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Chebyshev–Hankel matrices and the splitting approach for centrosymmetric Toeplitz-plus-Hankel matrices ☆

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Abstract

Centrosymmetric Toeplitz-plus-Hankel matrices are investigated on the basis of their "splitting property", which is their similarity to the direct sum of two special Toeplitz-plus-Hankel matrices. These matrices can be considered as Hankel matrices (moment matrices) in bases of Chebyshev polynomials and are called *Chebyshev–Hankel matrices*. Chebyshev–Hankel matrices have similar properties like Hankel matrices. This concerns inversion formulas and fast algorithms. A superfast algorithm for solving Chebyshev–Hankel and centrosymmetric Toeplitz-plus-Hankel systems is presented that is based on real trigonometric transforms. The main tool of investigation is the interpretation of Chebyshev–Hankel matrices as matrices of restricted multiplication operators with respect to Chebyshev bases. © 2001 Elsevier Science Inc. All rights reserved.

Keywords: Toeplitz matrix; Hankel matrix; Chebyshev polynomials; Fast algorithms

1. Introduction

In this paper we consider real Toeplitz-plus-Hankel matrices $A=[t_{i-j}+h_{i+j}]_{i,j=0}^{n-1}$ that are centrosymmetric, i.e., which have the property $J_nAJ_n = A$, where J_n denotes the $n \times n$ counteridentity

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$$J_n = \left.\begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}\right\} n \,.$$

In particular, symmetric Toeplitz matrices are centrosymmetric Toeplitz-plus-Hankel matrices.

A vector $x \in \mathbb{R}^n$ is called symmetric if $J_n x = x$ and skewsymmetric if $J_n x = -x$. The symmetric and skewsymmetric vectors form invariant subspaces of a centrosymmetric matrix. Therefore, a centrosymmetric matrix is similar to 2×2 block diagonal matrix, where the 2 diagonal blocks correspond to the restriction of the matrix (actually the corresponding linear operators) to the subspaces of symmetric and skewsymmetric vectors. We shall refer to this fact as the *splitting property* of centro-symmetric matrices.

The splitting property, despite not always formulated in this form, was observed and applied in several fields. Its first application was most likely the reduction of the moment problem on the unit circle, i.e. the trigonometric moment problem, with real data to a moment problem on the interval $[-1, 1]$ (see [1]). Then it was used in [3] for an efficient root location test and by engineers in signal processing and seismology (see [6]). Concerning solution of positive definite Toeplitz systems the splitting property was first exploited by Delsarte/Genin [7,8]. These authors constructed the so-called *split Levinson* and *Schur algorithms* which require only about half of the computational amount of the classical Levinson and Schur algorithms.

Differing from the approach of Delsarte/Genin, we investigate the matrices obtained after splitting in more detail. At the first glance they are less structured than the original matrix. But it turns out that they actually have structure properties similar to Hankel matrices. Since they are related to Chebyshev polynomials, they will be called *Chebyshev–Hankel matrices* or briefly *CH-matrices*. We distinguish four kinds of CH-matrices related to four kinds of Chebyshev polynomials.

It is well known that the inverse of a Hankel matrix is a Bezoutian. In Section 4 it will be shown that the inverse of a CH-matrix is some Chebyshev polynomial analogue of a Bezoutian. This fact can be utilized for multiplication of the inverse of an $m \times m$ CH-matrix by a vector with the help of six trigonometric transforms of length m, in particular this means with $\mathrm{O}(m \log m)$ complexity. As a consequence, the same complexity estimation holds also for inverses of centrosymmetric Toeplitz-plus-Hankel matrices. Note that in [15] it is shown, also using the splitting property but some different arguments, that matrix-vector multiplication by the inverse of a general Toeplitz-plus-Hankel can also be carried out with six transforms only.

In Section 5, we present $\mathrm{O}(m^2)$ complexity algorithms for LU-factorization of an $m \times m$ strongly nonsingular CH-matrix and its inverse. These algorithms can also be used to compute the quantities occurring in the inversion formulas for CH-matrices. They lead to algorithms for centrosymmetric Toeplitz-plus-Hankel systems. For the case of a symmetric Toeplitz matrix, a comparison with the split Levinson and Schur algorithms in [7,8] reveals that one step of our algorithms is, in principle, equivalent to two steps of algorithms of Delsarte/Genin, but the complexity is essentially the same.

In Section 6, we show that to the algorithm described in Section 5 a divide-and-conquer strategy can be applied which leads to an algorithm for computing the quantities in the inversion formula with complexity $O(m \log^2 m)$. Together with fast matrix–vector multiplication for inverse CH-matrices, this leads to an algorithm with this complexity for solving CH-systems. Since a centrosymmetric Toeplitz-plus-Hankel systems is equivalent to two CH-systems of half the size, this will also provide a new superfast algorithm for centrosymmetric Toeplitz-plus-Hankel systems, in particular symmetric Toeplitz systems. Note that in the pure Toeplitz case it is sufficient to solve one of the two CH-systems. The solution of the other one can be obtained from it in $O(n)$ operations. This will be explained in the forthcoming paper [16].

For Toeplitz systems $O(m \log^2 m)$ complexity algorithms were presented in [2,4,5, 9,13,19,21] and other papers. All these algorithms are based on the Fast Fourier Transform. Differently, our algorithm is based on real trigonometric transforms. Since the complexity for trigonometric transforms is essentially less than that one for complex FFT, even less than real FFT (see [20]), our algorithm should be, if the implementation details are worked out properly, faster than previous ones.

The algorithms in Sections 5 and 6 can be applied only if the matrix is strongly non-singular and even in this case they can be rather unstable. A way out of this situation is to use an equivalent interpolation formulation of CH-equations, which allow the application of pivoting. This is described in Section 7. The interpolation interpretation also provides more freedom in the choice of the size of the problem, so that one can achieve a size which is a power of 2, which is convenient for fast trigonometric transforms. Note that a different interpolation interpretation for symmetric Toeplitz systems was presented in [11].

CH-matrices R have a displacement structure (see [14,17]). More precisely, tridiagonal matrices U and V can be found [1] such that the matrix $RU - VR$ has rank 2. Matrices with this property were studied in [12] and inversion algorithms were obtained. We do not follow this approach because it gives less insight into the structure, though it is more general.

2. Symmetric-skewsymmetric splitting

Proposition 2.1. *A centrosymmetric Toeplitz-plus-Hankel matrix A admits a representation $A = T + J_n S$, where T and S are symmetric Toeplitz matrices.*

Proof. Suppose that $A = [a_{ij}]_{i,j=0}^{n-1}$, where $a_{ij} = t_{i-j} + h_{i+j}$. If A is centrosymmetric, then $a_{n-1-i,n-1-j} = a_{ij}$ for all $i, j = 0, \ldots, n-1$. That means

[1] U and V are given in (3.4) and (3.5).

$$t_{j-i} + h_{2n-2-i-j} = t_{i-j} + h_{i+j}.$$

Choosing $i = 0$ we obtain $t_j + h_{2n-2-j} = t_{-j} + h_j$ and choosing $j = 0$ we obtain $t_{-i} + h_{2n-2-i} = t_{-i} + h_i$. Comparing these relations we obtain $t_{-i} = t_i$ and $h_{2n-2-i} = h_i$. We set $s_i = h_{n-1-i}$. Then $s_{-i} = s_i$. Hence $S = [s_{i-j}]$ is symmetric and $A = T + J_n S$. □

A consequence of this proposition is the fact that a centrosymmetric Toeplitz-plus-Hankel matrix is also symmetric.

A vector $u \in \mathbb{R}^n$ is called *symmetric* if $J_n u = u$ and *skewsymmetric* if $J_n u = -u$. The subspaces of symmetric and skewsymmetric vectors are invariant under a centrosymmetric matrix. Therefore, centrosymmetric matrices are similar to the direct sum of two matrices of about half the size. The similarity is realized by the unitary matrices

$$Q_{2m+1} = \frac{1}{\sqrt{2}} \begin{bmatrix} -J_m & 0 & I_m \\ 0 & \sqrt{2} & 0 \\ J_m & 0 & I_m \end{bmatrix}, \quad Q_{2m} = \frac{1}{\sqrt{2}} \begin{bmatrix} -J_m & I_m \\ J_m & I_m \end{bmatrix}.$$

Proposition 2.2. *The centrosymmetric Toeplitz-plus-Hankel matrix $A = T + J_n S$ admits a representation*

$$A = Q_n^{\mathrm{T}} \begin{bmatrix} R_- & 0 \\ 0 & R_+ \end{bmatrix} Q_n, \tag{2.1}$$

where

1. *Case $n = 2m + 1$.*

$$\left(D_{m+1} = \operatorname{diag} \left(\frac{1}{\sqrt{2}}, \underbrace{1, \ldots, 1}_{m} \right) \right)$$

$$R_+ = D_{m+1} [\, t_{i-j} + s_{i-j} + t_{i+j} + s_{i+j} \,]_{i,j=0}^{m} D_{m+1},$$

$$R_- = [\, t_{i-j} - s_{i-j} - t_{i+j+2} + s_{i+j+2} \,]_{i,j=0}^{m-1}.$$

2. *Case $n = 2m$.*

$$R_\pm = [t_{i-j} \pm s_{i-j} \pm t_{i+j+1} + s_{i+j+1}]_{i,j=0}^{m-1},$$

The proof is a straightforward calculation.

In that way the consideration of the $n \times n$ centrosymmetric Toeplitz-plus-Hankel matrix R is reduced to the consideration of two Toeplitz-plus-Hankel matrices with a special structure and size $[(n + 1)/2]$ and $[n/2]$. Here $[\cdot]$ denote the integer part. Next we investigate the structure of the matrices $R_\pm$ and show that they are related to polynomial multiplication operators.

3. Polynomial interpretation

In the following we use Chebyshev polynomials of first and second kind

$$T_k(t) = \cos k\theta, \quad U_k(t) = \frac{\sin(k+1)\theta}{\sin\theta} \quad (t = \cos\theta,\ k \in \mathbb{Z}),$$

and of third and fourth kind (see [18])

$$V_k(t) = \frac{\cos(k+1/2)\theta}{\cos\theta/2}, \quad W_k(t) = \frac{\sin(k+1/2)\theta}{\sin\theta/2}.$$

They possess the following symmetry properties:

$$T_{-k} = T_k, \quad U_{-k-1} = -U_{k-1}, \quad V_{-k} = V_{k-1}, \quad W_{-k} = -W_{k-1}.$$

Note also that

$$V_k = U_k - U_{k-1}, \quad W_k = U_k + U_{k-1}.$$

All polynomials $Z_k = T_k, U_k, V_k, W_k$ satisfy the difference equation $Z_{k+1} - 2tZ_k + Z_{k-1} = 0$ with different initial conditions which are:

$$\begin{aligned} &T_0 = 1, \quad T_1 = t, \quad U_{-1} = 0,\\ &U_0 = 1, \quad V_{-1} = V_0 = 1, \quad W_{-1} = -W_0 = -1. \end{aligned}$$

We need the following relations for products that are well-known for first and second kind polynomials and can easily be verified for third and fourth kind polynomials:

$$\begin{aligned} 2T_iT_k &= T_{i+k} + T_{i-k} = T_{i+k} + T_{k-i},\\ 2T_iU_k &= U_{i+k} + U_{k-i} = U_{i+k} - U_{i-k-2},\\ 2T_iV_k &= V_{i+k} + V_{k-i} = V_{i+k} + V_{i-k-1},\\ 2T_iW_k &= W_{i+k} + W_{k-i} = W_{i+k} - W_{i-k-1}. \end{aligned} \tag{3.1}$$

Let $\mathbb{R}^m[t]$ denote the space of all real polynomials in t with degree $\leqslant m-1$. We consider the operator of multiplication by a polynomial $f(t)$ of degree $n-1$

$$\mathscr{M}_m(f)x(t) = f(t)x(t) \tag{3.2}$$

mapping $\mathbb{R}^m[t]$ to $\mathbb{R}^{m+n-1}[t]$. Let the polynomial $f(t)$ be given in its first kind Chebyshev expansion

$$f(t) = a_0 + 2\sum_{k=1}^{n-1} a_k T_k(t). \tag{3.3}$$

We evaluate the matrices of the operator $\mathscr{M}_m(f)$ with respect to different Chebyshev bases. To unify the presentation we introduce the notations

$$e_k^1(t) = T_k(t), \quad e_k^2(t) = U_k(t), \quad e_k^3(t) = V_k(t), \quad e_k^4(t) = W_k(t),$$

for all k with the exception that we define

$$e_0^1(t) = \frac{1}{\sqrt{2}}.$$

Proposition 3.1. *For $\nu = 1, 2, 3, 4$, the matrix $M_m^\nu(f)$ of the operator $\mathscr{M}_m(f)$ with respect to the basis $\{e_k^\nu(t)\}$ is given by*

$$\begin{aligned}
M_m^1(f) &= D_{m+n-1}\,[a_{|i-j|} + a_{i+j}]\,D_m,\\
M_m^2(f) &= [a_{|i-j|} - a_{i+j+2}],\\
M_m^3(f) &= [a_{|i-j|} + a_{i+j+1}],\\
M_m^4(f) &= [a_{|i-j|} - a_{i+j+1}],
\end{aligned}$$

for $i = 0, \ldots, m+n-2,\ j = 0, \ldots, m-1,$ where we set $a_i = 0$ for $i \geqslant n$, and D_m is defined as in Proposition 2.2.

Proof. In view of (3.1), we have for $j \neq 0$

$$\begin{aligned}
\mathscr{M}_m(f)e_j^1(t) &= a_0T_j + \sum_{k=1}^{n-1} a_k(T_{j+k} + T_{j-k})\\
&= \sum_{i=j}^{j+n-1} a_{i-j}T_i + \sum_{i=1}^{n-1-j} a_{i+j}T_i + \sum_{i=1}^{j-1} a_{j-i}T_i + a_j\\
&= \sum_{i=1}^{j+n-1} (a_{|i-j|} + a_{i+j})e_i^1(t) + \sqrt{2}\,a_j e_0^1(t).
\end{aligned}$$

Furthermore,

$$\mathscr{M}_m(f)e_0^1(t) = a_0 e_0^1(t) + \sqrt{2}\sum_{i=1}^{n-1} a_i e_i^1(t).$$

From these two relations the first assertion follows. The others are proved in the same way. □

We introduce $m \times m$ matrices $R_m^\nu(f)$ $(\nu = 1, 2, 3, 4)$ by

$$R_m^\nu(f) = [I_m 0 \quad]\ M_m^\nu(f).$$

The matrix $R_m^\nu(f)$ will be called $m \times m$ *CH-matrix of the* νth *kind with the symbol* $f(t)$.

As an example, let us mention the four kinds of CH-matrices with symbol $f(t) = 2t$, which is important for the algorithms in Section 5:

$$
R_m^1(2t) = \begin{bmatrix} 0 & \sqrt{2} & & & & \\ \sqrt{2} & 0 & 1 & & & \\ & 1 & 0 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}, \tag{3.4}
$$

$$
R_m^2(2t) = \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix},
$$

$$
R_m^3(2t) = \begin{bmatrix} 1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}, \tag{3.5}
$$

$$
R_m^4(2t) = \begin{bmatrix} -1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}.
$$

Let P_m^ν denote the projection defined by

$$
P_m^\nu \sum_k x_k e_k^\nu(t) = \sum_{k=0}^{m-1} x_k e_k^\nu(t). \tag{3.6}
$$

Then $R_m^\nu(f)$ is the matrix of the operator

$$
\mathscr{R}_m^\nu(f)x(t) = P_m^\nu f(t)x(t)
$$

with respect to the basis $\{e_k^\nu(t)\}_{k=0}^{m-1}$.

Combining Propositions 2.1, 2.2 and 3.1 we obtain the following.

Theorem 3.1. *Let* $A = T + J_n S$, $T = [t_{i-j}]_{i,j=0}^{n-1}$, $S = [s_{i-j}]_{i,j=0}^{n-1}$ *be a centrosymmetric Toeplitz-plus-Hankel matrix*, $a_k^{\pm} = t_k \pm s_k$, *and*

$$f_{\pm}(t) = a_0^{\pm} + 2\sum_{k=1}^{n-1} a_k^{\pm} T_k(t).$$

Then the matrices $R_{\pm}$ *appearing in Proposition* 2.2 *are CH-matrices given by*

$$R_+ = R_{m+1}^1(f_+), \quad R_- = R_m^2(f_-)$$

if $n = 2m + 1$, *and*

$$R_+ = R_m^3(f_+), \quad R_- = R_m^4(f_-)$$

if $n = 2m$.

4. Inversion formulas for CH-matrices and CH-Bezoutians

We show in this section that the inverses of CH-matrices have a similar structure like the inverses of Hankel matrices. They are Bezoutians with the respect to Chebyshev bases.

For $\nu = 1, 2, 3, 4$, let $\ell^{\nu}(t)$ denote the column vector $[e_i^{\nu}(t)]_{i=0}^{m-1}$. If $x \in \mathbb{R}^m$, then we set $x^{\nu}(t) = \ell^{\nu}(t)^{\mathrm{T}} x$. If A is any $m \times m$ matrix, then we denote by $A^{\nu}(t, s)$ the polynomial in two variables

$$A^{\nu}(t, s) = \ell^{\nu}(t)^{\mathrm{T}} A\, \ell^{\nu}(s).$$

Clearly, for any ν, the matrix A is completely defined by the polynomial $A^{\nu}(t, s)$.

The polynomials $A^{\nu}(t, s)$ are Chebyshev versions of the generating function of a matrix which is defined as follows. We set $\ell(t) = (t^i)_{i=0}^{m-1}$. The generating function of an $m \times m$ matrix A is, by definition, the bivariate polynomial

$$A(t, s) = \ell(t)^{\mathrm{T}} A\, \ell(s).$$

We introduce $m \times m$ matrices E^{ν} whose ith column e_i^{ν} is the coefficient vectors of the Chebyshev polynomial $e_i^{\nu}(t)$, i.e., $e_i^{\nu}(t) = \ell(t)^{\mathrm{T}} e_i^{\nu}$. Then we have $\ell^{\nu}(t)^{\mathrm{T}} = \ell(t)^{\mathrm{T}} E^{\nu}$.

Definition. Let $u, v \in \mathbb{R}^{m+1}$. The $m \times m$ matrix $B = \mathrm{Bez}^{\nu}(u, v)$ with

$$B^{\nu}(t, s) = \frac{u^{\nu}(t)v^{\nu}(s) - v^{\nu}(t)u^{\nu}(s)}{t - s}$$

will be called *CH-Bezoutian of the* νth *kind* (or briefly *CH-Bezoutian*) of u and v.

CH-Bezoutians are related to classical Bezoutians which are defined as follows. If $u, v \in \mathbb{R}^{m+1}$, then the $m \times m$ matrix $B = \mathrm{Bez}\,(u, v)$ with

$$B(t,s)=\frac{u(t)v(s)-v(t)u(s)}{t-s}$$

is called (Hankel) Bezoutian of u and v.

According to the definition, we have

$$\ell^\nu(t)^{\mathrm{T}}\mathrm{Bez}^\nu(u,v)\ell^\nu(s)=\ell(t)^{\mathrm{T}}\mathrm{Bez}(u^\nu,v^\nu)\ell(s).$$

Thus

$$E^\nu\mathrm{Bez}^\nu(u,v)(E^\nu)^{\mathrm{T}}=\mathrm{Bez}\,(u^\nu,v^\nu).$$

We consider a non-singular CH-matrix $R=R_m^\nu(f)$ ($\nu\in\{1,2,3,4\}$), and besides this matrix the $(m-1)\times(m+1)$ matrix $\tilde{R}$ obtained from $R_{m+1}^\nu(f)$ after cancelling the last two rows. This matrix has full rank and therefore a two-dimensional kernel. Any basis of this kernel is called *fundamental system*. We select a "canonical" fundamental system in the following way. Suppose that u' and v' are the solution of

$$Ru'=-h,\quad Rv'=e,$$

where e denotes the last unit vector in $\mathbb{R}^m$ and h the last column of $R_{m+1}^\nu(f)$ canceling the last component. Then the vectors

$$u=\begin{bmatrix}u'\\1\end{bmatrix},\quad v=\begin{bmatrix}v'\\0\end{bmatrix},$$

form a basis of the kernel of $\tilde{R}$. We call $\{u,v\}$ or $\{u^\nu(t),v^\nu(t)\}$ the *canonical fundamental system of R*.

Theorem 4.1. *Let R be a non-singular CH-matrix of the* νth *kind and let* $\{u,v\}$ *be the canonical fundamental system of R. Then*

$$R^{-1}=\frac{1}{2}\,\mathrm{Bez}^\nu(u,v).\tag{4.1}$$

Proof. We fix a real number c and consider the polynomial

$$l_c(t)=\sum_{i=0}^{m-1}e_i^\nu(c)e_i^\nu(t).$$

Then the coefficient of $e_i^\nu(t)$ in the expansion of $2(t-c)l_c(t)$ is equal to $e_{i-1}^\nu(c)-2ce_i^\nu(c)+e_{i+1}^\nu(c)$ for $i=1,\dots,m-2$ and for $i=1$, $\nu\neq 1$, which is 0. The first part of the expansion will be, for $\nu\neq 1$,

$$e_0^\nu(c)e_{-1}^\nu(t)+(e_1^\nu(c)-2ce_0^\nu(c))e_0^\nu(t)+\cdots$$

Since $e_{-1}^2=0$, the coefficient of $e_0^2(t)$ will be 0, and in view of $e_{-1}^3=e_0^3=1$ and $e_{-1}^4=-e_0^4=-1$ the same is true for $\nu=3$ and $\nu=4$. A simple calculation shows that, for $\nu=1$, the coefficients of $e_0^1(t)$ and $e_1^1(t)$ in the expansion are 0. Summing up, we obtained that for all $\nu=1,2,3,4$ the coefficients of $e_i^\nu(t)$ in the expansion of the polynomial $2(t-c)l_c(t)$ in the basis $\{e_i^\nu(t)\}$ vanish for $i=0,\dots,m-2$.

Let, for fixed c, $x_c(t)$ be the polynomial given by $x_c(t) = \ell^\nu(c)^{\mathrm{T}} R^{-1} \ell^\nu(t)$. Then, by definition, we have $P_m^\nu f(t) x_c(t) = (\ell^\nu(c))^\nu(t)$. We consider the polynomial $u_c(t) = 2(t-c)x_c(t)$. Then $P_{m-1}^\nu f(t) u_c(t) = P_{m-1}^\nu 2(t-c)(\ell^\nu(c))^\nu(t)$ which vanishes, as it was shown above. That means the coefficient vector belongs to the two-dimensional kernel of $\tilde{R}$, which is spanned by u and v. Therefore,

$$2(t-c)x_c(t) = \alpha(c)u^\nu(t) - \beta(c)v^\nu(t)$$

for some functions $\alpha(c)$ and $\beta(c)$. Since the left-hand side is a polynomial in c the right-hand side is polynomial in c, too. Changing the roles of c and t and taking into account that R_m^ν is symmetric we obtain that $\alpha(c) = av^\nu(c)$ and $\beta(c) = au^\nu(c)$ for some constant a. Applying $\mathrm{Bez}^\nu(u, v)$ to the last unit vector, we obtain that a is actually equal to 1. This proves the theorem. □

Clearly, the identity matrix I_m is a CH-matrices that is equal to its inverses. The canonical fundamental system is equal to $\{e_{m-1}^\nu(t), e_m^\nu(t)\}$. From this we obtain the following well-known Christoffel–Darboux formulas:

$$I_m^\nu(t,s) = \sum_{k=0}^{m-1} e_k^\nu(t) e_k^\nu(s) = \frac{e_m^\nu(t)e_{m-1}^\nu(s) - e_{m-1}^\nu(t)e_m^\nu(s)}{2(t-s)}.$$

Let us mention one more example. The matrix J_m is a CH-matrix of second, third and fourth kind. The corresponding canonical fundamental systems are $\{U_0, U_m\}$, $\{V_0, V_m - V_{m-1}\}$, and $\{W_0, W_m + W_{m-1}\}$, respectively. J_m is not a CH-matrix of first kind, but the modified matrix $J_m \mathrm{diag}\,(\sqrt{2}, 1, \ldots, \sqrt{2})$ is. Its canonical fundamental system is $\{\frac{1}{2}T_0, T_m - T_{m-2}\}$. From this, several summation formulas for Chebyshev polynomials can be derived.

The importance of Theorem 4.1 for practical calculation consists in the fact that CH-Bezoutians can be represented with the help of trigonometric transforms. Let us explain this briefly. For more information we refer to [15][2] (see also [10]).

The trigonometric transforms we have in mind are defined as follows:

- DST-1 *and* DCT-1

$$\mathscr{S}_1 = \left[\sin\frac{(i+1)(j+1)\pi}{m}\right]_{i,j=0}^{m-2}, \quad \mathscr{C}_1 = \left[\cos\frac{ij\pi}{m}\right]_{i,j=0}^{m}.$$

- DST-2 *and* DCT-2

$$\mathscr{S}_2 = \left[\sin\frac{(i+1)(2j+1)\pi}{2m}\right]_{i,j=0}^{m-1}, \quad \mathscr{C}_2 = \left[\cos\frac{i(2j+1)\pi}{2m}\right]_{i,j=0}^{m-1}.$$

- DST-3 *and* DST-3

$$\mathscr{S}_3 = \mathscr{S}_2^{\mathrm{T}}, \quad \mathscr{C}_3 = \mathscr{C}_2^{\mathrm{T}}.$$

[2] A more detailed version of this paper is in preparation.

- DST-4 *and* DCT-4

$$\mathscr{S}_4 = \left[\sin\frac{(2i+1)(2j+1)\pi}{4m}\right]_{i,j=0}^{m-1},$$
$$\mathscr{C}_4 = \left[\cos\frac{(2i+1)(2j+1)\pi}{4m}\right]_{i,j=0}^{m-1}.$$

For all these transforms $\mathrm{O}(m\log m)$ complexity algorithms do exist (see, for example [20]).

Obviously, the columns of the matrices of the trigonometric transforms are, up to some multiplicative constants, equal to the the values of the vectors $e_k^{\nu}(t)$ at the Chebyshev nodes

$$\sigma_j = \cos\frac{j\pi}{m}, \quad \rho_j = \cos\frac{(2j+1)\pi}{2m}.$$

That means, if $\mathrm{Bez}^{\nu}(u,v)$ is multiplied from the right and the left by appropriate trigonometric transforms one obtains a matrix of the form

$$\left[\frac{\alpha_i\beta_j - \gamma_i\delta_j}{\rho_i - \sigma_j}\right] = D_1\Omega D_2 + D_3\Omega D_4,$$

where D_1, D_2, D_3, D_4 are diagonal matrices and

$$\Omega = \left[\frac{1}{\rho_i - \sigma_j}\right].$$

The matrix Ω, in its turn, can be represented with the help of trigonometric transforms. In fact,

$$\Omega = -\Lambda_1^{-1}\mathscr{S}_4\mathscr{C}_3\Lambda_2^{-1},$$

where

$$\Lambda_1 = \mathrm{diag}\left[\sin\frac{(2j+1)\pi}{4m}\right]_{j=0}^{m-1}, \quad \Lambda_2 = \mathrm{diag}\left[\cos\frac{j\pi}{2m}\right]_{j=0}^{m-1}$$

(see [15]). In that way a vector can be multiplied by a CH-Bezoutians with the help of six transforms of length m plus $\mathrm{O}(m)$ operations. For preprocessing the data, assuming that u and v are given, four transforms of length m are required.

5. Algorithms for LU-factorization

In this and the next sections, we consider only strongly non-singular CH-matrices $R = R_m^{\nu}(f) = [r_{ij}]_{i,j=0}^{m-1}$. That means we assume that the principal subsections $[r_{ij}]_{i,j=0}^{k-1}$ are non-singular for $k = 1, \ldots, m$. This covers, in particular, the case when R is positive definite.

We present fast algorithms for the construction of the LU-factorization of R and its inverse. More precisely, we construct an upper triangular matrix $U=[u_{ij}]_{i,j=0}^{m-1}$ and a lower triangular matrix $L=[l_{ij}]_{i,j=0}^{m-1}$ satisfying

$$RU = L \quad \text{and} \quad u_{ii} = 1 \quad (i = 0, \dots, m-1). \tag{5.1}$$

In polynomial language this can be written in the form

$$f(t)u_k(t) = l_k(t), \tag{5.2}$$

where

$$u_k(t) = \sum_{i=0}^{k} u_{ik} e_i^{\nu}(t), \quad l_k(t) = \sum_{i=k}^{m+n-2} l_{ik} e_i^{\nu}(t).$$

Theorem 5.1. *The columns of U and L in* (5.1) *can be computed via the recursion*

$$u_{k+1}(t) = (2t - \beta_k)u_k(t) - \alpha_k u_{k-1}(t),$$

$$l_{k+1}(t) = (2t - \beta_k)l_k(t) - \alpha_k l_{k-1}(t),$$

where $k = 1, \dots, m-2,$

$$\alpha_k = \frac{l_{kk}}{l_{k-1,k-1}}, \quad \beta_k = \frac{l_{k-1,k-1}l_{k+1,k} - l_{kk}l_{k,k-1}}{l_{kk}l_{k-1,k-1}}$$

with the exception that for $\nu = 1$

$$\alpha_1 = \frac{\sqrt{2}l_{11}}{l_{00}}, \quad \beta_1 = \frac{l_{21} - \alpha_1 l_{10}}{l_{11}}.$$

This theorem can be proved by straightforward verification.

The first columns are given by

$$u_0(t) = e_0^{\nu}(t), \quad l_0(t) = f(t)e_0^{\nu}(t) = \sum_{i=0}^{n+m-2} l_{i0} e_i^{\nu}(t)$$

for $\nu = 1, 2, 3, 4$, and the second columns by

$$u_1(t) = (2t - \beta_0^{\nu})u_0(t), \quad l_1(t) = (2t - \beta_0^{\nu})l_0(t)$$

if $\nu \neq 1$ and by

$$u_1(t) = \frac{1}{\sqrt{2}}(2t - \beta_0^{\nu})u_0(t), \quad l_1(t) = \frac{1}{\sqrt{2}}(2t - \beta_0^{\nu})l_0(t)$$

if $\nu = 1$, where

$$\beta_0^1 = \frac{\sqrt{2}\,l_{10}}{l_{00}}, \quad \beta_0^2 = \frac{l_{10}}{l_{00}}, \quad \beta_0^3 = \frac{l_{10} + l_{00}}{l_{00}}, \quad \beta_0^4 = \frac{l_{10} - l_{00}}{l_{00}}.$$

The recursions can easily be translated into vector language with the help of the matrix representations (3.4) and (3.5).

Note that the canonical fundamental system of R producing the inverse matrix R^{-1} is given by

$$u(t) = u_m(t), \quad v(t) = \frac{1}{l_{m-1,m-1}} u_{m-1}(t),$$

where $u_m(t)$ is also obtained by the recursion of Theorem 5.1.

The algorithm emerging from the theorem is a hybrid Levinson–Schur type algorithm. It is in particular convenient for parallel computation and has $\mathrm{O}(m)$ complexity if m processors are available.

It is possible to calculate only the columns of the upper factor U, and the quantities l_{ij} for $0 \leqslant j - i \leqslant 1$ as some inner products of rows of R and the u_k. This leads to a Levinson type algorithm. It is also possible to calculate only the lower factor L, which results in a pure Schur type algorithm. In this case the solution of a system $Rx = b$ will be obtained by backward substitution.

Due to the splitting property discussed in Section 2, the algorithms for the factorization of R can be used for the solution of systems with a centrosymmetric coefficient matrix $Ax = b$. It is obvious that the Toeplitz-plus-Hankel matrix A is strongly non-singular if and only the two CH-matrices obtained after splitting are strongly non-singular.

6. Superfast algorithm

It is well known that two polynomials in Chebyshev expansion can be multiplied with the help of two trigonometric transforms and the complexity of trigonometric transforms is $\mathrm{O}(m \log m)$. Due to this fact, it is efficient to apply a divide-and-conquer strategy to the algorithm emerging from Theorem 5.1. This leads to a $\mathrm{O}(m \log^2 m)$ complexity algorithm to compute the canonical fundamental system of a CH-matrix. Recall from Section 4 that with the help of a given fundamental system a system of equations $Rx = b$ can be solved with $\mathrm{O}(m \log m)$ complexity.

We introduce 2×2 matrix polynomials

$$U_k(t) = \begin{bmatrix} u_k(t) & u_{k-1}(t) \\ l_k(t) & l_{k-1}(t) \end{bmatrix}, \quad \Theta_k(t) = \begin{bmatrix} 2t - \beta_k & 1 \\ -\alpha_k & 0 \end{bmatrix}$$

Then the relation in Theorem 5.1 can be written in the form

$$U_{k+1}(t) = U_k(t)\Theta_k(t). \tag{6.1}$$

We define, for $j > k$

$$\Theta_{kj}(t) = \Theta_k(t)\Theta_{k+1}(t) \cdots \Theta_{j-1}(t).$$

Then, for $j > i > k$,

$$\Theta_{kj}(t) = \Theta_{ki}(t)\Theta_{ij}(t), \quad U_j(t) = U_k(t)\Theta_{kj}(t). \tag{6.2}$$

In order to achieve complexity $\mathrm{O}(m\log^2 m)$ it is important to carry out the calculations not with the complete polynomials $l_k(t)$ but only with the relevant part of them. We define

$$l_k^j(t)=\sum_{i=k}^{j} l_{ik}e_i^\nu(t).$$

Obviously, $\Theta_{k,k+1}(t)$ can be computed from $l_{k-1}^k(t)$ and l_k^{k+1}, $\Theta_{k,k+2}(t)$ can be computed from $l_{k-1}^{k+2}(t)$ and $l_k^{k+3}(t)$, and, in general, $\Theta_{kj}(t)$ from $l_{k-1}^{2j-k-2}(t)$ and $l_k^{2j-k-1}(t)$. Furthermore, the following is true for $k<i<j$:

$$\left[l_i^{2j-i-1}(t)\quad l_{i-1}^{2j-i-2}(t)\right]=\left[P_{2j-i}^\nu\, h_{i-1}(t)\quad P_{2j-i-1}^\nu\, h_i(t)\right],$$

where

$$\left[h_i(t)\quad h_{i-1}(t)\right]=\left[l_k^j(t)\quad l_{k-1}^j(t)\right]\Theta_{ki}(t). \tag{6.3}$$

Here P_i^ν is defined as in (3.6).

This leads to the following recursive procedure.

Input: $[l_{k-1}^{2j-k-2}(t)\ l_k^{2j-k-1}(t)]$, *Output*: $\Theta_{kj}(t)$

1. If $j=k+1$, then apply Theorem 5.1.
2. Otherwise choose i with $k<i<j$ and carry out the following steps:
   (a) Apply the procedure for $[l_{k-1}^{2i-k-2}(t)\quad l_k^{2i-k-1}(t)]$. The output is $\Theta_{ki}(t)$.
   (b) Compute $[l_{i-1}^{2j-i-2}(t)\quad l_i^{2j-i-1}(t)]$ by (6.3).
   (c) Apply the procedure for $[l_{i-1}^{2j-i-2}(t)\quad l_i^{2j-i-1}(t)]$. The output is $\Theta_{ij}(t)$.
   (d) Compute $\Theta_{kj}(t)=\Theta_{ki}(t)\Theta_{ij}(t)$ using fast trigonometric transforms.

It is convenient to choose *i* close to the average of *j* and *k*. Proceeding in this way the problem to compute $\Theta_{kj}(t)$ is reduced to two subproblems of about half the size plus $\mathrm{O}((j-k)\log(j-k))$ operations for polynomial multiplication. This ends up with complexity $\mathrm{O}((j-k)\log^2(j-k))$. In particular, $U_m(t)$ can be computed with $\mathrm{O}(m\log^2 m)$ operations.

7. Interpolation interpretation

The algorithms presented in Sections 5 and 6 work only for strongly non-singular matrices. From the algebraic point of view, this could be overcome by generalizing the concept of a fundamental system to singular CH-matrices and to transfer the approach of [13] from Hankel to CH-matrices. However, from numerical point of view this approach is unsatisfactory due to the instability of the corresponding algorithms. Another drawback concerning the superfast algorithm described in Section 6 is that it works efficiently only if the order *m* is a power of 2.

We show now that interpolation interpretation is the way to overcome both drawbacks. Let us demonstrate this for the vector $u = (u_k)_{k=0}^m$ $(u_m = 1)$ of the fundamental system of a CH-matrix of νth kind with symbol $f(t)$ (see Section 4). In polynomial language this means

$$P_m^\nu f(t)u(t) = 0, \tag{7.1}$$

where $u(t) = \sum_{k=0}^m u_k e_k^\nu(t)$.

In order to simplify the forthcoming presentations we generalize the definition (3.6) defining P_{ij}^ν by

$$P_{ij}^\nu \sum_k x_k e_k^\nu(t) = \sum_{k=i}^{j-1} x_k e_k^\nu(t),$$

and denote the range of P_{ij}^ν by $\mathscr{P}_{ij}^\nu$. Eq. (7.1) can be written in the form

$$f(t)u(t) = l(t) \tag{7.2}$$

with $l(t) \in \mathscr{P}_{m,m+n}^\nu$. We now fix an integer $N > m + (n/2)$, preferably a power of 2. Then $l(t)$ in (7.2) can be represented (uniquely) in the form

$$l(t) = T_N(t)z(t) - w(t),$$

where $w(t) \in \mathscr{P}_{m,m+n-N}^\nu$ and $z(t) \in \mathscr{P}_{0m}^\nu$. In fact, if $N > m + n - 1$, then we can choose $z(t) = 0$ and there is nothing to show. Suppose now that $N \leqslant m + n - 1$ and

$$l(t) = \sum_{k=m}^{m+n-1} l_k e_k^\nu(t).$$

Then $T_N(t)z(t) \in \mathscr{P}_{m,m+n}^\nu$ and $w(t) = T_N(t)z(t) - l(t) \in \mathscr{P}_{0N}^\nu$. We conclude that $w(t) \in \mathscr{P}_{mN}^\nu$.

We replace the left-hand side of (7.2) by this expression and evaluate both sides at the roots τ_k of $T_N(t)$, which are $\tau_k = \cos(((2k+1)\pi)/2N)$ $(k = 0, \ldots, N-1)$. Now the following can be verified.

Proposition 7.1. *Eq.* (7.1) *is equivalent to the existence of a polynomial* $w(t) \in \mathscr{P}_{mN}^\nu$ *such that* $(u(t), w(t))$ *satisfies the homogeneous interpolation conditions*

$$f(\tau_k)u(\tau_k) + w(\tau_k) = 0 \quad (k = 0, \ldots, N-1). \tag{7.3}$$

The interpolation problem (7.3) can be solved by standard algorithms for rational interpolation. It can also be speeded up to a superfast algorithm, as it is shown for the Hankel case in [21]. The advantage of the interpolation setting is that reordering the numbers τ_k will not destroy the structure of the problem, so that pivoting can be applied, which stabilizes the corresponding algorithm. Furthermore, N can be chosen as a power of 2, so that efficient algorithm for fast trigonometric transform can be applied. The details will be worked out elsewhere.

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References

[1] N.I. Akhiezer, The Classical Moment Problem, Oliver & Boyd, London, 1965.
[2] G.S. Ammar, W.B. Gragg, Superfast solution of real positive definite Toeplitz systems, SIAM J. Matrix Analysis Appl. 9 (1988) 61–76.
[3] Y. Bistritz, Zero location with respect to the unit circle of discrete-time linear system polynomials, Proc. IEEE 72 (1984) 1131–1142.
[4] R.R. Bitmead, B.D.O. Anderson, Asymptotically fast solution of Toeplitz and related systems of linear equations, Linear Algebra Appl. 34 (1980) 103–116.
[5] R. Brent, F. Gustavson, D. Yun, Fast solution of Toeplitz systems of equations and computation of Padé approximants, J. Algorithms 1 (1980) 259–295.
[6] K.P. Bube, R. Burridge, The one-dimensional inverse problem of reflection seismology, SIAM Rev. 25 (1983) 497–559.
[7] P. Delsarte, Y. Genin, The split Levinson algorithm, IEEE Trans. Acoust. Speech, Signal Process. ASSP-34 (1986) 470–477.
[8] P. Delsarte, Y. Genin, On the splitting of classical algorithms in linear prediction theory, IEEE Trans. Acoust. Speech, Signal Process. ASSP-35 (1987) 645–653.
[9] F. de Hoog, A new algorithm for solving Toeplitz systems of equations, Linear Algebra Appl. 88/89 (1987) 123–138.
[10] C. Di Fiore, P. Zellini, Matrix displacement decompositions and applications to Toeplitz linear systems, Linear Algebra Appl. 268 (1998) 197–226.
[11] G. Heinig, Transformation approaches for fast and stable solution of Toeplitz systems and polynomial equations, in: Proceedings of the International Workshop on the Recent Advances in Applied Mathematics, Kuwait, 4–7 May 1996, pp. 223–238.
[12] G. Heinig, W. Hoppe, K. Rost, Structured matrices in interpolation and approximation problems, Wiss. Zeitschr. d. TU Karl-Marx-Stadt 31 (1989) 2.
[13] G. Heinig, P. Jankowski, Parallel and superfast algorithms for Hankel systems of equations, Numer. Math. 58 (1990) 109–127.
[14] G. Heinig, K. Rost, Algebraic Methods for Toeplitz-like Matrices and Operators, Birkhäuser, Basel, 1984.
[15] G. Heinig, K. Rost, Representations of inverses of real Toeplitz-plus-Hankel matrices using trigonometric transformations, in: M. Griebel, S. Margena, P. Yalamov (Eds.), Large-scale Scientific Computation of Engineering and Enviromental Problems II, NNFM, vol. 73, Vieweg, Braunschweig, 2000, pp. 80–87.
[16] G. Heinig, K. Rost, Centro-symmetric and centro-skewsymmetric Toeplitz matrices and Bezoutians, Linear Algebra Appl. (submitted).
[17] T. Kailath, A. Sayed, Fast Reliable Algorithms for Matrices with Structure, SIAM, Philadelphia, PA, 1999.
[18] J.C. Mason, Chebyshev polynomials of the second third, and fourth kind in approximation, indefinite integration, and integral transforms, J. Comp. Appl. Math. 49 (1993) 169–178.
[19] M. Morf, Doubling algorithms for Toeplitz and related equations, in: Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing, Denver, 1980, pp. 954–959.
[20] M. Tasche, Fast algorithms for discrete Chebyshev–Vandermonde transforms and applications, Numer. Algorithms 5 (1993) 453–464.
[21] M. Van Barel, P. Kravanja, A stabilized superfast solver for indefinite Hankel systems, Linear Algebra Appl. 284 (1998) 335–355.