MATH 257 - Linear Algebra with Computational Applications

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1 Course Introduction

This is an introductory linear algebra course, which covers basic definitions and algorithms of the subject needed in higher levels of engineering, science, and economics.

This course introduces the mathematical theory along with how to implement it in the **Python programming language**

Prior Python experience is recommended

2 Introduction to Linear Systems

Linear Equations are in the form of

$$a_1x_1 + \cdots + a_nx_n = b$$

where $a_1, \ldots, a_n, b$ are numbers and $x_1, \ldots, x_n$ are variables.

For Example,

$$4x_1 - 5x_2 + 2 = x_1$$

is a linear equation because it can be rearranged to form an equation that is in the form of $a_1x_1 + \cdots + a_nx_n = b$

$$\begin{aligned} 4x_1 - 5x_2 + 2 &= x_1 \\ 4x_1 - x_1 - 5x_2 &= -2 \\ 3x_1 - 5x_2 &= -2 \end{aligned}$$

However,

$$x_2 = 2\sqrt{x_1} - 7$$

is **not** a linear equation because it cannot be expressed in the form of $a_1x_1 + \cdots + a_nx_n = b$

Linear Systems are collections of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.

A **solution** of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

2.1 Example Problem - Two equations in two variables

$$x_1 + x_2 = 1$$
$$-x_1 + x_2 = 0$$

What is a solution for this system of linear equations?

Solution - Use the **elimination method**

1. Add the two systems to eliminate the x_1 variable

$$2x_2 = 1$$
$$x_2 = \frac{1}{2}$$

2. Plug into the first equation to find the x_2 variable

$$x_1 + \frac{1}{2} = 1$$
$$x_1 = \frac{1}{2}$$

3. Thus $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$ is the only solution.

2.2 Does every system have a solution?

No! Observe the system

$$x_1 - 2x_2 = -3$$
$$2x_1 - 4x_2 = 8$$

Its process of solving is as follows

1. Multiply the first equation by 2 to eliminate the x_1 variable

$$2x_1 - 4x_2 = -6$$

2. Subtract the first equation from the second to cancel x_1

$$0 = 14$$

3. The equation $0 = 14$ is always false, so **no solutions** exist.

2.2.1 Example Problem

$$x_1 + x_2 = 3$$
$$-2x_1 - 2x_2 = -6$$

How many solutions are there to this system of equations?

Solution

1. Multiply the first equation by -2 to eliminate x_1

$$-2x_1 - 2x_2 = -6$$

2. Both the first and second equation are the same. Subtracting the two in order to cancel out x_1 will result in

$$0 = 0$$

3. This means both equations have the same solutions. Therefore, the system is said to have **infinitely many solutions**.

2.3 Theorem 1

A linear system has either **one unique solution**, **no solution**, or **infinitely many solutions**.

Definition - the **solution set** of a linear system is the set of all solutions of the linear system. Two linear systems are **equivalent** if they have the same solution set.

The general strategy is to replace one system with an equivalent system that is easier to solve.

2.3.1 Example Problem

Transform this linear system into another easier equivalent system

$$x_1 - 3x_2 = 1$$
$$-x_1 + 5x_2 = 3$$

Solution - Add the first equation to the second equation

$$x_1 - 3x_2 = 1$$
$$2x_2 = 4$$

$x_2 = 2$ and $x_1 = 1 + 3(2) = 7$

3 Matrices and Linear Systems

Definition - An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Example Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \begin{bmatrix} 1+\sqrt{5} \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 3 & 4 \\ 5 & -6 & 7 & 8 \\ -9 & 10 & 11 & 12 \end{bmatrix}$$

In terms of the entries of A:

$$A = \begin{bmatrix} a_1 1 & a_1 2 & \dots & a_1 n \\ a_2 1 & a_2 2 & \dots & a_2 n \\ \vdots & \vdots & \ddots & \vdots \\ a_m 1 & a_m 2 & \dots & a_m n \end{bmatrix}$$

where a_{ij} is in the ith row and jth column

Definition - For a linear system, we define the **coefficient** and **augmented** matrix as follows:

Linear System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Coefficient Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Augmented Matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

3.1 Example Problem

Determine the coefficient matrix and augmented matrix of the linear system

$$\begin{aligned} x_1 - 3x_2 &= 1 \\ -x_1 + 5x_2 &= 3 \end{aligned}$$

Solution - The coefficient matrix would be

$$\begin{bmatrix} 1 & -3 \\ -1 & 5 \end{bmatrix}$$

and the augmented matrix would be

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ -1 & 5 & 3 \end{array}\right]$$

3.2 Elementary Row Operation

An **elementary row operation** is one of the following

- Replacement - add a multiple of one row to another row: $R_i \to R_i + cR_j$, where $i \neq j$.
- Interchange - Interchange two rows: $R_i \leftrightarrow R_j$
- Scaling - Multiply all entries in a row by a nonzero constant: $R_i \to cR_i$, where $c \neq 0$

3.2.1 Example Problem

Give several examples of elementary row operations

Solution

- Replacement

$$R_2 \to R_2 + 3R_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

- Interchange

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Scaling

$$R_2 \to 3R_2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Elementary row operations can undo or **reverse** each other. For example, the elementary row operation $R_3 \to R_3 - 3R_1$ reverses the row operation of $R_3 \to R_3 + 3R_1$

$$R_3 \to R_3 + 3R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every row operation is reversible. Above showed an example of reversing the replacement operator. Similarly, the scaling operator $R_2 \to cR_2$ is reversed by the scaling operator $R_2 \to \frac{1}{c}R_2$. Row interchange $R_1 \leftrightarrow R_2$ is reversible by performing it twice.

Two matrices are **row equivalent** if one matrix can be transformed into the other matrix by a sequence of elementary row operations.

3.3 Theorem 1

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

4 Echelon forms of matrices

Definition - A matrix is in echelon form or **row echelon form** when

1. All **nonzero rows** (rows with at least one nonzero element) are above any rows of all zeroes
2. The **leading entry** (the first nonzero number from the left) of a nonzero row is always strictly to the right of the leading entry of the row above it.

4.1 Example

The following matrices achieve row echelon form

$$\begin{bmatrix} 2 & -2 & 3 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 & 2 & 0 & 5 \\ 0 & 2 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.2 RREF - Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if it is in row echelon form **and**

- The leading entry in each nonzero row is 1
- Each leading entry is the only nonzero entry in its column

4.2.1 Examples

The following matrices are in RREF

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 2 & 5 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 1 & \frac{1}{2} & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -3 & 4 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

4.3 Theorem 1

Each matrix is row-equivalent to one and only one matrix in reduced echelon form.

Definition - We say a matrix B is the **reduced echelon form** (RREF) of a matrix if A and B are row-equivalent and B is in reduced echelon form.

4.3.1 Example Problem

Is each matrix also row-equivalent to one and only one matrix in echelon form?

Solution - No! For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ are row-equivalent and both in echelon form.

4.4 Calculating RREF

Find the rref of matrix $\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \end{bmatrix}$

Solution - to achieve RREF, the leading entry of each nonzero row needs to be 1 and each leading entry is the only nonzero entry in the column

1. $R_2 \to R_2 - R_1$
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{bmatrix}$$

2. $R_1 \to \frac{1}{3}R_1$, $R_2 \to \frac{1}{2}R_2$
$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{bmatrix}$$

3. $R_1 \to R_1 + 3R_2$
$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{bmatrix}$$

4.5 Pivot Position

The position of a leading entry in an echelon form of a matrix. A **pivot column** is a column that contains a pivot position

4.5.1 Example Problem

Locate the pivot columns of the following matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution

1. $R_1 \leftrightarrow R_3$
$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

2. $R_2 \to R_2 + R_1$
$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

3. $R_3 \to R_3 + 1.5R_2$
$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

The columns of **1**, **2**, and **4** are the pivot columns of A.

4.6 Pivot Variables

Basic Variable (Pivot Variable) - A variable that corresponds to a pivot column in the coefficient matrix of a linear system. A **free variable** is a variable that is not a pivot variable.

4.6.1 Example Problem

Consider the augmented matrix and system. Determine the basic and free variables.

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \qquad \begin{aligned} x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 8x_4 &= 5 \\ x_5 &= 7 \end{aligned}$$

Solution - The first, third and fifth columns are pivot columns. Therefore, x_1, x_3, and x_5 are basic variables and x_2, x_4 are free variables.