7

Polar Coordinates and Graphs

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. So far we have been using rectangular (or cartesian) coordinates, which are directed distances from two perpendicular axes. However, as we will see, this is not always the easiest coordinate system to work in. In this chapter, we describe a coordinate system introduced by Newton, called the polar coordinate system, which is more convenient for many purposes.

7.1 Polar Coordinates

A polar coordinate system in a plane consists of a fixed point O, called the pole (or origin), and a ray emanating from the pole, called the polar axis. In such a coordinate system, we represent each point P in the plane by the ordered pair (r, θ) where r is the distance from the pole O to the point P and θ is an angle from the polar axis to the ray OP, Figure 7.1 . The number r is called the radial coordinate of P and the number θ the angular coordinate (or polar angle) of point P.

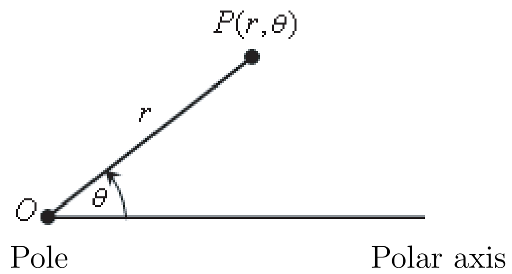


Figure 7.1 A polar coordinate system

We use the convention that an angle is positive if measured in the counter clockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 7.2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on the opposite side of O. If $r > 0$, the point (r, θ) lies in the same quadrant as θ. If $r < 0$, it lies in the quadrant on the opposite side of the pole.

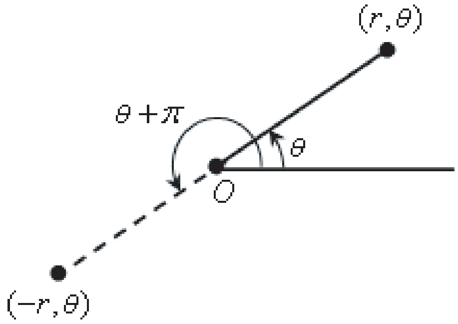


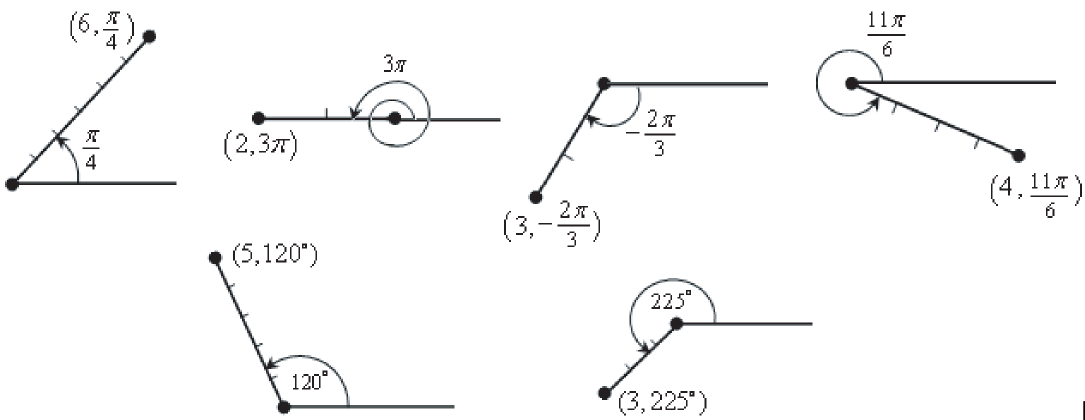
Figure 7.2 The point $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

Example 7.1.1 Plot the points whose polar coordinates are given.

(a) $(6, \frac{\pi}{4})$ (b) $(2, 3\pi)$ (c) $(3, -\frac{2\pi}{3})$ (d) $(4, \frac{11\pi}{6})$

(e) $(5, 120°)$ (f) $(3, 225°)$

Solution



■

Note Polar coordinates differ from rectangular coordinates in that any point has more than one representation in polar coordinates. The polar coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point. More generally, this point has the polar coordinates $(r, \theta + n\pi)$ for any even integer n, and the coordinates $(-r, \theta + n\pi)$ for any odd integer n. Thus the polar coordinates pair

$$\left(2, \tfrac{\pi}{3}\right), \quad \left(-2, \tfrac{4\pi}{3}\right), \quad \left(2, \tfrac{7\pi}{3}\right), \quad \left(-2, -\tfrac{2\pi}{3}\right)$$

all represent the same point.

7.2 Relating Polar and Rectangular Coordinates

The connection between polar and rectangular coordinates can be seen in Figure 7.3, which the pole corresponds to the origin and the polar axis coincides with the positive x axis. If the point P has rectangular coordinates (x, y) and polar coordinates (r, θ) then, see Figure 7.3,

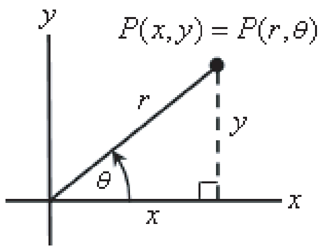


Figure 7.3 The connection between polar and rectangular coordinates

we have, for converting polar coordinates to rectangular coordinates,

$$\boxed{x = r\cos\theta\,, \quad y = r\sin\theta}$$

and, for converting rectangular coordinates to polar coordinates,

$$\boxed{r^2 = x^2 + y^2\,, \quad \tan\theta = \frac{y}{x}\,.}$$

Example 7.2.1 Convert the following points into the given coordinate system.

1 $(2, \frac{\pi}{3})$ into rectangular coordinates

Solution With $r = 2$ and $\theta = \frac{\pi}{3}$, we have that

$$x = 2\cos\tfrac{\pi}{3} = 2\big(\tfrac{1}{2}\big) = 1$$

$$y = 2\sin\tfrac{\pi}{3} = 2\big(\tfrac{\sqrt{3}}{2}\big) = \sqrt{3}\,.$$

Thus, in rectangular coordinates, the point is $(1, \sqrt{3})$. ■

2 $(1, -1)$ into polar coordinates

Solution With $x = 1$ and $y = -1$, we have that

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{-1}{1}\,.$$

The point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\frac{\pi}{4}$ or $\frac{7\pi}{4}$. Thus in polar coordinates, one possible answer is $(\sqrt{2}\,, -\frac{\pi}{4})$, another is $(\sqrt{2}\,, \frac{7\pi}{4})$. ■

Example 7.2.2 Convert the given equations from rectangular to polar coordinates.

1 $x = 2$

Solution

$$x = 2$$

$$r\cos\theta = 2$$

$$r = 2\sec\theta$$

■

2 $y = -1$

Solution

$$y = -1$$

$$r\sin\theta = -1$$

$$r = -\csc\theta$$

■

3 $x^2 + y^2 = 25$

Solution

$$\begin{aligned} x^2 + y^2 &= 25 \\ r^2 &= 25 \\ r &= \pm 5 \end{aligned}$$ ■

4 $x^2 - y^2 = 9$

Solution

$$\begin{aligned} x^2 - y^2 &= 9 \\ r^2(\cos^2\theta - \sin^2\theta) &= 9 \\ r^2(\cos 2\theta) &= 9 \\ r^2 &= 9\sec 2\theta \end{aligned}$$ ■

5 $2x - 5x^3 = 1 + xy$

Solution

$$\begin{aligned} 2x - 5x^3 &= 1 + xy \\ 2r\cos\theta - 5r^3\cos^3\theta &= 1 + r^2\cos\theta\sin\theta \end{aligned}$$ ■

6 $x^2(x^2 + y^2) = y^2$

Solution

$$\begin{aligned} x^2(x^2 + y^2) &= y^2 \\ r^4\cos^2\theta &= r^2\sin^2\theta \\ r^2(r^2\cos^2\theta - \sin^2\theta) &= 0. \end{aligned}$$

So

$$r^2 = 0 \quad \text{or} \quad r^2 = \tan^2\theta\,.$$

We can see that $r^2 = 0$ is included in $r^2 = \tan^2\theta$, for $\theta = 0$. Thus $r^2 = \tan^2\theta$. ■

Example 7.2.3 Convert the given equations from polar to rectangular coordinates.

1 $r = 3$

Solution

$$\begin{aligned} r &= 3 \\ r^2 &= 9 \\ x^2 + y^2 &= 9 \end{aligned}$$ ■

2 $r = 2\cos\theta$

Solution

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x \\ x^2 - 2x + y^2 &= 0 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$ ■

3 $r = -8\sin\theta$

Solution

$$\begin{aligned} r &= -8\sin\theta \\ r^2 &= -8r\sin\theta \\ x^2 + y^2 &= -8y \\ x^2 + y^2 + 8y &= 0 \\ x^2 + (y+4)^2 &= 16 \end{aligned}$$ ■

4 $\theta = \frac{\pi}{4}$

Solution

$$\begin{aligned} \theta &= \frac{\pi}{4} \\ \tan^{-1}\left(\frac{y}{x}\right) &= \frac{\pi}{4} \\ \frac{y}{x} &= \tan\frac{\pi}{4} \\ \frac{y}{x} &= 1 \\ y &= x \end{aligned}$$ ■

5 $\quad r = 2\sec\theta$

Solution

$$\begin{aligned} r &= 2\sec\theta \\ \frac{r}{\sec\theta} &= 2 \\ r\cos\theta &= 2 \\ x &= 2 \end{aligned}$$ ■

6 $\quad r = \dfrac{4}{2\cos\theta + 3\sin\theta}$

Solution

$$\begin{aligned} r &= \frac{4}{2\cos\theta + 3\sin\theta} \\ 2r\cos\theta + 3r\sin\theta &= 4 \\ 2x + 3y &= 4 \end{aligned}$$ ■

7 $\quad r\cos\left(\theta - \frac{\pi}{3}\right) = 3$

Solution

$$\begin{aligned} r\cos\left(\theta - \tfrac{\pi}{3}\right) &= 3 \\ r\left(\cos\theta\cos\tfrac{\pi}{3} + \sin\theta\sin\tfrac{\pi}{3}\right) &= 3 \\ r\left(\tfrac{1}{2}\cos\theta + \tfrac{\sqrt{3}}{2}\sin\theta\right) &= 3 \\ r\cos\theta + \sqrt{3}\,r\sin\theta &= 6 \\ x + y\sqrt{3} &= 6 \end{aligned}$$ ■

8 $\quad r = \sin\theta + \cos\theta$

Solution

$$\begin{aligned}
r &= \sin\theta + \cos\theta \\
r^2 &= r\sin\theta + r\cos\theta \\
x^2 + y^2 &= y + x \\
x^2 - x + y^2 - y &= 0 \\
\left(x - \tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2}\right)^2 &= \tfrac{1}{4} + \tfrac{1}{4} \\
\left(x - \tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2}\right)^2 &= \tfrac{1}{2}
\end{aligned}$$

■

7.3 Graphs of Polar Equations

A polar equation is an equation of the form

$$F(r, \theta) = 0\,, \quad r = f(\theta).$$

in which r and θ appear as variables.

A point is said to lie on the graph of a polar equation if at least one of its sets of polar coordinates satisfies the equation. Other sets of polar coordinates for the same point may not satisfy the equation.

Example 7.3.1 Consider the polar equation

$$\theta = \frac{\pi}{4}\,.$$

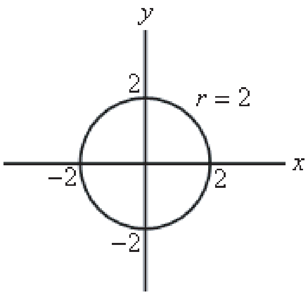
Solution The graph of this equation contains all points whose angular coordinate θ is $\frac{\pi}{4}$, and whose radial coordinate r may be positive, negative, or zero. Thus the graph is the straight line through the origin. This straight line has many other polar coordinates, such as $\theta = \frac{5\pi}{4}, -\frac{3\pi}{4}$. But only one of its infinitely many sets of polar coordinates satisfies $\theta = \frac{\pi}{4}$. ■

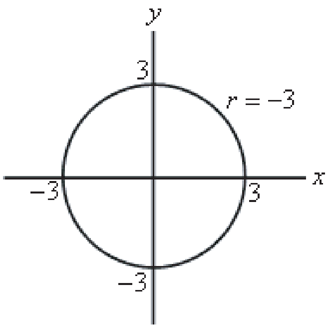
We now consider several examples of graphs of some simple polar equations.

Example 7.3.2 Sketch the graphs of

$$r = 2\,, \quad r = -3\,.$$

Solution





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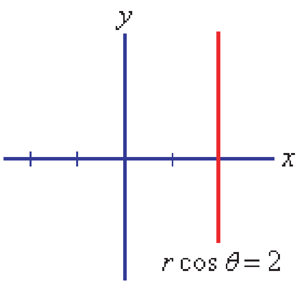
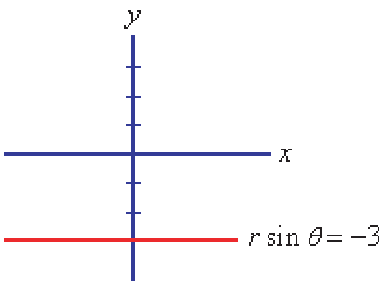
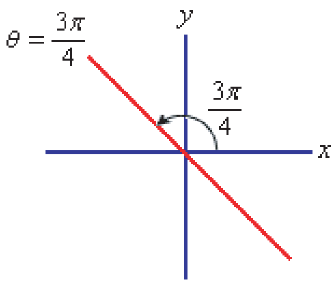
Example 7.3.3 Sketch the graphs of

$$r\cos\theta = 2\,, \quad r\sin\theta = -3\,, \quad \theta = \frac{3\pi}{4}\,.$$

Solution







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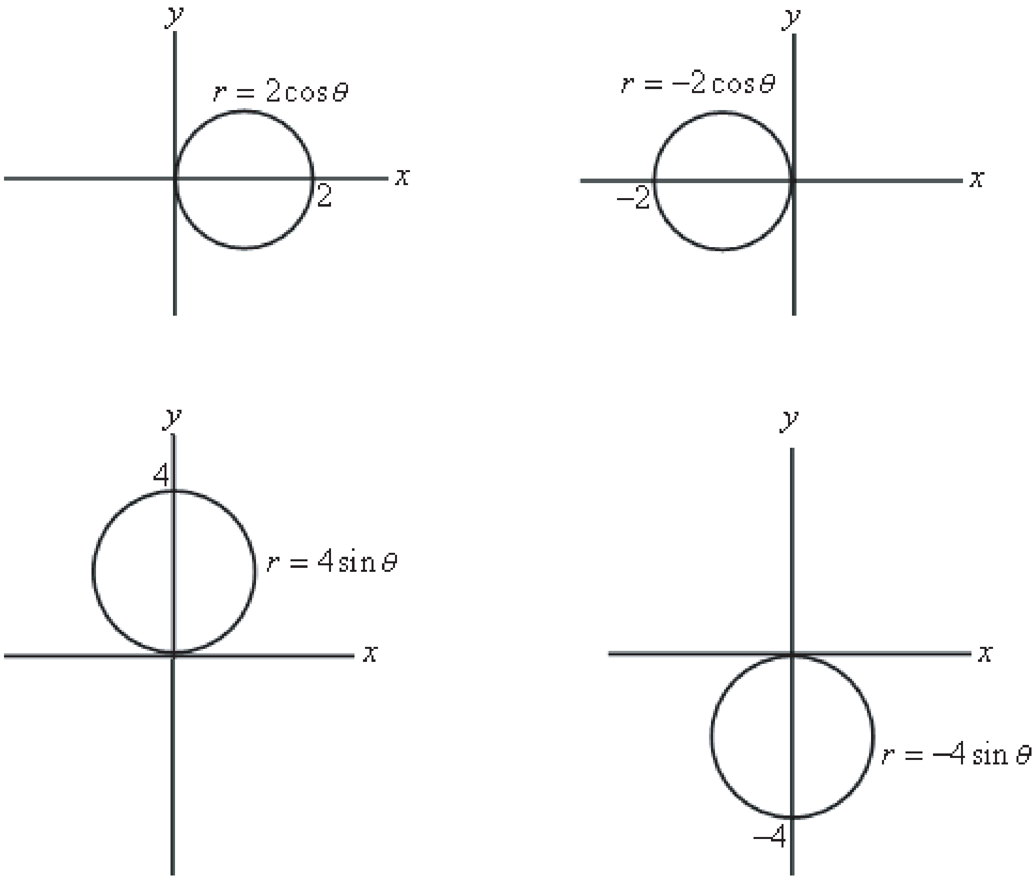
From Examples 7.3.2 and 7.3.3, we conclude that

Equation	Graph
$r = r_0$	circle of radius $\lvert r_0 \rvert$ centered at origin
$r\cos\theta = a$	vertical line at $x = a$
$r\sin\theta = b$	horizontal line at $y = b$
$\theta = \theta_0$	line through the origin making an angle θ_0 with the polar axis

Example 7.3.4 Sketch the graphs of

$$r = 2\cos\theta\,, \quad r = -2\cos\theta\,, \quad r = 4\sin\theta\,, \quad r = -4\sin\theta.$$

Solution



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From Example 7.3.4, we conclude that

Equation of circle with radius $a > 0$	Center	
	rectangular coordinates	polar coordinates
$r = 2a\cos\theta$	$(a, 0)$	$(a, 0)$
$r = -2a\cos\theta$	$(-a, 0)$	(a, π)
$r = 2a\sin\theta$	$(0, a)$	$(a, \frac{\pi}{2})$
$r = -2a\sin\theta$	$(0, -a)$	$(a, -\frac{\pi}{2})$

Example 7.3.5 Graph the set of points whose polar coordinates satisfy the given conditions.

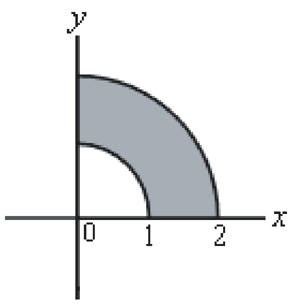
(a) $1 \le r \le 2\,, \quad 0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2\,, \quad \theta = \dfrac{\pi}{4}$

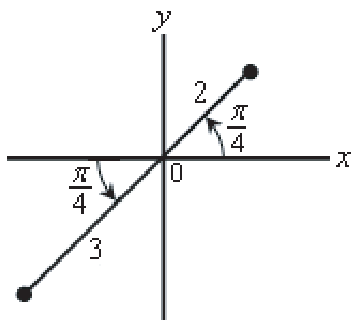
(c) $r \le 0\,, \quad \theta = \dfrac{\pi}{4}$

(d) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)

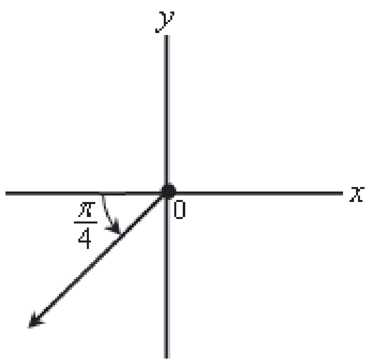
Solution



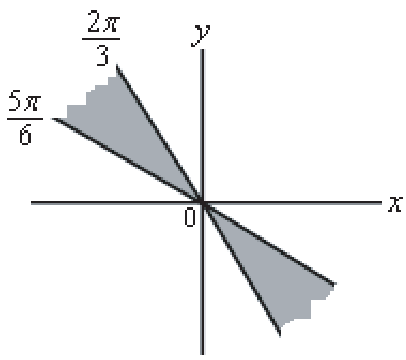
$1 \le r \le 2\,, \quad 0 \le \theta \le \dfrac{\pi}{2}$



$-3 \le r \le 2\,, \quad \theta = \dfrac{\pi}{4}$



$r \le 0\,, \quad \theta = \dfrac{\pi}{4}$

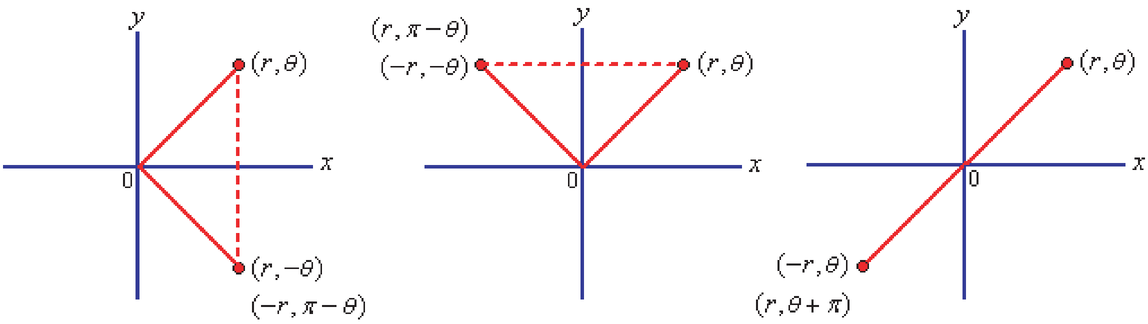


$\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)

■

7.3.1 Symmetry

It is helpful to note how we can often recognize certain symmetry properties, in polar coordinates, which is suggested by Figure 7.4.



About the x-axis About the y-axis About the origin

Figure 7.4 Three tests for symmetry

From Figure 7.4 we conclude that

Symmetry about the	The polar equation is equilalent if replacing
x-axis	θ by $-\theta$
y-axis	θ by $\pi - \theta$
origin	θ by $\theta + \pi$ or r by $-r$

Note A graph that is symmetric both about the x-axis and about the y-axis is also symmetric about the origin.

Example 7.3.6 Show that the graph of $r = 2\cos\theta$ is symmetric about the x-axis.

Solution To test for symmetry about the x-axis, we replace θ by $-\theta$. This yields

$$r = 2\cos(-\theta) = 2\cos\theta\,.$$

Thus replacing θ by $-\theta$ does not change the equation. ■

Example 7.3.7 Show that the graph of $r = 3\sin\theta$ is symmetric about the y-axis.

Solution To test for symmetry about the y-axis, we replace θ by $\pi - \theta$. This yields

$$r = 3\sin(\pi - \theta) = 3\sin\theta\,.$$

Thus replacing θ by $\pi - \theta$ does not change the equation. ■

Example 7.3.8 Show that the graph of $r = 2\cos 2\theta$ is symmetric about the x-axis and y-axis.

Solution To test for symmetry about the x-axis, we replace θ by $-\theta$. This yields

$$r = 2\cos(-2\theta) = 2\cos 2\theta\,.$$

Thus replacing θ by $-\theta$ does not change the equation.

To test for symmetry about the y-axis, we replace θ by $\pi - \theta$. This yields

$$r = 2\cos 2(\pi - \theta) = 2\cos(2\pi - 2\theta) = 2\cos 2\theta\,.$$

Thus replacing θ by $\pi - \theta$ does not change the equation.

Note See the graph of $r = 2\cos 2\theta$ in Example 7.3.17 on page 167. ■

Example 7.3.9 Show that the graph of $r^2 = a^2\cos 2\theta$ is symmetric about the x-axis, y-axis, and the origin.

Solution

Note See the graph of $r^2 = a^2\cos 2\theta$ on page 168. ■

7.3.2 Interesting Polar Graphs

We will discuss some interesting but simple polar graphs as follows

Limaçons

Equations with any of the four forms

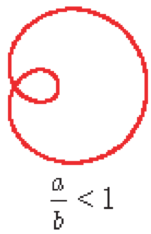
$$r = a \pm b\cos\theta\,, \quad r = a \pm b\sin\theta$$

where $a > 0$ and $b > 0$ represent polar curves called limaçons. There are four possible shapes for a limaçon that are determined by the ratio a/b.

1 For $\frac{a}{b} < 1$, equations of the four forms

$$r = a \pm b\cos\theta \quad , \quad r = a \pm b\sin\theta$$

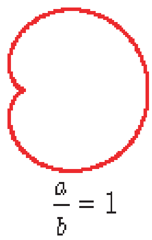
are called *limaçons with inner loop.*



2 For $\frac{a}{b} = 1$, equations of the four forms

$$r = a \pm b\cos\theta \quad , \quad r = a \pm b\sin\theta$$

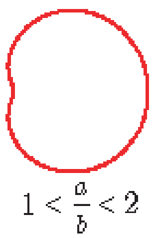
are called *cardioids.*



3 For $1 < \frac{a}{b} < 2$, equations of the four forms

$$r = a \pm b\cos\theta \quad , \quad r = a \pm b\sin\theta$$

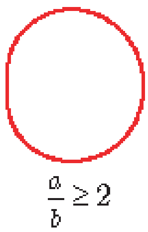
are called *dimpled limaçons.*



4 For $\frac{a}{b} \geq 2$, equations of the four forms

$$r = a \pm b\cos\theta \quad , \quad r = a \pm b\sin\theta$$

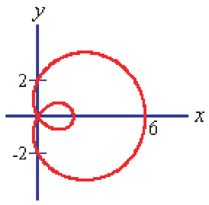
are called *convex limaçons.*



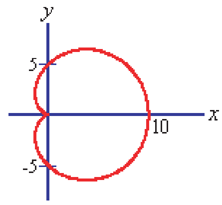
Example 7.3.10 Sketch the graphs of

$$r = 2 + 4\cos\theta\,, \quad r = 5 + 5\cos\theta\,, \quad r = 8 + 6\cos\theta\,, \quad r = 8 + 4\cos\theta.$$

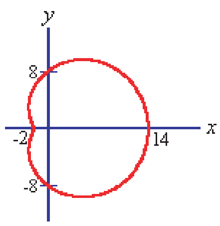
Solution



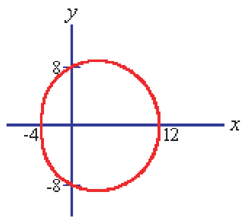
$r = 2 + 4\cos\theta$



$r = 5 + 5\cos\theta$



$r = 8 + 6\cos\theta$



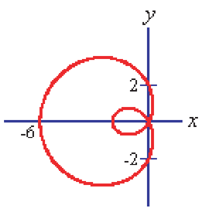
$r = 8 + 4\cos\theta$

Note Only limaçons with inner loop, and cardioids that contain the origin. ■

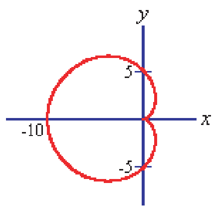
Example 7.3.11 Sketch the graphs of

$$r = 2 - 4\cos\theta\,, \quad r = 5 - 5\cos\theta\,, \quad r = 8 - 6\cos\theta\,, \quad r = 8 - 4\cos\theta.$$

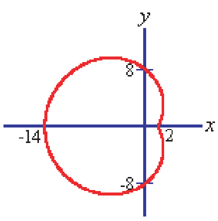
Solution



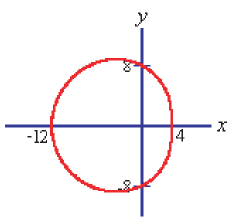
$r = 2 - 4\cos\theta$



$r = 5 - 5\cos\theta$



$r = 8 - 6\cos\theta$

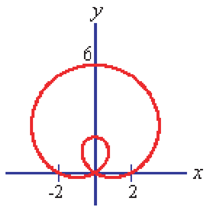


$r = 8 - 4\cos\theta$ ■

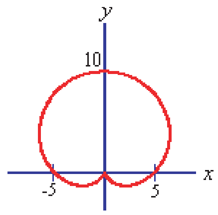
Example 7.3.12 Sketch the graphs of

$$r = 2 + 4\sin\theta\,, \quad r = 5 + 5\sin\theta\,, \quad r = 8 + 6\sin\theta\,, \quad r = 8 + 4\sin\theta.$$

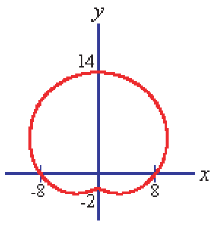
Solution



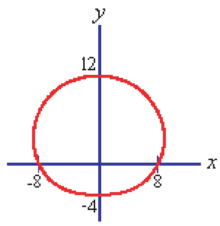
$r = 2 + 4\sin\theta$



$r = 5 + 5\sin\theta$



$r = 8 + 6\sin\theta$

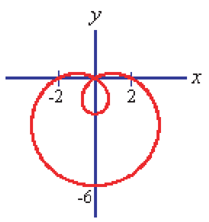


$r = 8 + 4\sin\theta$ ■

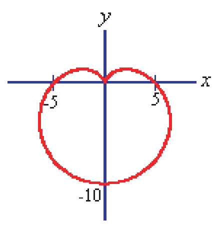
Example 7.3.13 Sketch the graphs of

$$r = 2 - 4\sin\theta\,, \quad r = 5 - 5\sin\theta\,, \quad r = 8 - 6\sin\theta\,, \quad r = 8 - 4\sin\theta.$$

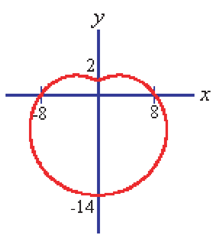
Solution



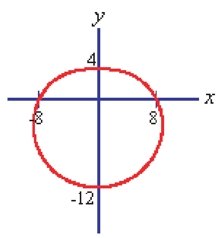
$r = 2 - 4\sin\theta$



$r = 5 - 5\sin\theta$



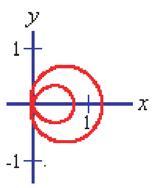
$r = 8 - 6\sin\theta$



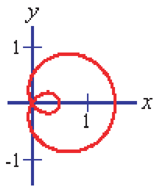
$r = 8 - 4\sin\theta$ ■

Example 7.3.14 Sketch the graphs of $r = a + \cos\theta$ with the constant a varying from 0.25 to 3 in steps of 0.25.

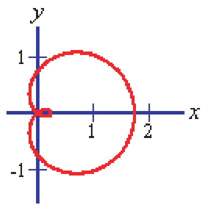
Solution



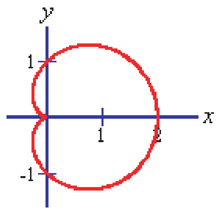
$a = 0.25$



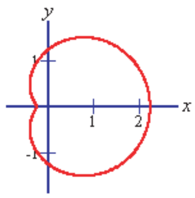
$a = 0.5$



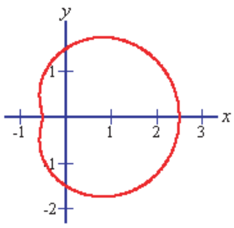
$a = 0.75$



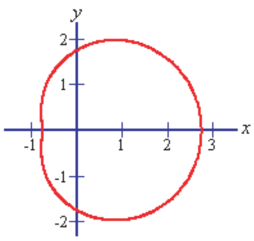
$a = 1$



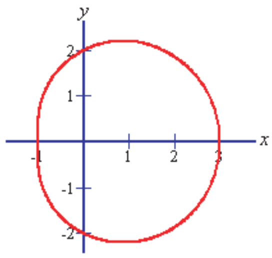
$a = 1.25$



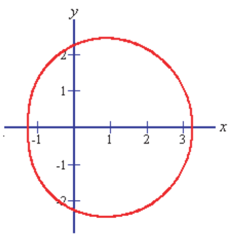
$a = 1.5$



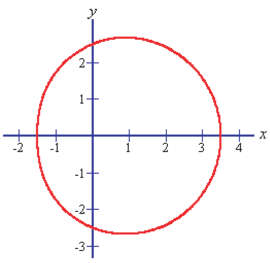
$a = 1.75$



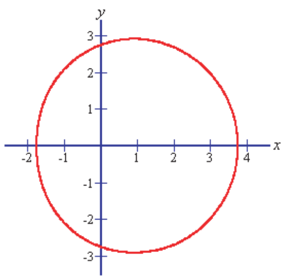
$a = 2$



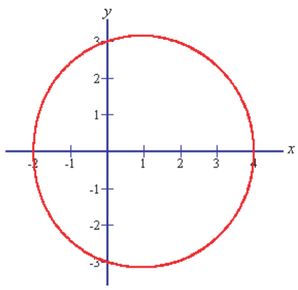
$a = 2.25$



$a = 2.5$



$a = 2.75$



$a = 3$

As a increases from the starting value of 0.25 , the inner loops get smaller and smaller until the cardioid is reached at $a = 1$. As a increases further, the limaçons evolve through the dimpled limaçon into the convex limaçon. ■

Note Limaçon is derived from the Latin word *limax* for a snail-like creature that is commonly called a slug.

Rose Curves

Equations of the form

$$r = a\cos n\theta\,, \quad r = a\sin n\theta$$

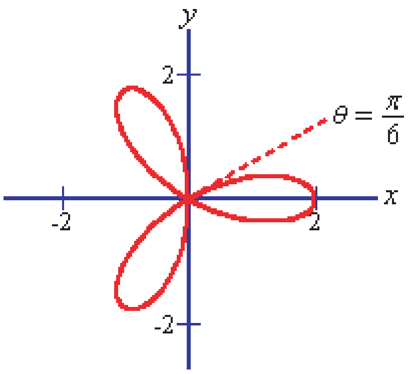
where n is an integer, represent flower-shaped curves called roses. These can be broken up into the following two cases.

1 The rose consists of n equally spaced leaves (petals) of radius $|a|$ if n is odd.

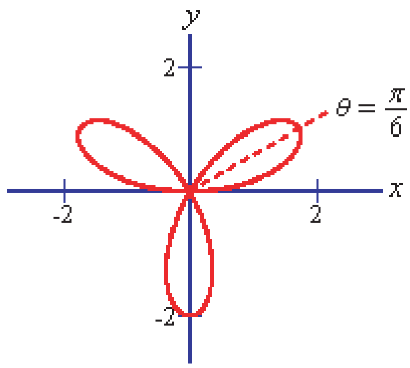
2 The rose consists of $2n$ equally spaced leaves (petals) of radius $|a|$ if n is even.

Example 7.3.15 Sketch the graphs of $r = 2\cos 3\theta\,,\ r = 2\sin 3\theta\,.$

Sol



$r = 2\cos 3\theta$

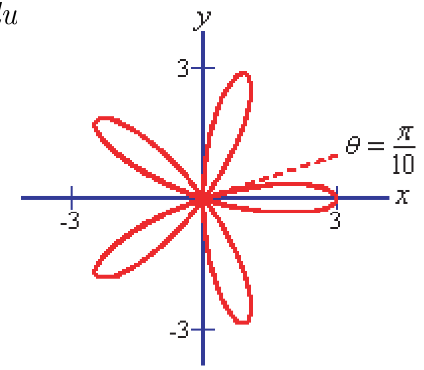


$r = 2\sin 3\theta$

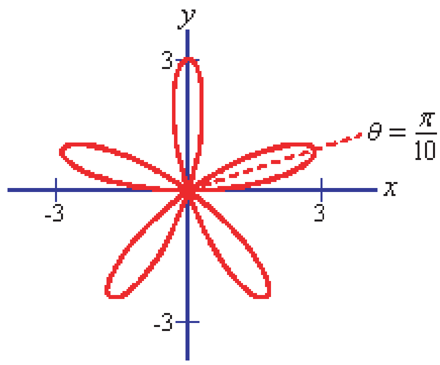
Note What do the graphs of $r = -2\cos 3\theta\,,\ r = -2\sin 3\theta$ look like ? ■

Example 7.3.16 Sketch the graphs of $r = 3\cos 5\theta\,,\ r = 3\sin 5\theta\,.$

Solu



$r = 3\cos 5\theta$

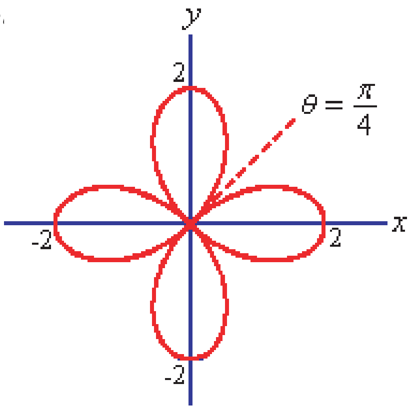


$r = 3\sin 5\theta$

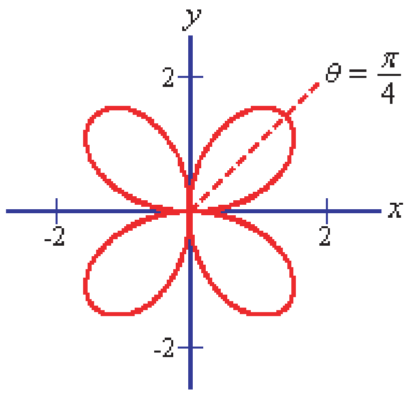
Note What do the graphs $r = -3\cos 5\theta\,,\ r = -3\sin 5\theta$ of look like ? ■

Example 7.3.17 Sketch the graphs of $r = 2\cos 2\theta$, $r = 2\sin 2\theta$.

So



$r = 2\cos 2\theta$

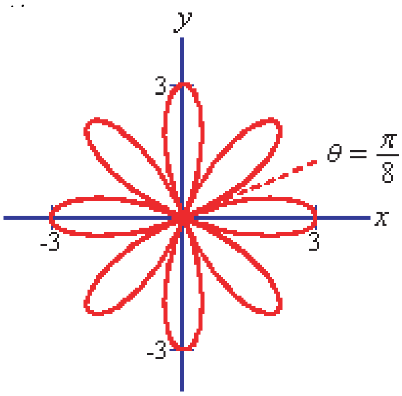


$r = 2\sin 2\theta$

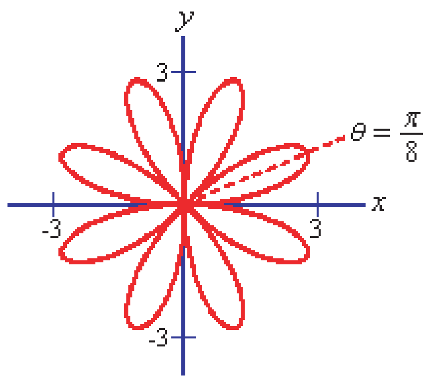
Note What do the graphs of $r = -2\cos 2\theta$, $r = -2\sin 2\theta$ look like ? ■

Example 7.3.18 Sketch the graphs of $r = 3\cos 4\theta$, $r = 3\sin 4\theta$.

Sol



$r = 3\cos 4\theta$



$r = 3\sin 4\theta$

Note What do the graphs of $r = -3\cos 4\theta$, $r = -3\sin 4\theta$ look like ? ■

Remark For the rose curves

$$r = a\cos n\theta\,, \qquad r = a\sin n\theta$$

where n is an integer. We conclude that

1 The angle between each axis of rose curves is $\dfrac{360^\circ}{n}$ if n is odd.

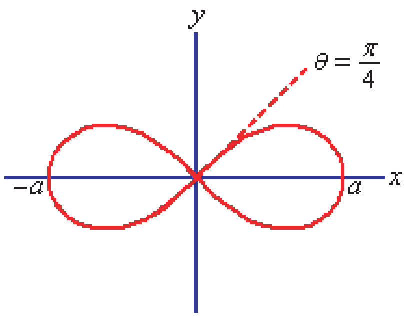
2 The angle between each axis of rose curves is $\dfrac{360^\circ}{2n}$ if n is even.

Lemniscates

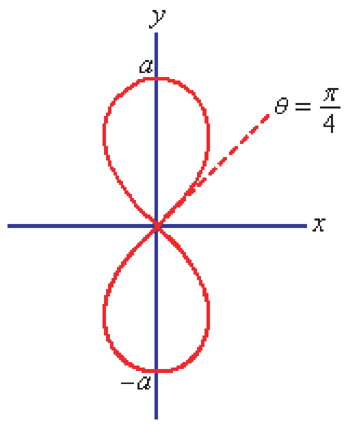
Equations of the form

$$r^2 = a^2 \cos 2\theta\,, \quad r^2 = -a^2 \cos 2\theta\,, \quad r^2 = a^2 \sin 2\theta\,, \quad r^2 = -a^2 \sin 2\theta$$

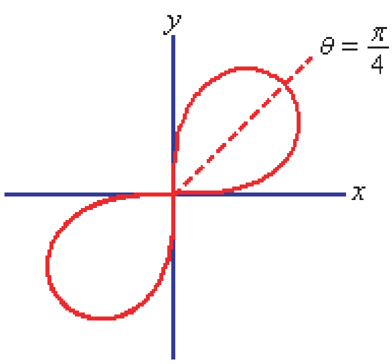
represent propeller-shaped graphs called lemniscates as follows.



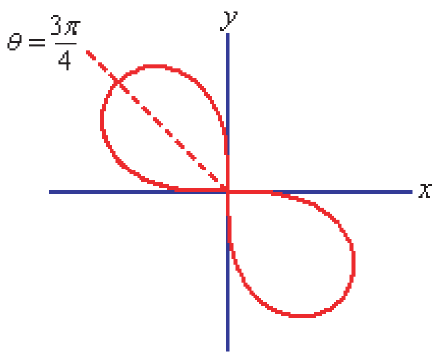
$r^2 = a^2 \cos 2\theta$



$r^2 = -a^2 \cos 2\theta$



$r^2 = a^2 \sin 2\theta$



$r^2 = -a^2 \sin 2\theta$

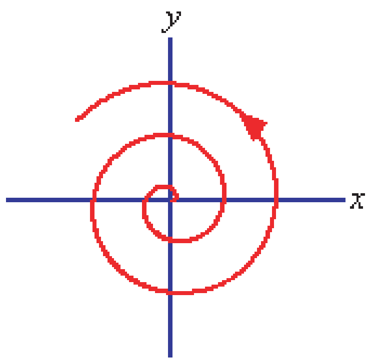
Note Lemniscate is derived from the Greek word *lemniscos* for a looped ribbon resembling the number 8.

Spirals

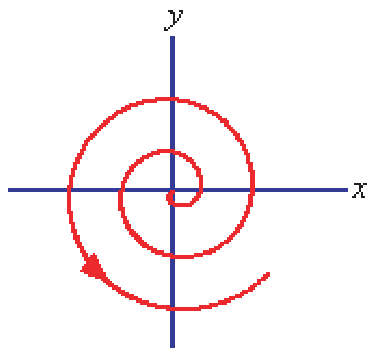
Equations of the form

$$r = a\theta\,, \quad r = -a\theta\,, \quad r = e^{a\theta}\,, \quad r = e^{-a\theta}$$

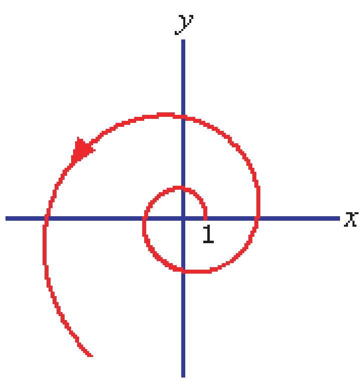
where $a > 0$ and $\theta \geq 0$, represent curves called spirals as follows.



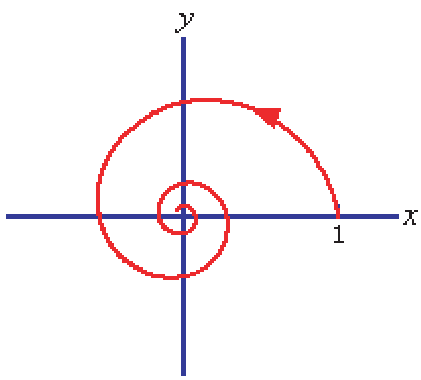
$r = a\theta\ , a > 0$



$r = -a\theta\ , a > 0$



$r = e^{a\theta}$, $a > 0$



$r = e^{-a\theta}$, $a > 0$

Note What do the graphs look like when $a > 0$ and $\theta \leq 0$?

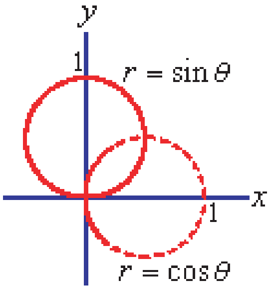
7.4 Finding Points Where Polar Graphs Intersect

The fact that we can represent a point in different ways in polar coordinates makes extra care necessary in deciding when a point lies on the graph of a polar equation and in determining the points in which polar graphs intersect. The problem is that a point of intersection may satisfy the equation of one curve with polar coordinates that are different from the ones with which it satisfies the equation of another curve. Thus solving the equations of two curves simultaneously may not identify all their points of intersection. *The only sure way to identify all the points of intersection is to graph the equations.*

Example 7.4.1 Find the points of intersection of $r = \cos\theta$ and $r = \sin\theta$.

Solution From the graph of two curves, point of intersection is $(0,0)$. Find another point from

$$\begin{aligned}\sin\theta &= \cos\theta\\ \tan\theta &= 1\\ \theta &= \frac{\pi}{4}.\end{aligned}$$

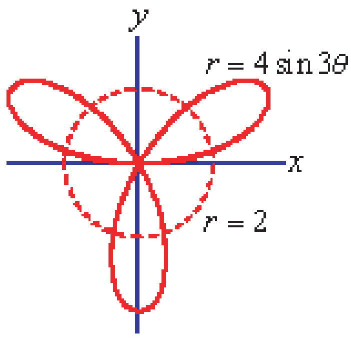


Thus intersection points are $(0,0)$, $(\frac{1}{\sqrt{2}}, \frac{\pi}{4})$. ∎

Example 7.4.2 Find the points of intersection of $r = 4\sin 3\theta$ and $r = 2$.

Solution From the graph of two curves, there are 6 intersection points.

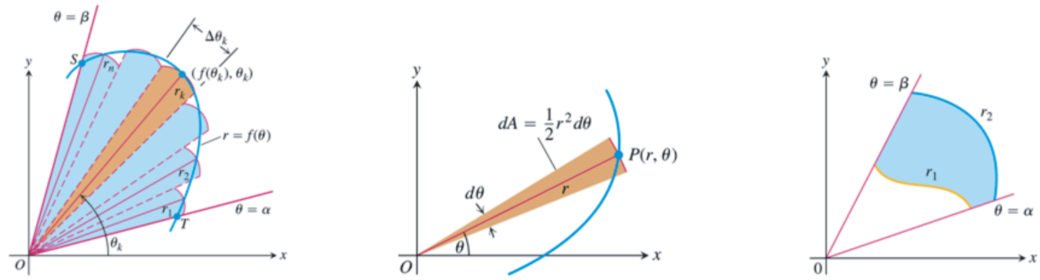
$$\begin{aligned}4\sin 3\theta &= 2\\ \sin 3\theta &= \frac{1}{2}\\ 3\theta &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}\\ \theta &= \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}.\end{aligned}$$



Thus intersection points are $(2, \frac{\pi}{18})$, $(2, \frac{5\pi}{18})$, $(2, \frac{13\pi}{18})$, $(2, \frac{17\pi}{18})$, $(2, \frac{25\pi}{18})$, $(2, \frac{29\pi}{18})$. ∎

7.5 Area in Polar Coordinates

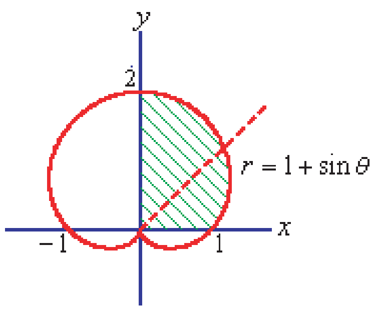
The area A of the region bounded by $0 \le r_1(\theta) \le r_2(\theta)$ and $\alpha \le \theta \le \beta$ is



$$A = \int_\alpha^\beta \frac{1}{2}(r_2)^2 d\theta - \int_\alpha^\beta \frac{1}{2}(r_1)^2 d\theta = \frac{1}{2}\int_\alpha^\beta \left[(r_2)^2 - (r_1)^2\right] d\theta\,.$$

Example 7.5.1 Find the area of the region bounded by the cardioid $r = 1 + \sin\theta$ in the first quadrant.

Solution

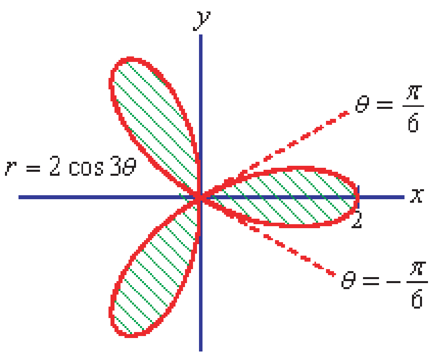


$$\begin{aligned}
A &= \frac{1}{2}\int_\alpha^\beta \left[(r_2)^2 - (r_1)^2\right] d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \left[(1+\sin\theta)^2 - 0^2\right] d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} (1 + 2\sin\theta + \sin^2\theta)\, d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \left[1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\right] d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \left[\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta\right] d\theta \\
&= \frac{1}{2}\left[\frac{3\theta}{2} - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \\
&= \frac{1}{2}\left[\frac{3\pi}{4} - 0 - 0 - (0 - 2 - 0)\right] \\
&= 1 + \frac{3\pi}{8} \quad \text{square units}
\end{aligned}$$

■

Example 7.5.2 Find the area of the region bounded by the curve $r = 2\cos 3\theta$.

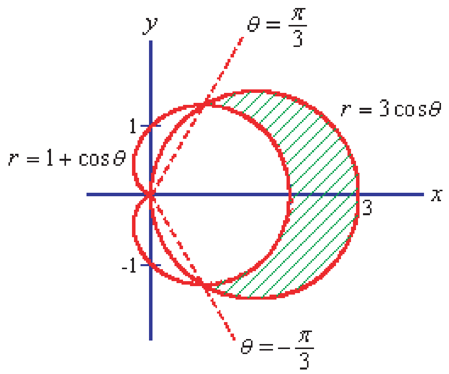
Solution



$$\begin{aligned} A &= \frac{6}{2}\int_0^{\pi/6} (2\cos 3\theta)^2\, d\theta \\ &= 3\int_0^{\pi/6} 4\cos^2 3\theta\, d\theta \\ &= 6\int_0^{\pi/6} (1+\cos 6\theta)\, d\theta \\ &= 6\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} \\ &= 6\left[\frac{\pi}{6} + 0 - (0+0)\right] \\ &= \pi \quad \text{square units} \end{aligned}$$ ■

Example 7.5.3 Find the area that lies outside the cardioid $r = 1 + \cos\theta$ and inside the circle $r = 3\cos\theta$.

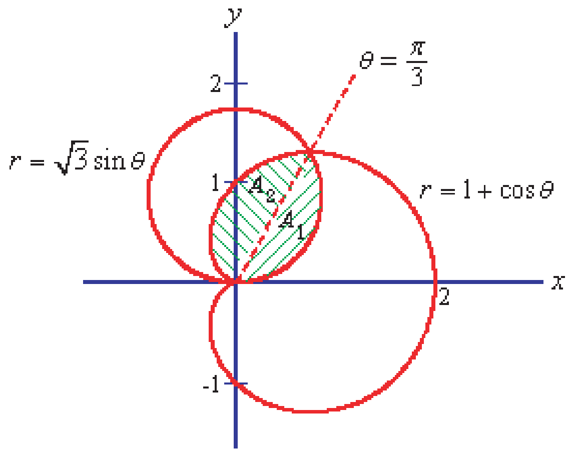
Solution



$$\begin{aligned} A &= \frac{2}{2}\int_0^{\pi/3} \left[(3\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta \\ &= \int_0^{\pi/3} \left(9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta\right) d\theta \\ &= \int_0^{\pi/3} \left(8\cos^2\theta - 2\cos\theta - 1\right) d\theta \\ &= \int_0^{\pi/3} (4 + 4\cos 2\theta - 2\cos\theta - 1)\, d\theta \\ &= \int_0^{\pi/3} (3 + 4\cos 2\theta - 2\cos\theta)\, d\theta \\ &= 3\theta + 2\sin 2\theta - 2\sin\theta \Big|_0^{\pi/3} \\ &= \pi + \sqrt{3} - \sqrt{3} - (0+0-0) \\ &= \pi \quad \text{square units} \end{aligned}$$ ■

Example 7.5.4 Find the area of the region bounded by the cardioid $r = 1 + \cos\theta$ and the circle $r = \sqrt{3}\sin\theta$.

Solution

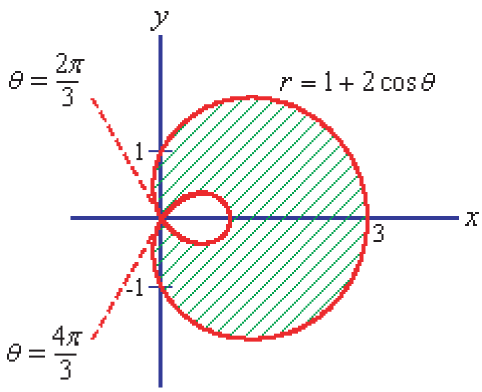


$$
\begin{aligned}
A &= A_1 + A_2 \\
&= \frac{1}{2}\int_0^{\pi/3} (\sqrt{3}\sin\theta)^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi} (1+\cos\theta)^2 d\theta \\
&= \frac{3}{2}\int_0^{\pi/3} \sin^2\theta\, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi} (1 + 2\cos\theta + \cos^2\theta) d\theta \\
&= \frac{3}{2}\int_0^{\pi/3} \frac{1-\cos 2\theta}{2}\, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi} \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right) d\theta \\
&= \frac{3}{4}\int_0^{\pi/3} (1-\cos 2\theta)\, d\theta + \frac{1}{2}\int_{\pi/3}^{\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta \\
&= \frac{3}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/3} + \frac{1}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\pi/3}^{\pi} \\
&= \frac{3}{4}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} - (0-0)\right] + \frac{1}{2}\left[\frac{3\pi}{2} + 0 + 0 - \frac{\pi}{2} - \sqrt{3} - \frac{\sqrt{3}}{8}\right] \\
&= \frac{\pi}{4} - \frac{3\sqrt{3}}{16} + \frac{\pi}{2} - \frac{9\sqrt{3}}{16} \\
&= \frac{3\pi}{4} - \frac{3\sqrt{3}}{4} = \frac{3}{4}(\pi - \sqrt{3}) \quad \text{square units}
\end{aligned}
$$

■

Example 7.5.5 Find the area bounded by each loop of the limaçons with equation $r = 1 + 2\cos\theta$.

Solution



Let A_1 be the area of the outer loop and A_2 be the area of the inner loop.

$$
\begin{aligned}
A &= A_1 - A_2 \\
&= \frac{2}{2}\int_0^{2\pi/3} (1 + 2\cos\theta)^2\, d\theta - \frac{2}{2}\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta \\
&= \int_0^{2\pi/3} (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta - \int_{2\pi/3}^{\pi} (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta \\
&= \int_0^{2\pi/3} (3 + 4\cos\theta + 2\cos 2\theta)\, d\theta - \int_{2\pi/3}^{\pi} (3 + 4\cos\theta + 2\cos 2\theta)\, d\theta \\
&= \Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_0^{2\pi/3} - \Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{2\pi/3}^{\pi} \\
&= 2\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2} - \left(3\pi - 2\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) \\
&= 2\pi + \frac{3\sqrt{3}}{2} - \pi + \frac{3\sqrt{3}}{2} \\
&= \pi + 3\sqrt{3} \quad \text{square units}
\end{aligned}
$$

■

Exercise 6.1

1 Plot the points whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

1.1	$(1, \pi/2)$	1.2	$(-2, \pi/4)$
1.3	$(3, 2)$	1.4	$(3, 0)$
1.5	$(2, -\pi/7)$	1.6	$(-1, -\pi/2)$

2 Find the rectangular coordinates of the points whose polar coordinates are given.

2.1	$(6, \pi/6)$	2.2	$(7, 2\pi/3)$
2.3	$(-6, -5\pi/6)$	2.4	$(0, -\pi)$
2.5	$(7, 17\pi/6)$	2.6	$(-5, 0)$

3 Plot the points with the given rectangular coordinates and find two sets of polar coordinates for each.

3.1	$(-1, 1)$	3.2	$(1, -\sqrt{3})$
3.3	$(0, 3)$	3.4	$(-1, 0)$

4 Graph the set of points whose polar coordinates satisfy the given equations and inequalities.

4.1	$r = 2$	4.2	$0 \le r \le 2$
4.3	$r \ge 1$	4.4	$0 \le \theta \le \pi/6, \quad r \ge 0$
4.5	$\theta = 2\pi/3, \quad r \le -2$	4.6	$\theta = \pi/3, \quad -1 \le r \le 3$
4.7	$0 \le \theta \le \pi, \quad r = 1$	4.8	$0 \le \theta \le \pi, \quad r = -1$
4.9	$\theta = \pi/2, \quad r \le 0$	4.10	$\pi/4 \le \theta \le 3\pi/4, \quad 0 \le r \le 1$
4.11	$-\pi/4 \le \theta \le \pi/4, \quad -1 \le r \le 1$	4.12	$0 \le \theta \le \pi/2, \quad 1 \le \lvert r \rvert \le 2$

5 Replace the polar equation by an equivalent rectangular equation.

5.1 $r\sin\theta = 0$

5.2 $r = 4\csc\theta$

5.3 $r\cos\theta + r\sin\theta = 1$

5.4 $r^2 = 4r\sin\theta$

5.5 $r^2\sin 2\theta = 2$

5.6 $r = e^{r\cos\theta}\csc\theta$

5.7 $r\sin\theta = \ln r + \ln\cos\theta$

5.8 $r^2 = -4r\cos\theta$

5.9 $r = 2\cos\theta + 2\sin\theta$

6 Replace the rectangular equation by an equivalent polar equation.

6.1 $x = 7$

6.2 $x = y$

6.3 $x^2 + y^2 = 4$

6.4 $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

6.5 $y^2 = 4x$

6.6 $x^2 + (y-2)^2 = 4$

7 Find the points of intersection of the pairs of curves.

7.1 $r = 1 + \cos\theta\,, \quad r = 1 - \cos\theta$

7.2 $r = \cos\theta\,, \quad r = 1 - \cos\theta$

7.3 $r^2 = \sin 2\theta\,, \quad r^2 = \cos 2\theta$

7.4 $r = 1\,, \quad r = 2\sin 2\theta$

8 Find the area of the region described.

8.1 The region inside the limaçon $r = 4 + 2\cos\theta$.

8.2 The region enclosed by the lemniscate $r^2 = a^2\cos 2\theta$.

8.3 The region inside the cardioid $r = 2(1+\cos\theta)$ and outside the circle $r = 3$.

8.4 The region inside the circle $r = 10$ and to the right of the line $r = 6\sec\theta$.

8.5 The region between the curves $r = \cos 2\theta$ and $r = 2 + \sin\theta$.

8.6 The region shared by the curves $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

Answers to Exercise 6.1

1.1 $(1, \frac{5\pi}{2}), (-1, \frac{3\pi}{2})$

1.2 $(2, \frac{5\pi}{4}), (-2, \frac{9\pi}{4})$

1.3 $(3, 2+2\pi), (-3, 2+\pi)$

1.4 $(3, 2\pi), (-3, \pi)$

1.5 $(2, \frac{13\pi}{7}), (-2, \frac{6\pi}{7})$

1.6 $(1, \frac{\pi}{2}), (-1, \frac{3\pi}{2})$

2.1 $(3\sqrt{3}, 3)$

2.2 $(-\frac{7}{2}, \frac{7\sqrt{3}}{2})$

2.3 $(3\sqrt{3}, 3)$

2.4 $(0, 0)$

2.5 $(-\frac{7\sqrt{3}}{2}, \frac{7}{2})$

2.6 $(-5, 0)$

3.1 $(\sqrt{2}, \frac{3\pi}{4}), (\sqrt{2}, -\frac{5\pi}{4})$

3.2 $(2, -\frac{\pi}{3}), (-2, \frac{2\pi}{3})$

3.3 $(3, \frac{\pi}{2}), (3, \frac{5\pi}{2})$

3.4 $(1, \pi), (-1, 0)$

5.1 $y = 0$

5.2 $y = 4$

5.3 $x + y = 1$

5.4 $x^2 + (y-2)^2 = 4$

5.5 $xy = 1$

5.6 $y = e^x$

5.7 $y = \ln x$

5.8 $(x+2)^2 + y^2 = 4$

5.9 $(x-1)^2 + (y-1)^2 = 2$

6.1 $r\cos\theta = 7$

6.2 $\theta = \frac{\pi}{4}$

6.3 $r^2 = 4$ or $r = 2$

6.4 $r^2(4\cos^2\theta + 9\sin^2\theta) = 36$

6.5 $r\sin^2\theta = 4\cos\theta$

6.6 $r = 4\sin\theta$

7.1 $(0, 0), (1, \frac{\pi}{2}), (1, \frac{3\pi}{2})$

7.2 $(0, 0), (\frac{1}{2}, \pm\frac{\pi}{3})$

7.3 $(0, 0), (\pm\frac{1}{\sqrt[4]{2}}, \frac{\pi}{8})$

7.4 $(1, \frac{\pi}{12}), (1, \frac{5\pi}{12}), (1, \frac{7\pi}{12}), (1, \frac{11\pi}{12})$
$(1, \frac{13\pi}{12}), (1, \frac{17\pi}{12}), (1, \frac{19\pi}{12}), (1, \frac{23\pi}{12})$

8.1 18π

8.2 a^2

8.3 $\frac{9\sqrt{3}}{2} - \pi$

8.4 $100\cos^{-1}\left(\frac{3}{5}\right) - 48$

8.5 4π

8.6 $\frac{5\pi}{2}$

Exercise 6.2

1 Use a graphing utility to generate the polar graph. Be sure to choose the interval of domain θ so that a complete graph is generated.

1.1 $r = \sin\frac{\theta}{2}$, $r = \cos\frac{\theta}{2}$

1.2 $r = \sin\frac{3\theta}{2}$, $r = \cos\frac{3\theta}{2}$

1.3 $r = \sin\frac{5\theta}{4}$, $r = \cos\frac{5\theta}{4}$

1.4 $r = \sin\frac{5\theta}{8}$, $r = \cos\frac{5\theta}{8}$

1.5 $r = \sin\frac{\theta}{3}$, $r = \cos\frac{\theta}{3}$

1.6 $r = \sin\frac{2\theta}{3}$, $r = \cos\frac{2\theta}{3}$

1.7 $r = \sin\frac{4\theta}{3}$, $r = \cos\frac{4\theta}{3}$

1.8 $r = \sin\frac{5\theta}{3}$, $r = \cos\frac{5\theta}{3}$

1.9 $r^2 = \sin 2\theta$, $r^2 = \cos 2\theta$

1.10 $r^2 = \sin 3\theta$, $r^2 = \cos 3\theta$

1.11 $r = \sec 3\theta$

1.12 $r = \theta\sin\theta$

1.13 $r = \dfrac{1}{\theta}$, $r = \sqrt{\theta}$, $r = \dfrac{1}{\sqrt{\theta}}$

1.14 $r = e^{\cos\theta} - 2\cos 4\theta + \sin^3\frac{\theta}{4}$

2 Use a graphing utility to generate the polar graph with the constant a varying from 0 to 2 in steps of 0.25. Be sure to choose the interval of domain θ so that a complete graph is generated.

2.1 $r = a + \sin 2\theta$, $r = a + \cos 2\theta$

2.2 $r = a + \sin 3\theta$, $r = a + \cos 3\theta$

2.3 $r = a + \sin\frac{\theta}{3}$, $r = a + \cos\frac{\theta}{3}$

2.4 $r = a + \sin\frac{2\theta}{3}$, $r = a + \cos\frac{2\theta}{3}$

2.5 $r = a + \sin\frac{\theta}{4}$, $r = a + \cos\frac{\theta}{4}$

2.6 $r = a + \sin\frac{3\theta}{4}$, $r = a + \cos\frac{3\theta}{4}$

3 In the late seventeenth century, the Italian astronomer Giovanni Domenico Cassini (1625-1712) introduced the family of curves called Cassini ovals

$$(x^2 + y^2 + a^2)^2 - 4a^2x^2 = b^4 \tag{7.1}$$

where $a > 0, b > 0$ in his studies of the relative motions of the Earth and the Sun.

3.1 Show that equation (7.1) in polar equation is

$$r^4 + a^4 - 2a^2r^2\cos 2\theta = b^4.$$

3.2 Use a graphing utility to generate the polar graph with $a < b$, $a = b$, $a > b$.