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Solution

(a) $$\lim_{x \to 5} (2x^2 - 3x + 4) = \lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} 4 \qquad \text{(by Laws 2 and 1)}$$

$$= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4 \qquad \text{(by 3)}$$

$$= 2(5^2) - 3(5) + 4 \qquad \text{(by 9, 8, and 7)}$$

$$= 39$$

(b) We start by using Law 5, but its use is full (a) by substituting 5 for x. Similarly, direct substi-
see that the limits of the numerator and denom
tor is not 0.

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} (x^3 + 2x^2}{\lim_{x \to -2} (5 - 3x}$$

$$= \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -}}{\lim_{x \to -2} 5 -}$$

$$= \frac{(-2)^3 + 2(-2)^2}{5 - 3(-2)}$$

$$= -\frac{1}{11}$$

DIRECT SUBSTITUTION PROPERTY If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

However, not all limits can be evaluated by direct substitution. Thus the following are some technique of limits evalution.

- Factorization method
- Rationalization method
- Sqeezing method

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Example

Find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

Solution

Let $f(x) = (x^2 - 1)/(x - 1)$. We can't find the limit by substituting $x = 1$

So, we can evaluate it by factorization numerator and denominators, and cancel like terms.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

Example

Find $\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

Solution

We can't apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned}
\lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \to 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} \\
&= \lim_{t \to 0} \frac{(t^2 + 9) - 9}{t^2\left(\sqrt{t^2 + 9} + 3\right)} = \lim_{t \to 0} \frac{t^2}{t^2\left(\sqrt{t^2 + 9} + 3\right)} \\
&= \lim_{t \to 0} \frac{1}{\sqrt{t^2 + 9} + 3} = \frac{1}{\sqrt{\lim_{t \to 0} (t^2 + 9)} + 3} = \frac{1}{3 + 3} = \frac{1}{6}
\end{aligned}$$

Theorem (Squeeze theorem)

If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except

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possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $$\lim_{x \to a} g(x) = L$$

Example

Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

Solution

First note that we **cannot** use

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 \cdot \lim_{x \to 0} \sin \frac{1}{x}$$

Because, $\lim_{x \to 0} sin\frac{1}{x}$ does'n exist. But we know that:

$-1 \le sin\frac{1}{x} \le 1$ and multiply it by x^2, we get

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

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One sided limits

Left-hand limit

Definition

We write $\lim_{x \to a^-} f(x) = L$

and say the left-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to $'a'$ and x less than $'a'$

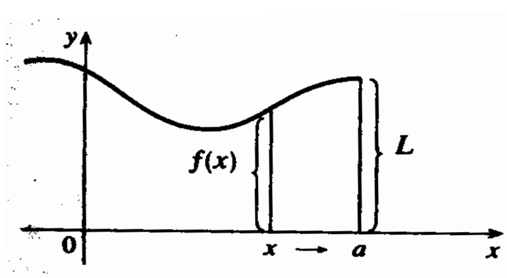
Right-hand limit

Definition

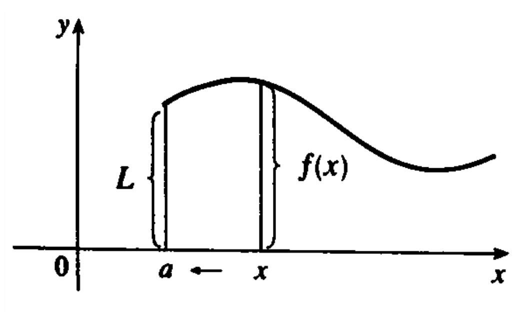
We write $\lim_{x \to a^+} f(x) = L$

and say the right-hand limit of $f(x)$ as x approaches a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to $'a'$ and x greater than $'a'$

Graphical representation of Left-hand limit and Right-hand limit



(a) $\lim_{x \to a^-} f(x) = L$



(b) $\lim_{x \to a^+} f(x) = L$

By comparing limit definition with Left-hand limit and Right-hand limit, the following is true.

$if \lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

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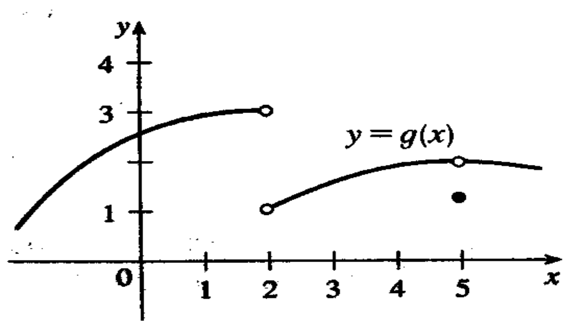
Example

The graph of a function $\boldsymbol{g}$ is shown in the following figure. Use it evaluate the following limits.

$$(a) \lim_{x\to 2^-} g(x) \quad (b) \lim_{x\to 2^+} g(x) \quad (c) \lim_{x\to 2} g(x)$$
$$(d) \lim_{x\to 5^-} g(x) \quad (e) \lim_{x\to 5^+} g(x) \quad (f) \lim_{x\to 5} g(x)$$



Solution:

From the given graph

$(a)\ \lim_{x\to 2^-} g(x) = 3$ $\quad (b)\ \lim_{x\to 2^+} g(x) = 1$

(c) Since the left and right limits are different, so we conclude that $\lim_{x\to 2} g(x)$ does not exist.

The graph also shows that

$(d) \lim_{x\to 5^-} g(x) = 2$ $\quad (e)\ \lim_{x\to 5^+} g(x) = 2$

(f) Since, the left and right limits are same, so we have

$$\lim_{x\to 5} g(x) = 2$$

Tip: Evaluate the following limits

a. $\lim_{x\to 2^+} \sqrt{x-2}$ b. $\lim_{x\to 2^-} \sqrt{x-2}$ c. $\lim_{x\to 1^+} \sqrt{1-x}$ d. $\lim_{x\to 2^-} \sqrt{4-x^2}$

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Infinite Limits and Limit at Infinity and Asymptotes

Infinite Limits:

Definition:

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Example

Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

Solution

As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very

Large (See the table in the margin). Thus the values of $f(x)$ do not approach a number.

Therefore, $\lim_{x \to 0} \left(\frac{1}{x^2}\right)$ does not exist.

To indicate the kind of behavior exhibited in Example we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

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Definition:

Let f be defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

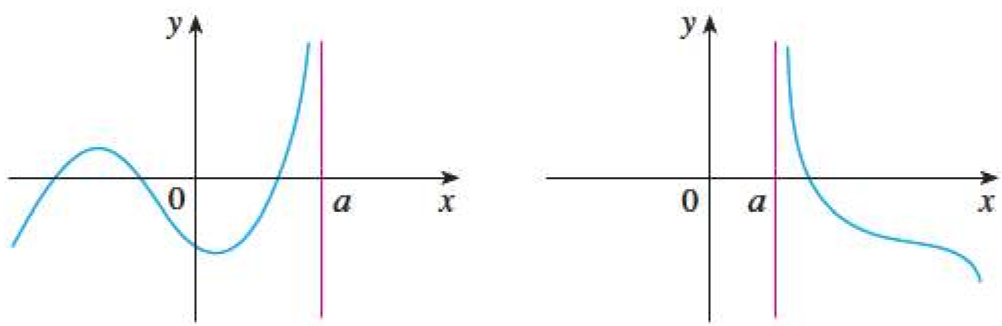
Note:

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Graphically,



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$

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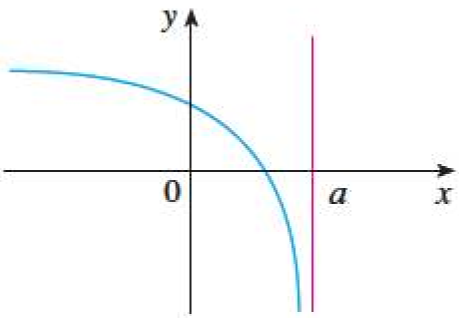
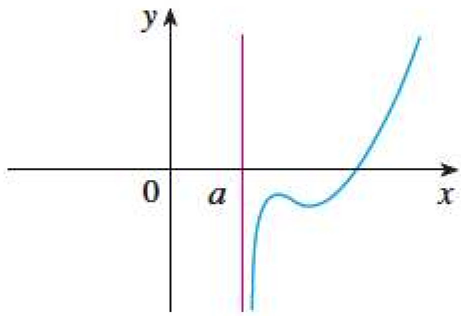


(c) $\lim_{x \to a^-} f(x) = -\infty$



(d) $\lim_{x \to a^+} f(x) = -\infty$

Definition:

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Example: Determine the following infinite limits

a. $\lim_{x \to 5^+} \frac{6}{x-5}$

b. $\text{Lim}_{x \to -2^+} \frac{x-1}{x^2(x+2)}$

c. $\lim_{x \to 3^-} \frac{x^2-9}{x-3}$

d. $\lim_{x \to 3^-} \frac{x-3}{x^2-9}$

Solution:

a. $\lim_{x \to 5^+} \frac{6}{x-5} = \infty$

$\Rightarrow x = 5$ is vertical asymptote

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b. $\text{Lim}_{x\to-2^+}\frac{x-1}{x^2(x+2)} = -\infty$

$\Longrightarrow x = -2$ is vertical asymptote

c. $\lim_{x\to3^-}\frac{x^2-9}{x-3} = \lim_{x\to3^-}\frac{(x-3)(x+3)}{x-3} = \lim_{x\to3^-}(x+3) = 3+3 = 6$

$\Longrightarrow x = 3$ is not vertical asymptote

Limit at Infinity

Definition :

If the values of $f(x)$ eventually get closer and closer to a number L as x increases without bound, then we write

$$\lim_{x\to+\infty} f(x) = L$$

Similarly, if the values of $f($ $\lim_{x\to-\infty} f(x) = L$ loser and closer to a number L as x decreases without bound, then

Example

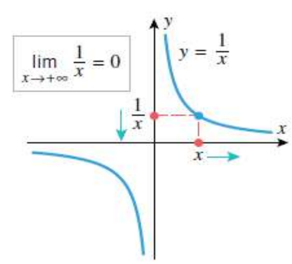
as x increases without bound, the values of $f(x) = 1/x$ are positive,

but get closer and closer to 0; and as x decreases without bound,

the values of $f(x) = 1/x$ are negative, and also get cl
limiting behaviors by writing

$$\lim_{x\to+\infty}\frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{x} = 0$$

Graphically,



x	1	10	100	1000	10,000	⋯
$f(x)$	1	0.1	0.01	0.001	0.0001	⋯

x increasing without bound

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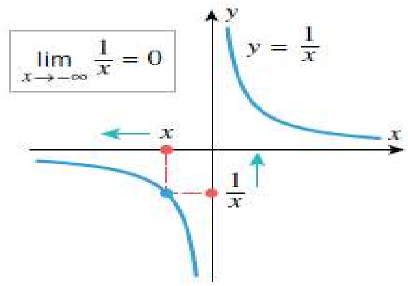


x	···	−10,000	−1000	−100	−10	−1
$f(x)$	···	−0.0001	−0.001	−0.01	−0.1	−1

x decreasing without bound

Definition:

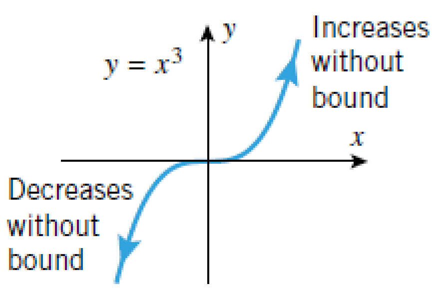
A line $y = L$ is called a
function f if
$\lim_{x \to +\infty} f(x) = L$ or

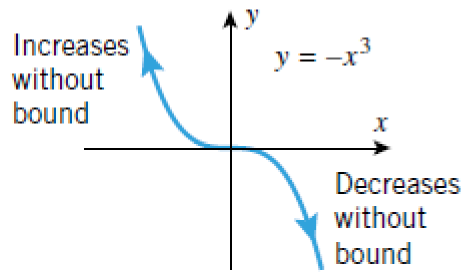
Limits at infinity can fail to exist for various reasons. One possibility is that the values of $f(x)$ may increase or decrease without bound as $x \to +\infty$ or as $x \to -\infty$. For example, the values of $f(x) = x^3$ increase without bound as $x \to +\infty$ and decrease without bound as

$x \to -\infty$; and for $f(x) = -x^3$ the values decrease without bound as $x \to +\infty$ and increase

without bound as $x \to -\infty$ We denote this by writing

$$\lim_{x \to +\infty} x^3 = +\infty, \quad \lim_{x \to -\infty} x^3 = -\infty, \quad \lim_{x \to +\infty} (-x^3) = -\infty, \quad \lim_{x \to -\infty} (-x^3) = +\infty$$





Example: Evaluate the following limits

a. $\lim_{x\to\infty} \frac{2x+3}{1-4x}$ b. $\lim_{x\to\infty} \frac{x^2-3}{x+6}$ c. $\lim_{x\to\infty} \frac{x+6}{x^2-3}$ d. $\lim_{x\to-\infty} \frac{2x+3}{1-4x}$

Solution:

a. $\lim_{x\to\infty} \frac{2x+3}{1-4x} = \lim_{x\to\infty} \frac{\frac{2x+3}{x}}{\frac{1-4x}{x}} = \frac{\lim_{x\to\infty}\left(2+\frac{3}{x}\right)}{\lim_{x\to\infty}\left(\frac{1}{x}-4\right)} = \frac{2+0}{0-4} = -\frac{1}{2}$

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$\Longrightarrow y = -\frac{1}{2}$ is horizontal asymptote

b. $\lim_{x\to\infty} \frac{x^2-3}{x+6} = \lim_{x\to\infty} \frac{\frac{x^2-3}{x}}{\frac{x+6}{x}} = \frac{\lim_{x\to\infty}\left(x-\frac{3}{x}\right)}{\lim_{x\to\infty}\left(1+\frac{6}{x}\right)} = \frac{\infty-0}{1+0} = \infty$

The limit doesn't exist.

$\Longrightarrow$ The graph of $f(x)$ has no horizontal asymptote

c. $\lim_{x\to\infty} \frac{x+6}{x^2-3} = \lim_{x\to\infty} \frac{\frac{x+6}{x^2}}{\frac{x^2-3}{x^2}} = \frac{\lim_{x\to\infty}\left(\frac{1}{x}+\frac{6}{x^2}\right)}{\lim_{x\to\infty}\left(1-\frac{3}{x^2}\right)} = \frac{0+0}{1-0} = 0$

$\Longrightarrow y = 0$ is horizontal asymptote

7.2. Continuity and one sided Continuity

Definition

$$A\ function\ f\ is\ \boldsymbol{continuous\ at\ a\ number\ a}\ if$$

$$\lim_{x\to a} f(x) = f(a)$$

Note that the above definition implicitly requires three things if f is continuous at a:

i. $f(a)\ is\ defined$ (i.e., a is in the domain of f)

ii. $\lim_{x\to a} f(x)$ exists

iii. $\lim_{x\to a} f(x) = f(a)$

Note: Instead of showing the three conditions, it is sufficient to show $\lim_{x\to a} f(x) = f(a)$. If this (third) condition is satisfied, then the $f(x)$ is continuous at a.

If f is not continuous at a, we say that f is **discontinuous at $\boldsymbol{a}$**.

Example

Show that the function $f(x) = \frac{x+1}{2x^2-1}$ is continuous at $a = 4$

Solution

Given $f(x) = \frac{x+1}{2x^2-1}$

$$f(a) = f(4) = \frac{5}{31} \text{ and}$$

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$$\lim_{x\to a} f(x) = \lim_{x\to 4} \frac{x+1}{2x^2-1} = \frac{5}{31}$$

Since, $\lim_{x\to 4} \frac{x+1}{2x^2-1} = \frac{5}{31} = f(4)$

Therefore $f(x) = \frac{x+1}{2x^2-1}$ is continuous at $a = 4$

Example

Show that $f(x)$ is discontinuous at $a = 2$ where

$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & if\ x \neq 2 \\ 1 & x = 2 \end{cases}$$

Solution

$f(2) = 1$ and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x - 2} = \lim_{x\to 2} x + 1 = 3.$$

But $\lim_{x\to 2} f(x) \neq f(2)$

Therefore, the function $f(x)$ is not continuous at 2.

Example:

Determine the value of a so that the piecewise defined function

$$f(x) = \begin{cases} x+3, & \text{if } x > 2 \\ ax-1. & \text{if } x \leq 2 \end{cases} \text{ is continuous on } (-\infty, \infty).$$

Solution If f is continuous on $(-\infty, \infty)$, then f must be continuous at $x = 2$.

$$\Rightarrow \lim_{x\to 2^+} f(x) = f(2) \Rightarrow \lim_{x\to 2^+} (x+3) = a(2) - 1 \Rightarrow 5 = 2a - 1 \Rightarrow a = 3$$

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Example:

Let $f(x)=\begin{cases} ax+b, \text{if } x\leq -2 \\ 2x+a, \text{if } -2<x\leq 3 \\ ax^2-bx+4, \text{if } x>3 \end{cases}$

If f is a continuous function, find the values of a and b.

Solution f should be continuous at $x=-2$ and $x=3$ because f is a continuous function.

i f is continuous at $x=-2$

$\Rightarrow \lim_{x\to -2^+} f(x)=f(-2) \Rightarrow \lim_{x\to 2^+}(2\times(-2)+a)=(a(-2)+b) \Rightarrow -4+a=-2a+b$

$\Rightarrow$ $3a-b=4$ . . . . *equation* (1)

ii f is continuous at $x=3$

$\Rightarrow \lim_{x\to 3^+} f(x)=f(3) \Rightarrow \lim_{x\to 3^+}(ax^2-bx+4)=2(3)+a$

$\Rightarrow$ $9a-3b+4=6+a$

$\Rightarrow$ $8a-3b=2$ . . . . *equation* (2)

Solving the system of equations

$\begin{cases} 3a-b=4 \\ 8a-3b=2 \end{cases}$ gives $a=10$ and $b=26$.

One-sided continuity

Definition :

A function f is **continuous from the right at a number a** if

$$\lim_{x\to a^+} f(x)=f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x\to a^-} f(x)=f(a)$$

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Definition:

A function f is **continuous on an interval** if it is continuous at every number in the interval. (At an endpoint of the interval we understand continuous to mean continuous from the right or continuous from the left).

Example:

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Solution

If $-1 < a < 1$, then using the limit rules, we have

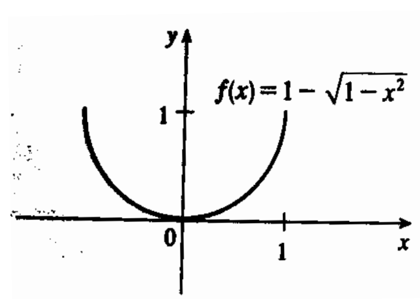
$$\lim_{x \to a} f(x) = \lim_{x \to a}(1 - \sqrt{1 - x^2})$$

$$= 1 - \lim_{x \to a} \sqrt{1 - x^2}$$

$$= 1 - \sqrt{\lim_{x \to a}(1 - x^2)}$$

$$= 1 - \sqrt{1 - a^2} = f(a)$$

Thus, by definition f is continuous at a if $-1 < a < 1$.



We must also calculate the right-hand limit at -1 and the left-hand limit at 1.

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+}(1 - \sqrt{1 - x^2})$$

$$= 1 - \sqrt{\lim_{-1^+}(1 - x^2)}$$

$$= 1 = f(-1)$$

So f is continuous from the right at -1.

Similarly, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(1 - \sqrt{1 - x^2})$

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$$= 1 - \sqrt{\lim_{1^-}(1 - x^2)}$$

$$= 1 = f(1)$$

$\Longrightarrow f$ is continuous from the left at 1.

Therefore, according to above definition, f is continuous on $[-1, 1]$

The graph of f is sketched in above figure. It is the lower half of the circle $x^2 + (y-1)^2 = 1$.

Theorem: If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ 2. $f - g$ 3. cf 4. fg 5. $\frac{f}{g}$ $\quad if\ g(a) \neq 0$

Theorem

i. Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$

ii. Any rational function is continuous wherever it is defined; that is, it is Continuous on its domain.

Theorem:

If g is continuous at a and f is continuous at $g(a)$, then $(fog)(x) = f\big(g(x)\big)$ is continuous at a.

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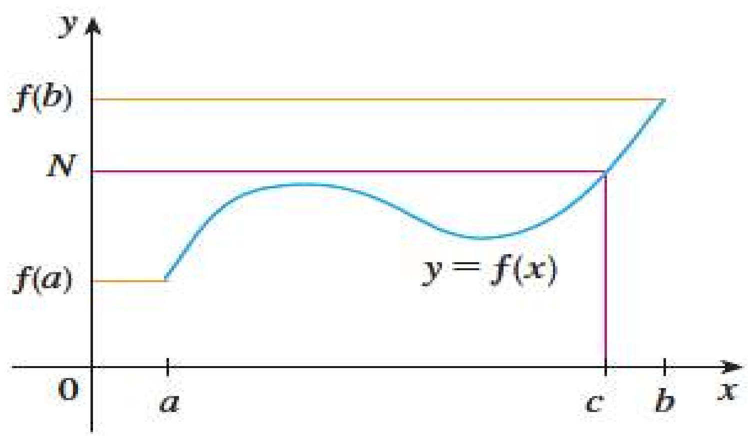
7.3. The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number strictly between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that

$$f(c) = N$$

The Intermediate value theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ *and* $f(b)$. It is illustrated in the following figure.



Example:

Let $f(x) = 3x + 5$. Find a real number c with $2 < c < 5$ such that $f(c) = 13$

Solution

f is continuous on $(-\infty, \infty)$ because, it is polynomial function; hence continuous on [2, 5].

Since, $N = 13$, $f(2) = 11$ and $f(5) = 20$ and the condition $f(2) < 13 < f(5)$ is true.

$$\Longrightarrow f(c) = 13$$

$$\Longrightarrow 3(c) + 5 = 13 \Rightarrow 3c = 8$$

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$$\Rightarrow c = 8/3 \in (2, 5)$$

Therefore, the required value of c is $c = 8/3$.

Example:

Let $f(x) = x^2 - x + 5$. Find a real number c with $-3 < c < 2$ such that $f(c) = 10$

Solution:

f is continuous on $(-\infty, \infty)$ since it is polynomial function; hence continuous on [-3, 2].

Also, $f(2) < 10 < f(-3)$ since $f(2) = 7 \text{ and } f(-3) = 17$

Therefore, there exist a number c with $-3 < c < 2$ such that $f(c) = 10$.

$f(c) = 10$

$$\Rightarrow c^2 - c + 5 = 10 \Rightarrow c^2 - c - 5 = 0$$

$$\Rightarrow c = \frac{1 \pm \sqrt{21}}{2}. \text{ But } \frac{1+\sqrt{21}}{2} > 2$$

Therefore, the only answer is $c = \frac{1-\sqrt{21}}{2}$.

Theorem: $\lim_{x \to 0} \frac{\sin x}{x} = 1$, where x is in radians.

Example 1 Evaluate each of the following limits.

a	$\lim_{x\to 0}\frac{\sin(3x)}{x}$	**b**	$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$	**c**	$\lim_{x\to\infty} x^2\sin\left(\frac{1}{x^2}\right)$
d	$\lim_{x\to 0}\frac{x}{\sin x}$	**e**	$\lim_{x\to 0}\frac{\tan x}{\sin x}$	**f**	$\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}$
g	$\lim_{x\to 0}\frac{\sin^3 x}{x^3}$	**h**	$\lim_{x\to 0}\frac{1-\cos x}{x^2}$	**i**	$\lim_{x\to 1}\frac{\sin(x-1)}{1-x+x^2-x^3}$

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Solution

a $$\lim_{x\to 0}\frac{\sin(3x)}{x}=\lim_{x\to 0}\frac{3\sin(3x)}{(3x)}=3\lim_{x\to 0}\frac{\sin(3x)}{3x}=3$$

b $$\lim_{x\to\infty}x\sin\left(\frac{1}{x}\right)=\lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}=\lim_{y\to 0}\left(\frac{\sin y}{y}\right)=1,\text{ where } y=\frac{1}{x}$$

c $$\lim_{x\to\infty}x^2\sin\left(\frac{1}{x^2}\right)=\lim_{x\to\infty}\left(\frac{\sin\frac{1}{x^2}}{\frac{1}{x^2}}\right)=\lim_{y\to 0}\left(\frac{\sin(y)^2}{y^2}\right)=1.\quad \textit{Why?}$$

d $$\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}\left(\frac{1}{\left(\frac{\sin x}{x}\right)}\right)=1.\quad \textit{Why?}$$

e $$\lim_{x\to 0}\frac{\tan x}{\sin x}=\lim_{x\to 0}\frac{\left(\frac{\tan x}{x}\right)}{\left(\frac{\sin x}{x}\right)}=\frac{\lim_{x\to 0}\frac{\tan x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}}=\frac{1}{1}=1.\quad \textit{Why?}$$

f $$\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)}=\lim_{x\to 0}\frac{\left(3\frac{\sin(3x)}{3x}\right)}{\left(4\frac{\sin(4x)}{4x}\right)}=\frac{3}{4}.\quad \textit{Why?}$$

g $$\lim_{x\to 0}\frac{\sin^3 x}{x^3}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3=1$$

Theorem: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$ and $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x=e$

Example 3 Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+100}$

Solution $$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+100}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{100}=e\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)\right)^{100}=e.\quad \textit{Why?}$$

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Example 4 Evaluate $\lim\limits_{x\to\infty}\left(1+\dfrac{9}{x}\right)^{x}$

Solution Let $\dfrac{1}{y} = \dfrac{9}{x}$, then $x = 9y$.

Thus, $\lim\limits_{x\to\infty}\left(1+\dfrac{9}{x}\right)^{x} = \lim\limits_{y\to\infty}\left(1+\dfrac{1}{y}\right)^{9y} = \left[\lim\limits_{y\to\infty}\left(1+\dfrac{1}{y}\right)^{y}\right]^{9} = e^{9}$. Why?

Example 6 Evaluate $\lim\limits_{x\to-\infty}\left(\dfrac{5x+1}{5x-3}\right)^{1-4x}$

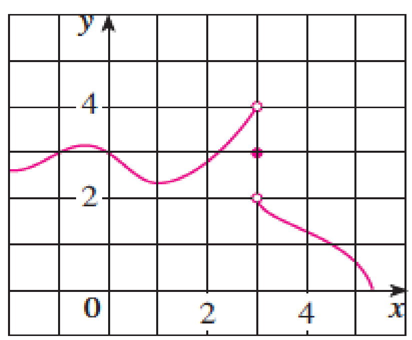
Solution $\lim\limits_{x\to-\infty}\left(\dfrac{5x+1}{5x-3}\right)^{1-4x} = \lim\limits_{x\to-\infty}\left(\dfrac{5x-3}{5x+1}\right)^{4x-1} = \left(\dfrac{1+\dfrac{-3}{5x}}{1+\dfrac{1}{5x}}\right)^{4x-1} = \left(\dfrac{e^{\frac{-3}{5}}}{e^{\frac{1}{5}}}\right)^{4} = e^{-3.2}$

Review Exercise

1. The graph of a function $\boldsymbol{f}$ is shown in the following figure. Use it to state the values(if exist). If it does not exist, explain why?

(a) $\lim\limits_{x\to0} f(x)$ (b) $\lim\limits_{x\to3^-} f(x)$ (c) $\lim\limits_{x\to3^+} f(x)$

(d) $\lim\limits_{x\to3} f(x)$ (e) $f(3)$

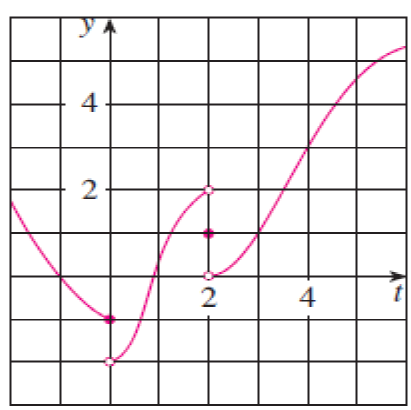


2. The graph of a function $\boldsymbol{g}$ is shown in the following figure. Use it to state the values(if exist). If it does not exist, explain why?

(a) $\lim\limits_{t\to0^-} g(t)$ (b) $\lim\limits_{t\to0^+} g(t)$ (c) $\lim\limits_{t\to0} g(t)$

(d) $\lim\limits_{t\to2^-} g(t)$ (e) $\lim\limits_{t\to2^+} g(t)$ (f) $\lim\limits_{t\to2} g(t)$

(g) $g(2)$ (h) $\lim\limits_{t\to4} g(t)$



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3. Evaluate the following limits

   a. $\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}$

   b. $\lim_{x\to 4}\frac{|x-4|}{x+4}$

   c. $\lim_{x\to 5}\frac{x^2-3x-10}{x^2-10x+25}$

   d. $\lim_{x\to 0} x\cos\left(\frac{12}{x}\right)$

   e. $\lim_{x\to 3}\frac{x-3}{x^2-6x+9}$

   f. Let $f(x)=\begin{cases} x-1, & x\le 3 \\ 3x-7, & x>3 \end{cases}$. Find $\lim_{x\to 3^+} f(x)$, $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3} f(x)$

   g. $\lim_{x\to\infty}\frac{\sqrt{x^2+2}}{3x-6}$

4. What is the value of k if $\lim_{x\to 1} f(x)$ exists? Where $f(x)=\begin{cases} x^2+3k, & x>1 \\ -2x-1, & x\le 1 \end{cases}$.

Chapter 8: Derivative and its Application

8.1. Definition of derivative and its geometric interpretation

Definition: Let $\boldsymbol{a}$ be a number in the domain of a function f. if $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists; we call this limit the **derivative of $\boldsymbol{f}$ at $\boldsymbol{a}$** and denoted it by $\boldsymbol{f'(a)}$, so that $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ if the limit is exists we say that $\boldsymbol{f}$ **has a derivative at $\boldsymbol{a}$** ,that $\boldsymbol{f}$ **is differentiable at $\boldsymbol{a}$** ,or that $\boldsymbol{f'(a)}$ **exists.**

- $f'(a)$ is the slope of the line tangent to the graph of f at $(a, f(a))$.

Notation: The derivative of $\boldsymbol{f}$ *at* $\boldsymbol{a}$ is denoted by $\boldsymbol{f'(a)}$, which is read as $\boldsymbol{f}$ prime of $\boldsymbol{a}$.If $f'(\boldsymbol{a})$ exists, then we say that $\boldsymbol{f}$ *has* a derivative at $\boldsymbol{a}$ or $\boldsymbol{f}$ is differentiable at $\boldsymbol{a}$.

Example 1. Find the derivative of each of the following functions at the given number.

a. $f(x) = 4x + 5;\ x_o = 2$

b. $f(x)=\frac{1}{4}x^2+x\ ;\ x_0=-1$

c. $f(x)=x^3-9x\ ;\ x_0=1/3$

d. $f(x)=\sqrt{x}\ ;\ x_0=4$

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Solution Using the Definition,

$$f'(x_0) = \lim_{x \to x_o} \frac{f(x) - f(x_o)}{x - x_o}, \text{ you obtain,}$$

a $$f'(2) = \lim_{x \to 2} \frac{(4x+5)-(4(2)+5)}{x-2} = \lim_{x \to 2} \frac{4x-8}{x-2} = \lim_{x \to 2} \frac{4(x-2)}{x-2} = 4.$$

b $$f'(-1) = \lim_{x \to -1} \frac{\frac{1}{4}x^2 + x - \left(\frac{1}{4}(-1)^2 - 1\right)}{x-(-1)} = \lim_{x \to -1} \frac{\frac{1}{4}x^2 + x + \frac{3}{4}}{x+1}$$

$$= \lim_{x \to -1} \frac{\frac{1}{4}(x + 3)\ (x + 1)}{x + 1} = \frac{1}{4} \lim_{x \to -1} (x + 3) = \frac{1}{2}.$$

c $$f'\left(\frac{1}{3}\right) = \lim_{x \to \frac{1}{3}} \frac{x^3 - 9x - \left(\left(\frac{1}{3}\right)^3 - 9\left(\frac{1}{3}\right)\right)}{x - \frac{1}{3}} = \lim_{x \to \frac{1}{3}} \frac{\left(x^2 + \frac{1}{3}x - \frac{80}{9}\right)\left(x - \frac{1}{3}\right)}{x - \frac{1}{3}}$$

$$= \left(\frac{1}{3}\right)^2 + \frac{1}{3} \times \frac{1}{3} - \frac{80}{9} = \frac{-26}{3}.$$

d $$f'(x) = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}$$

The different notations for the derivative

Recall the functional notation and the delta notation for the gradient of a graph at apoint. The following are some other notations for the derivatives.

If $y = f(x)$, then $f'(x) = \frac{dy}{dx}, \ \frac{d}{dx} f(x), \ D(f\ (x))$

Using these notations, we have

$$f'(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0} = \frac{d}{dx}\ f(x)\Big|_{x = x_0} = D\left(f\left(x_0\right)\right)$$

Definition;

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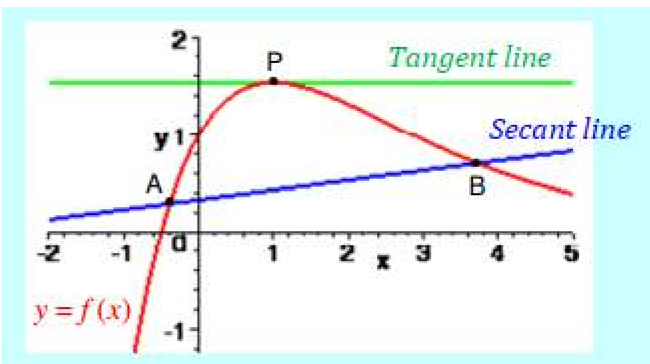
1. **A** function f is differentiable on (a,∞) if f is differentiable on (a, ∞) and the one side limit$\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ exists.
2. A function f is differentiable on $(-\infty,a)$ if f is differentiable on $(-\infty, a)$and the one side limit.$\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$exists.

Definition; Secant line and tangent line

A line which intersects a (continuous) graph in exactly two points is said to be **a secant line**.

A line, which touches a graph at exactly one point, is said to be a **tangent line** at that point. The intersection point is said to be the **point of tangency**. The slope of the graph of a function at a point **P** is the slope of the tangent line at point **P**.



8.2. Rules of differentiation

Derivative of power function

Theorem: Power Rule Differentiation

Let $f(x) = x^n$, where n is positive integer. Then $f'(x) = nx^{n-1}$.

Example 2. Find the derivatives of each of the following functions.

a $f(x) = x^4$ b $f(x) = x^{10}$ c $f(x) = x^{95}$ d $f(x) = x^{102}$

Solution:

a $f'(x) = (x^4)' = 4x^3$ b $f'(x) = (x^{10})' = 10x^9$

c $f'(x) = 95x^{94}$ d $f'(x) = 102x^{101}$

Theorem: Let $f(x) = x^r$; where r is a real number. Then $f'(x) = rx^{r-1}$

Example 3. Find the derivatives of each of the following functions.

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a $f(x) = x^{\frac{1}{2}}$ **b** $f(x) = x^{\frac{3}{5}}$ **c** $f(x) = x^{\pi}$

d $f(x) = x^{-4.05}$ **e** $f(x) = x^{\sqrt{2}}$ **f** $f(x) = x^{e-3}$

Solution:

a $f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$ **b** $f'(x) = \frac{3}{5}x^{\frac{3}{5}-1} = \frac{3}{5x^{\frac{2}{5}}}$

c $f'(x) = \pi x^{\pi-1}$ **d** $f'(x) = -4.05x^{-5.05}$

e $f'(x) = \sqrt{2}\, x^{\sqrt{2}-1}$ **f** $f'(x) = (e-3)\, x^{e-4}$

8.2.1. Derivative of a Sum or Difference of Two Functions

Theorem: If f and g are differentiable at x_o, then $f+g$ and $f-g$ are also differentiable at x_o, and

1) $(f+g)'(x_o) = f'(x_o) + g'(x_o)$……… The sum rule.
2) $(f-g)'(x_o) = f'(x_o) - g'(x_o)$……… The difference rule.

Example 4. Find the derivatives of each of the following functions**.**

a $f(x) = \sqrt{x} + 3^x$ **b** $h(x) = x^{\frac{1}{3}} + \log_2 x$ **c** $k(x) = e^x - \cos x$

Solution

a $f'(x) = \left(\sqrt{x}\right)' + \left(3^x\right)' = \frac{1}{2\sqrt{x}} + 3^x \ln 3.$

b $h'(x) = \left(x^{\frac{1}{3}}\right) + (\log_2 x)' = \frac{1}{3}x^{\frac{-2}{3}} + \frac{1}{x\ln 2} = \frac{1}{3x^{\frac{2}{3}}} + \frac{1}{x\ln 2}.$

c $k'(x) = (e^x)' - (\cos x)' = e^x - (-\sin x) = e^x + \sin x.$

8.2.2. Derivatives of Product and Quotient of Function

Theorem: If f and g are differentiable at x_o, then fg and its derivative is given as follows:

$$(fg)'(x_o) = f'(x_o)g(x_o) + f(x_o)g'(x_o)$$

Theorem: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$, for all x at which both f and g are differentiable.

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Example 5. Find the derivatives of each of the following using the product rule

a $f(x) = x\sin x$ **b** $f(x) = x^2 \cos x$

c $f(x) = (x^2 - 5x + 1)e^x$ **d** $f(x) = \sqrt{x}\log_2 x$

Solution

a $f'(x) = (x\sin x)' = (x)'\sin x + x(\sin x)' = 1 \times \sin x + x(\cos x)$

$= \sin x + x\cos x$

b $f'(x) = (x^2\cos x)' = (x^2)'\cos x + x^2(\cos x)' = 2x\cos x + x^2(-\sin x)$

$= 2x\cos x - x^2\sin x$

C and d exercise

Theorem: If f and g are differentiable functions and $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable for all x at which f and g are differentiable with

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Example 5. Find the derivatives of each of the following function at the given number.

a $f(x) = \dfrac{x}{x+5}$ at $x = 1$

b $f(x) = \tan x$ at $x = \dfrac{\pi}{3}$

c $f(x) = \dfrac{\ln x}{x}$ at $x = e$

Solution Using the quotient rule we obtain,

a $f'(x) = \dfrac{(x+5)(x)' - x(x+5)'}{(x+5)^2} = \dfrac{x+5-x}{(x+5)^2} = \dfrac{5}{(x+5)^2}$

$\Rightarrow f'(1) = \dfrac{5}{(1+5)^2} = \dfrac{5}{36}$

b $f'(x) = \left(\dfrac{\sin x}{\cos x}\right)' = \dfrac{\cos x(\sin x)' - \sin x(\cos x)'}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} = \sec^2 x \Rightarrow f'\left(\dfrac{\pi}{3}\right) = \sec^2\left(\dfrac{\pi}{3}\right) = 4$

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8.3. Derivatives of special functions (exponential, logarithmic and trigonometric)

Derivatives of Trigonometric Functions

Theorem; Derivatives of sine and cosine function

- If $f(x) = sinx\ , then\ f'(x) = cosx$
- If $f(x) = cosx\ , then\ f'(x) = -sinx$

Proof: exercise

Derivatives of exponential function

Theorem; derivative of exponential functions

If $f(x) = a^x\ ; a > 0\ , then\ f'(x) = a^x lna$

Proof: exercise

Example 2: Find the derivative of each of the following exponential functions.

a) $f(x) = 4^x$ c) $f(x) = \sqrt{5^x}$

b) $f(x) = e^{x+3}$ d)$(x) = \sqrt[3]{e^x}$

Derivatives of logarithmic functions

Theorem; derivative of logarithmic function

If $f(x) = lnx\ , x > 0\ , then\ f'(x) = \frac{1}{x}$

Proof: exercise

Corollary; if $(x) = \log_a x\ , x > 0\ , a > 0\ and\ a \neq 1\ , then\ f'(x) = \frac{1}{xlna}$

Proof:

$$f(x) = \log_a x = \frac{lnx}{lna} \Rightarrow f'(x) = \frac{1}{lna}(lnx)' = \frac{1}{xlna}$$

Example 6. Find the derivatives of each of the following logarithmic functions

a) $f(x) = \log_2 x$ c) $f(x) = \log_{\frac{1}{5}} x$

b) $f(x) = \log x$ d) $f(x) = \log x^3$

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Solution

a $f(x) = \log_2 x \Rightarrow f'(x) = \dfrac{1}{x \ln 2}$

b $f(x) = \log x \Rightarrow f'(x) = \dfrac{1}{x \ln 10}$

c $f(x) = \log_{\frac{1}{5}} x \Rightarrow f'(x) = \dfrac{1}{x \ln\left(\frac{1}{5}\right)} = -\dfrac{1}{x \ln 5}$

d $f(x) = \log(x^3) \Rightarrow f(x) = 3\log x \Rightarrow f'(x) = \dfrac{3}{x \ln 10}$

The Chain Rule

Theorem: Let g be differentiable at x_o and f is differentiable at $g(x_o)$. Then $f \circ g$ is differentiable at x_o and $(f \circ g)'(x_o) = f'(g(x_o)).g'(x_o)$

Example 7. Let $h(x) = \sin(3x + 1)$ .Evaluate $h'(\frac{\pi-2}{6})$

Example 8. Find the derivatives of $f(x) = \sqrt{1 + x^2}$ at $x = 2$

Derivatives of Composite Functions

If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x with $(f \circ g)'(x) = f'(g(x)).g'(x)$

Example 9. Find the derivative of $f(x) = e^{x^2+x+3}$

Example 10. Find the derivative of $f(x) = (x + 5)^6$

The chain rule using the notation $\frac{dy}{dx}$

Let $y = f(u)$ and $u = g(x)$. Then,

$$y = f(g(x)),\ \frac{dy}{du} = f'(u) \text{ and } \frac{du}{dx} = g'(x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(f(g(x))\right) = f'\left(g(x)\right).\, g'(x) = f'(u).\, \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}.$$

Therefore, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

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8.4. Extreme values

Maximum and minimum values of functions

One of the principal goals of calculus is to investigate the behavior of various functions. As part of this investigation, you will be laying the groundwork for solving a large class of problems that involve finding the maximum or minimum value of a function, if it exists. Such problems are often called optimization problems.

Definition

Let f be a function defined on set S.
If for some c in S
$f(c) \geq f(x)$ for every x in S, then $f(c)$ is called an absolute maximum of f on S.
If $f(c) \leq f(x)$ for every x in S, then $f(c)$ is called an absolute minimum of f on S.
The absolute maximum and absolute minimum of f on S are called extreme values or the absolute extreme values of f on S.

Sometimes we just use the terms maximum and minimum instead of absolute maximum and absolute minimum, if the context is clear.

Example

1.Given a set $S = \{0, 1, 2, 3, 4, 5\}$ and $f(x) = 2x + 3$
a Find $S' = \{f(x) | x \in S\}$ **b** What is the largest element of S'?
c What is the smallest element of S'?

Solution:

a. $S'=\{3,5,7,9,11,13\}$, since
for x=0 ,f(x)=2.0+3=3 x=1 ,f(x)=2.1+3=5
x=2 ,f(x)=2.2+3=7 x=3 ,f(x)=2.3+3=9
x=4 ,f(x)=2.4+3=11 x=5 ,f(x)=2.5+3=13

b. 13 is the largest element of S' ,thus $f(5)$ is an absolute maximum of f on S'.

c. 3 is the smallest element of S' ,thus $f(5)$ is an absolute minimum of f on S'

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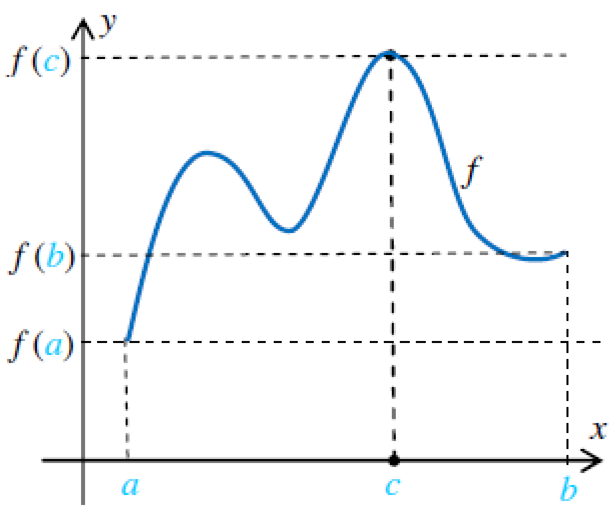
How one can be sure whether a given function f has maximum and minimum values on a given interval.

Actually, if a function f is continuous on a closed bounded interval, it can be shown that both the absolute maximum and absolute minimum must occur. This result, called the **extreme value theorem**, plays an important role in the application of derivatives.

Extreme-value theorem

Let a function f be continuous on a closed, bounded interval $[a, b]$. Then f has both absolute maximum and absolute minimum values on $[a, b]$.

To illustrate this theorem, let's consider the following graph of a function on the interval $[a, b]$.



Note that this theorem does not tell us where and how to find the maximum and minimum values on $[a, b]$; it simply asserts that a continuous function on a closed and bounded interval has extreme values.

Now let as discuss how and where to find the maximum and minimum value of f on $[a, b]$. To this end, we need to define relative extreme values and critical numbers.

Sometimes there are extreme values even when the conditions of the theorem are not satisfied, but if the conditions hold, the existence of extreme value is guaranteed.

Note that the maximum value of a function occurs at the highest point on its graph and the minimum value occurs at the lowest point.

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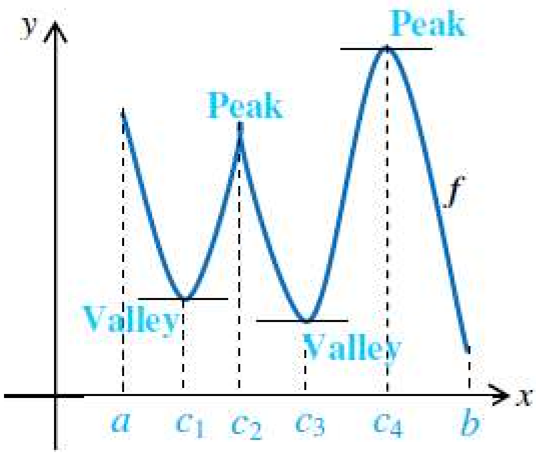
Definition

A function f is said to have a relative

a. maximum at a number c in an open interval I, if $f(c) \geq f(x)$ for all x in I.

b. minimum at a number c in an open interval I, if $f(c) \leq f(x)$ for all x in I.

The relative maxima and relative minima are called relative extrema.



Observe from the above graph that, the extrema of a continuous function occur either at end points of the interval or at points where the graph has a "peak" or a "valley" (points where the graph is higher or lower than all nearby points).

For example, the function f in the above Figure has peaks at $(c2, f(c2)), (c4, f(c4))$ and valleys at $(c1, f(c1)), (c3, f(c3))$. Peaks and valleys are what you call relative extrema.

Theorem

If a continuous function f has a relative extremum at c, then either $f'(c) = 0$ or f has no derivative at c.

The converse is not true.For example if $f(x) = x^3$ $f'(0) = 0$

However, 0 is neither a local minima nor maxima. It is in fact a *saddle point*.

Definition

Let c be in the domain of f. Then if $f'(c) = 0$ or f has no derivative at c, then c is said to be a **critical number** of f.

Example

Find the critical numbers of the given functions.

1.$f(x) = x^3 + x^2$

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2.$f(x) = 2\cos x + x$

3.$f(x) = 3x^5 - 20x^3$

Solution :

1.$f'(x) = 3x^2 + 2x$ which is defined for every real numbers.

Thus the critical number of the function is c such that

$f'(c) = 0$

$f'(c) = 3c^2 + 2c = 0$

$= c(3c + 2) = \quad 0, implies \quad C = 0 \; or \; 3c + 2 = 0 \qquad C = 0 \; or \; c = \frac{-2}{3}$ which are in the domain of the function.

To find the absolute extrema of a continuous function f on $[a, b]$:

Step 1 Compute $f'(x)$ and find critical numbers of f on (a, b).

Step 2 Evaluate f at the endpoints a, b and at each critical number.

Step 3 Compare the values in Step 2.

Thus by comparing the values of f in step 3 you have:

- ✓ the largest value of f is the absolute maximum of f on $[a, b]$.
- ✓ the smallest value of f is the absolute minimum of f on $[a, b]$.

Find the absolute extrema of the following functions on the indicated interval.

1. $f(x) = x^3 + x^2$ on [-2,1].

2. $f(x) = x^4 - 2x^2 + 3$ on [-1,2].

Rolle's theorem

Let f be function that satisfies the following three conditions:

a f is continuous on the closed interval $[a, b]$

b f is differentiable on the open interval (a, b)

c $f(a) = f(b)$

Then, there is a number c in (a, b) such that $f'(c) = 0$

Exercise

Verify that each of the following functions satisfies the three conditions of Rolle's Theorem on the given interval. Then, find all values of c that satisfy the conclusion of Rolle's Theorem.

1. $f(x) = x^2 - 4x + 1$ 0n $[0,4]$
2. $f(x) = x\sqrt{x + 6}$ 0n $[-6,0]$

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3. $f(x) = \sin 2\pi x$ on [-1,1].

The mean-value theorem

Let f be a function that satisfies the following conditions:

1. f is continuous on the closed interval $[a, b]$

2 . f is differentiable on the open interval (a, b)

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or equivalently $f(b) - f(a) = f'(c)(b - a)$

Exercise

Verify that the following functions satisfy the conditions of the mean-value theorem on the given interval. Then find all values of c that satisfy the conclusion of the Mean-value theorem.

1. $f(x) = \frac{x}{x+2}$ 0n $[1,4]$
2. $f(x) = 3x^2 + 2x + 5 \quad [-1,1]$

Definition

Let f be a function on an interval I.

- **i.** If for any x_1, x_2 in $I, x_1 < x_2$ implies $f(x_1) \leq f(x_2)$ f is said to be increasing on I.
- **ii.** If for any x_1, x_2 in $I, x1 < x_2$ implies $f(x_1) \geq f(x_2)$ f is said to be decreasing on I.
- **iii.** If for any x_1, x_2in $I, x_1 < x_2$ implies $f(x_1) < f(x_2)$ f is said to be strictly increasing on I.
- **iv.** If for any x_1, x_2 in I, $x_1 < x_2$ implies $f(x_1) > f(x_2)$ f is said to be strictly decreasing on I.

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Example 9 Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Solution $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$

You are going to find intervals in which $f'(x)$ is positive or negative. Use sign charts for this purpose, as follows:

		-1		0		2	
$12x$	$- - -$		$- - -$	0	$+ + + + + +$		$+ + +$
$x-2$	$- - -$		$- - -$		$- - - - - -$	0	$+ + +$
$x+1$	$- - -$	0	$+ + +$		$+ + + + + +$		$+ + +$
$f'(x)$	$- - -$	0	$+ + +$	0	$- - - - - -$	0	$+ + +$

From the sign chart one can see that

i $f'(x) \geq 0$ on $[-1, 0]$ and $[2, \infty)$ and $f'(x) = 0$ only at $x = -1$, 0 and $x = 2$, thus f is strictly increasing on $[-1, 0]$ and $[2, \infty)$.

ii $f'(x) \leq 0$ on $(-\infty, -1]$ and $[0, 2]$ and $f'(x) = 0$ only at $x = -1$, 0 and 2, thus f is strictly decreasing on $(-\infty, -1]$ and $[0, 2]$.

First derivative test for local extreme values of a function

Suppose that c is a critical number of a continuous function, then

a. If f' changes sign from positive to negative at c, then f has a local maximum at c.

b. If f' changes sign from negative to positive at c, then f has a local minimum at c.

c. If f' does not change sign at c (that is, f' is positive on both sides of c or negative (on both sides), then f has neither local maximum nor minimum at c.

Exercise

Find the local maximum and minimum values of the function:

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1 $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ **2** $g(x) = x + 2\sin x$ for $0 \le x \le 2\pi$

Solution

1 $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x + 1)$

From the sign chart in Example 9 one can see that

$f'(x) \le 0$ on $(-\infty, -1]$ and $[0, 2]$ and $f'(x) = 0$ only at $x = -1$, 0 and 2.

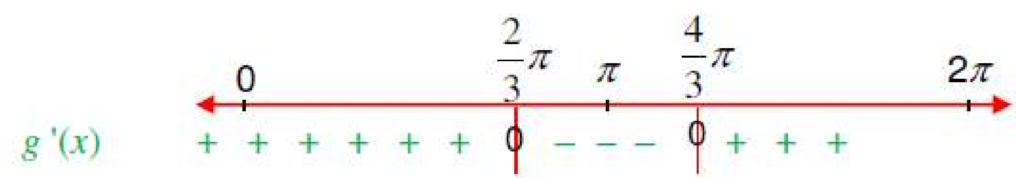
Thus, f is strictly decreasing on $(-\infty, -1]$ and $[0, 2]$.

a f' changes sign from negative to positive at -1 and 2

Hence both $f(-1) = 0$ and $f(2) = -27$ are local minimum value.

b f' changes sign from positive to negative at 0 and hence $f(0) = 5$ is the local maximum value.

2 $g'(x) = 1 + 2\cos x$, $g'(x) = 0 \Rightarrow \cos x = -\dfrac{1}{2} \Rightarrow x = \dfrac{2\pi}{3}$ or $x = \dfrac{4\pi}{3}$ in $[0, 2\pi]$



a g' changes sign from positive to negative at $\dfrac{2}{3}\pi$ and hence

$g\left(\dfrac{2}{3}\pi\right) = \dfrac{2}{3}\pi + 2\sin\left(\dfrac{2}{3}\pi\right) = \dfrac{2}{3}\pi + \sqrt{3}$ is a local maximum value

b g' changes sign from negative to positive at $\dfrac{4}{3}\pi$ and hence

$g\left(\dfrac{4}{3}\pi\right) = \dfrac{4}{3}\pi + 2\sin\dfrac{4}{3}\pi = \dfrac{4}{3}\pi - \sqrt{3}$ is a local minimum value.

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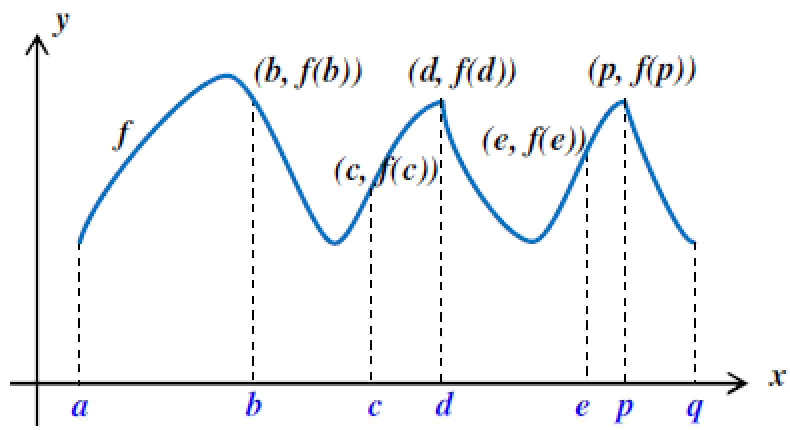
Definition

If the graph of a function lies above all of its tangents on an interval I, then it is called **concave upward** on I.

If the graph of a function lies below all of its tangents on an interval I, then it is called **concave downward** on I.

Consider the following graph.



On the intervals $(a, b), (c, d), (e, p)$ the graph is concave downward.

On the intervals $(b, c), (d, e)$ and (p, q) the graph is concave upward.

Note:

Points $(b, f\ (b)), (c, f\ (c)), (d, f\ (d)), (e, f\ (e))$ and $(p, f(p)$ are points on the graph at which concavity changes either from concave up to concave down or from concave down to concave up. Such types of points are called inflection points.

Definition

A point on a curve is called an inflection point, if the curve changes either from concave up to concave down or from concave down to concave up.

Concavity test

Let f be a function which is twice differentiable on an interval I, then

- **a.** If $f''(x) > 0$ for all x in I, the graph of f is concave upward on I.
- **b.** If $f''(x) < 0$ for all x in I, the graph of f is concave downward on I.

Another application of the second derivative is the following test for maximum and minimum values. It is a consequence of the concavity test.

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The second derivative test

Suppose f is twice differentiable and f'' is continuous at c

a. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

b. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

$f''(c) > 0$ near c and so f is concave upward near c. This means that the graph of f lies above its horizontal tangent at c and so f has a local minimum at c.

$f''(c) < 0$, near c and so f is concave downward near c. This means that the graph of f lies below its horizontal tangent at c and so f has a local maximum at c.

Example 12 Discus the behaviour of the curve $f(x) = x^4 - 4x^3$ with respect to concavity, points of inflection, local maximum and minimum.

Solution $f'(x) = 4x^3 - 12x^2 \Rightarrow f'(x) = 4x^2(x-3)$

Thus $f'(x) = 0 \Rightarrow 4x^2(x-3) = 0 \quad \Rightarrow x = 0$ or $x = 3$

Now $f''(x) = 12x^2 - 24x$, $f''(0) = 0$ and $f''(3) = 36 > 0$

Since $f'(3) = 0$ and $f''(3) = 36 > 0$, $f(3) = -27$ is a local minimum value by the second derivative test.

Since $f''(0) = 0$, the second derivative test gives no information about the critical number 0. But since $f'(x) < 0$ for $x < 0$ and also for $0 < x < 3$, the first derivative test tells us that f does not have a local extreme value at 0.

To determine intervals of concavity and inflection points we use the following sign chart:

		0		2	
$12x$	− − − − − −	0	+ + + + + +		+ + +
$x-2$	− − − − − −		− − − − − −	0	+ + +
$f''(x)$	+ + + + + +	0	− − − − − −	0	+ + +

The points with coordinates (0, 0) and (2, –16) are inflection points.

The graph of f is concave upward on $(-\infty, 0)$ and $(2, \infty)$ and concave downward on (0, 2).

NOTE

The second derivative test is inconclusive when $f''(c) = 0$. In other words, at such a point, there might be a maximum, there might be a minimum, or there might be neither. This test also fails when $f''(c)$ does not exist. In such cases, the first derivative test must be used. In fact, even when both tests apply, the first derivative test is often the easier one to use.

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8.5. Maximization and minimization problems

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: the minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

Generally such a problem will have the following mathematical form: Find the largest (or smallest) value of $f(x)$ when $a \leq x \leq b$.Sometimes a or b are infinite, but frequently the real world imposes some ***constraint*** on the values that x may have.

Such a problem differs in two ways from the relative maximum and minimum problems we encountered when graphing functions: We are interested only in the function between a and b, and we want to know the largest or smallest value that $f(x)$ takes on, not merely values that are the largest or smallest in a small interval. That is, we seek not a relative maximum or minimum but a *global* (or *absolute*) maximum or minimum.

As a summary from what you have seen in solving problems by the application of differential calculus, the greatest challenge is often to convert the real-life word problem into a mathematical maximization or minimization problem, by setting up the function that is to be maximized or minimized. The following guideline adapted to particular situation may help.

1. Understand the problem

The first step is to read the problem carefully until it is clearly understood. Ask yourself:

What is the unknown? What are the given quantities? What are the given conditions?

2. Draw a diagram (if necessary)

In most problems, it is useful to draw a diagram and identify the given and required quantities on the diagram.

3. Introduce notation

Assign a symbol to the quantity that is to be maximized or minimized and select symbols for the unknowns. Decide what the variables are and in what units their values are being measured. For example, A for area in square metres, r for radius in inches, C for cost in Euros, Birr, Dollar….

In other words, if the problem does not introduce these variables, you need to do so.

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4. Express the quantity, which is going to be optimized in terms of the unknowns.

5. If the quantity which is going to be optimized is expressed as a function of more than one unknown in step 4, use the given information to find relationships (in the form of equations) between these unknowns. Then use these equations to eliminate all but one of the unknown in the expression. Thus, the quantity which is going to be optimized will be expressed as a function of one unknown.

6.Find the critical points of f. Compare all critical values and endpoints to determine the absolute extrema of f.

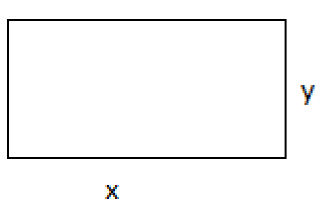
7. Provide your solution meaningfully, which includes unit(s).

Example

1. Find the dimensions of the rectangle of largest area having fixed perimeter.
2. Find two nonnegative real numbers whose sum is 18 and whose product is maximum.
3. A cylindrical can is to be made to hold 10 litres of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.
4. A box with square base is to hold a volume 200. The bottom and top are formed by folding in flaps from all four sides, so that the bottom and top consist of two layers of cardboard. Find the dimensions of the box that requires the least material. Also, find the ratio of height to side of the base.
5. Marketing tells you that if you set the price of an item at \$10 then you will be unable to sell it, but that you can sell 500 items for each dollar below \$10 that you set the price. Suppose your fixed costs total \$3000, and your marginal cost is \$2 per item. What is the most profit you can make?

Solution

1. Suppose we have the following rectangle



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Then the area is given by

$A(x, y) = xy$.

and the perimeter (for some fixed constant P) is

$p = 2x + 2y$ where we have that $x, y \geq 0$.

We will use the constraint on the perimeter to write y as a function x :

$$2x + 2y = p \Rightarrow y = \frac{p}{2} - x$$

Therefore, we can write the area as a function of only one variable:

$A(x) = x\left(\frac{p}{2} - x\right) = \frac{px}{2} - x^2$.

with domain $0 \leq x \leq \frac{p}{2}$.. Differentiating, we find

$$A'(x) = \frac{p}{2} - 2x.$$

This means that $x = \frac{p}{4}$ is the only critical point. Now compare:

$$A(0) = 0,\ A\left(\frac{p}{4}\right) = \frac{p^2}{16},\ A\left(\frac{p}{2}\right) = 0.$$

We conclude that the area is maximized when the dimensions are $x = y = \frac{p}{4}$.

2. There are many pairs of numbers whose sum is 18. For instance,

$$(1, 17), (2\ , 16), (3, 15), (4, 14), (5, 13), (6, 12), (7, 11), (8, 10), (9, 9),$$

$(5.2, 12.8), (6.5, 11.5), \ldots$, etc.

All these pairs have different products, and you cannot list all such pairs and find all the products. As a result, you fail to get the maximum product in doing this.

Instead of listing such pairs and products you take two variables say x and y such that $x \geq 0, y \geq 0$, and $x + y = 18$ with the product xy maximum.

Since $x + y = 18$, then $y = 18 - x. (0 \leq x \leq 18, 0 \leq y \leq 18)$

Thus you want to maximize $x\ (18 - x) = 18x - x^2$.

Consider $f\ (x) = 18x - x^2$, which is continuous on $[0, 18]$ and differentiable on $(0, 18)$.

$$f\ '(x) = 18 - 2x\ f\ '(x) = 0 \Rightarrow x = 9$$

The maximum occurs either at end points or at critical numbers. Thus evaluating the values of the function at critical numbers and end points, you get,

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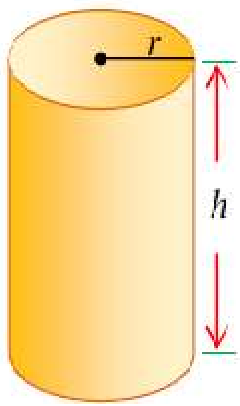
$f(0) = 0, f(18) = 0$ and $f(9) = 81$

Comparing these values, $x = 9$ gives the maximum product. Hence $x = 9$ and $y = 9$ are the two real numbers whose sum is 18 and whose product is maximum.

3.



In order to minimize the cost of the metal, you have to minimize the total surface area of the cylinder. You see that the sides are made from a rectangular sheet with dimensions $2\pi r$ (circumference of the base circle) and h. So the total surface area is given by

$$A = 2\pi r^2 + 2\pi rh$$

To eliminate h you use the fact that the volume is given as:

V =10 litres = 10,000 cm^3.

$$\Rightarrow \pi r^2 h = 10000 \qquad \Rightarrow h = \frac{10000}{\pi r^2}$$

$$\Rightarrow A(r) = 2\pi r^2 + 2\pi r\left(\frac{10000}{\pi r^2}\right) = 2\pi r^2 + \frac{20,000}{r} \Rightarrow A'(r) = 4\pi r - \frac{20,000}{r^2}$$

$$A'(r) = 0 \Rightarrow 4\pi r - \frac{20,000}{r^2} = 0 \Rightarrow r = 10\left(\sqrt[3]{\frac{5}{\pi}}\right)$$

Applying the second derivative test $A''(r) = 4\pi + \dfrac{40,000}{r^3} > 0$ for any $r > 0$, and hence $r = 10\left(\sqrt[3]{\dfrac{5}{\pi}}\right)$ gives the minimum value.

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Chapter 9: Integration and its Application

Introduction to Integral Calculus

Integration is the reverse process of differentiation. It is the process of finding the function itself when its derivative is known. Differential calculus deals with rate of change of functions, whereas integral calculus deals with total size or value such as areas enclosed by curves, volumes of revolution, lengths of a curves, total mass, total force, etc. Differential calculus and integral calculus are connected by a theorem called the fundamental theorem of calculus.

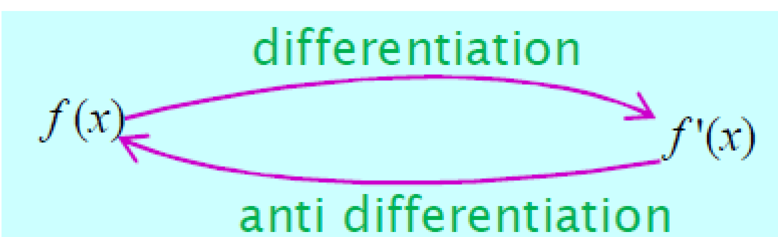
In integral calculus there are two kinds of integrations which are called the indefinite integral or the anti-derivative and the definite integral. The indefinite integral or the anti-derivative involves finding the function whose derivative is known. The definite integral, denoted by $\int_a^b f(x)dx$ is informally defined to be the signed area of the region in the xy-plane bounded by the curve $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.

N.B: One of the main goals of this unit is to examine the theory of integral calculus and introduce you to its numerous applications in science and engineering.

9.1. Integration as Reverse Process of Differentiation

The Concept of Indefinite Integral

Definition1. The process of finding $f(x)$ from its derivative $f'(x)$ is said to be anti-differentiation or integration. $f(x)$ is said to be the anti-derivative of $f'(x)$.



Integration is the reverse operation of differentiation.

Definition 2. The set of all anti-derivatives of a function $f(x)$ is called the indefinite integral of $f(x)$. The indefinite integral of $f(x)$ is denoted by $\int f(x)dx$ read as "the integral of $f(x)$ with respect to x".

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- ✓ The symbol $\int$ is said to be the integral sign.
- ✓ The function $f(x)$ is said to be the integrand of the integral.
- ✓ dx denotes that the variable of integration is x.
- ✓ If a function has an integral, then it is said to be integrable.
- ✓ If F '$(x) = f(x)$, then $\int f(x)dx = F(x) + c$
- ✓ $\int f(x)dx$ is read as, " the integral of $f(x)$ with respect to x".
- ✓ c is said to be the constant of integration.

Exercise 1:

a. $\int \frac{d}{dx}(4x^2 + 2x + 6)dx$

c. $\int \frac{d}{dx}(x + 2)dx$

b. $\int \frac{d}{dx}(x^5 + c)dx$

Integrals of constant, power, exponential and logarithmic functions and simple trigonometric functions

i. Integrals of constant and power function

N.B:

a. Let k be a constant and $n \neq -1$, then $\int k\, x^n\, dx = \dfrac{k}{n+1} x^{n+1} + c$.

b. $$\int k(ax+b)^n\ dx = \frac{k}{a(n+1)}\ (ax+b)^{n+1} + c,\ n \neq -1 \text{ and } a \neq 0.$$

Example 1. Solve the following integrals

a. $\int x^{-\frac{4}{3}}\sqrt{x}\, dx$

c. $\int (3x + 5)^3\, dx$

b. $\int (3x + 5)^3\sqrt{3x + 5}\, dx$

d. $4\sqrt[3]{(1 - 4)^5}$

ii. Integration of exponential functions

Remember that $\frac{d}{dx}e^x = e^x$, hence $\int e^x\, dx = e^x + c$ and $\int ke^x\, dx = ke^x + c$.

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Similarly for $a > 0$, $\frac{d}{dx}a^x = a^x lna \Longrightarrow \frac{1}{lna}\frac{d}{dx}a^x = a^x$, hence $\int a^x lna\, dx = a^x + c$ and

Note:

$$\int ka^x dx = \frac{k}{\ln a}a^x + c \quad \text{and} \quad \int a^{kx} dx = \frac{a^{kx}}{k\ln a} + c$$

$\int a^x\, dx = \frac{a^x}{lna} + c$; $a > 0$ and $a \neq 1$.

Exrcise 1. Evaluete

a. $\int 5^{x+1} dx$

b. $\int \frac{4e^x}{e^{4x+1}} dx$

c. $\int \sqrt{5}^{x+1} dx$

d. $\int 5e^{-1+2x} dx$

Note:

If k is a constant, then $\int \frac{k}{x} dx = k\ln|x| + c.$

iii. Integrals simple trigonometric functions

You know that $\int \frac{d}{dx} f(x)dx = f(x) + c.$ Therefore, using the derivatives of simple trigonometric functions

a. $\int \frac{d}{dx}(sinx)dx = \int cosx\, dx = sinx + c$

b. $\int \frac{d}{dx}(cosx)dx = \int sinx\, dx = -cosx + c$

Similarly, solve

a. $\int csc^2 x\, dx$

b. $\int secx\, tanx\, dx$

c. $\int csc^2 x\, dx$

d. f. $\int cscx\, cotx\, dx$

e. $\int kcscx\, cotx\, dx$

f. $\int sin(ax + b)\, dx$

Example 1. Evaluate

a. $\int sin(4x - 1)dx$

b. $\int sec^2(2x + 1)dx$

c. $\int 3cos(4x + \frac{\pi}{3})dx$

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9.2. Techniques of Integration

i. **Integration by Substitution**

This method is based on a change of variable equation and the chain rule. The change of the variable is helpful to make unfamiliar integral form to the integral form you can recognize. **Integration by substitution** is possible to transform a difficult integral to an easier integral by using a substitution.

Exercise 1. Evaluate

a. $\int \cos(ax+b)dx$

b. $\int \frac{1}{1-2x}dx$

c. $\int x^2\sqrt{1+2x^3}dx$

d. $\int \frac{sin\sqrt{x}}{\sqrt{x}}dx$

ii. **Integration by partial fractions**

Let's say that we want to evaluate $\int \frac{P(x)}{Q(x)}dx$, where $\frac{P(x)}{Q(x)}$ a proper rational fraction is. In such cases, it is possible to write the integrand as a sum of simpler rational functions by using partial fraction decomposition. Post this, integration can be carried out easily. The following image indicates some simple partial fractions which can be associated with various rational functions:

	Factor in the Denominator	Corresponding term in the Partial Fraction
1	$ax+b$	$\frac{A}{ax+b}$, where A is constant
2	$(ax\ +\ b)^k$	$\frac{A_1}{(ax+b)}+\frac{A_1}{(ax+b)^2}+\cdots+\frac{A_1}{(ax+b)^k}$ Where $A_1, A_{2,} \dots, A_k$
3	(ax^2+bx+c) With $b^2-4ac<0$	$\frac{Ax+B}{ax^2+bx+c}$, where A is constant
4	$(ax^2+bx+c)^{\boldsymbol{k}}$ With $b^2-4ac<0$	$\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$ Where $A_1, A_{2,} \dots, A_k$ *and* $B_1, B_2, \dots, B_k$ are constants

Example 1. Evaluate

a. $\int \frac{(3x-2)}{(x+1)^2\,(x+3)}dx$

c. $\int \frac{3x-1}{(x-1)\,(x-2)\,(x-3)}dx$

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b. $\int \frac{x^2+1}{x^2-5x+6}dx$

d. $\int \frac{dx}{(x+1)(x-2)}$

iii. **Integration by parts**

Note:

Let u and v be functions of x i.e. $u = u(x)$ and $v = v(x)$.

Then, $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \Rightarrow u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$

$$\Rightarrow \int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dx}dx \Rightarrow \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

In short, $\int u\,dv = uv - \int v\,du$

Exercise 1. Evaluate

c. $\int xe^x dx$

c. $\int x^2e^{3x}dx$

Note:

If $a > 0$ and $a \neq 1$,

$$\int \log_a x\,dx = \int \frac{\ln x}{\ln a}dx = \frac{1}{\ln a}\int \ln x\,dx$$
$$= \frac{1}{\ln a}(x\ln x - x) + c$$

d. $\int lnxdx$

d. $\int xcosxdx$

Exercise 1. Evaluate

a. $\int x^2 log_3^x dx$

c. $\int log(3x+1)dx$

b. $\int e^x sinxdx$

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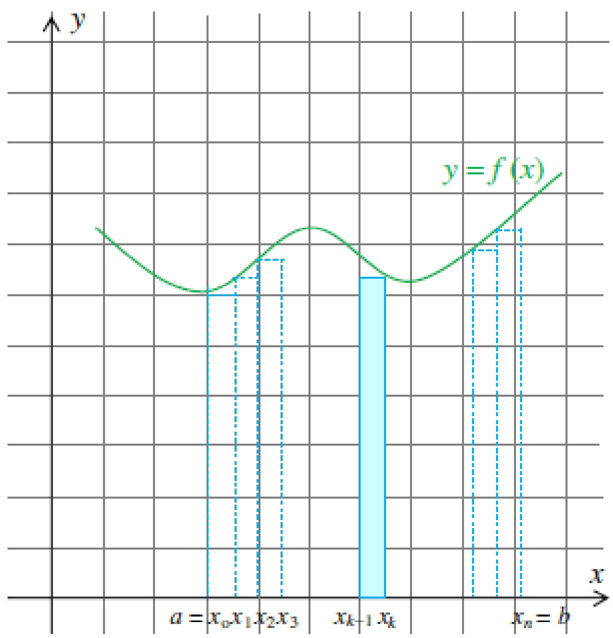
9.3. Definite Integral and Fundamental Theorem of Calculus

The Area of a Region under a Curve

Consider a function f which is non-negative and continuous on $[a, b]$. Then the area under the curve of $y = f(x)$ and the x-axis between the lines $x = a$ and $x = b$ is calculated as follows. Divide the interval $[a, b]$ into n sub intervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$ each of length $\Delta x = \frac{b-a}{n}$

Let n rectangles; each of width $\frac{b-a}{n}$ be inscribed in the region as shown in Figure.



Let $\in [x_{k-1}, x_k]$ such that $f(z_k)$ is the height of the k^{th} rectangle.

Let ΔA_k be the area of the k^{th} rectangle.

Then, $\Delta A_k = \left(\frac{b-a}{n}\right) f(z_k)$

Let A be the sum of the n – rectangles.

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Then,

$$A = \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} \left(\frac{b-a}{n}\right) f(z_k) \; = \left(\frac{b-a}{n}\right) \sum_{k=1}^{n} f(z_k)$$

The area A of the region is the limiting value of A, when $n \to \infty$

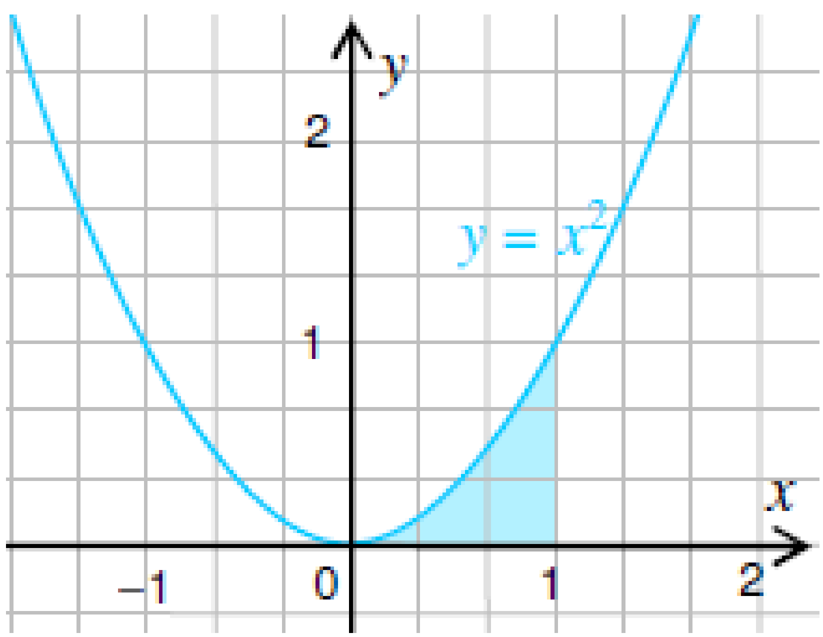
$$\text{i.e. } A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{k=1}^{n} f(z_k)$$

Definition 3

1. The sum $\sum_{k=1}^{n} f(z_k)\Delta x$ is said to be the integral sum of the function f in the interval $[a, b]$.
2. If $\lim_{n\to\infty} \sum_{k=1}^{n} f(z_k)\Delta x$ exists and is equal to I, then I is said to be the definite integral of f over the interval $[a, b]$ and is denoted by $I = \int_a^b f(x)\, dx$. a and b are said to be the lower and upper limits of integration, respectively.

Example 1 Find the area of the region enclosed by the graph of $f(x) = x^2$ and the x-axis between the lines $x = 0$ and $x = 1$.

Solution



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Using the definition, calculate the area of the region as follows.

$$A=\int_a^b f(x)\,dx \Rightarrow \int_a^b x^2\,dx=\lim_{n\to\infty}\sum_{k=1}^{n} f(z_k)\Delta x$$

Where $\Delta x=\dfrac{1-0}{n}=\dfrac{1}{n}$ and $z_k=\dfrac{k-1}{n} \Rightarrow f(z_k)=\left(\dfrac{k-1}{n}\right)^2$

$$\Rightarrow \sum_{k=1}^{n} f(z_k)\Delta x=\sum_{k=1}^{n}\left(\frac{k-1}{n}\right)^2\left(\frac{1}{n}\right)=\frac{1}{n^3}\sum_{k=1}^{n}(k-1)^2$$

$$=\frac{1}{n^3}\left[0+1+2^2+3^2+\ldots+(n-1)^2\right]$$

$$=\frac{1}{n^3}\frac{(n-1)(n)(2(n-1)+1)}{6}=\frac{1}{6n^3}\left[2n^3-3n^2+n\right]$$

$$=\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}$$

$$\Rightarrow A=\int_a^b x^2dx=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}\right)=\frac{1}{3}$$

Theorem: Estimate of the definite integral

If the function f is continuous on $[a,b]$, then $\lim_{n\to\infty} f(z_i)\Delta x$ exists.

That is, the definite integral $\int_a^b f(x)\,dx$ exists.

Example 2 Show that $\int_0^{\frac{\pi}{2}} \sin x\,dx$ exists.

Solution $f(x)=\sin x$ is continuous on $\left[0,\ \dfrac{\pi}{2}\right]$.

Thus, by the above theorem, the definite integral exists.

Example 3 Show that $\int_{-1}^{2}\dfrac{1}{x}dx$ doesn't exist.

Solution $f(x)=\dfrac{1}{x}$ is discontinuous at $x=0$.

$\Rightarrow$ f is not continuous on $[-1,2] \Rightarrow \int_{-1}^{2}\dfrac{1}{x}dx$ doesn't exist.

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Fundamental Theorem of Calculus

Fundamental Theorem of calculus is the statement which asserts that differentiation and integration are inverse operations of each other. The next theorem allows you to evaluate the definite integral by using the anti-derivative of the function to be integrated.

Theorem: Fundamental theorem of calculus

If f is continuous on the closed interval $[a, b]$ and F is an anti-derivative (or indefinite integral) of f. That is, $F'(x) = f(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\, dx = F(b) - F(a)$

Example 4 Evaluate $\int_1^4 x\, dx$

Solution This value is calculated using the definition of definite integrals.

Here you use the fundamental theorem of calculus.

The indefinite integral,

$$F(x) = \int x\, dx = \frac{x^2}{2} + c$$

$$\Rightarrow \int_1^4 x\, dx = F(4) - F(1) = \left(\frac{4^2}{2} + c\right) - \left(\frac{1^2}{2} + c\right) = \frac{15}{2}$$

Observe that evaluating the definite integral using the integral sum is lengthy and complicated as compared to using the fundamental theorem of calculus.

Note:

In evaluating $F(b) - F(a)$, the constant of integration cancels out.

Therefore, you write $F(x)\Big|_a^b$ to mean $F(b) - F(a)$

Example 5 Evaluate $\int_1^3 \left(x^3 + x + 1\right) dx$

Solution $\int_1^3 \left(x^3 + x + 1\right) dx = \frac{x^4}{4} + \frac{x^2}{2} + x\Big|_1^3 = \left(\frac{3^4}{4} + \frac{3^2}{2} + 3\right) - \left(\frac{1}{4} + \frac{1}{2} + 1\right)$

$$= \frac{81}{4} + \frac{9}{2} + 3 - \frac{1}{4} - \frac{1}{2} - 1 = \frac{80}{4} + \frac{8}{2} + 2 = 26$$

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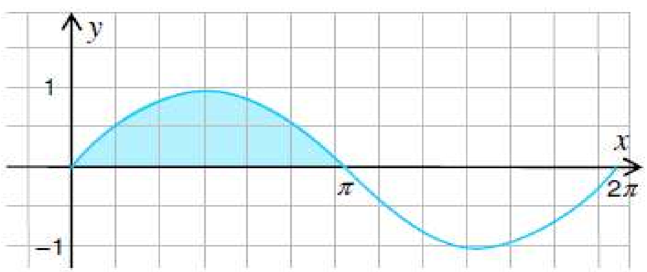
Example 6 Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\, dx$

Solution $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\, dx = -\cos x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = -\left[\cos\frac{\pi}{3} - \cos\frac{\pi}{6}\right] = -\left[\frac{1}{2} - \frac{\sqrt{3}}{2}\right] = \frac{\sqrt{3}-1}{2}$$

Example 7 Find the area of the region bounded by the arc of the sine function between $x = 0$ and $x = \pi$.

Solution The area A of this region identified to be the value of the definite integral $\int_0^{\pi} \sin x\, dx$.

$$\Rightarrow A = \int_0^{\pi} \sin x\, dx = -\cos x \Big|_0^{\pi} = -[\cos\pi - \cos 0] = -[-1-1] = 2$$



Properties of the definite integral

If f and g are continuous on $[a, b]$, $k \in \mathbb{R}$ and $c \in [a, b]$ then

1. $\int_a^a f(x)\, dx = 0$
2. $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$
3. $\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$
4. $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$
5. $\int_a^b kf(x)\, dx = k\int_a^b f(x)\, dx$

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Example 10 Evaluate each of the following integrals using the above properties.

a $\int_{3}^{3}(x^{3}+1)\,dx$ **b** $\int_{\frac{\pi}{4}}^{\pi}\sin x\,dx$ **c** $\int_{1}^{2}\left(x-\frac{1}{x^{2}}\right)^{2}dx$

d $\int_{1}^{\sqrt{2}}\frac{x}{x^{2}+1}dx+\int_{\sqrt{2}}^{5}\frac{x}{x^{2}+1}dx$ **e** $\int_{-1}^{1}e^{\pi+3x}\,dx$

Change of variable

In evaluating the indefinite integral $\int x\,dx = F(x)$, the methods you have been using are: substitution, partial fractions and integration by parts.

In the substitution method, the composition function fog is the anti-derivative of $(fog).g'$.

$$\Rightarrow \int_{a}^{b} f\big(g(x)\big), g'(x)\,dx = F(g(b)) - F\big(g(a)\big)$$

To evaluate the definite integral by the method of substitution, you transform the integrand as well as the limits of integration.

For this process you have the following theorem.

Theorem: Change of variables

If the function f is continuous on a closed interval $[c, d]$, the substitution function $u = g(x)$ is differentiable on $[a, b]$ with $g(a) = c$ and $g(b) = d$, then

$$\int_{a}^{b} f\big(g(x)\big), g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 11 Evaluate the integral $\int_{1}^{2} x e^{(x^{2})}dx$

Solution Using integration by substitution,

$$u = x^{2}, \frac{du}{dx} = 2x \Rightarrow \frac{1}{2}\,du = x\,dx$$

As x varies from 1 to 2, $u = g(x)$ varies from $g(1) = 1$ to $g(2) = 2^{2} = 4$.

$$\int_{1}^{2} xe^{x^{2}}\,dx = \frac{1}{2}\int_{1}^{4} e^{u}\,du = \frac{1}{2}e^{u}\Big|_{1}^{4} = \frac{1}{2}\left(e^{4}-e\right)$$

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Example 12 Evaluate the integral $\int_{-3}^{1} x\sqrt{2x^2+5}\,dx$.

Solution Here, $u = g(x) = 2x^2 + 5$, $g(-3) = 2(-3)^2 + 5 = 23$,

$g(1) = 2(1)^2 + 5 = 7$

$$\frac{du}{dx} = \frac{d}{dx}\left(2x^2+5\right) = 4x \Rightarrow \frac{1}{4}du = x\,dx$$

$$\int_{-3}^{1} x\sqrt{2x^2+5}\,dx = \frac{1}{4}\int_{23}^{7}\sqrt{u}\,du = \frac{1}{4}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_{23}^{7} = \frac{1}{6}\left(7\sqrt{7} - 23\sqrt{23}\right).$$

Example 13 Evaluate $\int_{0}^{\frac{\pi}{3}} \cos^3 x \sin x\,dx$

Solution The derivative of $\cos x$ is $-\sin x$ which is a factor of the integrand.

Hence, $u = g(x) = \cos x$.

$\Rightarrow \; -du = \sin x\,dx$.

$$\int_{0}^{\frac{\pi}{3}} \cos^3 x \sin x\,dx = -\int_{g(0)}^{g\left(\frac{\pi}{3}\right)} u^3\,du = -\int_{1}^{\frac{1}{2}} u^3\,du = -\frac{u^4}{4}\Bigg|_{1}^{\frac{1}{2}} = -\left(\frac{1}{64} - \frac{1}{4}\right) = \frac{15}{64}$$

9.4. Area of a region under a curve and between two curves

The Area between Two Curves

You calculated the area of some regions under the graphs of a non-negative function f on $[a, b]$, when the definite integral $\int_a^b f(x)\,dx$ was defined. However the focus was to evaluate the integral rather than to calculate area. Here, you use this concept of area in order to determine the area of a region whose upper and lower boundaries are graphs of continuous functions on a given closed interval $[a, b]$.

Example 1 Find the area of the region bounded by the graph of the function $f(x) = x^2 - 3x + 2$ and the x-axis between $x = 0$ and $x = 3$.

Solution Look at the graph of f between $x = 0$ and $x = 3$.

Let A_1, A_2 and A_3 be the areas of the parts of the region between $x = 0$ and $x = 1$, $x = 1$ and $x = 2$ and $x = 2$ and $x = 3$, respectively.

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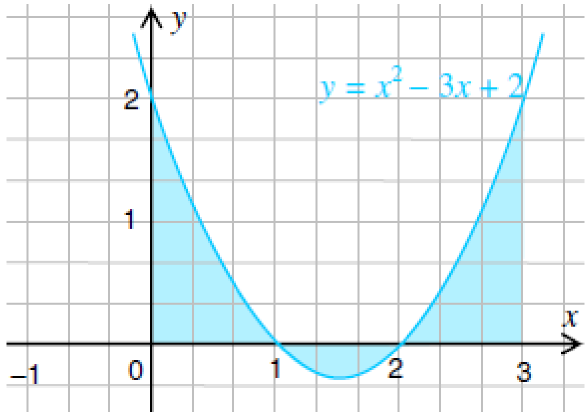

The part of the region between $x = 1$ and $x = 2$ is below the x-axis.

$$\Rightarrow A_2 = -\int_1^2 \left(x^2 - 3x + 2\right) dx = -\left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Bigg|_1^2 = 4 - \frac{23}{6} = \frac{1}{6}$$

Whereas, $A_1 = \int_0^1 (x^2 - 3x + 2)\, dx = \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Bigg|_0^1 = \frac{5}{6}$ and

$$A_3 = \int_2^3 \left(x^2 - 3x + 2\right) dx = \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\Bigg|_2^3 = \frac{5}{6}$$

Therefore, the area A of the region is

$$A = A_1 + A_2 + A_3 = \frac{11}{6}$$

What would have happened, if you had simply tried to calculate A as

$$A = \int_0^3 \left(x^2 - 3x + 2\right) dx\,?$$

Example 2 Find the area of the region enclosed by the graph of $f(x) = \sin x$ and the x-axis between $x = -\frac{\pi}{2}$ and $x = 2\pi$.

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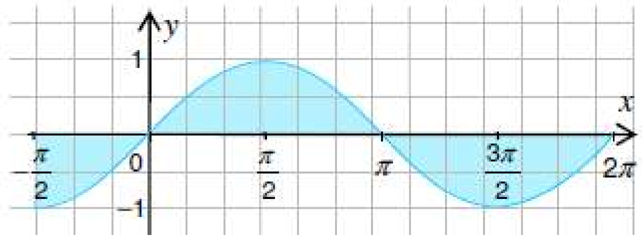
Solution



From the graph you have the area A of the region

$$A = -\int_{-\frac{\pi}{2}}^{0} \sin x\,dx + \int_{0}^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$$

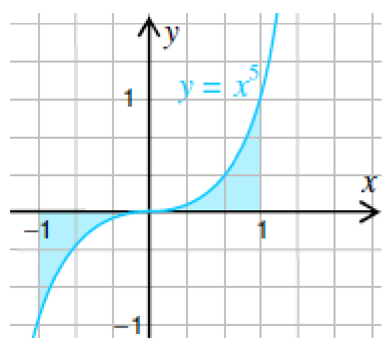
$$= -\cos x\Big|_{0}^{\frac{-\pi}{2}} - \cos x\Big|_{0}^{\pi} + \cos x\Big|_{\pi}^{2\pi} = \cos x\Big|_{-\frac{\pi}{2}}^{0} + \cos x\Big|_{\pi}^{0} + \cos x\Big|_{\pi}^{2\pi}$$

$$= \cos 0 + (\cos 0 - \cos \pi) + (\cos (2\pi) - \cos (\pi))$$

$$= 1 + 1 - (-1) + 1 - (-1) = 5$$

Example 3 Find the area of the region bounded by the graph of $f(x) = x^5$ and the x-axis between $x = -1$ and $x = 1$.

Solution



From the symmetry of the region, you have the area

$$A = 2\int_{0}^{1} x^5\,dx = 2\left(\frac{x^6}{6}\right)\Big|_{0}^{1} = \frac{1}{3}$$

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Theorem:

Suppose f and g are continuous functions on $[a, b]$ with $f(x) \geq g(x)$ on $[a, b]$. The area A bounded by the curves of $y = f(x)$ and $y = g(x)$ between the lines $x = a$ and $x = b$ is

$$A = \int_a^b (f(x) - g(x))\, dx$$

Example 6 Find the area of the region enclosed by the curves $g(x) = x^2 - x - 6$ and $f(x) = x - 3$.

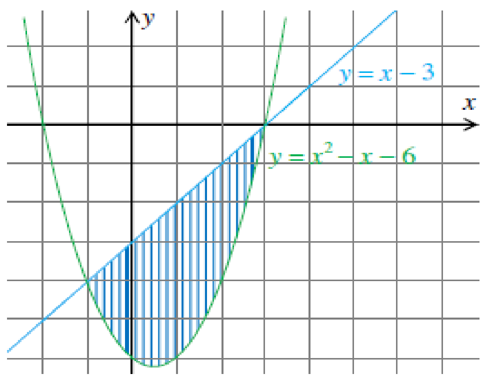
Solution The first step is to draw the graphs of both functions using the same axes.

You solve the equation $f(x) = g(x)$ to get the intersection points of the graphs.

$$x^2 - x - 6 = x - 3$$

$$\Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = 3 \text{ or } x = -1$$

$f(x) \geq g(x)$ on $[-1, 3]$



✍Note:
The height of each infinitesimal rectangle within the shaded region is equal to $(f(x) - g(x))$.

$\Rightarrow$ The area of the region is

$$A = \int_{-1}^{3} \left((x-3) - (x^2 - x - 6)\right) dx = \int_{-1}^{3} \left(-x^2 + 2x + 3\right) dx = -\frac{x^3}{3} + x^2 + 3x \Big|_{-1}^{3}$$

$$= \frac{-27}{3} + 9 + 9 - \left(-\left(\frac{-1}{3}\right) + 1 + 3(-1)\right) = 9 - \left(\frac{1}{3} + 1 - 3\right) = \frac{32}{3}$$

Example 7 Find the area of the region enclosed by the graphs of $f(x) = 4 - x^2$ and $g(x) = 6 - x$ between the lines $x = -2$ and $x = 3$.

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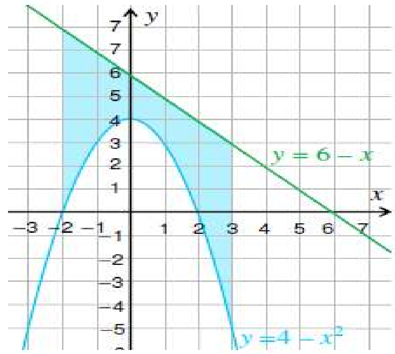
Solution The first step is to draw the graphs of both functions using the same coordinate axes.



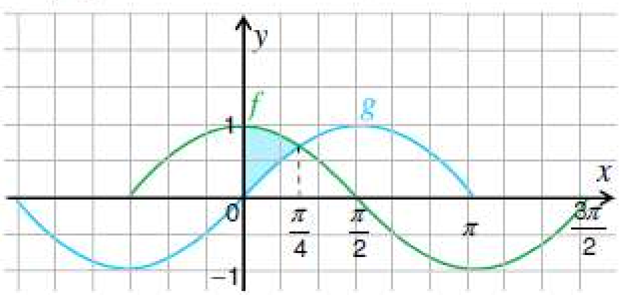
$g(x) \geq f(x)$ on $[-2, 3]$

$\Rightarrow$ The Area, $A = \int_{-2}^{3} \left((6-x)-(4-x^2)\right) dx = \int_{-2}^{3} (x^2 - x + 2)\, dx$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 2x \Big|_{-2}^{3} = \frac{27}{3} - \frac{9}{2} + 6 - \left[\frac{-8}{3} - \frac{4}{2} - 4\right] = \frac{115}{6}$$

Example 8 Find the area of the region in the first quadrant which is enclosed by the y-axis and the curves of $f(x) = \cos x$ and $g(x) = \sin x$.

Solution Look at the graphs of both functions.



The curves meet at $x = \frac{\pi}{4}$ and $\cos x \geq \sin x$ on $\left[0, \frac{\pi}{4}\right]$.

Therefore the required area is

$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx = \sin x - (-\cos x) \Big|_0^{\frac{\pi}{4}}$$

$$= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - (\sin 0 + \cos 0) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 = \sqrt{2} - 1$$

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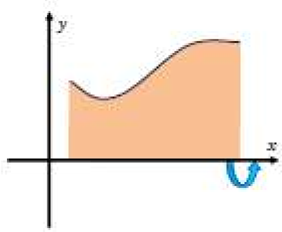
Exercise

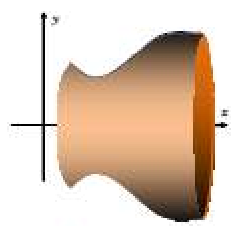
1. Find the area enclosed by the graphs of $f(x) = x^3 - x$ and $g(x) = 4 - x^2$.
2. Find the area of the region enclosed by the curves of $f(x) = 2x^2 - x^3$ and $g(x) = x^2 - 1$.
3. Find the area enclosed by the graph of $f(x) = |x|$ and the x-axis between the vertical lines $x = -4$ and $x = 3$.
4. Determine the area of the region enclosed by the graphs of $x = -y^2$ and $x = 9 - 2y^2$.
5. Find the area of the region enclosed by the graph of $y = x^2 + 1$ and the line $y = 5$.

Volume of Revolution

In this section, you will apply integral calculus to determine the volume of a solid by considering cross sections. Suppose a region rotates about a straight line as shown in the figure below. Then a solid figure, called a solid of revolution, will be formed.

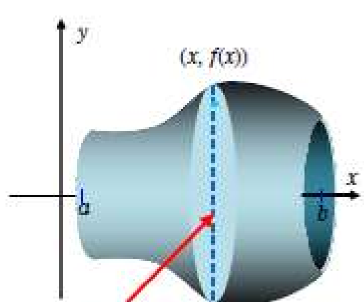


a A region to be rotated (Two dimensional)



b A solid of revolution (Three dimensional)

The volume of a solid of revolution is said to be a volume of revolution. The line about which the area rotates is an axis of symmetry. Now, consider the following solid of revolution generated by revolving the region between the curve $y = f(x)$ and the x – axis from $x = a$ to $x = b$.



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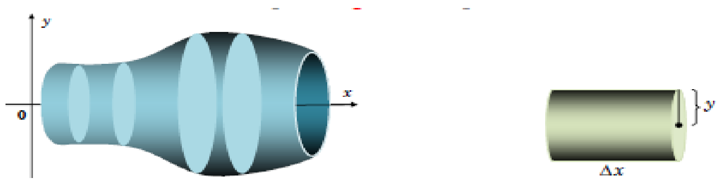
Every cross section which is perpendicular to the x −axis at x is a circular region with radius $r = f(x)$. Thus, the area of the cross section is $\pi r^2 = \pi(f(x))^2$.

How to determine the volume of a solid of revolution

Divide the solid of revolution into n equally spaced cross sections which are perpendicular to the axis of rotation [See the Figure Below].



As the cuts get close enough, then the sections so obtained will approximately be a cylindrical solid.

Let V_K be the volume of the k^{th} sections, then

$$V_K = \pi r^2 h, \text{ where } r = f(x_k) \text{ and } h = \Delta x$$

$$\Rightarrow V_k = \pi\, (f(x_k))^2\, \Delta x$$

Let ΔV be the sum of the volumes of the n sections.

Then, $\Delta V = \sum_{k=1}^{n} v_k$.

The volume V of the solid of revolution is

$$V = \lim_{\Delta x \to 0} \Delta V = \lim_{n \to \infty} \sum_{k=1}^{n} V_k = \lim_{n \to \infty} \sum_{k=1}^{n} \pi\big(f(x_k)\big)^2 \Delta x = \int_a^b \pi\big(f(x)\big)^2 dx$$

Exercise

1. Find the volume generated when the area bounded by the line $y = x$ and the x-axis from $x = 0$ to $x = 3$ is rotated about the x-axis.
2. Find the volume of the solid generated by revolving the region bounded by the graph of $y = x^2$ and the x-axis between $x = 0$ and $x = 1$ about the x-axis.
3. The area bounded by the graph of $y = x^2 + 1$ and the line $y = 4$ rotates about the y-axis, find the volume of the solid generated.
4. Find the volume of the solid of revolution about the x-axis, when the region enclosed by $y = e^x - 1$ and the x-axis from $x = \ln\left(\frac{1}{2}\right)$ to $x = \ln(2)$ rotates.
5. Using the volume of a solid of revolution, show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

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Unit Summary

1. *Anti-derivative or Indefinite integral*

Let $f\ (x)$ be a function, then

- ✓ $F(x)$ is said to be an antiderivative of $f\ (x)$ if $F'^{(x)} = f\ (x)$.
- ✓ The set of anti-derivatives of $f\ (x)$ is said to be the indefinite integral of $f(x)$.
- ✓ The indefinite integral of $f\ (x)$ is denoted by $\int f\ (x)\, dx$.
- ✓ If $F\ (x)$ and $G\ (x)$ are anti-derivatives of $f\ (x)$, then the difference between $F\ (x)$ and $G\ (x)$ is a constant.

2. *The Integral of Some Functions*

The Integral of power functions

i. $\int x^r\, dx = \frac{x^{r+1}}{r+1} + c;\ \ r \neq -1$

ii. If $r = -1$, then $\int \frac{1}{x} dx = \ln x + c$

iii. $\int k\, x^r\, dx = k \int x^r\ dx$

The Integral of trigonometric functions

i. $\int \cos x\, dx = \sin |x| + c$

ii. $\int \sin x\, dx = -\cos |x| + c$

iii. $\int \sec^2 x\, dx = \tan x + c$

iv. $\int \sec x \tan x\ dx = \sec x + c$

v. $\int \tan x\, dx = -\ln |\cos x| + c$

vi. $\int \csc x \cot x\ dx = -\csc x + c$

vii. $\int \csc^2 x\, dx = -\cot x + c$

The Integral of exponential functions

i. $\int e^x\, dx = e^x + c$

ii. $\int \log_a x\, dx = \frac{1}{\ln a}(x \ln x - x) + c$

3. *The Integral of a sum or difference of functions*

i. $\int k\, f(x)\, dx = k \int f(x)\ dx$

ii. $\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx$

4. *Techniques of Integration*

Integration by substitution

$\int f\big(g(x)\big).\, g'(x)\ dx = \int f(u)\ du;$ where $u = g(x)$.

i. $\int f'(x) f(x)\ dx = \frac{\big(f(x)\big)^2}{2} + c$

ii. $\int \frac{f'(x)}{f(x)}\ dx = \ln |f(x)| + c$

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Integration by parts

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

5. *Fundamental Theorem of Calculus*

If $f(x) = F'(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

6. *Properties of definite integrals*

i. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

ii. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

iii. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$

iv. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

v. $\int_a^a f(x)\,dx = 0$

vi. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx;\ \ a \leq c \leq b$

vii. If $u = g(x)$, $\int_a^b f\big(g(x)\big).g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

7. *Applications of the definite integral*

i. The area A bounded by two continuous curves $y = f(x)$ and $y = g(x)$ on $[a, b]$ with $f(x) \geq g(x)\ \forall x \in [a, b]$ is

$$A = \int_a^b \big(f(x) - g(x)\big)\,dx$$

ii. The volume V of a solid of revolution generated by revolving the region bounded by $y = f(x)$ and $y = g(x)$ with $f(x) \geq g(x)\ \forall x \in [a, b]$ about the x-axis is

$$V = \pi\int_a^b \Big(\big(f(x)\big)^2 - \big(g(x)\big)^2\Big)\,dx$$

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Chapter 10: Further on Trigonometry

10.1. The Functions $y = sec\ x$, $y = cosecx$ and $y = cot\ x$

The three fundamental trigonometric functions of acute angle θ are defined as follows.

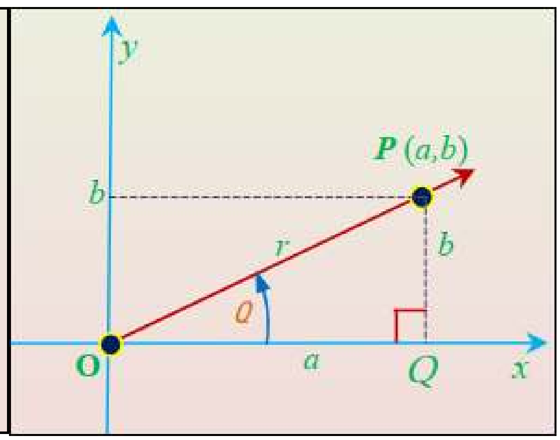
Name of Function	Abbreviation	Value at θ
sine	sin	$\sin\ \theta = \frac{opp}{hyp}$
cosine	cos	$\cos\ \theta = \frac{adj}{hyp}$
tangent	tan	$\tan\ \theta = \frac{opp}{adj}$

Considering the standard right- angled triangle and looking at the ratios these basic trigonometric functions represents in relation to the angle Q, you obtain:

$$\sin Q = \frac{OPP}{HYP} = \frac{b}{r}$$

$$\cos Q = \frac{ADJ}{HYP} = \frac{a}{r}$$

$$\tan Q = \frac{OPP}{ADJ} = \frac{b}{a}$$



- There are six trigonometric functions. The reciprocal of the ratios that define the sine, cosine, tangent functions are used to define the remaining three trigonometric functions. These reciprocal functions of acute angle θ are defined as follows.

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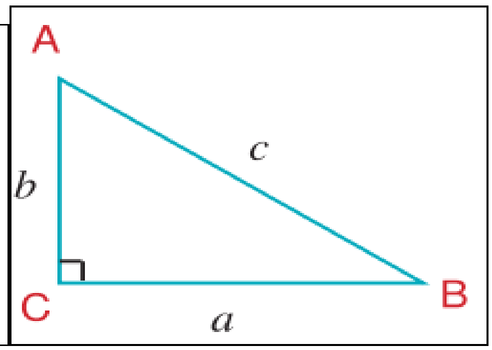
Name of Function	Abbreviation	Value at θ
cosecant	csc	$\csc\theta = \dfrac{hyp}{opp}$
secant	sec	$\sec\theta = \dfrac{hyp}{adj}$
cotangent	cot	$\cot\theta = \dfrac{adj}{opp}$

✓ The relationship of these trigonometric functions in a standard right angled triangle is shown below:

$$\csc A = \frac{c}{a} = \frac{1}{\sin A}$$

$$\sec A = \frac{c}{b} = \frac{1}{\cos A}$$

$$\cot A = \frac{b}{a} = \frac{1}{\tan A}$$



Example 1 Given the triangle below, find:

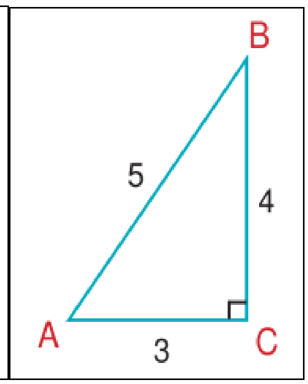
a $\cot A$ **b** $\csc B$

c $\sec A$ **d** $\csc A$

Solution

a $\cot A = \frac{3}{4}$ **b** $\csc B = \frac{5}{3}$

c $\sec A = \frac{5}{3}$ **d** $\csc A = \frac{5}{4}$



Graphs of $\boldsymbol{y = \sec x}$, $\boldsymbol{y = cosecx}$ **and** $\boldsymbol{y = \cot x}$

In Grade 10, you studied the graphs of the sine, cosine and tangent functions. In this topic you will study graphs of remaining trigonometric functions.

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1 If $f(x) = \csc x$, then $D_f = \{x \in \mathbb{R}: x \neq k\pi, k \in \mathbb{Z}\}$

Range = $(-\infty, -1] \cup [1, \infty)$

Period, $P = 2\pi$

2 If $f(x) = \sec x$, then $D_f = \left\{x \in \mathbb{R} : x \neq \frac{(2k+1)\pi}{2} ; k \in \mathbb{Z}\right\}$

Range = $(-\infty, -1] \cup [1, \infty)$

Period, $P = 2\pi$

3 If $f(x) = \cot x$, then $D_f = \{x \in \mathbb{R} : x \neq k\pi, k \in \mathbb{Z}\}$

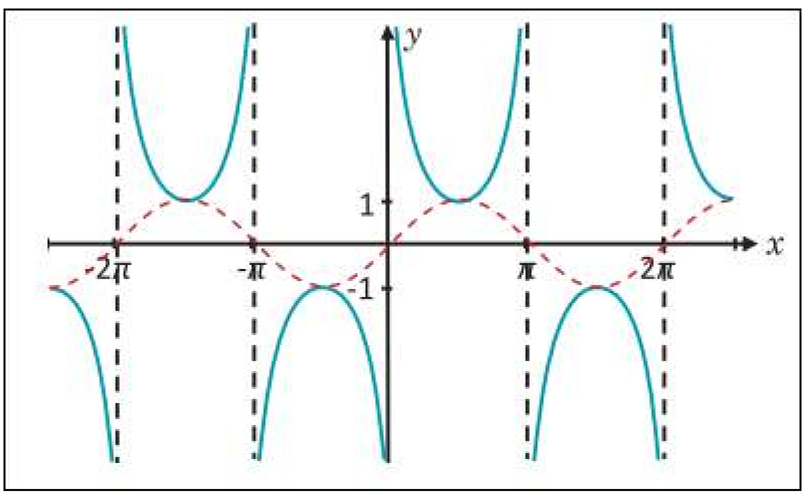
Range = $\mathbb{R}$

Period, $P = \pi$

Here you want to draw graph of

$f(x) = csc\ x$

- ✓ The graph of $y = csc\ x$ has a vertical asymptote at the point where the graph of sine function crosses the x-axis.



Graph of $y = csc\ x$

- ✓ Applying the same techniques as for the cosecant function, we can draw the graphs of secant and cotangent functions as follows.

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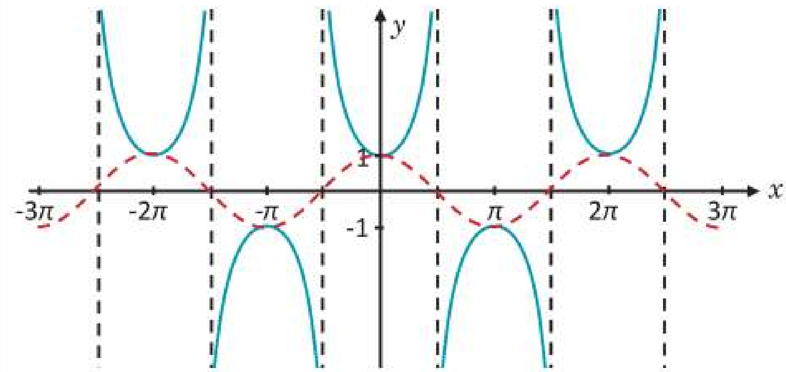
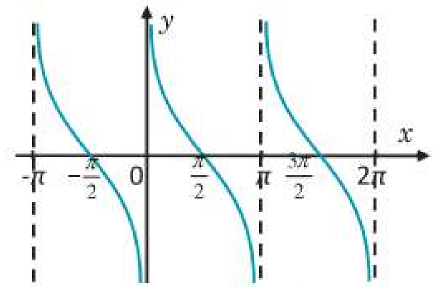


Graph of $y = \sec x$



Graph of $y = \cot x$

1 Determine each of the following values without the use of tables or calculators.

a $\sec\left(\frac{\pi}{4}\right)$ **b** $\csc\left(-\frac{\pi}{2}\right)$ **c** $\cot\left(\frac{-3\pi}{4}\right)$

d $\sec\left(\frac{\pi}{3}\right)$ **e** $\csc\left(-\frac{\pi}{6}\right)$ **f** $\cot\left(\frac{5\pi}{6}\right)$

g $\sec\left(\frac{2\pi}{3}\right)$ **h** $\csc\left(\frac{7\pi}{3}\right)$ **i** $\cot\left(\frac{7\pi}{6}\right)$

j $\cot(-\pi)$ **k** $\sec\left(\frac{5\pi}{2}\right)$ **l** $\csc(3\pi)$

2 Determine the largest interval $I \subseteq [0, 2\pi]$ on which

a $f(x) = \csc x$ is increasing. **b** $f(x) = \sec x$ is increasing.

c $f(x) = \cot x$ is increasing.

3 Simplify each of the following expressions.

a $\sec x \sin x$ **b** $\tan x \csc x$ **c** $1 + \frac{\tan x}{\cos x}$

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10.2. Inverse of Trigonometric Functions

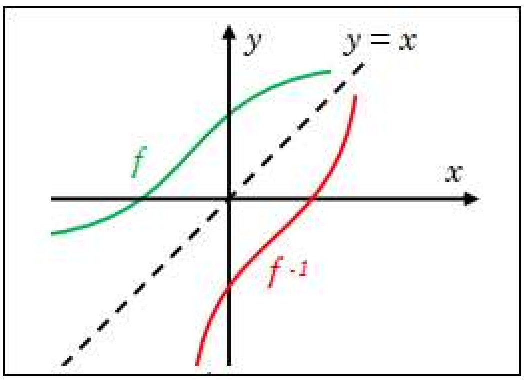
You now need to define inverses of the trigonometric functions, starting with a brief review of the general concept of inverse functions.

Facts about inverse functions

For f a one-to-one function and f^{-1} its inverse:

1. If (a, b) is an element of f, then (b, a) is an element of f^{-1}, and conversely.
2. Range f = Domain of f^{-1}
3. Domain of f = Range of f^{-1}

The graph of f^{-1} is obtained by reflecting the graph of f in the line $\boldsymbol{y = x}$.



Definition 9.1 Inverse sine or Arcsine function

The inverse sine or arcsine function, denoted by $\sin^{-1}$ or arcsin, is defined by

$\sin^{-1}x = y$ or $\arcsin x = y$, if and only if $x = \sin y$ for $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$

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☑ Remark:

1 The inverse sine function is the function that assigns to each number x in $[-1, 1]$ the unique number y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $x = \sin y$.

2 Domain of $\sin^{-1} x$ is $[-1, 1]$ and Range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

3 From the definition, you have

$\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ $\quad\quad$ $\sin^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

☑ Caution:

$\sin^{-1} x$ is different from $(\sin x)^{-1}$ and $\sin x^{-1}$;

$(\sin x)^{-1} = \dfrac{1}{\sin x}$ and $\sin x^{-1} = \sin\left(\dfrac{1}{x}\right)$

Example 1 Calculate $sin^{-1}\, x$ for

a $x = 0$ **b** $x = 1$ **c** $x = \frac{\sqrt{3}}{2}$ **d** $x = -1$

Solution

a $\sin^{-1}(0) = 0$ since $\sin 0 = 0$ and $0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

b $\sin^{-1}(1) = \frac{\pi}{2}$ since $\sin\left(\frac{\pi}{2}\right) = 1$ and $\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

c $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

d $\sin^{-1}(-1) = -\frac{\pi}{2}$ since $\sin\left(-\frac{\pi}{2}\right) = -1$ and $-\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

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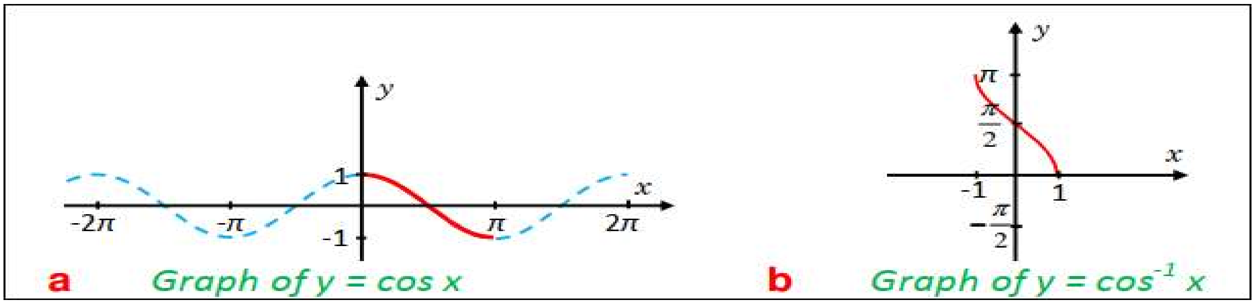
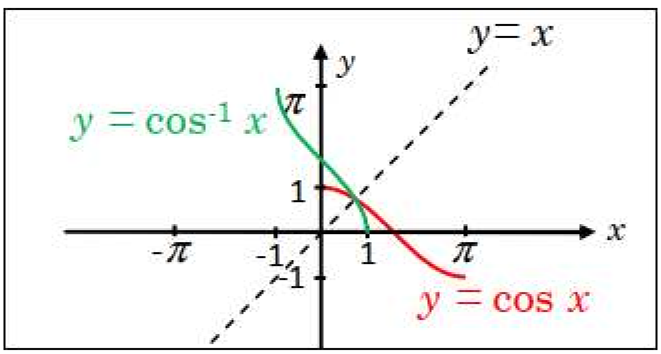
Inverse of cosine function

- ✓ You know that $cos\ x$ is not one-to-one. Note, however, that $cos\ x$ decreases from 1 to –1 in the interval [0, π]. Thus if $y = cos\ x$ and x is restricted in the interval $[0, \pi]$, then for every y in [–1, 1], there is a unique x such that $cos\ x = y$.



Use this restricted cosine function to define the inverse cosine function. Reflecting the graph of $y = cos\ x$ on $[0, \pi]$ in the line $y = x$, gives the graph of $f(x) = cos^{-1} x$ as shown below:



Definition 9.2

The inverse cosine or arccosine function, denoted by $\cos^{-1}$ or arccos, is defined by

$\cos^{-1} x = y$, if and only if $x = \cos y$, for $0 \leq y \leq \pi$.

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☑ Remark:

1 Domain of $\cos^{-1}x$ is $[-1, 1]$ and Range of $\cos^{-1}x$ is $[0, \pi]$

2 From the definition, you have

$$\cos(\cos^{-1}x) = x, \text{ if } -1 \le x \le 1.$$

$$\cos^{-1}(\cos x) = x, \text{ if } 0 \le x \le \pi.$$

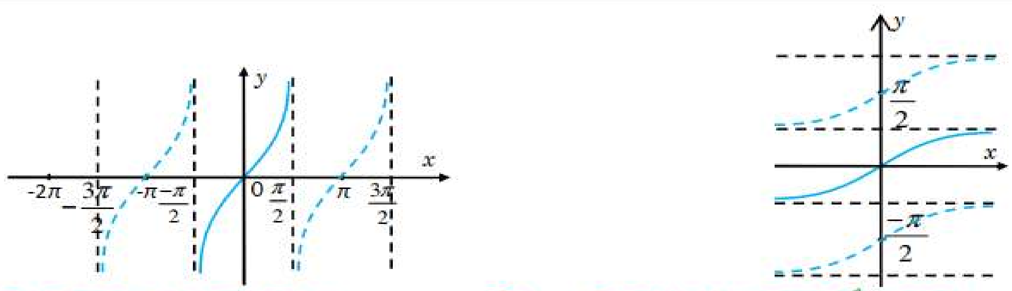
Example 2 Calculate $cos^{-1}\, x$ for

a $x = 0$ **b** $x = 1$ **c** $x = \frac{\sqrt{3}}{2}$ **d** $x = -1$

Inverse of tangent function

✓ The function $tan\, x$ is not one-to-one on its domain as it can be seen from its graph.

To get a unique x for a given y, you restrict x to the interval ,$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.



a *Graph of $y = tan\, x$* **b** *Graph of $y = tan^{-1}\, x$ and range = $\{y \in \mathbb{R}: y \neq (2n-1)\frac{\pi}{2}; n \in \mathbb{N}\}$.*

Definition 9.3

The inverse tangent function is a function denoted by $\tan^{-1}x$ or arctan x that assigns to each real number x the unique number y in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $x = \tan y$.

Reflecting the graph of $y = tan\, x$ in the line $y = x$ gives the graph of $f(x) = tan^{-1}x$ as shown in below:

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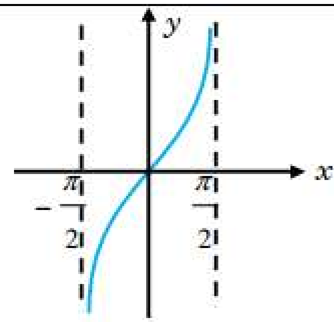
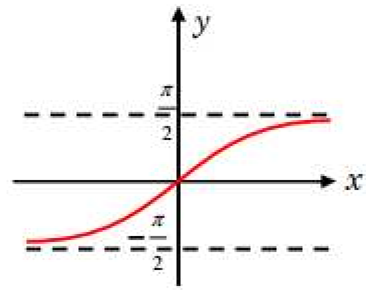


a *Graph of $y = \tan x$*



b *Graph of $y = \tan^{-1} x$*

☑ Remark:

1 Domain of $\tan^{-1}x$ is $(-\infty, \infty)$ and Range of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

You stress that $\frac{\pi}{2}$ is not in the range of $\tan^{-1}x$ because $\tan \frac{\pi}{2}$ is not defined.

2 From the above definition, you have,

$\tan(\tan^{-1} x) = x$ for all real x

$\tan^{-1}(\tan x) = x$, if $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Example 3 Compute (in radians).

***a*.** $tan^{-1}(0)$ ***b*.** $tan^{-1}(\sqrt{3})$ ***c*.** $tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Solution

a $\tan^{-1}(0) = 0$ because $\tan(0) = 0$ and $0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$ because $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

c $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ because $\tan -\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$ and $-\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Inverse cotangent, secant, and cosecant functions

Here, the definitions of the inverse cotangent, secant and cosecant functions are given.

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Definition 9.4

i The inverse cotangent function $\cot^{-1}$ or arccot is defined by

$y = \cot^{-1} x$, if and only if $x = \cot y$ where $0 < y < \pi$ and $-\infty < x < \infty$.

ii The inverse secant function $\sec^{-1} x$ or arcsec x is defined by

$y = \sec^{-1} x$, if and only if $x = \sec y$ where $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$, $|x| \ge 1$.

iii The inverse cosecant function $\csc^{-1} x$ or arccsc x is defined by

$y = \csc^{-1} x$, if and only if $x = \csc y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$, $|x| \ge 1$.

Example 5 Find the exact values of

a $\cot^{-1}\left(\sqrt{3}\right)$ **b** $\sec^{-1}(2)$ **c** $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

Solution

a $y = \cot^{-1}\left(\sqrt{3}\right) \Rightarrow \cot y = \sqrt{3}$ and $0 < y < \pi \Rightarrow y = \frac{\pi}{6}$

b $\sec^{-1}(2) = \frac{\pi}{3}$ because $\sec\left(\frac{\pi}{3}\right) = 2$ and $0 < \frac{\pi}{3} < \frac{\pi}{2}$

c $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

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1	Find the exact values of each of the following expressions without using a calculator or tables.					
	a $\sin^{-1}\left(-\frac{1}{2}\right)$		**b** $\cos^{-1}(3)$		**c** $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$	
	d $\csc^{-1}\left(-\frac{2}{\sqrt{3}}\right)$		**e** $\sec^{-1}\left(\sqrt{2}\right)$		**f** $\cot^{-1}(-1)$	
	g $\cos\left(\sin^{-1}\left(\frac{12}{13}\right)\right)$		**h** $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right)$		**i** $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$	
	j $\arccos\left(\cos\left(\frac{5\pi}{6}\right)\right)$		**k** $\cos\left(\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$		**l** $\tan\left(\tan^{-1}\sqrt{3}\right)$	
	m $\tan\left(\arcsin\left(\frac{\sqrt{3}}{2}\right)\right)$		**n** $\cos^{-1}\left(\tan\left(\frac{-\pi}{4}\right)\right)$			
2	Express each of the following expressions in terms of x.					
	a $y = \sin(\arctan x)$		**b** $y = \cos(\arcsin x)$		**c** $y = \tan(\arccos x)$	
3	Prove each of the following identities.					
	a $\tan^{-1}(-x) = -\tan^{-1} x$		**b** $\operatorname{arcsec} x = \arccos\left(\frac{1}{x}\right)$ for $\lvert x\rvert \geq 1$			

10.3. Graphs of Some Trigonometric Functions

Consider functions such as $y = a\sin(kx + b) + c$ and $y = a\cos(kx + b) + c$. These functions are used in the analysis of sound waves, x-rays, electric circuits, vibrations, spring-mass systems, etc.

Recall that if f is a periodic function, the amplitude of f is given by

$$|a| = \frac{Maximum\ value\ of\ f - Minimum\ value\ of\ f}{2}$$

Example: Find the amplitudes of each of the following functions

a) $f(x) = \sin x$

b) $g(x) = -\cos x$

c) $h(x) = 0.25\sin x$

d) $k(x) = |\sin x|$

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Note the following:

- ✓ The graph of $y = a \sin x$ still crosses the x-axis where the graph of $y = \sin x$ crosses the x-axis, because $a \times 0 = 0$.
- ✓ Since the maximum value of $\sin x$ is 1, the maximum value of $a \sin x$ is $|a| \times 1 = |a|$. The constant $|a|$, the amplitude of the graph of $y = a \sin x$, indicates the maximum deviation of the graph of $y = a \sin x$ from the x-axis.
- ✓ The period of $y = a \sin x$ is also 2π, since $a \sin(x + 2\pi) = a \sin x$.

The graph of $f(x) = \sin(kx)$

Example: Sketch graphs of $f(x) = \sin(2x)$ and $g(x) = \sin(x)$ on the interval $[0,2\pi]$ and note that the period of $y = \sin(2x)$ is π.

For $0 \leq x \leq 2\pi$,

- $\sin(2x) = 0 \Longrightarrow 2x = 0, \pi, 2\pi \Longrightarrow x = 0, \frac{\pi}{2}, \pi$

The graph of $f(x) = \sin(2x)$ crosses the x-axis at $(0,0), \left(\frac{\pi}{2}, 0\right)$ *and* $(\pi, 0)$.

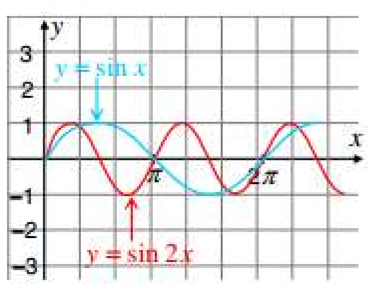
- $\sin(2x) = 1 \Longrightarrow 2x = \frac{\pi}{2} \Longrightarrow x = \frac{\pi}{4}$.

The function attains its maximum value at $\frac{\pi}{4}$.

- $\sin(2x) = -1 \Longrightarrow 2x = \frac{3\pi}{2} \Longrightarrow x = \frac{3\pi}{4}$.

The function attains its minimum value at $\frac{3\pi}{4}$.

From all these, the curve of $f(x) = \sin(2x)$can be drawn together with $y = \sin x$.



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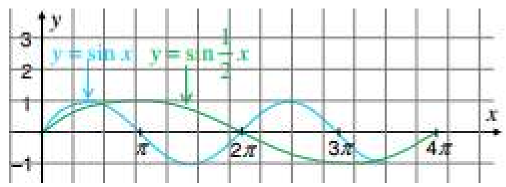
The period of $y = \sin(2x)$is π. It has two complete cycles on $[0,2\pi]$.

Similarly, for $0 \leq x \leq 4\pi$, the function $f(x) = sin\left(\frac{1}{2}x\right)$ has the following properties.

- The graph of f crosses the x-axis at at $(0,0), (2\pi, 0)$ and $(4\pi, 0)$.
- The graph has a peak at $(\pi, 1)$.
- The graph has a valley at $(3\pi, -1)$.

Now we can sketch the graph of $f(x) = \sin\left(\frac{1}{2}x\right)$ and $y = \sin x$ as follows



Remark:

a) The period of $y = \sin(kx)$is $\frac{2\pi}{|k|}$.

b) A similar investigation shows that the period of $y = \cos(kx)$ is $\frac{2\pi}{|k|}$.

Graphs of $y = a\sin(kx)$ and $y = acos\ (kx)$

The above discussions may lead you to the following procedures of drawing graphs

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Procedures for drawing graphs

Step 1: Determine the period $P = \frac{2\pi}{|k|}$ and the amplitude $|a|$.

Step 2: Divide the interval $[0, P]$ along the x-axis into four equal parts:

$x = 0, \frac{P}{4}, \frac{P}{2}, \frac{3P}{4}, P$

Step 3: Draw the graph of the points corresponding to $x = 0, \frac{P}{4}, \frac{P}{2}, \frac{3P}{4}, P$.

x	0	$\frac{P}{4}$	$\frac{P}{2}$	$\frac{3P}{4}$	P
$a\sin(kx)$	0	a	0	$-a$	0
$a\cos(kx)$	a	0	$-a$	0	a

Step 4: Connect the points found in Step 3 by a sine wave.

Step 5: Repeat this one cycle of the curve as required.

Example: Use the above procedure to sketch the graphs of the functions and indicate the amplitude and the period.

a) $y = 2\sin(3x)$

b) $y = -3\cos\left(\frac{2}{3}x\right)$

c) $f(x) = \frac{2}{3}\sin x$

d) $f(x) = 5\sin\left(\frac{2}{3}x\right)$

e) $f(x) = \frac{1}{2}\cos\left(-\frac{3}{2}x\right)$

Graphs of $y = a\sin(kx + b) + c$ and $y = a\cos(kx + b) + c$

Consider the function $y = \text{a}\sin(kx + b) + c$

$$\Rightarrow y - c = \text{a}\sin\left(k\left(x + \frac{b}{k}\right)\right)$$

This is simply the function $y = \text{a}\sin(kx)$after it has been shifted $-\frac{b}{k}$ units in the x-direction and c units in the y-direction.

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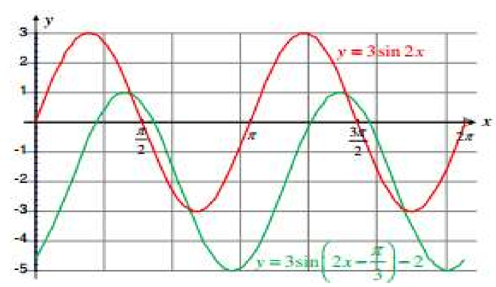
- If $\frac{b}{k} > 0$, the graph is shifted to the negative x-direction and if $\frac{b}{k} < 0$, the graph is shifted to the positive direction.
- If $c > 0$, the graph is shifted to the positive y-direction and if $c < 0$, the graph is shifted to the negative y-direction.

Example: Draw the graph of $y = 3\sin(2x - \frac{\pi}{3}) - 2$.

Rewriting the equation in the form

$$y - 2 = 3\sin\left(2\left(x - \frac{\pi}{6}\right)\right)$$

Thus the graph of the function is obtained by shifting the graph of $y = 3\sin(2x)$ in the positive x-direction by $\frac{\pi}{6}$ units and 2 units in the negative y-direction as shown in the figure.



Procedures for drawing graphs

Assume that $k > 0$. (*If $k < 0$, use the symmetric properties of sine and cosine*).

Step 1: Determine the period, $P = \frac{2\pi}{k}$, the amplitude $= |a|$ and phase shift $= -\frac{b}{k}$

Step 2: Divide the interval $\left[\frac{-b}{k}, \frac{-b}{k} + \frac{2\pi}{k}\right]$ along the x-axis into four equal parts.

The length of each interval will be $\frac{\pi}{2k}$. Why? Explain!

The dividing values of x are:

$x = \frac{-b}{k}$, $x = \frac{-b}{k} + \frac{\pi}{2k}$, $x = -\frac{b}{k} + \frac{\pi}{k}$, $x = \frac{-b}{k} + \frac{3\pi}{2k}$ and $x = \frac{-b}{k} + \frac{2\pi}{k}$

Step 3: Draw the graph of the points corresponding to those values of x.

x	$-\frac{b}{k}$	$\frac{-b}{k} + \frac{\pi}{2k}$	$\frac{-b}{k} + \frac{\pi}{k}$	$\frac{-b}{k} + \frac{3\pi}{2k}$	$\frac{-b}{k} + \frac{2\pi}{k}$
$a\sin(kx + b)$	0	a	0	$-a$	0
$a\cos(kx + b)$	a	0	$-a$	0	a

Step 4: Connect the points found in Step 3 by a sine wave.

Step 5: Repeat this portion of the graph indefinitely to the left and to the right every $\frac{2}{k}\pi$ units on the x-axis.

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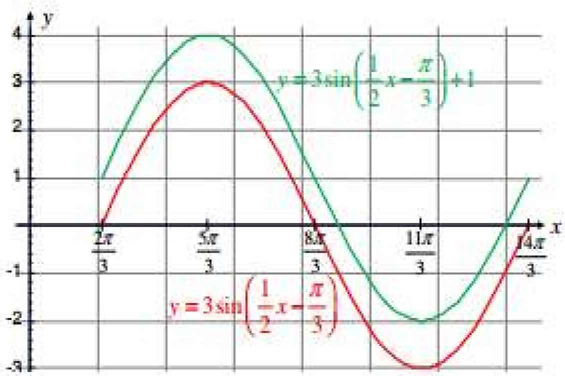
Example: Draw the graph of

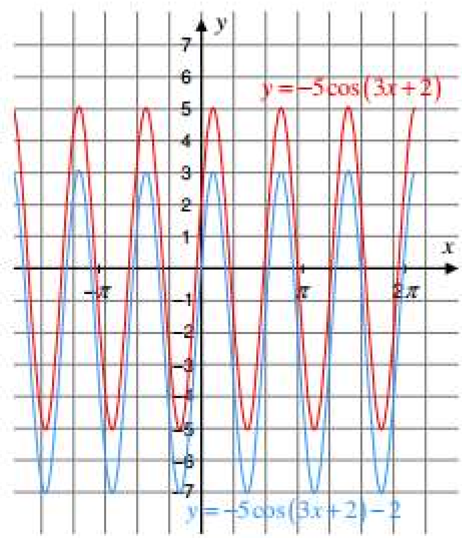
a) $f(x) = 3\sin\left(\frac{1}{2}x - \frac{\pi}{3}\right) + 1$

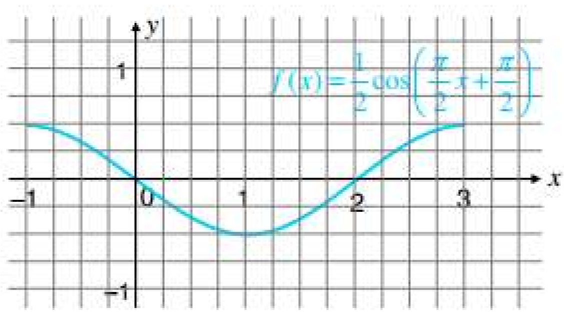
b) $f(x) = -5\cos(3x + 2) - 2$

c) $f(x) = \frac{1}{2}\cos(\frac{\pi}{2}x + \frac{\pi}{2})$ for one cycle

Solution:







10.4. Applications of Graphs in Solving Trigonometric Equations

Example:

1. Draw the graphs of $f(x) = \tan x$ and the line $y = 1$using the same coordinate system. Using the graphs

a) Determine the particular solution in the range $-\frac{\pi}{2} < x < \frac{\pi}{2}$ that satisfies the equation $\tan x = 1$.

b) Find the general solution of the equation $\tan x = 1$

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c) If x_1 is a particular solution of the equation $\tan x = t$, (t in $\mathbb{R}$) in the range $-\frac{\pi}{2} < x < \frac{\pi}{2}$, determine the general solution in terms of x_1 and π.

2. Draw the graph of $y = \frac{1}{2}$ and $y = \cos x$ using the same coordinate system. Determine a particular solution of the equation $\cos x = \frac{1}{2}$ in the range $-\pi \leq x \leq \pi$.
3. Determine the general solution of $\cos x = \frac{1}{2}$ using the particular solutions, n and 2π.
4. Solve $\tan x = -\frac{1}{\sqrt{3}}$.
5. Solve $\cos x = -\frac{\sqrt{3}}{2}$
6. Solve $\sin x = \frac{\sqrt{2}}{2}$
7. Solve $\sin(4x) = -\frac{1}{2}$ Ans: S.S=$\{\frac{11\pi}{24} + \frac{n\pi}{2}, \frac{7\pi}{24} + \frac{n\pi}{2}\}$

Note: $\sin x_1 = \sin(\pi - x_1) \Longrightarrow x_2 = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$. Also, if x_1 is a particular solution in the interval $[0,2\pi]$, then the general solution set of the equation $\sin x = b$, $|b| \leq 1$ is $\{(-1)^n x_1 + n\pi\}$.

Exercise: Find the general solution set for the following equations

a) $\cos x = \frac{\sqrt{3}}{2}$

b) $\cos(2x) + \sin^2 x = 0$

c) $\sin(6x) = \frac{\sqrt{3}}{2}$

d) $\sin^2 x - \sin x \cos x = 0$ over $[0,2\pi]$

e) $\cos x = \frac{\sqrt{3}}{2}$ and $\tan x = -\frac{\sqrt{3}}{3}$ on $[0,2\pi]$

f) $2\sin^2 x + \cos^2 x - 1 = 0$ on $[0,2\pi]$

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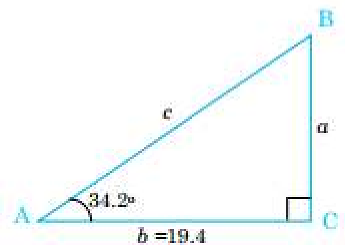
10.5. Applications of trigonometric functions

Many applied problems can be solved by using right angle triangle trigonometry. To solve a triangle means to find the lengths of all its sides and the measures of all its angles. First solve a right angled triangle.

Example: Solve the right angled triangle shown below for all unknown sides and angles.



Answer: $B = 55.8°$, $a = 13.18$, $c = 23.46$.

In many situations, trigonometric functions can be used to determine a distance that is difficult to measure directly. Two such cases are illustrated below.

a

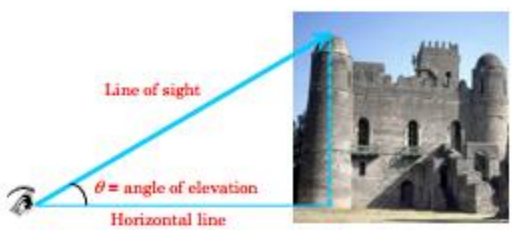


Figure 9.30

b

Each angle is formed by two lines: a horizontal line and a line of sight. If the angle is measured upward from the horizontal, as in a, then the angle is called an angle of elevation. If it is measured downward as in b, it is called an angle of depression.

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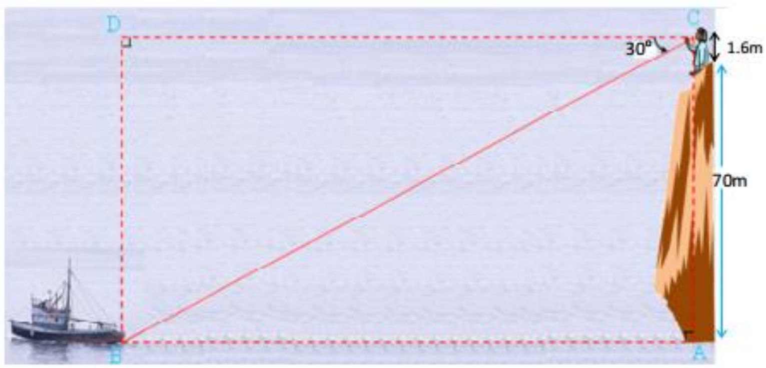
Example: A surveyor is standing 50 m from the base of a large tree, as shown below. The surveyor measures the angle of elevation to the top of the tree as 15°. How tall is the tree if the surveyor is 1.72m tall?

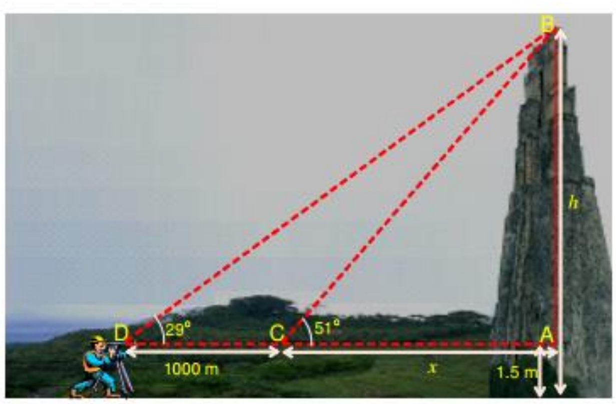
Answer: The tree is about 15m tall.

Example: A woman standing on top of a cliff spots a boat in the sea, as given in Figure below. If the top of the cliff is 70 m above the water level, her eye level is 1.6 m above the top of the cliff and if the angle of depression is 30°, how far is the boat from a point at sea level that is directly below the observer?



Answer: The boat is $71.6\sqrt{3}$m

Example: In order to measure the height of a hill, a surveyor takes two sightings from a transit 1.5m high. The sightings are taken 1000m apart from the same ground elevation. The first measured angle of elevation is 51°, and the second is 29°. To the nearest meter, what is the height of the hill (above ground level)?



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Answer: AB=1007m

The trigonometric functions can also be used to solve triangles that are not right angled triangles. Such triangles are called oblique triangles. Any triangle, right or oblique, can be solved if at least one side and any other two measures are known.

In order to solve oblique triangles, you need the law of sines and the law of cosines. The law of sines applies to the first three situations listed above. The law of cosines applies to the last two situations.

Law of sines: In any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where a, b, c are sides corresponding to angles A, B,C .

Example: In ΔEFG, FG=4.56, $m(\sphericalangle E) = 43°$ and $m(\sphericalangle G) = 57°$. Solve the triangle.

Law of cosines: In any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Remark: When the included angle is right, the law of cosines is reduced to the Pythagorean theorem.

Example: Solve ΔABC if $a = 32, c = 48\ and\ m(\sphericalangle B) = 125.2°$

Trigonometric formulae for the sum and differences

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1 Sine of the Sum and the Difference

- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$

2 Cosine of the Sum and Difference

- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$

3 Tangent of the Sum and Difference

- $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

Example: Find the exact values of

a) $\sin 75°$ and $\sin 15°$. Answer: $\sin 75° = \frac{\sqrt{2}+\sqrt{6}}{4}$ and $\sin 15° = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$

b) $\cos 105°$. Answer: $\cos 105° = \frac{\sqrt{2}}{4}(1-\sqrt{3})$

c) $\tan 150°$. Hint: 150=180-30

d) $\tan 195°$. Hint: 195=150+45

Double angle and half angle formulas

1 Double Angle Formula.

- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x$
- $\tan(2x) = \dfrac{2\tan x}{1-\tan^2 x}$

2 Half Angle Formula

- $\cos^2\left(\frac{x}{2}\right) = \frac{1+\cos x}{2}$; $\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$
- $\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos x}{2}$; $\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$
- $\tan^2\left(\frac{x}{2}\right) = \frac{1-\cos x}{1+\cos x}$ for $\cos x \neq -1$;

$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$$

The sign is determined by the quadrant that contains $\frac{x}{2}$.

Further, $\cos(2x) = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

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Example: Find the exact values of

a) $\sin\frac{\pi}{8}$ b) $\cos 15°$ c) $\tan\frac{\pi}{8}$

Answer: a) $\sin^2\frac{\pi}{8} = \frac{1-\cos\frac{\pi}{4}}{2} = \frac{2-\sqrt{2}}{4}$ b) $\cos 15° = \frac{\sqrt{2+\sqrt{3}}}{2}$ c) $\tan\frac{\pi}{4} = 1 = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$. This leads to the solution $\tan\frac{\pi}{8} = \sqrt{2} - 1$.

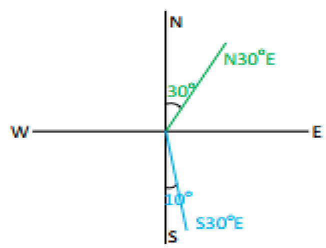
Applications

I. Navigations

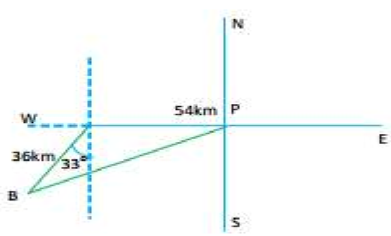
A bearing is an acute angle between a line of travel or line of sight and the north-south line. Bearings are usually given angles in degrees such as east or west of north, so that $N\theta E$ is read as θ east of north, and so on.

Example: The two bearings in fig below are respectively

a) $N30°E$ b) $S10°E$



Example: A ship leaves a port and travels 54km due west. It then changes course and sails 36km on a bearing $S33°W$. How far is it from the port at this point?



Answer: The ship is about 80km from the port.

II. Optics

A light ray is refracted as it passes from a first medium into a second medium according to the equation

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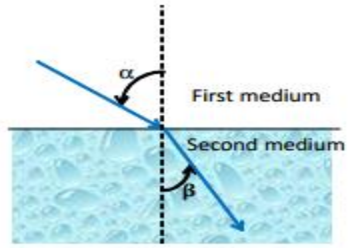
$$\frac{\sin\alpha}{\sin\beta} = \mu$$

where α is the angle of incidence and β is the angle of refraction. The Greek letter μ is the index of refraction of the second medium with respect to the first.



Example: The index of refraction of water with respect to air is μ =1.33. Determine the angle of refraction, if a ray of light passes through water with an angle of incidence $\alpha = 30°$. Answer: $\beta = 22.1°$

III. Simple harmonic motion

Definition: A harmonic function is a function that can be written in the form

$$f(t) = a\cos\omega t + b\sin\omega t$$

Note that $a\cos\omega t + b\sin\omega t = A\cos(\omega t - \delta)$ or $(= A\sin(\omega t + \emptyset)$

where $A = \sqrt{a^2 + b^2}$, $(\cos\delta, \sin\delta) = (\frac{a}{A}, \frac{b}{A})$ and $(\cos\emptyset, \sin\emptyset) = (\frac{b}{A}, \frac{a}{A})$.

The frequency of the function is the number of complete periods per unit time. Since $y = A\cos(\omega t - \delta)$ or $A\sin(\omega t + \emptyset)$ returns to the same y value in one period equal to $\frac{2\pi}{\omega}$ time units, the natural frequency of a function is $\frac{\omega}{2\pi}$. Units of frequency are $\frac{cycles}{sec}$ also called Hertz.

Example: Given the equation for simple harmonic motion $d = 6\cos\frac{3}{4}\pi t$ find

a) The maximum displacement
b) The frequency
c) The value of d when $t = 4$
d) The least positive value of t for which $d = 0$.

Example: Niddle C is struck on a piano with amplitude of $a = 2$. The frequency of niddle C is 264cycles/sec. Write an equation for the resulting sound wave.

Solution: with $a = 2$, we have $y = 2\sin\omega t$. But frequency $= \frac{\omega}{2\pi} = 264$.

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