UIUC MATH 257 - Linear Algebra with Computational Applications

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0 MATH 257 - Course Overview

MATH 257 is an introductory linear algebra course administered by the **University of Illinois Urbana-Champaign (UIUC)**, which covers basic definitions and algorithms of the subject needed in higher levels of engineering, science, and economics.

This course has a bulk emphasis on introducing the mathematical theory behind the field of Linear Algebra, along with a gentle introduction to how Linear Algebra can be applied to various computer programs and algorithms. This includes understanding how to implement various Linear Algebra related concepts and applying them to real world scenarios in the **Python programming language**

Therefore, prior Python experience is recommended

This document covers MATH 257 for the Spring 2025 semester!

0.1 Prequisites, Future Courses, and MATH 257 Resources

The **syllabus** for MATH 257 Spring 2025

Prerequisites

- MATH 220/MATH 221 (Calculus I)
- CS 101 (Introduction to Computing) - **note that any other equivalent computing course can also be completed**,
  - For CS majors, this can include CS 124 (Introduction to Computer Science I) or CS 128 (Introduction to Computer Science II)
  - For ECE majors, this can include ECE 120 (Intro to Computing) or ECE 220 (Computer Systems and Programming)

Future Courses - CS Majors may take CS 357 (Numerical Methods), which expands upon the computational application of the concepts taught in this course. ECE majors can take the equivalent MATH 357.

0.2 About the Author

Hello, my name is Anirudh Konidala and I am a UIUC student studying Computer Science and Education. MATH 257 was definitely a big struggle for me and I never fully studied the material well enough to ace the midterms

Therefore, when it came to the last few weeks before the MATH 257 final exam, I decided to compile a big set of notes on both the **mathematical** and **programming** portion of this course, so that I could ace this course and pass the course with a decent grade.

I hope this content can also help others ace MATH 257, so that they don't have to bomb the midterms/exams like I did.

0.3 Overview

Linear algebra is the branch of mathematics that deals with vector spaces and linear transformations between them. It focuses on concepts like vectors, matrices, systems of linear equations, determinants, eigenvalues, and eigenvectors.

0.4 Book Structure

This book is structured with regards to the Lecture Videos and Modules from Canvas. There are 46 chapters, each representing various essential Linear Algebra concepts taught in MATH 257. Each section corresponds to the appropriate Module lecture video on Canvas. If there is a lack of Canvas access, you can view the module videos on Mediaspace. Each chapter/module is the contents/table of contents.

- Midterm 1 - Chapters/Modules 1 - 11
- Midterm 2 - Chapters/Modules 12 - 23
- Midterm 3 - Chapters/Modules 24 - 38
- Final Exam - Chapters/Modules 1 - 46

Contents

1 Introduction to Linear Systems

Linear Equations are in the form of

$$a_1x_1 + \cdots + a_nx_n = b$$

where $a_1, \ldots, a_n, b$ are numbers and $x_1, \ldots, x_n$ are variables.

For Example,

$$4x_1 - 5x_2 + 2 = x_1$$

is a linear equation because it can be rearranged to form an equation that is in the form of $a_1x_1 + \cdots + a_nx_n = b$

$$\begin{aligned} 4x_1 - 5x_2 + 2 &= x_1 \\ 4x_1 - x_1 - 5x_2 &= -2 \\ 3x_1 - 5x_2 &= -2 \end{aligned}$$

However,

$$x_2 = 2\sqrt{x_1} - 7$$

is **not** a linear equation because it cannot be expressed in the form of $a_1x_1 + \cdots + a_nx_n = b$

Linear Systems are collections of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.

A **solution** of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

1.1 Example Problem - Two equations in two variables

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 + x_2 &= 0 \end{aligned}$$

What is a solution for this system of linear equations?

Solution - Use the **elimination method**

1. Add the two systems to eliminate the x_1 variable

$$\begin{aligned} 2x_2 &= 1 \\ x_2 &= \frac{1}{2} \end{aligned}$$

2. Plug into the first equation to find the x_2 variable

$$\begin{aligned} x_1 + \frac{1}{2} &= 1 \\ x_1 &= \frac{1}{2} \end{aligned}$$

3. Thus $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$ is the only solution.

1.2 Does every system have a solution?

No! Observe the system

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$

Its process of solving is as follows

1. Multiply the first equation by 2 to eliminate the x_1 variable
$$2x_1 - 4x_2 = -6$$
2. Subtract the first equation from the second to cancel x_1
$$0 = 14$$
3. The equation $0 = 14$ is always false, so **no solutions** exist.

1.2.1 Example Problem

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$

How many solutions are there to this system of equations?

Solution

1. Multiply the first equation by -2 to eliminate x_1
$$-2x_1 - 2x_2 = -6$$
2. Both the first and second equation are the same. Subtracting the two in order to cancel out x_1 will result in
$$0 = 0$$
3. This means both equations have the same solutions. Therefore, the system is said to have **infinitely many solutions**.

1.3 Three Types of Linear Systems

A linear system comes in three forms. It has either **one unique solution**, **no solution**, or **infinitely many solutions**.

The **solution set** of a linear system is the set of all solutions of the linear system. Two linear systems are **equivalent** if they have the same solution set.

The general strategy is to replace one system with an equivalent system that is easier to solve.

1.3.1 Example Problem

Transform this linear system into another easier equivalent system

$$\begin{aligned} x_1 - 3x_2 &= 1 \\ -x_1 + 5x_2 &= 3 \end{aligned}$$

Solution - Add the first equation to the second equation

$$\begin{aligned} x_1 - 3x_2 &= 1 \\ 2x_2 &= 4 \end{aligned}$$

$x_2 = 2$ and $x_1 = 1 + 3(2) = 7$

2 Matrices and Linear Systems

Definition - An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Example Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 + \sqrt{5} \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 3 & 4 \\ 5 & -6 & 7 & 8 \\ -9 & 10 & 11 & 12 \end{bmatrix}$$

In terms of the entries of A:

$$A = \begin{bmatrix} a_1 1 & a_1 2 & \dots & a_1 n \\ a_2 1 & a_2 2 & \dots & a_2 n \\ \vdots & \vdots & \ddots & \vdots \\ a_m 1 & a_m 2 & \dots & a_m n \end{bmatrix}$$

where a_{ij} is in the ith row and jth column

Definition - For a linear system, we define the **coefficient** and **augmented** matrix as follows:

Linear System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Coefficient Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Augmented Matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

2.1 Example Problem

Determine the coefficient matrix and augmented matrix of the linear system

$$\begin{aligned} x_1 - 3x_2 &= 1 \\ -x_1 + 5x_2 &= 3 \end{aligned}$$

Solution - The coefficient matrix would be

$$\begin{bmatrix} 1 & -3 \\ -1 & 5 \end{bmatrix}$$

and the augmented matrix would be

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ -1 & 5 & 3 \end{array}\right]$$

2.2 Elementary Row Operation

An **elementary row operation** is one of the following

- **Replacement -** add a multiple of one row to another row: $R_i \to R_i + cR_j$, where $i \neq j$.
- **Interchange -** Interchange two rows: $R_i \leftrightarrow R_j$
- **Scaling -** Multiply all entries in a row by a nonzero constant: $R_i \to cR_i$, where $c \neq 0$

2.2.1 Example Problem

Give several examples of elementary row operations

Solution

- Replacement

$$R_2 \to R_2 + 3R_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

- Interchange

$$R_1 \leftrightarrow R_3$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Scaling

$$R_2 \rightarrow 3R_2$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Elementary row operations can undo or **reverse** each other. For example, the elementary row operation $R_3 \rightarrow R_3 - 3R_1$ reverses the row operation of $R_3 \rightarrow R_3 + 3R_1$

$$R_3 \rightarrow R_3 + 3R_1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every row operation is reversible. Above showed an example of reversing the replacement operator. Similarly, the scaling operator $R_2 \rightarrow cR_2$ is reversed by the scaling operator $R_2 \rightarrow \frac{1}{c}R_2$. Row interchange $R_1 \leftrightarrow R_2$ is reversible by performing it twice.

Two matrices are **row equivalent** if one matrix can be transformed into the other matrix by a sequence of elementary row operations.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

3 Echelon forms of matrices

Definition - A matrix is in echelon form or **row echelon form** when

1. All **nonzero rows** (rows with at least one nonzero element) are above any rows of all zeroes
2. The **leading entry** (the first nonzero number from the left) of a nonzero row is always strictly to the right of the leading entry of the row above it.

The following matrices achieve row echelon form

$$\begin{bmatrix} 2 & -2 & 3 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 & 2 & 0 & 5 \\ 0 & 2 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3.1 RREF - Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if it is in row echelon form **and**

- The leading entry in each nonzero row is 1
- Each leading entry is the only nonzero entry in its column

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 2 & 5 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 1 & \frac{1}{2} & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -3 & 4 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Each matrix is row-equivalent to one and only one matrix in reduced echelon form.

A matrix B is the **reduced echelon form** (RREF) of a matrix if A and B are row-equivalent and B is in reduced echelon form.

3.1.1 Example Problem

Is each matrix also row-equivalent to one and only one matrix in echelon form?

Solution - No! For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ are row-equivalent and both in echelon form.

3.2 Calculating RREF

Find the rref of matrix $\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \end{bmatrix}$

Solution - to achieve RREF, the leading entry of each nonzero row needs to be 1 and each leading entry is the only nonzero entry in the column

1. $R_2 \to R_2 - R_1$

$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{bmatrix}$$

2. $R_1 \to \frac{1}{3}R_1$, $R_2 \to \frac{1}{2}R_2$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{bmatrix}$$

3. $R_1 \to R_1 + 3R_2$

$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{bmatrix}$$

3.3 Pivot Position

The position of a leading entry in an echelon form of a matrix. A **pivot column** is a column that contains a pivot position

3.3.1 Example Problem

Locate the pivot columns of the following matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution

1. $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

2. $R_2 \to R_2 + R_1$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

3. $R_3 \to R_3 + 1.5R_2$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

The columns of **1**, **2**, and **4** are the pivot columns of A.

3.4 Pivot Variables

Basic Variable (Pivot Variable) - A variable that corresponds to a pivot column in the coefficient matrix of a linear system. A **free variable** is a variable that is not a pivot variable.

3.4.1 Example Problem

Consider the augmented matrix and system. Determine the basic and free variables.

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \qquad \begin{aligned} x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 8x_4 &= 5 \\ x_5 &= 7 \end{aligned}$$

Solution - The first, third and fifth columns are pivot columns. Therefore, x_1, x_3, and x_5 are basic variables and x_2, x_4 are free variables.

4 Gaussian Elimination

The idea behind **Gaussian Elimination** is to solve linear systems for the pivot variables in terms of free variables (if any) in the equation
Specifically, Gaussian Elimination is an **algorithm** or process used to solve linear systems backed behind matrices

1. Write down the augmented matrix
2. Find the RREF (reduced row echelon form) of the matrix
3. Write down the equations corresponding to the RREF
4. Express pivot variables in terms of free variables

4.1 Example

Find the general solution of

$$\begin{aligned} 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

The solution simply involves following the Gaussian Elimination algorithm

1. Write down the augmented matrix

$$\left[\begin{array}{ccccc|c} 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

2. Find the RREF of the matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{array}\right]$$

3. Write down equations corresponding to the RREF

$$\begin{aligned} x_1 &= -2x_3 + 3x_4 + 5x_5 = -4 \\ x_2 &= -2x_3 + 2x_4 + x_5 = -3 \end{aligned}$$

4. Express pivot variables in terms of free variables

$$\begin{aligned} x_1 &= 2x_3 - 3x_4 - 5x_5 - 4 \\ x_2 &= 2x_3 - 2x_4 - x_5 - 3 \\ x_3, x_4, x_5 &= \text{free} \end{aligned}$$

4.2 Consistent Linear Systems

A linear system is **consistent** if and only if an echelon form of the augmented matrix has no row of the form $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & b \end{array}\right]$, where b is nonzero.

Linear systems are consistent when

1. There is a **unique** solution (no free variables)
2. Infinitely **many** solutions (at least one free variable)

4.2.1 Example

If a linear system has an augmented matrix of $\left[\begin{array}{cc|c} 3 & 4 & -3 \\ 3 & 4 & -3 \\ 6 & 8 & -5 \end{array}\right]$, what can be inferred about the number of solutions in the system?

Solution

Convert matrix to echelon form $\left[\begin{array}{cc|c} 3 & 4 & -3 \\ 3 & 4 & -3 \\ 6 & 8 & -5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$

There is no solution because of the $\left[\begin{array}{cc|c} 0 & 0 & 1 \end{array}\right]$ row (the bottom row in the echelon form of the original matrix)

As a linear equation, this row is equivalent to $0x_1 + 0x_2 = 1$, which is an equation that has no solution!

5 Linear Combinations

Consider the $m \times n$ matrices $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$, and $B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \ddots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$

5.1 Sum and Scalar Product

The **sum** of $A + B$ would be

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

Similarly, the **product** cA for a scalar c is

$$\begin{bmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \dots & \dots & \ddots & \dots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{bmatrix}$$

5.1.1 Adding Matrices Example

$$\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1+2 & 0+3 \\ 5+3 & 2+1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 8 & 3 \end{bmatrix}$$

5.1.2 Scalar Multiplying Matrices Example

$$5\begin{bmatrix} 2 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5\cdot 2 & 5\cdot 1 & 5\cdot 0 \\ 5\cdot 3 & 5\cdot 1 & 5\cdot -1 \end{bmatrix} = \begin{bmatrix} 10 & 5 & 0 \\ 15 & 5 & -5 \end{bmatrix}$$

5.2 Row and Column Vectors

Column Vectors are $m \times 1$-matrices, while **Row Vectors** are $1 \times n$-matrices

5.2.1 Examples of Row and Column Vectors

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

5.3 Transpose

If A is $m \times n$, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A. In terms of matrix elements: $(A^T)_{ij} = A_{ji}$

5.3.1 Figuring out the Transpose of a Matrix

What is the transpose of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$?

In a transpose, the rows of the original matrix simply become the columns of the transposed matrix

$$A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

5.4 Span

The **linear combination** of $m \times n$-matrices $A_1, A_2, \ldots, A_p$ with **coefficients** $c_1, c_2, \ldots, c_p$ is defined as

$$c_1A_1 + c_2A_2 + \cdots + c_pA_p$$

For example, for $m \times n$-matrices A_1 and A_2, some examples of linear combinations of these two matrices are

$$3A_1 + 2A_2 \quad A_1 - 2A_2 \quad \frac{1}{3}A_1 = \frac{1}{3}A_1 + 0A_2$$

The **span** $(A_1, \ldots, A_p)$ is defined as the set of all linear combinations of $A_1, \ldots, A_p$, or simply

$$\text{span}(A_1, \ldots, A_p) := \{c_1A_1 + c_2A_2 + \cdots + c_pA_p : c_1, \ldots, c_p \text{ scalars}\}$$

The **set** of all column vectors of length m is represented as $\mathbb{R}^m$

For example, let $a_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, and $b = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$. Is b a linear combination of a_1, a_2?

Find x_1, x_2 such that

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

1. Form the system of equations

$$\begin{aligned} x_1 + 4x_2 &= -1 \\ 0x_1 + 2x_2 &= 8 \\ 3x_1 + 14x_2 &= -5 \end{aligned}$$

2. Form the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 2 & 8 \\ 3 & 14 & -5 \end{array}\right]$$

3. Compute the echelon form

$$\left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 2 & 8 \\ 0 & 2 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 2 & 8 \\ 0 & 0 & -10 \end{array}\right]$$

With this echelon form matrix, it can be concluded that b is not a linear combination of a_1 and a_2 because the system is inconsistent

This means that there is no solution that exists for the bottom row linear equation

$$0x_1 + 0x_2 = -10$$

Therefore, this means that there is no solution that exists for the derived linear system **and** the original vector equation, $x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$

Geometrically speaking, this means that b is not in the span of a_1 and a_2

5.5 TTK - Things to Know

Solving linear systems is the same as finding linear combinations!

A vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

has the same solution set as the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} a_1 & a_2 & \cdots & a_n & b \end{array}\right]$$

In particular, b can be generated by a linear combination of $a_1, a_2, \cdots, a_n$ if and only if there is a solution to the linear system corresponding to the augmented matrix

A matrix is defined in terms of its **colums** or **rows**

$$A := \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \quad \text{or} A := \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$$

6 Matrix Vector Multiplication

Suppose x is a vector in $\mathbb{R}^m$ and $A = \begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$ an $m \times n$-matrix. The product Ax is defined by

$$Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

- Ax is a linear combination of the columns of A using the entries in x as coefficients.
- Ax is only defined if the number of entries of x is equal to the number of columns of A

6.1 Example Problem

If $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \end{bmatrix}$, and $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, determine Ax and Bx

$$Ax = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$Bx = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 21 \end{bmatrix}$$

6.2 Example Problem 2

Consider the vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Find a 2×2 matrix A such that (x_1, x_2) is a solution to the above equation if and only if

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}?$$

Take $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Form a linear equation with A using a linear combination Ax, where x represents the column vector of unknown variables

$$Ax = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is the vector expression that matches the problem statement and we know that it is equivalent to the column vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is also equivalent to to the column vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Therefore one possible 2×2 matrix is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

6.3 Equivalent Formations of a Linear System

Let $A = \begin{bmatrix} a_1, \cdots, a_n \end{bmatrix}$ be an $m \times n$-matrix and b in $\mathbb{R}^m$. The the following are equivalent

- $(x_1, x_2, \ldots, x_n)$ is a solution of the vector equation, $x_1 a_1 + x_2 a_2 + \cdots x_n a_n = b$
- $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to the matrix equation, $Ax = b$
- $(x_1, x_2, \ldots, x_n)$ is a solution of the system with augmented matrix, $\left[\begin{array}{c|c} A & b \end{array}\right]$

The notation for the system of equations with augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ will be written as $Ax = b$

6.4 Matrices as Machines

Let A be a $m \times n$ matrix.

1. Input: n-component vector $x \in \mathbb{R}^m$
2. Output: m-component vector $b = Ax \in \mathbb{R}^m$

For example, consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. What does this machine do?

Solution

1. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be our input

$$Ax = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

2. Therefore, the machine A switches the entries of the vector x

Geometrically speaking, this machine reflects across the $x_1 = x_2$-line

6.5 Composition of Machines

Let A be an $m \times n$ matrix and B be an $k \times l$ matrix. Now we can compose the two machines

However, this composition only works for some k, l, m, n. For which?

Solution

- If A is an $m \times n$-matrix and x in $\mathbb{R}^n$, then Ax is in $\mathbb{R}^m$
- In order to calculate $B(Ax)$ when then need B to have m columns.
- So we need $l = m$. Both n and k can be arbitrary.

6.5.1 Worked Example

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ be as before. Is $A(Bx) = B(Ax)$?

Solution No, projection and reflection do not commute!

$$A(B\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad B(A\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = B\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

7 Matrix Multiplication

Let A be an $m \times n$ matrix and let $B = \begin{bmatrix} b_1 \dots b_p \end{bmatrix}$ be an $n \times p$-matrix. We define

$$AB := \begin{bmatrix} Ab_1 & Ab_2 & \dots & Ab_p \end{bmatrix}$$

Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$

Solution

$$Ab_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 6\begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -7 \end{bmatrix} = -3\begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} - 7\begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

These are the columns of the product matrix, where $AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$

Note that Ab_1 and Ab_2 are linear combinations of the columns of A. This means that each column of AB is a linear combination of the columns of A using coefficients from the corresponding columns of B

7.1 Worked Example

Let A be an $m \times n$ matrix and let B be an $n \times p$-matrix. We define

$$AB := \begin{bmatrix} Ab_1 & Ab_2 & \dots & Ab_p \end{bmatrix}$$

If C is a 4×3 and D is a 3×2, are CD and DC defined? What are their sizes or dimensions?

Solution

1. The product AB can only be defined if B has as many rows as A has columns

2. If this is the case, then AB has as many rows as A and as many columns as B

3. Therefore, CD is defined and has a 4×2 dimension, while DC is not defined

Recall that matrices can be thought of as machines

- Let B be $n \times p$: input $x \in \mathbb{R}^p$, output $c = Bx \in \mathbb{R}^n$
- Let A be $m \times n$: input $y \in \mathbb{R}^n$, output $b = Ay \in \mathbb{R}^m$

Compute $(AB)x$ and $A(B(x))$. Are these the same?

Solution

$$Bx = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$$

$$A(Bx) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix} = (x_1 + 2x_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (x_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

$$(AB)x = \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 Ab_1 + x_2 Ab_2 = x_1 (1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}) + x_2 (2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

$$= x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then for every $x \in \mathbb{R}^p$

$$A(Bx) = (AB)x$$

7.2 Row Column Rule

Let A be $m \times n$ and B be $n \times p$ such that

$$A = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}, \text{ and } B = \begin{bmatrix} C_1 & \cdots & C_p \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} R_1C_1 & \cdots & R_1C_p \\ R_2C_1 & \cdots & R_2C_p \\ R_mC_1 & \cdots & R_mC_p \end{bmatrix} \text{ and } (AB)_{ij} = R_iC_j = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

7.2.1 Example Problem

Let $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined

$$AB = \begin{bmatrix} 2 \cdot 2 + 3 \cdot 0 + 6 \cdot 4 & 2 \cdot (-3) + 3 \cdot 1 + 6 \cdot (-7) \\ -1 \cdot 2 + 0 \cdot 0 + 1 \cdot 4 & -1 \cdot (-3) + 0 \cdot 1 + 1 \cdot (-7) \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$$

7.3 Outer Product Rule

Let A be $m \times n$ and B be $n \times p$ such that

$$A = \begin{bmatrix} C_1 \cdots C_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$$

Then

$$AB = C_1R_1 + \cdots + \cdots + C_nR_n$$

7.3.1 Example Problem

Let $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined

Solution

$$\begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 24 & -42 \\ 4 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$$

8 Properties of Matrix Multiplication

The **identity matrix** I_n of size n is defined as

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Let A be an $m \times n$ matrix and B and C be matrices for which the indicated sums nad products are defined

1. $A(BC) = (AB)C$ (associative law of multiplication)
2. $A(B + C) = AB + AC, (B + C)A = BA + CA$ (distributive laws)
3. $r(AB) = (rA)B = A(rB)$ for every scalar r
4. $A(rB + sC) = rAB + sAC$ for every scalars r, s (linearity of matrix multiplication)
5. $I_m A = A = AI_n$ (identity for matrix multiplication)

8.1 Matrix Identities vs Real Number Identities

While matrix multiplication properties are analogous to that of real numbers, not all properties of real numbers hold for matrices

For Example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Determine AB and BA. Are these matrices the same?

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

No. $AB \neq BA$. These matrices are not the same. Matrix multiplication is not commutative!

8.2 Transpose Property of Matrices

Have $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. What is $(AB)^T$? What about $A^T B^T$ and $B^T A^T$?

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 10 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$

$$A^T B^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$

The transpose of a product is the product of transposes in **opposite order**

$$(AB)^T = B^T A^T$$

8.3 Powers of Matrices

Let $A^k = A \cdots A$ for k-times; that is A^k is obtained by multiplying A $k-times$ with itself

For which matrices A does A^k make sense? If A is $m \times n$ what can m and n be?

To be able to multiply A by any $m \times n$-matrix, we need that $m = n$

Determine $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3$

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix}$$

Higher powers of matrices are more difficult to calculate using this method!

9 Elementary Matrices

Let A be a 3×3-matrix. What happens to A if you multiply it by one of E_1, E_2, and E_3?

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + 2a_{11} & a_{32} + 2a_{12} & a_{33} + 2a_{13} \end{bmatrix}$$

If an elementary row operation is performed on an $m \times n$-matrix A, the resulting matrix can be written as EA, where the $m \times m$-matrix E is created by performing the same row operations on I_m

An **elementary matrix** is one that an elementary row operation can be performed upon the identity matrix

Let A, B be two $m \times n$ matrices and row-equivalent. Then there is a sequence $m \times m$-elementary matrices $E_1, \ldots, E_l$ such that

$$E_l \ldots E_1 A = B$$

Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find two elementary matrices E_1, E_2 such that $E_2 E_1 A = B$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B$$

Set $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$$E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B$$

Recall that

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Find elementary matrices E_1^{-1} and E_2^{-1} such that $A = E_1^{-1} E_2^{-1} = B$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B$$

Set $E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Notice how the row operations were reversed to go from matrix B to A

10 Inverse of a Matrix

The inverse of a real number a is denoted as a^{-1}. For example, $7^{-1} = \frac{1}{7}$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

Not all real numbers have inverse. 0^-1 is not well defined, since there is no real number b such that $0 \cdot b = 1$

Recall that the identity matrix I_n is the $n \times n$-matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the inverse of A

10.1 Example Problem

What is the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

Elementary matrices are invertible because row operations are reversible. So the inverse matrix is the elementary matrix corresponding to $R_2 \to R_2 - 5R_1$: $\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Let A be an invertible matrix, then its inverse C is unique

Assume B and C are both inverses of A. Then

$$BA = AB = I_n$$
$$CA = AC = I_n$$

Thus, $B = BI_n = BAC = I_nC = C$

10.2 Inverse Properties

Suppose A and B are invertible. Then

1. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1})
2. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. A^T is invertible and $(A^T)^-1$ $(A^{-1})^T$

Proofs

$$AA^{-1} = I = A^{-1}A$$

$$(B^{-1}A^{-1})(AB) = B^{-1}IB = B^{-1}B$$
$$(AB)(B^{-1}A^{-1}) = AI^{-1} = AA^{-1} = I$$

$$A^T(A^{-1})^T = (A^-1A)^T = I^T = I$$

10.3 Multiplying the inverse of A

A^{-1} is denoted as the inverse of A. Multiplying by A^{-1} is like "dividing by A"

- Writing $\frac{A}{B}$ is unclear whether this means AB^{-1} or $B^{-1}A$, and these two matrices are completely different

If $AB = I$, then $A^{-1} = B$ and so $BA = I$

Similarly, not all $n \times n$ matrices are invertible. For example, the 2×2 matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not invertible

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \neq I_2$$

Recall that the identity 2×2 matrix (I_2) is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The second row of matrix $\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \end{bmatrix}$, the second row of the identity matrix. Therefore, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ can't be invertible

Suppose that A is an invertible $n \times n$ matrix. Then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$

Proof

The vector $A^{-1}b$ is a solution, because

$$A(A^{-1}b) = (AA^{-1})b = I_n b = b$$

Suppose there is another solution w, then $Aw = b$. Thus

$$w = I_n w = A^{-1}Aw = A^{-1}b$$

Additionally, A must have n pivots because otherwise $Ax = b$ would not have a solution of each b

11 Computing the Inverse

A 1×1 matrix $\begin{bmatrix} a \end{bmatrix}$ is invertible when $a \neq 0$ and its inverse is $\begin{bmatrix} \frac{1}{a} \end{bmatrix}$

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible

Proof

$$\begin{aligned} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \frac{1}{ad - bc} \begin{bmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Let A be an $n \times n$. The following are equivalent:

- A is invertible
- the RREF of A is I_n

Proof

Suppose A can be row-reduced to the identity matrix

$$A = A_0 \rightsquigarrow A_1 \rightsquigarrow \cdots \rightsquigarrow A_m = I_n$$

Thus there are elementary matrices $E_1, \cdots, E_m$ such that

$$E_m E_{m-1} \cdots E_1 A = I_n$$

Thus

$$A^{-1} = E_m E_{m-1} \cdots E_1 = E_m E_{m-1} \cdots E_1 I_n$$

This boils down to the ideaw where we suppose A is invertible. Every sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}

11.1 Algorithm

Place A and I side by side to form an augmented matrix of $\begin{bmatrix} A \mid I \end{bmatrix}$

This becomes a $n \times 2n$ matrix (Big Augmented Matrix), instead of $n \times (n + 1)$

Perform row operations on this matrix (which will produce identical operations on A and I).

By Theorem: $\begin{bmatrix} A \mid I \end{bmatrix}$ will row reduce to $\begin{bmatrix} I \mid A^{-1} \end{bmatrix}$

11.1.1 Example

Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists

$$\left[A \mid I\right] = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$

12 LU Decomposition

An $n \times n$ matrix A is

- **upper triangular** when in the form of

$$\begin{bmatrix} 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{bmatrix}$$

- **lower triangular** when in the form of

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & \ddots & \vdots \\ * & * & * & * & * \end{bmatrix}$$

L is typically referred to as the lower triangular matrix, while U is the upper triangular matrix.

The product of the two matrices L and U make up the original matrix A, where $A = LU$

12.1 Calculating L and U

Given matrix,

$$A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

calculate its LU Decomposition

Solution

- Form the U matrix first, which must have zeroes in the lower corner of the matrix
$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 4 \\ 0 & -4 \end{bmatrix}$$
where
$$U = \begin{bmatrix} 2 & 4 \\ 0 & \frac{4}{3} \end{bmatrix}$$
- L forms based off the inverse coefficients that were used to create U
  - Since U was formed by multiplying $\frac{1}{3}$, that is the value that gets filled in
$$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & \frac{4}{3} \end{bmatrix}$$

13 Solving systems using LU Decomposition

Systems can be solved using the decomposed matrices L and U, where $Lc = b$ and $Ux = c$, where c and x are some form of vector

13.0.1 Example

Given the following matrices
$$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & \frac{4}{3} \end{bmatrix}$$
find a solution to the system such that

14 Inner Product and Orthogonality

The **inner product** of v, $w \in \mathbb{R}^n$ is
$$v \cdot w = v^T w$$

- If $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, then $v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$
- $(v \cdot w)^T = v \cdot w$ because matrices are 1 x 1. Therefore
$$v^T w = (v^T w)^T = w^T (v^T)^T = w^T v$$
$$v^T w = w^T v$$

- $v \cdot v = 0$ if and only if $v = 0$
- Let u, v, and w be vectors in $\mathbb{R}^n$ and let c be any scalar
  - $u \cdot v = v \cdot u$
  - $(u + v) \cdot w = u \cdot w + v \cdot w$
  - $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
  - $u \cdot u \geq 0$, and $u \cdot u = 0$ if and only if $u = 0$

14.1 Important Terms

Let $v, w \in \mathbb{R}^n$

- **norm** (length) of $v = ||v|| = \sqrt{v \cdot v} = \sqrt{v_1^2 + \cdots + v_n^2}$
- **distance** between v and $w = ||v - w||$
- **unit vectors** in $\mathbb{R}^n$ are vectors of length 1

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow v \cdot v = 5 \text{ and } ||v|| = \sqrt{5}$$

  The example above, given that $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, is not a unit vector since its norm or length does not equal 1

- **normalization** includes the process of converting any vector in $\mathbb{R}^n$ to a unit vector. Suppose that

$$u = \frac{v}{||v||} = \frac{v}{\sqrt{5}} = 1$$

  u is the resulting unit vector, which occurred by normalizing v.

14.1.1 Example

Compute $|| \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} ||$ and dist$(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$

$$|| \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} || = \sqrt{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

and dist$(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = || \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} || = \sqrt{6}$

14.2 Orthogonality

- Let $v, w \in \mathbb{R}^n$; v and w are **orthogonal** if $v \cdot w = 0$
- If v and w are both $\in \mathbb{R}^n$ and non-zero, then they are orthogonal only if they are perpendicular (form a right angle)
- A set of vectors in $\mathbb{R}^n$ is **pairwise orthogonal** if each pairing of them is orthogonal. Such set is called an **orthogonal set**.

14.2.1 Example

Find a non-zero $v \in \mathbb{R}^3$ such that $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}$ and v form an orthogonal set

Orthogonal sets occur when multiplying v and and any vector in the set of vectors equal to 0. Therefore,

$$\begin{bmatrix}1\\1\\0\end{bmatrix} \cdot v = \begin{bmatrix}1\\-1\\0\end{bmatrix} \cdot v = 0$$

Assume that vector v contains elements $\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$. Therefore,

$$v_1 + v_2 = 0 \text{ and } v_1 - v_2 = 0$$

Solving the system leads to $v_1 = 0$ and $v_2 = 0$. Therefore, a possible vector for vector v could be

$$v = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

14.2.2 Orthonormal Sets

An **orthonormal set** occurs when the set is an orthogonal set and all vectors in the set are unit vectors

14.2.3 Example

Let $v_1 = \begin{bmatrix}1\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\-1\end{bmatrix}$. Since the vectors are not orthonormal (norm $\neq 1$), we need to normalize in order to get an orthonormal set

$$u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

15 Subspaces

A non-empty subset H of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

- $u, v \in H$, then the sum $u + v \in H$
- $u \in H$ and c is scalar, then $cu \in H$

Therefore, if $v_1, v_2, \cdots, v_m \in \mathbb{R}^n$, then $\text{Span}(v_1, v_2, \cdots, v_m)$ is a subspace of $\mathbb{R}^n$

Let $u = c_1 v_1 + \cdots + c_m v_m$, and $w = d_1 v_1 + \cdots + d_m v_m$. Valid subspaces include

$$u + w = c_1 v_1 + \cdots + c_m v_m + d_1 v_1 + \cdots + d_m v_m = (c_1 + d_1) v_1 + \cdots + (c_m + d_m) v_m$$

$$cu = c(c_1 v_1 + \cdots + c_m v_m) = cc_1 v_1 + \cdots + cc_m v_m$$

15.1 Examples

Is $H = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} : x \in \mathbb{R} \right\}$ a subspace of $\mathbb{R}^2$?

1. Let $\begin{bmatrix} a \\ a \end{bmatrix}, \begin{bmatrix} b \\ b \end{bmatrix}$ be in H.

$$\begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix} \in H$$

2. Similarly,

$$c \begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} ca \\ ca \end{bmatrix} \in H$$

3. Yes, H is a subspace of $\mathbb{R}^2$

2. Let $Z = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$. Is Z a subspace of $\mathbb{R}^2$?

Yes!

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+0 \end{bmatrix} \in Z; \text{ and } c \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c0 \\ c0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in Z$$

16 Column Spaces and Nullspaces

Column Space - the set of all linear combinations of the columns of $m \times n$ matrix A, typically written as Col(A).

If $A = [a_1, a_2, \cdots, a_n]$, then $\text{Col}(A) = \text{span}(\text{a}_1, a_2, \cdots, a_n)$

16.1 Example

Describe the columns space of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\text{Col}(A) = \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) \rightarrow$ the x-axis of in $\mathbb{R}^2$

16.2 Column Space - TTK (Things to Know)

Col(A) is a subspace of $\mathbb{R}^m$ for $m \times n$ matrix A. This is true because Col(A) is a span, where spans are indeed subspaces. In fact, the column space is a subspace of $\mathbb{R}^m$ because the columns of A are in $\mathbb{R}^m$.

Let A be an $m \times n$ matrix and $b \in \mathbb{R}^m$. Then b is in Col(A) if and only if the linear system $Ax = b$ has a solution.

16.3 Example

Let $A =[\text{a}_1, \cdots, a_n]$. Suppose $Ax = b$, where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

$$b = Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Therefore, b is a linear combination of the columns of $A \leftrightarrow Ax = b$ is consistent.

If A and B are two row-equivalent matrices, is Col(A) = Col(B)?

No! Take $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then, $\text{Col}(A) = \text{span}(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) \neq \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ = Col(B)

16.4 Nullspace

Nullspace - the set of all solutions to $Ax = 0$ for $m \times n$ matrix A, notated as Nul(A), where $\text{Nul}(A) = \{v \in \mathbb{R}^n : Av = 0\}$

The nullspace of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. We can prove this because Nul(A) is non-empty since $0 \in \text{Nul}(A)$. Suppose $Au = 0$ and $Av = 0$. Then,

$$A(u + v) = Au + Av = 0 + 0 = 0$$

$$A(cu) = c(Au) = c(0) = 0$$

Therefore, Nul(A) is closed under addition and scalar multiplication.

16.5 Nullspace Examples

1. Let $H = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} : v_1 + v_2 - v_3 = 0 \right\}$. Find a matrix A such that $H = \text{Nul}(A)$

$v_1 + v_2 - v_3 = 0$ if and only if $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$. Thus, $\text{Nul}(\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = H)$.

2. Let $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$. Find two vectors ($v$ and w), such that $\text{Nul}(A) = \text{span}(v, w)$.

1. Let $u \in$Nul(A). Then

$$A \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0 \leftrightarrow u_1 + u_2 - u_3 = 0$$

$$u_1 = -u_2 + u_3$$

2. Therefore,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -u_2 + u_3 \\ u_2 \\ u_3 \end{bmatrix} = u_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

3. As a result, $\text{Nul}(A) = \text{span}(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix})$, where $v = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. Is there a matrix B, such that $\text{Nul}(A) = \text{Col}(B)$? Yes! Recall that $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$

1. $\text{Nul}(A) = \text{span}(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix})$

2. B's column space equals the nullspace when matrix B represents all vectors that consist of the nullspace of A. Therefore, $B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Let A be an $m \times n$ matrix, where $b, w \in \mathbb{R}^m$, such that $Aw = b$. Then, $\{v \in \mathbb{R}^n : Av = b\} = w+$ Nul(A).

For Example, if $A = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$ and $b = 1$, where $A \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T = b$, show how $\{v \in \mathbb{R}^n : Av = b\} = w+$ Nul(A)

$$Av = b, \text{ where } v^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where $\text{Nul}(A) = \text{span}(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} . \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix})$. Therefore,

$$\{v \in \mathbb{R}^n : Av = b\} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \text{span}(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix})$$

17 Vector Spaces

- Column Vectors in $\mathbb{R}^n$ allow you to take linear combinations of them
- There are many mathematical objects $X, Y, \ldots$ for which a linear combination $cX + dY$ makes sense ,and have the usual properties of linear combinations in $\mathbb{R}^n$

Vector Spaces - a collection of vectors, V, for which linear combinations make sense

Precisely, on V, there are two operations: addition and multiplication by scalars (real numbers).

17.1 Closure

Closures refer to the property that the result of adding two vectors or multiplying a vector by a scalar is also within the same vector space.

Consider that $u, v, w \in V$ and for all scalars $c, d \in \mathbb{R}$

- $u + v$ is in V (closed under addition), where $u + v = v + u$ (commutative property) and $(u + v) + w = u + (v + w)$ (associative property)
- There exists a zero vector 0_v in V such that $u + 0v = u$
- For each u in V, there exists a vector $-u$ in V satisfying $u + (-u) = 0v$
- $cu \in V$ (closed under scalar multiplication)
- $c(u + v) = cu + cv$ (distributive property) and $(c + d)u = cu + du$ (distributive property)
- $(cd)u = c(du)$
- $1u = u$

17.2 Vector Space Conceptual Examples

1. Prove how the set of function $\mathbb{R} \to \mathbb{R}$ is a vector space

   1. Proving a vector space includes verifying both closures under addition and scalar multiplication exist

      (a) Let f, g be two functions from $\mathbb{R}$ to $\mathbb{R}$

      $$(f+g)x = f(x) + g(x)$$

      (b) For scalar c, define cf by

      $$(cf)(x) = cf(x)$$

2. Prove that the set of all 2×2 matrices is a vector space.

   1. Verify with Addition

   $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

   2. Verify with Scalar Multiplication

   $$e \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix}$$

Let V be a vector space, such that a non-empty subset $W \subseteq V$ can only be a subspace of V when

- $u + v$ for all $u, v \in U$ (closed under addition)
- cu for all $u \in U$ and $c \in \mathbb{R}$ (closed under scalar multiplication)

17.3 Example Proofs

1. Show why the set of all symmetric 2×2 matrices is a subspace of the vector space of 2×2 matrices.

The set of symmetric matrices of a given size is non-empty since the zero matrix is symmetric. Let A, B be two symmetric 2×2 matrices. Therefore $A^T = A$ and $B^T = B$

$$(A+B)^T = A^T + B^T = A + B$$
$$(cA)^T = cA^T = cA$$

Therefore, the set is closed under addition and scalar multiplication.

2. Prove or disprove why the set of all invertible 2×2 matrices is a subspace of the vector space of 2×2 matrices.

We cannot prove this because the set is not closed under addition

1. Recall that invertible matrices occur when $AB = BA = I$, where A and B are two matrices and I is the identity matrix

   - 2×2 identity matrix, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   - A matrix is invertible if a matrix's determinant isn't zero. Take two invertible matrices, $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = I$.
   - $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This is the zero vector, which is not invertible!

3. Let $\mathbb{P}_n$ be the set of all polynomials of degree at most n, where

$$\mathbb{P}_n = \left\{ a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n : a_0, \ldots, a_n \in \mathbb{R} \right\}$$

Prove why it is a vector space. Is it a subspace of the vector space of all functions $\mathbb{R} \to \mathbb{R}$?

Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ and $q(t) = b_0 + b_1 t + \cdots + b_n t^n$ be two polynomials of degree at most n. and $q(t) = b_0 + b_1 t + \cdots + b_n t^n$ be two polynomials of degree at most n. Therefore,

$$(p + q)(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n$$

is also a polynomial of degree at most n and

$$(cp)(t) = (ca_0) + (ca_1)t + \cdots + (ca_t)t^n$$

is also a polynomial of degree at most n.

18 Linear Independence

Recall that vectors $v_1, \ldots, v_p$ are said to be linearly independent if the equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only the trivial solution (namely, $x_1 = x_2 = \cdots = x_p = 0$)

18.1 Verifying Linear Indepdence

Let vector $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Are v and w linearly indepedent?

No because it does not have a $(0, 0)$ solution! this is called linear dependence!

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \text{ and } x_1 + 2x_2 = 0, \text{ where } x_1 = -2x_2$$

Therefore, we can write the solution vector as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \text{ where } \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Simply speaking, the vectors can't be linearly indepdendent because $\begin{bmatrix} 2 \\ 2 \end{bmatrix} \in \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, indicating that $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is a possible linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which it is because $2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Therefore, vectors $v_1, \ldots, v_p$ are linear dependent if and only if there is $i \in \{1, \ldots, p\}$, such that $v_i \in \text{span}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p)$.

- Single non-zero vectors are always linear independent because $x_1 v_1 = 0$ only exists when $x_1 = 0$
- Two vectors v_1, v_2 are linearly independent if and only if neither of the vectors is a multiple of the other

$$x_1 v_1 + x_2 v_2 = 0, \text{ where } x_2 \neq 0 \to v_2 = -\frac{x_1}{x_2} v_1$$

- Vectors $v_1, \ldots, v_p$ containing the zero vector are linearly dependent. If

$$v_1 = 0 \to v_1 + 0v_2 + \cdots + 0v_p = 0$$

1. Determine if vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ are linearly independent.

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

forms the matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \end{array}\right]$$

Use RREF to get the solution of the system

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_3 = \text{free}; x_2 = -2x_3; x_1 = 3x_3$$

Let $x_3 = 1$

$$3\begin{bmatrix}1\\1\\1\end{bmatrix} - 2\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}-1\\1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Yes! They are linearly dependent. The third vector relies on the sum of the first two vectors. This happens because there is a free variable in their system.

Recall that this can be written as a matrix equation, $Ax = b$, where b is simply 0

$$\begin{bmatrix}1 & 1 & -1\\1 & 2 & 1\\1 & 3 & 3\end{bmatrix}\begin{bmatrix}3\\-2\\1\end{bmatrix} = 0$$

Therefore, the nullspace is what determines linear independence

Given that A is an $m \times n$ matrix. The following are equivalent

- The columns of A are linearly independent
- $Ax = 0$ has only the solution $x = 0$
- A has n pivots
- there are no free variables for $Ax = 0$

18.2 Linear Independence Conceptual Questions

1. Let $v_1, \ldots, v_n \in \mathbb{R}^m$. If $n > m$, then $v_1, \ldots, v_n$ are linearly dependent. Why?

Suppose A is the matrix that consists of vectors $v_1, \ldots, v_n$, which is an $m \times n$ matrix.

- There can be at most m pivots
- A cannot have n pivots because $n > m$
- Therefore, A's columns must be linearly dependent

2. Consider an $m \times n$ matrix A in echelon form. Why are the pivot columns of A linearly indepdent?

Suppose $A = \begin{bmatrix}a_1 & a_2 & a_3\end{bmatrix}$ and a_1, a_3 are the pivot columns The matrix $\begin{bmatrix}a_1 & a_3\end{bmatrix}$ has 2 pivots. This makes the vectors a_1, a_3 linearly independent.

19 Basis and Dimension

Basis - The a sequence of vectors $(v_1, \ldots, v_p)$ in vector space V, where

- $V = \text{span}(v_1, \ldots, v_p)$

- $(v_1, \ldots, v_p)$ are linearly independent

1. Are $(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix})$ and $(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix})$ bases of $\mathbb{R}^2$?

   1. Verify linear indepdence
   $$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$
   has two pivots and
   $$\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$
   has two pivots

   2. Determine the spanning set
   $$\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = \frac{a+b}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{a-b}{2}\begin{bmatrix}1\\-1\end{bmatrix}$$

19.1 Dimension

Every two bases in a vector space V contain the same numbers of vectors

Dimension - The number of vectors in a basis of V

The dimension of $\mathbb{R}^n$ is n

$$e_1 = \begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \cdots, e_n = \begin{bmatrix}0\\0\\0\\\vdots\\1\end{bmatrix}$$

where $(e_1, e_2, \ldots, e_n)$ is the basis of $\mathbb{R}^n$

$$[e_1 \ldots e_n] = I_n \to n \text{ pivots}, \begin{bmatrix}v_1\\\vdots\\v_n\end{bmatrix} = v_1e_1 + \cdots + v_ne_n$$

Therefore, $\dim \mathbb{R}^n = n$

Suppose that V has dimension d

- A sequence of d vectors in V are a basis if they span V
- A sequence of d vectors in V are a basis if they are linearly independent

19.2 Determining Basis

1. Determine whether ($\begin{bmatrix}1\\2\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\\3\end{bmatrix}$) is a basis of $\mathbb{R}^3$

The set has 3 elements, where dim $\mathbb{R}^3 = 3$. Therefore, it is a basis if and only if the vectors are linearly independent.

Form a matrix of the vectors and convert to row echelon form to determine pivots.

$$\begin{bmatrix}1 & 0 & 1\\2 & 1 & 0\\0 & 1 & 3\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 5\end{bmatrix}$$

Since each column contains a pivot, the three vectors are linearly independent. Therefore, it is indeed a basis of $\mathbb{R}^3$

A **basis** is the smallest set of vectors that span V, where removing any vector from it would mean it no longer spans V.

Produce a basis of $\mathbb{R}^2$ from the vectors, such that

$$v_1 = \begin{bmatrix}1\\2\end{bmatrix}, v_2 = \begin{bmatrix}1\\1\end{bmatrix}, v_3 = \begin{bmatrix}-.5\\-2\end{bmatrix}$$

To produce a basis, the minimal set of vectors must be linearly independent. Create a vector equation with two of the vectors.

$$\begin{bmatrix}1\\1\end{bmatrix} = a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}-.5\\-2\end{bmatrix}$$

$$\begin{cases} a - .5b = 1 \\ 2a - 2b = 1 \end{cases} \rightarrow b = 1, a = 1.5$$

This means that

$$v_2 = 1.5v_1 + v_3$$

Therefore, v_2 is a linear combination of v_1 and v_3, meaning that $\{v_1, v_3\}$ is the basis for $\mathbb{R}^2$ because the two vectors are linearly independent. 2. Produce a basis of $\mathbb{R}^2$ from the vector $\begin{bmatrix}2\\1\end{bmatrix}$

Any vector that is not in the span($\begin{bmatrix}2\\1\end{bmatrix}$). One such vector is $\begin{bmatrix}1\\0\end{bmatrix}$. Therefore, ($\begin{bmatrix}2\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$) is a basis.

20 Bases and Dimension for the four fundamental subspaces

Find a basis for Nul(A)

- Find teh parametric form of the solutions to $Ax = 0$
- Express solutions x as a linear combination of vectors with the free variables as coefficients
- Use these vectors as a basis of Nul(A)

For example, let $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 15 & 0 & 3 \end{bmatrix}$

$$\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & 13 & 0 \\ 0 & 0 & 1 & -2 & -5 & 0 \end{array} ccccc|c$$

$$\begin{cases} x_1 = -2x_2 - 5x_4 - 13x_5 \\ x_3 = 2x_4 + 5x_5 \end{cases}$$

where $Ax = 0$'s solutions are

$$\begin{bmatrix} -2x_2 - 5x_4 - 13x_5 \\ x_2 \\ 2x_4 + 5x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

where the basis is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -13 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$

20.1 Rank

Rank - the number of pivots a matrix has

Big Ideas

- Given matrix A with an $m \times n$ matrix with rank $r \to$ dim Nul$(A) = n - r$
- Suppose matrix $U = \begin{bmatrix} u_1 \cdots u_n \end{bmatrix}$ is a row echelon form of A.

  $$x_1u_1 + \cdots + x_nu_n = 0 \leftrightarrow Ax = 0 \leftrightarrow x_1a_1 + \cdots + x_na_n = 0$$

  because A and U are row equivalent.

Let A be an $m \times n$ matrix with rank r. The pivot columns of A form a basis of Col(A), where dim Col(A) $= r$

- this is because if U is the RREF of A, then the pivot columns of U and A must be linearly independent

For example, if $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}$, the basis of Col($A$) is simply the columns that contain a pivot

Let U be the matrix of A in echelon form

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 3 contain the pivot columns. Therefore, the basis of Col(A) are

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

where $u_2 = 2u_1$ and $u_4 = 4u_1 + 5u_3$

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ where $U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$, the RREF of A.

Col(A) $\neq$ Col(U) because row operations do not preseve the column space, but rather the row space. This means that Col(A^T) = Col(U^T)

$$\text{span}(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) \neq \text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

However, considering the transposes of A and U,

$$\text{span}(\begin{bmatrix} 1 \\ 3 \end{bmatrix}) = \text{span}(\begin{bmatrix} 1 \\ 3 \end{bmatrix})$$

Therefore, if A and B are two row equivalent matrices, then Col(A^T) = Col(B^T). Similarly, if A is an $m \times n$ matrix with a rank r, then the non-zero rows in echelon form form a basis of Col(A^T) and therefore dim(A^T) $= r$

If A is an $m \times n$ matrix with rank r, then

- dim Col(A) = dim Col(A^T) $= r$
- dim Nul(A) $= n - r$

20.2 Examples

Determine dim Nul(A^T) and dim Col(A) + dim Nul(A)?

$$\dim \text{Nul}(A^T) = m - r$$

$$\dim \text{Col}(A) + \dim \text{Nul}(A) = r + (n - r) = n$$

21 Orthogonal Complements

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$.

$$U = A \text{ in RREF} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{where Nul}(A) = \text{span}(\begin{bmatrix} -2 \\ 1 \end{bmatrix})$$

$$\text{Col}(A^T) = \text{span}(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$$

Given a subspace V of $\mathbb{R}^n$, the **orthogonal complement** of V, ($V^\perp$), is the subspace of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in V

$$W^\perp = \{v \in \mathbb{R}^n : v \cdot w = 0 \text{ for all } w \in W\}$$

Note that $(W^\perp)^\perp = W$

Let A be an $m \times n$ matrix. Nul(A) is the orthogonal complement of Col$(A^T)^\perp$, where Nul(A) = Col$(A^T)^\perp$

- $\text{Nul}(A)^\perp = \text{Col}(A^T)$
- $\text{Nul}(A^T) = \text{Col}(A)^\perp$

Let V be a subspace of $\mathbb{R}^n$. Then dim V + dim $V^\perp = n$

21.1 Examples

Find a basis of the orthogonal complement of span$(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix})$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A^T x = 0 \rightarrow x_1 = 0 \rightarrow x_2 = -x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} \to x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is the basis of $\text{Nul}(A^T)$

22 Coordinates

Let $(v_1, \ldots, v_p)$ be a basis of V. Then every vector w in V can be expressed uniquely as - $w = c_1 v_1 + \cdots + c_p w_p$

Let $\beta = (v_1, v_2, \ldots, v_p)$ be an ordered basis of V, and let $w \in V$. The coordinate vector, w_β of w with respect to the basis β is

$$w_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_p \end{bmatrix}, \text{ if } w = c_1 v_1 + c_2 v_2$$

Let $V = \mathbb{R}^2$ and consider the bases

22.1 Example

$$\beta = (b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$$

$$\epsilon = (e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

Let $w = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Determine w_β and w_ϵ

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore, $w_\beta = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Similarly,

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, $w_\epsilon = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

In $\mathbb{R}^n$, let e be the vector with a 1 in the i-th coordinate and 0's elsewhere. The standard basis of $\mathbb{R}^n$ is the ordered basis $\epsilon_n = (e_1, \ldots, e_n)$

22.2 Example

1. $\forall_v v \in \mathbb{R}^n$, we have $v = v_{\epsilon_n}$. Why?

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 e_1 + \cdots + v_n e_n \to v_{\epsilon n} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

2. Suppose basis $\beta = (b_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ of $\mathbb{R}^2$. Let $v \in \mathbb{R}^2$ be such that $v_\beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. What is v?

$$v_\beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \to v = 2b_1 + b_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

22.3 Change of Basis

Change of Basis - The matrix I_{C_β}, such that for all $v \in \mathbb{R}^n \to I_{C,\beta} v_\beta = v_C$, given that β and C are two bases of $\mathbb{R}^n$ Let $\beta = (b_1, \ldots, b_n)$ be a basis of $\mathbb{R}^n$. Then,

$$I_{\epsilon_n,\beta} = \begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix}$$

for all $v \in \mathbb{R}^n$

$$v = \begin{bmatrix} b_1 & \ldots & b_n \end{bmatrix} v_\beta$$

22.4 Change of Basis Examples

3. if $\beta = (b_1, \ldots, b_n)$ is a basis of $\mathbb{R}^n$, then what is i_{β,ϵ_n}

$$v = I_{\epsilon_n,\beta} v_\beta \to I^{-1}_{\epsilon_n,\beta} = v_\beta$$

2. If β and C are two bases of $\mathbb{R}^n$, what is $I_{\beta,C}$?

$$I_{B,\epsilon_n} I_{\epsilon_n,C} v_C = I_{\beta,\epsilon_n} v = v_\beta \to I_{\beta,C} = I_{\beta,\epsilon_n} I_{\epsilon_n,C}$$

An easier way to computer $I_{C,\beta}$

$$I_{C,\beta} = [(b_1)_C \ldots (b_n)_C] \text{ or } I_{\epsilon,\beta} = \begin{bmatrix} b_1 \ldots b_n \end{bmatrix}$$

23 Orthogonal and Orthonormal Bases

Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be non-zero and pairwise orthogonal. Then $v_1, \ldots, v_m$ are linearly independent. This implies that a set of n orthonormal vectors in $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$

Orthogonal/Orthonormal Basis - An orthogonal/orthonormal set of vectors that forms a basis

Let $\beta = (b_1, b_2, \ldots, b_n)$ be an orthogonal basis of $\mathbb{R}^n$ and let $v \in \mathbb{R}^n$. Then

$$v = \frac{v \cdot b_1}{b_1 \cdot b_1} b_1 + \cdots + \frac{v \cdot b_n}{b_n \cdot b_n} b_n$$

If β is orthonormal, then $b_i \cdot b_i = 1$ for $i = 1, \ldots, n$

23.1 Practice Problems

1. Let υ be the orthonormal basis $(u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix})$ of $\mathbb{R}^2$. Let $v = \begin{bmatrix}2\\3\end{bmatrix}$. Determine v_υ using the formula from the previous theorem!

$$v_\upsilon = \begin{bmatrix}u_1 \cdot v\\u_2 \cdot v\end{bmatrix} = \begin{bmatrix}\frac{5}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix}$$

2. Given the same υ, calculate the change of basis matrix I_{υ,ϵ_2}

$$I_{\upsilon,\epsilon_2} = I^{-1}_{\epsilon_2,\upsilon} = (\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix})^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} = I^T_{\epsilon_2,\upsilon}$$

Therefore, if $\upsilon = (u_1, \ldots, u_n)$ is a orthonormal basis of $\mathbb{R}^n$, then

$$I_{\upsilon,\epsilon_n} = \begin{bmatrix}u_1 \ldots u_n\end{bmatrix}^T$$

An $n \times n$ matrix Q is orthogonal if $Q^-1 = Q^T$. The columns of an orthogonal matrix form an orthonormal basis. This is because the product of Q and its transpose is the identity matrix.

24 Linear Transformation

Let V and W be vector spaces. A map $T : V \to W$ is a **linear transformation** if

$$T(av + bw) = aT(v) + bT(w)$$

for all $v, w \in V$ and all $a, b \in \mathbb{R}$

If $V = \mathbb{R}$ and $W = \mathbb{R}$, explain why $f(x) = 3x$ is linear and $g(x) = 2x - 2$ is not.

$$f(ax + by) = 3(ax + by) = 3ax + 3by = af(x) + bf(y)$$

$g(x) = 2x - 2$

$$g(0) + g(0) = 2 \cdot 0 - 2 + 2 \cdot 0 - 2 = -4 \neq -2 = g(0) = g(0+0)$$

This means that $T(0_v) = T(0 \cdot 0_v) = 0 \cdot T(0_v) = 0_w$

Let A be an $m \times n$ matrix, and consider the map $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(v) = Av$. Is this a linear transformation?

$$T(cv + dw) = cT(v) + dT(w)$$

Yes! because matrix multiplication is linear! Similarly, differentiation is linear as well

$$\frac{d}{dt}\begin{bmatrix} ap(t) \\ bq(t) \end{bmatrix} = a\frac{d}{dt}p(t) + b\frac{d}{dt}q(t)$$

If V, W are two vector spaces, then $T : V \to W$ is a linear transformation where $(v_1, \ldots, v_n)$ represents a basis of V. T is completely determined by the values of $T(v_1), \ldots, T(v_n)$

24.1 Examples

Let $T \to \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$. Find $T(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$.

$$\begin{aligned} T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) &= T(1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}) \\ &= T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) + 2T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) \\ &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \end{aligned}$$

24.2 Representing Linear Transformations as Matrices

Suppose $T : \mathbb{R}^m \to \mathbb{R}^m$ be a linear transformation. Then there is an $m \times n$ matrix A such that

(a) $T(v) = Av$, for all $v \in \mathbb{R}^n$

(b) $A = \begin{bmatrix} T(e_1) & T(e_2) & \ldots & T(e_n) \end{bmatrix}$, where $(e_1, e_2, \ldots, e_n)$ is the standard basis of $\mathbb{R}^n$

A represents the coordinate matrix of T with respect to the standard bases, which is formally notated as $T_{\epsilon_m \epsilon_n}$

24.3 Example

Let $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the "rotation over α radians (counterclockwise)" map, that is $T_\alpha(v)$ is the vector obtained by rotating v over angle α. Find the 2×2 matrix A_α, such that $T_\alpha(v) = A_\alpha v$ for all $v \in \mathbb{R}^2$

Figure out what happens when rotating the standard basis

Recall that the standard basis consists of $\{\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}\}$ due to the 2×2 identity matrix $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. Also recall from trigonometry that $x = cos(\alpha)$ and $y = sin(\alpha)$. On the point $(1, 0)$, cosine and sine are both positive.

However, when this point becomes rotated, to the point $(0, 1)$, it lands on the derivative of the point $(1, 0)$. This is because the derivative captures the rotation from the point $(1, 0)$ to $(0, 1)$, where $\frac{dx}{da} = -sin(\alpha)$ and $\frac{dy}{da} = cos(\alpha)$. Therefore,

$$T_\alpha(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}cos(\alpha)\\sin(\alpha)\end{bmatrix},\ T_\alpha(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}-sin(\alpha)\\cos(\alpha)\end{bmatrix}$$

$$A_\alpha = \begin{bmatrix}cos(\alpha) & -sin(\alpha)\\sin(\alpha) & cos(\alpha)\end{bmatrix}$$

25 Coordinate Matrix of a Linear Transformation

Let V, W be two vector space and let $\beta = (b_1, \ldots, b_n)$ be a basis of V and $C = (c_1, \ldots, c_m)$ be a basis of W. Let $T : V \to W$ be a linear transformation. Then there is a $m \times n$ matrix $T_{C,\beta}$ such that

(a) $T(v)_c = T_{C,\beta} v_\beta$

(b) $T_{c,\beta} = \begin{bmatrix}T(b_1)_c & T(b_2)_c & \ldots & T(b_n)_c\end{bmatrix}$

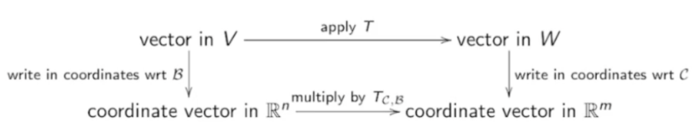


Figure 1: Example screenshot of the transformation diagram

The diagram above shows a linear transformation diagram when any linear transformation $T : V \to W$ interacts with bases. It explains how applying a linear transformation in abstract vector spaces is equivalent to multiplying coordinate vectors by a matrix in $\mathbb{R}^n$ when everything is expressed in terms of bases.

From the top arrow, left arrow, bottom arrow, and right arrow respectively,

- Apply the transformation T to a vector $\vec{v} \in V$, and get $T(\vec{v}) \in W$
- Write in coordintates with respect to β^n
- Multiply by $T_{c,\beta}$ and apply the matrix representation of T to the coordinate vector
- Write in coordinates with respect to C. Convert the transformed vector $T(\vec{v}) \in W$ to its coordinate form in $\mathbb{R}^m$

25.1 Examples

1. Let $D : \mathbb{P}_2 \to \mathbb{P}_1$ be given by $D(p(t)) = \frac{d}{dt}p(t)$. Consider the bases $\beta = (1, t, t^2)$ and $C = (1, t)$ of $\mathbb{P}_2$ and $\mathbb{P}_1$. Determine $D_{C,\beta}$.

$$D_{C,\beta} = \begin{bmatrix} D(1) & D(t) & D(t^2) \end{bmatrix}$$

$$\begin{aligned} D(1) &= 0 = 0 \cdot 1 + 0 \cdot t \\ D(t) &= 1 = 1 \cdot 1 + 0 \cdot t \\ D(t^2) &= 2t = 0 \cdot 1 + 2 \cdot t \end{aligned}$$

$$D_{C,\beta} = \begin{bmatrix} D(1)_c & D(t)_c & D(t^2)_c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

2. Consider $p(t) = 2 - t + 3t^2$ in $\mathbb{P}_2$. Compute $D(p(t))_C$ and $D_{C,\beta}p(t)_\beta$

$$D(2 - t + 3t^2) = -1 + 6t \to D(p(t))_c = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$$p(t)_\beta = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \to D_{C,\beta}p(t)_\beta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $T(v) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} v$. Consider the basis $\beta := (b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix})$. Compute $T_{\beta,\beta}$

$$\begin{aligned} T(b_1) &= T(\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2b_1 + 0b_2 \\ T(b_2) &= T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0b_1 + 4b_2 \end{aligned}$$

This results in the coordinate matrix $T_{\beta,\beta}$, where $T_{\beta,\beta} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

25.2 Change of Basis for a matrix of a linear transformation

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation and A and β be two bases of $\mathbb{R}^m$ and C, D be two bases of $\mathbb{R}^n$. Then

$$T_{C,A} = I_{C,D} T_{D,B} I_{B,A}$$

25.2.1 Example

Consider $\beta := D := \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ and $A := C := \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ as before. Le $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Determine $T_{C,A}$

$$T_{C,A} = I_{C,D} T_{D,B} I_{B,A}$$

$$I_{C,D} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$I_{\beta,A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Since β, D is the standard basis, $T_{D,\beta} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Therefore,

$$T_{C,A} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

26 Determinants

The **determinant** of a matrix is is a scalar, single numerical value, which can be used to determine key information about a matrix.

(a) The determinant states whether the matrix is invertible or not. When $det(A) \neq 0$, the matrix is invertible

(b) The determinant also states whether the matrix's corresponding system of equations has a solution, which only exists also when $det(A) \neq 0$

Note that the matrix's determinant is typically notated as det, while sometimes it may be seen as $||$

26.1 Calculating Determinants

Suppose A is a $n \times n$ square matrix.

(a) $det(A) = a$, when $n = 1$

(b) When $n = 2$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det(A) = ad - bc$

Furthermore, the determinant is the operation that assigns to each $n \times n$ matrix a number that satisfies the following conditions

- Normalization, where $det(I_n) = 1$
- Affected by elementary row operations
  - Replacement $\rightarrow$ adding a multiple of one row to another row does not change the determinant
  - Interchange $\rightarrow$ interchanging two different rows reverses the sign of the determinant
  - Scaling $\rightarrow$ multiplying all entries in a row by s, multiplies the determinant by s

26.1.1 Examples

1. Compute $det(\begin{bmatrix} 2 & 3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix})$.

Perform row operations to put matrix in row echelon form.

- $R_2 \rightarrow R_2 - \frac{1}{3}R_3$, $R_1 \rightarrow R_1 - \frac{1}{2}R_3 = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$
- $R_1 \rightarrow R_1 - 3R_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

The end result is the product of the left diagonal, in which that corresponding value is $2 \cdot 1 \cdot 6 = 12$.

26.2 Deriving ad - bc, the determinant of a 2×2 matrix

This ideal concept works for a 2×2 matrix as well, which is what derives the common formula $ad - bc$, the determinant of a 2×2 matrix.

Suppose matrix A, where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A in row echelon form would then be equivalent to

$$\begin{bmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{bmatrix}$$

where the determinant equals the left diagonal product. This works because of a row operation, which cancels out the c, $R_2 \to R_2 - \frac{c}{a}R_1$. Note that this only works when $a - 0$. If $a \neq 0$, then swap rows!

26.2.1 Example

1. Suppose A is a 3×3 matrix with $det(A) = 5$. What is $det(2A)$)?
   (a) A has three rows
   (b) multiplying each of them by 2 produces them $2A$.
   (c) $det(2A) = 2^3 det(A) = 40$

26.3 Important Determinant Concepts

Recall that $det(A) = 0$ if and only if A is not invertible. This means that if A and B are row equivalent, then $det(A) = 0 \leftrightarrow det(B) = 0$ because elementary row oeprations don't change whether the determinant is 0 or not.

Therefore, A can only be invertible if and only if A is row-equivalent to I_n and $det(A) \neq 0$.

Similarly, if A is invertible, then $det(A^{-1}) = \frac{1}{det(A)}$ and $det(A^T) = det(A)$.

Everything about determinants with respect to row operations applies to the same with matrix A^T's determinant

27 Cofactor Expansion

Let A be an $n \times n$ matrix, such that A_{ij} represents the matrix obtained from matrix A by deleting the i^{th} row and j^{th} column of A

Repeat this process to achieve a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which can be recalled has a determinant of $ad - bc$

This method is known as the **cofactor expansion** and it is used to evaluate the determinant of a bigger sized square matrix

27.1 Example Problem

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$. What is A_{23} and A_{43}?

$$A_{23} = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}$$
$$A_{43} = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 6 & 8 \\ 9 & 10 & 12 \end{bmatrix}$$

27.2 Using Cofactor Expansion to compute a matrix's determinant

Given that A is an $n \times n$ matrix, its (i, j)-cofactor is the scalar C_{ij}, which is defined by

$$C_{ij} = (-1)^{i+j} det(A_{ij})$$

Therefore, for every $i, j \in \{1, \ldots, n\}$,

$$\begin{aligned} det(A) &= a_{i1} + C_{i1} + a_{i2} + C_{i2} + \cdots + a_{in}C_{in} \\ &= a_{ij}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \end{aligned}$$

where the first expression represents the expansion across row i and the second expression represents the expansion across row j

27.2.1 Example Problem

Compute $det(\begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix})$ by cofactor expansion across row 1

(a) Calculate $C_{11} \rightarrow C_{11} = 1$. Take determinant of 2×2 matrix for the A_{11} cofactor

$$C_{11} \cdot A_{11} = 1 \cdot \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

$1 \cdot det(A_{11}) = 1 \cdot (-1(1) - 2(0)) = -1$

(b) Subtract the determinant of A_{12} $\cdot$C$_{12} \rightarrow C_{12} = 2$

$$C_{12} \cdot A_{12} = 2 \cdot \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$2 \cdot det(A_{12}) = 2 \cdot (3(1) - 2(2)) = -2$

(c) Add the determinant of A_{13} $\cdot$C$_{13} \rightarrow C_{13} = 0$

$$C_{11} \cdot det(A_{11}) - C_{12} \cdot det(A_{12}) + C_{13} \cdot det(A_{13}) = -1 + 2 = -1$$

This same process can be done for column expansion by using the same formula. To compute $det(A)$ using cofactor expansion down column 2

$$det(A) = -C_{12} \cdot det(A_{12}) + C_{22} \cdot det(A_{22}) + C_{32} \cdot det(A_{32})$$

$$-2(-1) + (-1)(1) - 0 = 1$$

Note that the cofactor expansion would not work for a large n square matrix. To compute the determinant of a large $n \times n$ matrix,

(a) one reduces to n determinants of size $(n-1) \times (n-1)$

(b) then $n(n-1)$ determinants of size $(n-2) \times (n-2)$

28 Eigenvectors and Eigenvalues

Eigenvector - A nonzero v, such that $Av = \lambda v$ for $n \times n$ matrix A

Eigenvalue - The λ scalar that is associated with the eigenvector v

28.1 Verifying Eigenvectors

1. Show that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 0 & -3 \\ -2 & -1 \end{bmatrix}$. Is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ an eigenvector?

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue, λ, -2

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \neq \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

No! $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not an eigenvector of A because it does not have an appropriate eigenvalue, λ, associated with it.

28.2 Finding Eigenvectors

1. Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$
$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is one eigenvector with eigenvalue, λ, 1

$$A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is another eigenvector with eigenvalue, λ, -1

2. For $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, find the eigenvectors and eigenvalues

$$B \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with $\lambda = 1$

$$B \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an eigenvector with $\lambda = 0$

28.3 Eigenspaces

Let λ be an eigenvalue of $m \times n$ matrix A. The **eigenspace** of A associated with λ is the set of eigenvectors of A with eigenvalue λ and the zero vector, where

$$\text{Eig}_\lambda(A) = \{v : Av = \lambda v\}$$

29 Computing Eigenvalues and Eigenvectors

If A is an $n \times n$ matrix and λ be a scalar, then λ is an eigenvalue of A if and only if $det(A - \lambda I) = 0$

If A be an $n \times n$ matrix, then $p_A(t) = det(A - tl)$ is a polynomial of degree n. Thus A has at most n eigenvalues, where $p_A(t)$ the **characteristic polynomial** of A.

29.1 Examples

Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Compute the eigenvalues of A

$$A = \lambda I = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = det(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}) = (3-\lambda)^2 - 1$$

$$= \lambda^2 - 6\lambda + 8 = 0 \quad = (\lambda - 4)(\lambda - 2) = 0$$

where $\lambda = 4, \lambda = 2$ are eigenvalues for A. Therefore,

$$Eig_\lambda = Nul(A - \lambda I)$$

2. What are the eigenspaces of A?

Let $\lambda_1 = 2$

$$A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \text{ RREF } = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to Nul(A - 2I) = span(\begin{bmatrix} -1 \\ 1 \end{bmatrix})$$

Let $\lambda_2 = 4$

$$A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \text{ RREF } \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \to Nul(A - 4I) = span(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$$

3. Find the eigenvalues and eigenspaces of $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$.

Notice that this matrix is in row echelon form. Therefore,

$$det(A - \lambda I) = \begin{bmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{bmatrix} = (3-\lambda)(6-\lambda)(2-\lambda)$$

Therefore, $\lambda = 2, 3, 6$. The eigenvalues of a triangular matrix are its diagonal entries.

$$\lambda = 2 \to A - 2I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2.5 \\ 0 & 0 & 0 \end{bmatrix} \to Nul(A - 2I) = span(\begin{bmatrix} 2 \\ -\frac{5}{2} \\ 1 \end{bmatrix})$$

$$\lambda = 3 \to A - 3I = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 10 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to Nul(A - 3I) = span(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix})$$

$$\lambda = 6 \to A - 6I = \begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 10 \\ 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to Nul(A - 6I) = span(\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix})$$

30 Properties of Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix, where λ is an eigenvalue of A

- The **algebraic multiplicity** of λ is its multiplicity as a root of the characteristic polynomial, which is the largest integer k such that $(t-\lambda)^k$ divides $p_A(t)$
- The **geometric multiplicity** of λ is the dimension of the eigenspace $Eig_\lambda(A)$ of λ

30.1 Example Problem

1. Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and determine their algebraic and geometric multiplicities

$$det(A - \lambda I) = det(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}) = (1-\lambda)^2$$

Therefore, $\lambda = 1$ is the only eigenvalue, which has an algebraic multiplicity of 2

$$\lambda = 1 : A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow Nul(A-I) = span(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$$

Therefore, there is only a dimension of 1 because the geometric multiplicity is 1

30.1.1 Connecting Eigenvectors to Linear Independence

Eigenvectors $v_1, \ldots, v_m$ with different corresponding eigenvalues of an $n \times n$ matrix A are linearly independent

30.2 Trace

Trace - the sum of the diagonal entries of matrix A, given that

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Mathematically speaking,

$$Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

30.2.1 Trace vs Determinant

Thet trace is the **sum** of the eigenvalues, $\lambda_1, \lambda_2 + \cdots + \lambda_n$, while the determinant is the **product** of the eigenvalues $\lambda_1 \cdot \lambda_2 \cdots \lambda_n$

Mathematically speaking,

$$Tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$
$$det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$$

30.2.2 Deriving the characteristic polynomial via the trace and determinant

The characteristic polynomial of $A \to det(A - \lambda I)$, which is also notated as $p(\lambda)$ is equivalent to

$$p(\lambda) = \lambda^2 - Tr(A)\lambda + det(A)$$

31 Markov Matrices

Markov Matrices or **stochastic matrices** are square matrices that have only non-negative entries, where the entries in each columnadd up to 1

Probability Vectors - also known as stochastic vectors and exist in $\mathbb{R}^n$ that have only non negative entries, which add up to 1

31.1 Markov Matrices Examples

$$\begin{bmatrix} .1 & .5 \\ .9 & .5 \end{bmatrix}, \begin{bmatrix} 0 & .25 & .4 \\ 1 & .25 & .2 \\ 0 & .5 & .4 \end{bmatrix}, \begin{bmatrix} .1 \\ .25 \\ .05 \\ .5 \\ .1 \end{bmatrix}$$

Note that a scalar constant multiplied by a vector can result in a probability vector. For example,

$$\frac{1}{10} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

results in a probability vector, while $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ does not

Suppose A is a Markov matrix and $v \in \mathbb{R}^n$ is a probability vector. Their product is also a probability vector!

31.2 Connecting markov matrices and probability vectors to eigenvalues

When A is a Markov matrix,

- 1 is an eigenvalue of A and every other eigenvalue λ of A satisfies $|\lambda| \leq 1$
- If A has only positive entries, then any other eigenvalue satisfies $|\lambda| < 1$

A **stationary** probability vector of a Markov matrix is a probability vector v that is an eigenvector of A corresponding to the eigenvalue 1 Suppose A is an $n \times n$ Markov matrix with only positive entries and $z \in \mathbb{R}^n$ be a probability vector. Then

$$\lim_{k \to \infty} A^k z$$

exists and z_∞ is a stationary probability vector of A, where $Az_\infty = z_\infty$

31.3 Markov Matrices example

Consider a fixed population of people with or without a job. Suppose each year, $\frac{1}{2}$ of those unemployed find a job, while $\frac{1}{10}$ of those employed lose their job. What is the unemployment rate in the long term equilibrium?

Let x_t be the percentage of population employed at time t and let y_t be the percentage of population unemployed at time t, where

$$\begin{bmatrix} x_t + 1 \\ y_t + 1 \end{bmatrix} = \begin{bmatrix} .9x_t + .5y_t \\ .1x_t + .5y_t \end{bmatrix} = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

Let $\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$ be the stationary probability vector, where

$$\begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix} \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix}$$

Determine $A - I$,

$$A - I = \begin{bmatrix} -0.1 & 0.5 \\ 0.1 & -0.5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} \rightarrow Nul(A - I) = span(\begin{bmatrix} 5 \\ 1 \end{bmatrix})$$

Since $x_\infty + y_\infty = 1 \rightarrow \begin{bmatrix} x_\infty \\ y_\infty \end{bmatrix} = \begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix}$

Therefore, $\frac{1}{6}$ is the umemployment rate in the long term equilibrium.

32 Diagonlization

A square matrix A is diagonalizable if there is an invertible matrix P and a diagonal matrix D, where $A = PDP^{-1}$

Suppose A is an $n \times n$ matrix that has n linearly independent eigenvectors $v_1, v_2, \dots, v_n$ with associated eigenvalues $\lambda_1, \dots, \lambda_n$. Then A is diagonalizable as PDP^{-1}, where

$$P = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \text{ and } D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

32.1 Diagonalization Problem

Diagonalize $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$

$det(A - \lambda I) = det(\begin{bmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{bmatrix}) = (6-\lambda)(3-\lambda) + 2 = \lambda^2 - 9\lambda + 20 = (\lambda-4)(\lambda-5)$

where $\lambda_1 = 4$ and $\lambda_2 = 5$

$$\lambda_1 = 4 \rightarrow A - 4I = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \rightarrow Nul(A - 4I) = span(\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix})$$

$$\lambda_2 = 5 \rightarrow A - 5I = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow Nul(A - 5I) = span(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$$

Therefore,

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, P = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & 1 \end{bmatrix}$$

where $A = PDP^{-1}$

32.2 Eigenbases and Change of Eigenbasis

Vectors $v_1, \dots, v_n$ form an **eigenbasis** of $n \times n$ matrix A, if $v_1, \dots, v_n$ form a basis of $\mathbb{R}^n$ and $v_1, \dots, v_n$ are all eigenvectors of A

Therefore,

- A has an eigenbasis
- A is diagonalizable
- The geometric multiplicities of all eigenvalues of A sum up to n

There also exists a diagonal matrix D, such that $A = I_{\epsilon_n, \beta} D I_{\beta, \epsilon_n}$ for $n \times n$ matrix A and eigenbasis $\beta = (v_1, \dots, v_n)$ for A

Therefore, Diagonalization is the **change of basis** to the eigenbasis!

33 Powers of Matrices

1. Suppose A has an eigenbasis, which allow A to raise to large powers easily! If $A = PDP^{-1}$, where D is a diagonal matrix, then for any m, $A^m = PD^mP^{-1}$. This makes it easy to find D^m

$$D^m = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^m = \begin{bmatrix} (\lambda_1)^m & & \\ & \ddots & \\ & & (\lambda_n)^m \end{bmatrix}$$

33.1 Examples

1. What is A^{100}, given that $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ with eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ with } \lambda_1 = \frac{1}{2}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ with } \lambda_2 = 1, v_3 = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} \text{ with } \lambda_3 = 2$$

Recall that P forms from the eigenvectors, D is a diagonal matrix that forms from the eigenvalues of the eigenvectors, and P^{-1} is the inverse matrix of P

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$A^{100} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}^{100} & 0 & 0 \\ 0 & 1^{100} & 0 \\ 0 & 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2^{100}} & 0 & 0 \\ (\frac{1}{2^{100}} - 1) & 1 & (6 \cdot 2^{100} - 6) \\ 0 & 0 & 2^{100} \end{bmatrix}$$

2. Let A be a 2×2 matrix where $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\frac{1}{2}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 1. Let $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Graphically determine how $A^n v$ behaves as $n \to \infty$

Recall that an eigenbasis forms when its eigenvectors are **linearly independent**.

Since $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a linear combination resulting from both eigenvectors, this means that the eigenvectors form an eigenbasis.

Recall that in an eigenvector, $A^n v = \lambda^n v$, which means that the eigenvalues can be substituted for A where

$$\lim_{n\to\infty} A^n v = \lim_{n\to\infty} \lambda^n v$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{2}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lim_{n\to\infty} 1^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

34 Matrix Exponential

For $n \times n$ matrix A, the **matrix exponential** e^{At} is defined as

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

34.1 Calculating Matrix Exponential

1. Compute e^{At} for $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$At = \begin{bmatrix} 2t & 0 \\ 0 & t \end{bmatrix}$$

where

$$(At)^k = \begin{bmatrix} (2t)^k & 0 \\ 0 & t^k \end{bmatrix}$$

and

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 1t \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} (2t)^2 & 0 \\ 0 & t^2 \end{bmatrix} + \cdots = \begin{bmatrix} e^2 t & 0 \\ 0 & e^t \end{bmatrix}$$

34.2 e^{At} definitions

Let A be an $n \times n$ matrix

- The series in the definition of e^{At} always converges
- $e^{At}e^{As} = e^{A(t+s)}$
- $e^{At}e^{-At} = I_n$
- $\frac{d}{dt}(e^{At}) = Ae^{At}$

34.3 Connecting Diagonalization with e^{At}

For $n \times n$ matrix A, such that $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D. Then

$$e^{At} = Pe^{Dt}P^{-1}$$

34.3.1 Example

1. Suppose $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$. Compute e^{At}

$$\begin{aligned} e^{At} &= Pe^{Dt}P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \end{bmatrix} \end{aligned}$$

35 Orthogonal Projections onto lines

Let $v, w \in \mathbb{R}^n$. The **orthogonal projection** of v onto the line spanned by w is

$$proj_w(v) = \frac{w \cdot v}{w \cdot w} w$$

Suppose $v, w \in \mathbb{R}^n$. Then $proj_w(v)$ is the point in span(w) closest to v; that is

$$dist(v, proj_w(v)) = min(u \in span(w))dist(v, u)$$

$v - proj_w(v)$ is known as the error term and it is in $span(w)^\perp$

$$\begin{aligned} v &= proj_w(v) + v - proj_w(v) \\ &= \in span(w) + \in span(w)^\perp \end{aligned}$$

Suppose $w \in \mathbb{R}^n$. Then for all $v \in \mathbb{R}^n$

$$proj_w(v) = (\frac{1}{w \cdot w} ww^T)v$$

35.1 Example Problem

Suppose $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What is the orthogonal projection matrix P onto $span(w)$. Use it to calculate the projections of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ onto $span(w)$

$$P = \frac{1}{w \cdot w} ww^T = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- If $v = \begin{bmatrix}1\\0\end{bmatrix}$, then $proj_w(v) = Pv = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}$
- If $v = \begin{bmatrix}1\\1\end{bmatrix}$, then $proj_w(v) = Pv = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} = v$
- If $v = \begin{bmatrix}1\\-1\end{bmatrix}$, then $proj_w(v) = Pv = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

36 Orthogonal Projections onto subspaces

Let W be a subspace of $\mathbb{R}^n$ and $v \in \mathbb{R}^n$. Then each v in $\mathbb{R}^n$ can be uniquely written as

$$v = \hat{v} + v^{\perp}$$

is the orthogonal projection of v onto W, $proj_W(v)$

If $(w_1, \ldots, w_m)$ is an orthogonal basis of W, then

$$proj_W(v) = (\frac{v \cdot w_1}{w_1 \cdot w_1})w_1 + \cdots + (\frac{v \cdot w_m}{w_m \cdot w_m})w_m$$

36.1 Example Problem

Let $W = span(\begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix})$ and $v = \begin{bmatrix}0\\3\\10\end{bmatrix}$. Then $proj_w(v)$

$$proj_w(v) = \frac{v \cdot w_1}{w_1 \cdot w_1}w_1 + \frac{v \cdot w_2}{w_2 \cdot w_2}w_2$$

$$\frac{\begin{bmatrix}0\\3\\10\end{bmatrix} \cdot \begin{bmatrix}3\\0\\1\end{bmatrix}}{\begin{bmatrix}3\\0\\1\end{bmatrix} \cdot \begin{bmatrix}3\\0\\1\end{bmatrix}}\begin{bmatrix}3\\0\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\3\\10\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix}}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}3\\0\\1\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}3\\3\\1\end{bmatrix}$$

$$v_{\perp} = \begin{bmatrix}0\\3\\10\end{bmatrix} - \begin{bmatrix}3\\3\\1\end{bmatrix} = \begin{bmatrix}-3\\0\\9\end{bmatrix}$$

36.2 Linear transformations + Orthogonal Projections

The projection map $proj_w : \mathbb{R}^n \to \mathbb{R}^n$ that sends v to $proj_w(v)$ is linear, where the matrix P_w is the matrix $(proj_W)_{\epsilon_n, \epsilon_n}$ that represents $proj_W$ with respect to the standard basis. P_w is the **orthogonal projection matrix** onto W.

36.2.1 Example Problems

1. Compute P_W for $W = span(\begin{bmatrix}3\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix})$

$$proj_W(\begin{bmatrix}1\\0\\0\end{bmatrix}) = \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}3\\0\\1\end{bmatrix}}{\begin{bmatrix}3\\0\\1\end{bmatrix}\cdot\begin{bmatrix}3\\0\\1\end{bmatrix}}\begin{bmatrix}3\\0\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{3}{10}\begin{bmatrix}3\\0\\1\end{bmatrix}$$

$$proj_W(\begin{bmatrix}0\\0\\1\end{bmatrix}) = \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}3\\0\\1\end{bmatrix}}{10}\begin{bmatrix}3\\0\\1\end{bmatrix} + \frac{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{1}{10}\begin{bmatrix}3\\0\\1\end{bmatrix}$$

$$proj_w(\begin{bmatrix}0\\1\\0\end{bmatrix}) = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$P_W = \begin{bmatrix}\frac{9}{10} & 0 & \frac{3}{10}\\ 0 & 1 & 0\\ \frac{3}{10} & 0 & \frac{1}{10}\end{bmatrix}$$

2. Let P_W be the orthogonal projection matrix W in $\mathbb{R}^n$ and let $v \in \mathbb{R}^n$.

- If $P_W v = v$, what can you say about v?
- If $P_W v = 0$, what can you say about v?

Suppose $v = w + u$, where $w \in W$ and $u \in W^\perp$. Then $w = P_W v$. Therefore, if $P_W v = v$, then $w = v$, meaning that $v \in W$ and if $P_W v = 0$, then $z = v$ and thus $v \in W^\perp$. 3. What is the orthogonal projection matrix $P_W^\perp$ for projecting onto $W^\perp$?

Let $v \in \mathbb{R}^n$. To show that Qv is the projection of v onto $W^\perp$, we need to check that $Qv \in W^\perp$ and $v - Qv \in (W^\perp)^\perp$

Since $P_W v$ is the projection matrix of v onto W. $Qv = v - P_W v \in W^\perp$. Since $v - Qv = P_W v$, $v - Qv \in W$, where $W = (W^\perp)^\perp$

37 Least Squares Solutions

Suppose $Ax = b$ is inconsistent, where there is no exact solution because b is not in the column space of A. Instead, the goal is to look for an x that minimizes this error, known as the **least squares solution**.

The **least squares solution** (or LSQ solution) of a system $Ax = b$ is a vector $\hat{x} \in \mathbb{R}^n$ such that

$$dist(A\hat{x}, b)) = min(x \in \mathbb{R}^n)dist(Ax, b)$$

Suppose A is an $m \times n$ matrix and $b \in \mathbb{R}^m$. Then $\hat{x}$ is an LSQ solution to $Ax = b$ if and only if $A\hat{x} = proj_{Col(A)}(b)$

37.1 Example Problems

1. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. What is the LSQ solution of $Ax = b$.

Determine $\hat{b}$ by figuring out the projection of b onto the column space of A.

Recall that the least squares solution, $\hat{x}$, gives the best fitting linear combination of A in order to determine $\hat{b}$, which approximates b.

To find this best fitting linear combination of A, we msut use the basis of the column space because that is what contains the minimal set of vectors that are unique and linearly independent in A

$$\hat{b} = proj_{Col(A)}(b) = \frac{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Then, solve for $A\hat{x} = \hat{b}$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

The solution to this matrix system was defined when computing $\hat{b}$, where $\hat{x} = \begin{bmatrix} \frac{1}{2} \\ \\ \frac{3}{2} \end{bmatrix}$. This is because the projection $\hat{b} = A\hat{x}$ calculates the unique coefficients needed to express $\hat{b}$ as a linear combination of the columns of A.

37.2 Normal Equations for Least Squares

(a) $\hat{x}$ is an LSQ solution to $Ax = b$ if and only if $A^T A\hat{x} = A^T b$

(b) $proj_{Col(A)}(b) = A(A^T A)^{-1}A^T b$ for $m \times n$ matrix A with linearly independent columns where $b \in \mathbb{R}^m$

37.2.1 More Examples

1. Let $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$. Find a LSQ solution of $Ax = b$.

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

The normal equations $A^T A\hat{x} = A^T b$ are $\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \hat{x} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$, where

$$\begin{cases} 17x + y = 19 \\ x + 5y = 11 \end{cases}$$

where $x = 1$ and $y = 2$ and $\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The projection of b onto $Col(A)$ is $A\hat{x} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$

38 Linear Regression

Recall that **linear regression** is a technique used to model the relationship between a dependent variable and one or more independent variables by fitting a straight line to the given data.

Linear Regression essentially finds the best fitting line through a scatterplot of points, which is ultimately the line that minimizese the total squared vertical distance between the observed data points and the line itself, where

$$y_i \approx \beta_0 + \beta_1 x_i$$

Or in matrix form,

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

The system is inconsistent, when the echelon form's system does not contain a clear solution

38.1 Example

Find β_1, β_2 such that the line $y = \beta_1 + \beta_2 x$ best fits the data

$$(x_1, y_1) = (2, 1), (x_2, y_2) = (5, 2)(x_3, y_3) \quad = (7, 3), (x_4, y_4) = (8, 3)$$

Setup a linear regression formula in matrix form and find its LSQ

$$\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

where $X^T X \hat{x} = X^T y$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

and $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ \\ \frac{5}{14} \end{bmatrix}$ with a least squares regression line of $y = \frac{2}{7} + \frac{5}{14}x$

The output, y is modeled as a linear combination of the input features present, which leads to a linear system that consists of a design matrix containing the linearly dependent data, where each linearly dependent variable consists of its own column on a design matrix

38.2 Formulating Regression of a Linear System

For example, if y depended on u and v linearly, then a design matrix with additional columns would have to be introduced, where each columns contains the linearly dependent variable

$$\begin{bmatrix} 1 & u_1 & v_1 \\ 1 & u_2 & v_2 \\ \vdots & \vdots & \vdots \\ 1 & u_n & v_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

39 Gram Schmidt

Gram Schmidt is a method used to find the orthogonal or orthonormal basis of a given subspace

Gram Schmidt uses a set of linearly independent vectors that span a given subspace to construct an orthogonal or orthonormal basis

39.1 Gram Schmidt orthonormalization

Every subspace of $\mathbb{R}^n$ has an orthonormal basis.

Given a basis $(a_1, \ldots, a_m)$ that produce an orthogonal basis of $(b_1, \ldots, b_m)$ and an orthonormal basis $(q_1, \ldots, q_m)$, then

b$_1 = a_1$, $q_1 = \frac{b_1}{||b_1||}$

b$_2 = a_2 - proj_{span(q_1)}(a_2)$ where $proj_{span(q_1)}(a_2) = (a_2 q_1)q_1, q_2 = \dfrac{b_2}{||b_2||}$

b$_3 = a_3 - proj_{span(q_1), span(q_2)}(a_3)$ where $proj_{span(q_1), span(q_2)}(a_3) = (a_3 \cdot q_1)q_1 + (a_3 \cdot q_2)q_2, q_3 = \dfrac{b_3}{||b_3||}$

...

where $span(q_1, \ldots, q_i) = span(a_1, \ldots, a_i)$ for $i = 1, \ldots, m$ and $q_j \notin span(a_1, \ldots, a_i)$ for all $j > i$

39.2 Gram Schmidt Example

Suppose $V = span(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix})$. Use Gram-Schmidt to figure out an orthonormal basis of V

Let $b_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $q_1 = \frac{b_1}{||b_1||} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$$b_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} - (\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}) \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$$

Gram Schmidt allows the normliazation of b_2, where $q_2 = \frac{b_2}{||b_2||} = \frac{1}{\sqrt{45}} \begin{bmatrix} -4 \\ -2 \\ 5 \end{bmatrix}$

39.3 QR decomposition

There exists an $m \times n$ matrix Q with orthonormal columns and an upper triangular $n \times n$ invertible matrix R, such that $A = QR$, given that A is an $m \times n$ matrix of rank n.

39.3.1 QR decomp example

Find the QR decomposition of $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}$

(a) Apply Gram Schmidt on columns of A

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \to q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) where $b_3 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} \to q_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

where $Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to R = Q^T A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

40 Spectral Theorem

- For symmetrical $n \times n$ matrix A, A has an orthonormal basis of eigenvectors
- For the same A, there is a diagonal matrix D and a matrix Q with orthonormal columns such that $A = QDQ^T$

40.1 Spectral Theorem v.s. Diagonalization

Diagonalization occurs when a matrix transforms into a diagonal matrix through a change in eigenbasis using its eigenvectors. It is written as $A = PDP^{-1}$ where, D represents the diagonal matrix of eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$ and P represents a matrix whose columns are the eigenvectors of A.

Spectral Theorem is a more powerful form of diagonalization, where a real symmetrical matrix can become diagonalizable with a *orthogonal* matrix, Q, whose columns are orthonormal eigenvectors of A with D representing a diagonal matrix of real eigenvalues, $\lambda_i \in \mathbb{R}^n$

Given that diagonalization is indeed a change in an eigenbasis, spectral theorem is a change in an orthogonal eigenbasis.

This makes spectral theorem easier to compute because the transpose is taken, rather than the inverse, unlike diagonalization

40.2 Example

1. Suppose A is an $n \times n$ matrix with an orthonormal eigenbasis, is it symmetric?

$$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A$$

This simply says that the $A^T = A$, which means A is symmetric, proving that A must be symmetric for spectral theorem to work.

2. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Write A as QDQ^T, where D is diagonal and Q has orthonormal columns.

Determine A's eigenvalues through the characteristic equation

$$det(A - \lambda I) = 0 - (3 - \lambda)^2 = 0, \text{ where } \lambda = 2, 4$$

Use the eigenvalues to find the eigenvectors, which can then be normalized to find the orthonormal eigenvectors.

$$\lambda_1 = 2 \to v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\lambda_2 = 4 \to v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Normalize to find the orthonormal eigenvectors, which merge into Q

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \text{ and } Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

40.3 Spectral Theorem scales eigenvectors

Recall that when a symmetric matrix A is diagonalized as $A = QDQ^T$, it is referred to as the change in orthonormal eigenbasis via Q^T, which scales each component by its corresponding eigenvalue (D) and then changing back to the original basis via A.

The A matrix scales the components of any vector along its eigenvector directions.

When expressing vector $\hat{v}$ as a linear combination of eigenvectors (for example, $\hat{v} = -\hat{q}_1 + \frac{1}{2}\hat{q}_2$), the eigenvectors are scaled by A. This saves calculations and can instead just scale eigenvector components by eigenvalues.

41 SVD - Singular Value Decomposition

A **singular value decomposition** of A is a decomposition where $A = U\Sigma V^T$, where A is an $m \times n$ matrix where

- U is an $m \times m$ matrix with orthonormal columns
- Σ is an $m \times n$ rectangular diagonal matrix with non-negative numbers on the diagonal
- V is an $n \times n$ matrix with orthonormal columns

The diagonal entries $\sigma_i = \Sigma_{ii}$ which are positive are called the **singular values** of A, which are usually arranged in decreasing order, that is $\sigma_1 \geq \sigma_2 \geq \ldots$

A Graphs

Terminology

- **Node** - an individual point/entity on a graph (also known as a vertex)
- **Edge** - a connection between two nodes
- **Graph** - a collection of nodes and edges

A proper graph consists of a set of nodes and a set of edges, formally notated as $(G = VE)$. This means that any collection of nodes and edges is formally conisdered as a graph, where

$$G = VE,\ \text{where V is the set of vertices, and E is the set of edges}$$

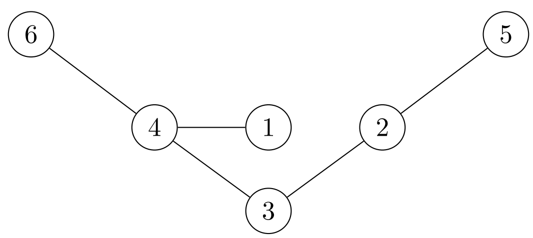


Figure 2: A small undirected graph on six nodes.

A.1 Adjacency Matrices

Consider Figure 1, which shows a graph of 6 nodes and 5 edges. Like any other graph, it can be represented as an adjacency matrix, $A = (a_{ij})$

$$a_{ij} = \begin{cases} 1 & i, j \text{ share common edge} \\ 0 & \text{else} \end{cases}$$

Adjacency Matrix - a square matrix used to represent the nodes and their connections on any given graph

a_{ij} tells us that there exists an edge between two nodes when nodes i and j share a common edge. Or mathematically speaking,

$$a_{ij} = 1 \leftrightarrow \exists e \in E : e = \{i, j\}\,,\ \text{where } E \text{ denotes the set of edges}$$

In the context of Figure 1, we can form an adjancency matrix based on the edges that connect our nodes, by following a series of steps.

(a) Identify the edges

(b) Create a square matrix that orders the rows and columns based on the number of nodes that exist in the graph.

- i represents the row node and j represents the column node
- Write 0 if there is no edge that connects between i and j, or 1 if there is an edge that connects between i and j

By following these steps, the graph from figure 1 results in an Adjacency Matrix, A where

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

A.2 Walks and Paths

A **walk** consists of a sequence of nodes, $v_0, v_1, v_2, \ldots$, where $v_i, v_{i+1}, \ldots$ share an edge and a **path** consists of a walk with distinct nodes in $G = (V, E)$

In a sequence of vertices, $W = (v_0, v_1, \ldots, v_k)$, a **walk** exists when there exists an edge connecting two nodes in every step along W

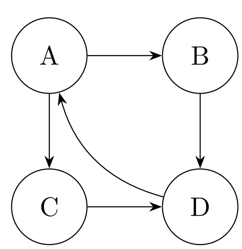
$$W \text{ is a walk } \leftrightarrow \forall i \in \{1, \ldots, k\}, \exists e \in E : e = \{v_{i-1}, v_i\}$$

Similarly speaking, let $P = (v_0, v_1, \ldots, v_k)$ represent a sequence of vertices in $G = (V, E)$. Then P is a path of length k if and only if $v_i \neq v_j$

$$P \text{ is a path } \leftrightarrow (\forall 1 \leq i \leq k, \{v_{i-1}, v_i\} \in E) \text{ and } (v_i \neq v_j) \text{ for all } 0 \leq i < j \leq k$$

A.3 Directed Graph

Directed Graph - a graph where each edge has its own orientation/direction.



The graph above represents a directed graph, with nodes A, B, C, and D with directed edges that point to other nodes.

A.3.1 Edge-Node Incidence Matrices

Since the edges are directed, we can describe how the edges connected to the nodes through an edge-node incidence matrix (or simply an incidence matrix).

Incidence Matrix - A matrix that describes how edges are connected to nodes on a graph, specifically speaking a directed graph

In an incidence matrix, there exists an adjacency matrix, a_{ij}, with the values -1, 0, and 1, where

$$a_{ij} = \begin{cases} -1 & \text{edge } i \text{ leaves node } j \\ +1 & \text{edge } i \text{ enters node } j \\ 0 & \text{otherwise} \end{cases}$$

This forms the incidence matrix, A, where

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

A.4 Proving Graph Theorems

Given a directed graph, G and its edge-node incidence matrix, A, then $\dim(\text{Nul}(A))$ = the number of connected components.

We can prove this theorem by using the 0 vector, where

$$\vec{0} = \vec{A}x = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ -x_2 + x_4 \\ -x_3 + x_4 \end{bmatrix}$$

Or simply speaking,

$$\vec{0} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ x_2 - x_3 \\ -x_2 + x_4 \\ -x_3 + x_4 \end{bmatrix}$$

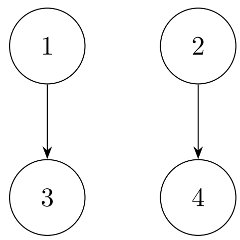
With this, we can derive the following solutions

$$x_1 = x_2$$
$$x_1 = x_3$$
$$x_2 = x_3$$
$$x_2 = x_4$$
$$x_3 = x_4$$

Where, the $\text{Nul}(A) = \text{span}(\begin{bmatrix}1\\1\\1\\1\end{bmatrix})$

A.4.1 Example Proof

Given the directed graph, G, where G represents the graph below



There are two connected components, where $Nul(A)$ forms the basis B, where $B = \{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\}$.

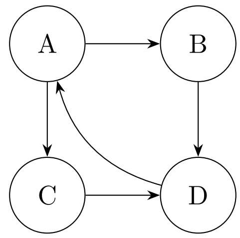
We can prove this basis by forming the resulting matrix and solving for its variables by setting it equal to the zero vector, $\vec{0}$

$$\begin{aligned}
&A\vec{x} = \vec{0} \\
&A = \begin{bmatrix}-1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix} \\
&\vec{0} = \begin{bmatrix}-x_1 + x_3\\ -x_2 + x_4\end{bmatrix} \\
&x_1 = x_3 \\
&x_2 = x_4
\end{aligned}$$

A.5 Cycles, Cycle Vectors, and Cycle Spaces

A **cycle** is a closed path in a graph, where its sequence of vertices and edges include the same vertex at the start and end of said sequence. Only the first and last vertices are equal

Consider the graph below,



Cycle Vector - A vectorized representation of a cycle in a graph, which records which edges are part of that cycle and in which direction they are traversed

Recall a_{ij}, where a_{ij} represents the connection between nodes i and j

$$a_{ij} = \begin{cases} -1 & \text{edge } i \text{ leaves node } j \\ +1 & \text{edge } i \text{ enters node } j \\ 0 & \text{otherwise} \end{cases}$$

A cycle, c, exists between $A \to C \to D \to A$. Its incidence matrix,

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

. Given the resulting cycle vector for c

$$c = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

By deriving the cycle vectors for every cycle, we get the **cycle space**, which is the span of cycle vectors.

Assume a generic graph G is a directed graph with an incidence matrix A. Prove that the cycle space of $G = Nul(A^T)$

Recall that the null space of a matrix is the set of all vectors, $\vec{x}$, such that $A\vec{x} = 0$.

$$\vec{0} = A^T\vec{y} = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \vec{y}$$

And therefore, the $RREF_A$, reduced row echelon form of A, is equal to

$$\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Use $RREF_A$ to form the solution vector, which results in

$$\begin{bmatrix} r \\ s \\ -s \\ r+s \\ -r-s \end{bmatrix}$$

This forms a nullspace, where $\text{Nul}(A^T) = \{ \begin{bmatrix} -r-s \\ r+s \\ r \\ -s \\ s \end{bmatrix}$, where $r, s \in \mathbb{R}\}$, which results in the cycle space of

$$span\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \}$$

B Linear Differential Equations