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Chapter 2 – Spatial Representation and Transformation

2.1 Position, Orientation and Frames

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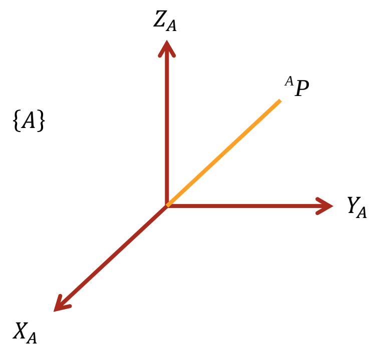
- Position Vector

$$^{A}P = \begin{bmatrix} P_X \\ P_Y \\ P_Z \\ \omega \end{bmatrix}$$

Usually $\omega = 1$



2.1 Position, Orientation and Frames

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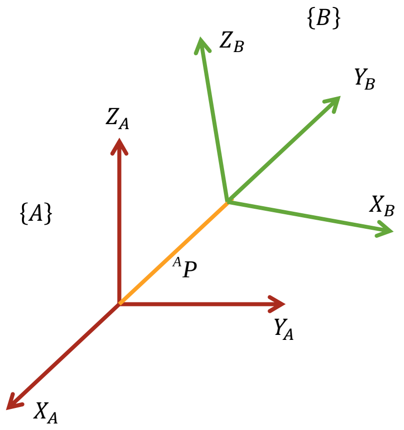
- Orientation Matrix



$$^AR_B = \left[\, {}^AX_B \quad {}^AY_B \quad {}^AZ_B \,\right]$$

$$R_A \cdot {}^AR_B = R_B$$

$$^AR_B = R_A^{-1} \cdot R_B = \begin{pmatrix} X_A^t \\ Y_A^t \\ Z_A^t \end{pmatrix} \left[X_B \quad Y_B \quad Z_B \right]$$

$$= \begin{pmatrix} X_B \cdot X_A & Y_B \cdot X_A & Z_B \cdot X_A \\ X_B \cdot Y_A & Y_B \cdot Y_A & Z_B \cdot Y_A \\ X_B \cdot Z_A & Y_B \cdot Z_A & Z_B \cdot Z_A \end{pmatrix}$$

2.1 Position, Orientation and Frames

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- Remarks:
  1. Elements on ^AR_B are called directional cosine
  2. And $^BR_A = {}^AR_B^T$

- Coordinate Frame:
  - Referring to $\{A\}$, coordinate frame $\{B\}$ can be defined as:

$$\{B\} = \left\{ {}^AR_B, {}^AP_{B\_origin} \right\}$$

2.2 Mapping Between Frames

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- Translated frame:

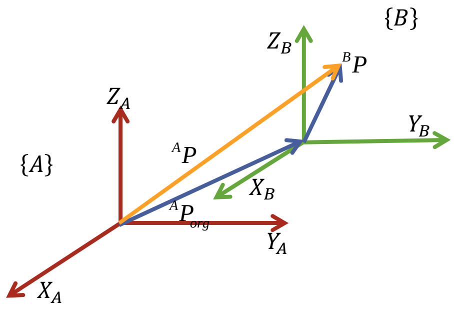
$${}^{A}P = {}^{B}P + {}^{A}P_{B\_origin}$$

- Rotated frame:

$${}^{A}P = {}^{A}R_{B} \, {}^{B}P$$

- General frame:

$${}^{A}P = {}^{A}R_{B} \, {}^{B}P + {}^{A}P_{B\_origin}$$



$$\begin{pmatrix} {}^{A}P \\ 1 \end{pmatrix} = \left(\begin{array}{c:c} {}^{A}R_{B} & {}^{A}P_{B\_origin} \\ \hdashline 0 & 1 \end{array}\right) \begin{pmatrix} {}^{B}P \\ 1 \end{pmatrix}$$

4x4 matrix (Homogenous transformation)

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2.3 Transformation

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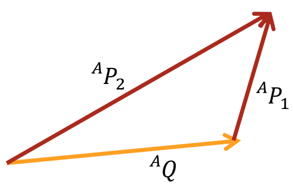
- Translation (commutative)

$${}^{A}P_{2} = {}^{A}P_{1} + {}^{A}Q$$

$${}^{A}P_{2} = Trans(Q) \, {}^{A}P_{1}$$

Prismatic revolute



$$Trans(P) + Trans(Q) = Trans(Q) + Trans(P)$$

$$Trans(Q) = \begin{pmatrix} 1 & 0 & 0 & q_x \\ 0 & 1 & 0 & q_y \\ 0 & 0 & 1 & q_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where $Q = (q_x q_y q_z)$

2.3 Transformation

- Rotation (not commutative)

$$^{A}P_2 = R_K(\theta)\,^{A}P_1 \qquad K: axis \quad \theta: angle$$

$$R_{k1}(\alpha) \bullet R_{k2}(\beta) \neq R_{k2}(\beta) \bullet R_{k1}(\alpha) \qquad except\ k_1 = k_2$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x' = x$$
$$y' = \cos\theta \cdot y - \sin\theta \cdot z$$
$$z' = \sin\theta \cdot y + \cos\theta \cdot z$$

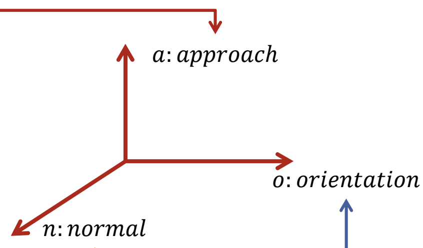
$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2.3 Transformation

- General

$$^{A}P_2 = T\,^{A}P_1$$

$$T = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$p: position$

2.3 Transformation

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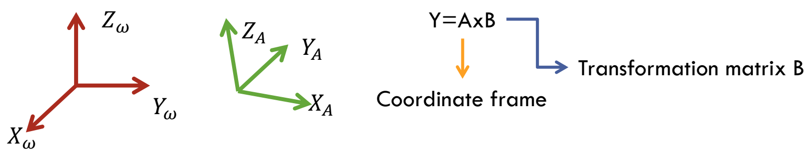
- Remarks:
  - **Y = A x B** (postmultiply), a transformation A by a second transformation B
  - ➔ the transformation described by B is with respect to the *frame of A*.



  - **Y = B x A** (premultiply), a transformation A by a second transformation B
  - ➔ the transformation described by B is with respect to the *base coordinate frame*.

2.3 Transformation

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- Compound

$${}^{B}P = {}^{B}T_{C} \bullet {}^{C}P$$

$${}^{A}P = {}^{A}T_{B} \bullet {}^{B}P$$

$${}^{A}P = {}^{A}T_{B} \bullet {}^{B}T_{C} \bullet {}^{C}P = {}^{A}T_{C}\,{}^{C}P$$

- Inversion

$${}^{B}T_{A} = {}^{A}T_{B}^{-1}$$

$$T = \begin{bmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{T^{-1} = \begin{pmatrix} n_x & n_y & n_z & -p\bullet n \\ o_x & o_y & o_z & -p\bullet o \\ a_x & a_y & a_z & -p\bullet a \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

2.3 Transformation

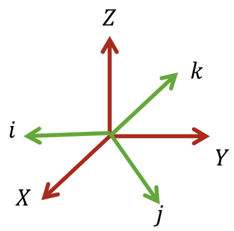
- General Rotation Transformation
  - Rotation about an arbitrary vector k from origin.
  - First find a coordinate frame C

$$C = \begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

k is the Z axis

i.e. $k = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$

$R_K(\theta) = R_{CZ}(\theta)$



Given a frame T, then we can find a frame X, s.t.

$$T = CX$$

$$\mathrm{X} = C^{-1}T$$

2.3 Transformation

- Rotating T about k is equivalent to rotating X around the Z axis of frame C

$$R_k(\theta)\mathrm{T} = \mathrm{C}R_z(\theta)\,\mathrm{X} = \mathrm{C}R_z(\theta)\mathrm{C}^{-1}T$$

$$R_k(\theta) = \mathrm{C}R_z(\theta)\mathrm{C}^{-1}$$

$$= \begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} n_x & n_y & n_z & 0 \\ o_x & o_y & o_z & 0 \\ a_x & a_y & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} \mathrm{K}_x^2 \mathrm{vers}\,\theta + \cos\theta & K_x K_y vers\theta - K_z\sin\theta & K_x K_z vers\theta + K_y\sin\theta & 0 \\ K_x K_y vers\theta + K_z\sin\theta & \mathrm{K}_y^2 \mathrm{vers}\,\theta + \cos\theta & K_y K_z vers\theta - K_x\sin\theta & 0 \\ K_x K_z vers\theta - K_y\sin\theta & K_y K_z vers\theta + K_x\sin\theta & \mathrm{K}_z^2 \mathrm{vers}\,\theta + \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

Where $vers\theta = 1 - cos\theta$ and $(k_x, k_y, k_z) = (a_x, a_y, a_z)$

2.4 Equivalent angle-axis of rotation

- Given a rotational transformation R

$$R = \begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Equate $R\ to\ R_k(\theta)$
- Define Trace (A)=sum of diagonal components of matrix A

$$Trace(R) = n_x + o_y + a_z + 1$$

$$= K_x^2 vers\theta + \cos\theta + K_y^2 vers\theta + \cos\theta + K_z^2 vers\theta + \cos\theta + 1 = 2 + 2\cos\theta$$

$$\therefore \cos\theta = \frac{1}{2}(n_x + o_y + a_z - 1)$$

2.4 Equivalent angle-axis of rotation

$$o_z - a_y = 2k_x s\theta$$

And

$$a_x - n_z = 2k_y s\theta$$

$$n_y - o_x = 2k_z s\theta$$

$$\therefore (o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2 = 4\sin^2\theta$$

$$\sin\theta = \pm\frac{1}{2}\sqrt{(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2}$$

Note: why it is not preferred to solve q from: $k_x = \frac{o_z - a_y}{2\sin\theta}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Choose positive sign, then $0 \le \theta \le 180°$

2.4 Equivalent angle-axis of rotation

- Another set:

$$\begin{cases} K_x^2 \operatorname{vers}\theta + \cos\theta = n_x \\ K_y^2 \operatorname{vers}\theta + \cos\theta = o_y \\ K_z^2 \operatorname{vers}\theta + \cos\theta = a_z \end{cases} \rightarrow \begin{cases} k_x = \pm\sqrt{\dfrac{n_x - \cos\theta}{1-\cos\theta}} \\ k_y = \pm\sqrt{\dfrac{o_y - \cos\theta}{1-\cos\theta}} \\ k_z = \pm\sqrt{\dfrac{a_z - \cos\theta}{1-\cos\theta}} \end{cases}$$

- From the largest element of n_x, o_y and a_z the largest of k_x, k_y and k_z can be determined

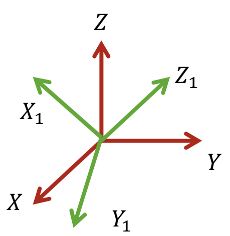
$$\text{If } n_x > \begin{cases} o_y \\ a_z \end{cases} \textit{then } k_x \textit{ is the largest} \quad \begin{matrix} n_y + o_x = 2k_x k_y vers\theta \\ o_z + a_y = 2k_y k_z vers\theta \\ n_z + a_x = 2k_z k_x vers\theta \end{matrix} \quad \begin{matrix} \therefore k_y = \dfrac{n_y + o_x}{2k_x vers\theta} \\ \therefore k_z = \dfrac{n_z + a_x}{2k_x vers\theta} \end{matrix}$$

2.5 Three Angle Rotation: Euler angles, RPY Angles

- Euler Angels:

$$Euler(\phi, \theta, \psi) = \underset{\text{Roll}}{R_Z(\phi)}\underset{\text{Pitch}}{R_Y(\theta)}\underset{\text{Yaw}}{R_Z(\psi)} \quad \text{(z,y,z)}$$

a) Rotate ϕ about z-axis
b) Rotate θ about new y-axis
c) Rotate ψ about new z-axis



2.5 Three Angle Rotation: Euler angles, RPY Angles

- Note: z−y−x or other combination are also possible.

$$=\begin{pmatrix}\cos\phi & -\sin\phi & 0 & 0\\ \sin\phi & \cos\phi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}*\begin{pmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{pmatrix}*\begin{pmatrix}\cos\psi & -\sin\psi & 0 & 0\\ \sin\psi & \cos\psi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

2.5 Three Angle Rotation: Euler angles, RPY Angles

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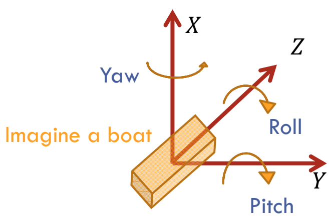
$$RPY(\phi,\theta,\psi)=Rz(\phi)Ry(\theta)Rx(\psi)$$

a) Rotate ψ about x-axis
b) Rotate θ about y-axis
c) Rotate ϕ about z-axis



2.5 Three Angle Rotation: Euler angles, RPY Angles

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- Euler transformation solution

$Euler(\phi,\theta,\psi)=T$

$$=\begin{pmatrix} \cos\phi\cos\theta\cos\psi-\sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi-\sin\phi\cos\psi & \cos\phi\sin\theta & 0 \\ \sin\phi\cos\theta\cos\psi+\cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi+\cos\phi\cos\psi & \sin\phi\sin\theta & 0 \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$=\begin{pmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\frac{a_y}{a_x}=\frac{\sin\phi\sin\theta}{\cos\phi\sin\theta}$$

$$\tan\phi=\frac{\sin\phi}{\cos\phi}=\frac{a_y}{a_x}=\frac{-a_y}{-a_x}$$

If $\theta=0$ then $a_x=a_y=0\rightarrow$ *degenerate* because "pitch" can have no answer
i.e. ϕ and ψ correspond to the same rotating axis.

It can be solved by setting $\phi=0$ or the previous value or any value then solving for ψ

2.5 Three Angle Rotation: Euler angles, RPY Angles

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- For Euler z-y-z
  a) Solution of ϕ:

  If $\theta\neq 0$ $\quad \Rightarrow \phi=\tan^{-1}\left[\frac{a_y}{a_x}\right]$ or $\Rightarrow \tan^{-1}\left[\frac{a_y}{a_x}\right]+180^o$

  b) Solution of θ:

  If $c\theta=a_z$ $\quad \Rightarrow s\theta=c\phi a_x+s\phi a_y=c\phi(c\phi s\theta)+s\phi(s\phi s\theta)$

  $$\theta=\tan^{-1}\left(\frac{s\theta}{c\theta}\right)=\tan^{-1}\left[\frac{c\phi a_x+s\phi a_y}{a_z}\right]$$

  c) Solution of ψ:

$$s\psi=-s\phi n_x+c\phi n_y=s\phi^2 s\psi+c\phi^2 s\psi$$
$$c\psi=-s\phi o_x+c\phi o_y$$

$$\psi=\tan^{-1}\frac{s\psi}{c\psi}=\tan^{-1}\left\{\frac{-s\phi n_x+c\phi n_y}{-s\phi o_x+c\phi o_y}\right\}$$

2.5 Three Angle Rotation: Euler angles, RPY Angles

- RPY transformation solution

$$RPY(\phi,\theta,\psi) = Rz(\phi)Ry(\theta)Rx(\psi)$$

$$= \begin{bmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & 0 \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.5 Three Angle Rotation: Euler angles, RPY Angles

- RPY transformation solution

When $\theta = \pm 90° \to \cos\theta = 0 \Rightarrow$ Degeneracy.

a) When $\theta \neq \pm 90^{\circ}$ $\phi = \tan^{-1}\dfrac{n_y}{n_x}$ or $\tan^{-1}\dfrac{n_y}{n_x} + 180^{\circ}$

b) $$\left.\begin{aligned} \sin\theta &= -n_z \\ \cos\theta &= \cos\phi n_x + \sin\phi n_y \end{aligned}\right\} \Rightarrow \theta = \tan^{-1}\left(\frac{-n_z}{\cos\phi n_x + \sin\phi n_y}\right)$$

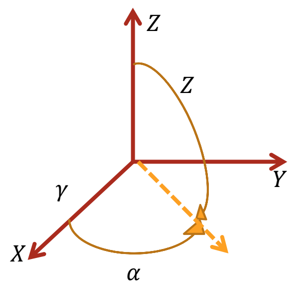
c) Solution ψ:

$$\left.\begin{aligned} \sin\psi &= \sin\phi a_x - \cos\phi a_y \\ \cos\psi &= -\sin\phi o_x + \cos\phi o_y \end{aligned}\right\} \Rightarrow \psi = \tan^{-1}\left(\frac{\sin\phi a_x - \cos\phi a_y}{-\sin\phi o_x + \cos\phi o_y}\right)$$

2.6 Specification of Position

- Cylindrical Coordinates:
  - $cyl(z, \alpha, \gamma) = trans(0,0,z) * R_z(\alpha) * Trans(\gamma, 0,0)$
  1) Translate γ along x-axis
  2) Rotate α about z-axis
  3) Translate z along z-axis

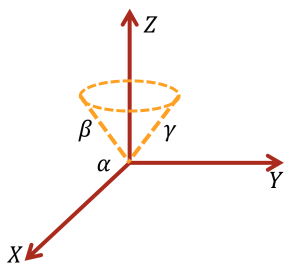
$$= \begin{pmatrix} c\alpha & -s\alpha & 0 & \gamma c\alpha \\ s\alpha & c\alpha & 0 & \gamma s\alpha \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{array}{cc} \begin{pmatrix} \gamma c\alpha \\ \gamma s\alpha \\ Z \\ 1 \end{pmatrix} & \begin{array}{c} x \\ y \\ z \\ \omega \end{array} \end{array}$$



2.6 Specification of Position

- Spherical Coordinates:
  - $S(\alpha, \beta, \gamma) = R_z(\alpha) * R_y(\beta) * Trans(0,0,\gamma)$
  1) Translate γ along z-axis
  2) Rotate β about y-axis
  3) Rotate α about z-axis

$$= \begin{pmatrix} c\alpha c\beta & -s\alpha & c\alpha s\beta & \gamma c\alpha s\beta \\ s\alpha c\beta & c\alpha & s\alpha s\beta & \gamma s\alpha s\beta \\ -s\beta & 0 & c\beta & \gamma c\beta \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{array}{cc} \begin{pmatrix} \gamma c\alpha s\beta \\ \gamma s\alpha s\beta \\ \gamma c\beta \\ 1 \end{pmatrix} & \begin{array}{c} x \\ y \\ z \\ \omega \end{array} \end{array}$$



2.6 Specification of Position

- Summary:

Translation	Rotation
p	$n\ o\ a$
$Cartesian(x, y, z)$	$R_k(\theta)$
$Cyl(z, \alpha, \gamma)$	$Euler(\phi, \theta, \psi)$
$Sph(\alpha, \beta, \gamma)$	$RPY(\phi, \theta, \psi)$