UIUC MATH 257 - Linear Algebra with Computational Applications

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1 Course Introduction

This is an introductory linear algebra course administered by the University of Illinois Urbana-Champaign (UIUC), which covers basic definitions and algorithms of the subject needed in higher levels of engineering, science, and economics.

This course introduces the mathematical theory along with how to implement it in the **Python programming language**

Prior Python experience is recommended

2 Introduction to Linear Systems

Linear Equations are in the form of

$$a_1x_1 + \cdots + a_nx_n = b$$

where $a_1, \ldots, a_n, b$ are numbers and $x_1, \ldots, x_n$ are variables.

For Example,

$$4x_1 - 5x_2 + 2 = x_1$$

is a linear equation because it can be rearranged to form an equation that is in the form of $a_1x_1 + \cdots + a_nx_n = b$

$$\begin{aligned} 4x_1 - 5x_2 + 2 &= x_1 \\ 4x_1 - x_1 - 5x_2 &= -2 \\ 3x_1 - 5x_2 &= -2 \end{aligned}$$

However,

$$x_2 = 2\sqrt{x_1} - 7$$

is **not** a linear equation because it cannot be expressed in the form of $a_1x_1 + \cdots + a_nx_n = b$

Linear Systems are collections of one or more linear equations involving the same set of variables, say, $x_1, x_2, \ldots, x_n$.

A **solution** of a linear system is a list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

2.1 Example Problem - Two equations in two variables

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 + x_2 &= 0 \end{aligned}$$

What is a solution for this system of linear equations?

Solution - Use the **elimination method**

1. Add the two systems to eliminate the x_1 variable
$$\begin{aligned} 2x_2 &= 1 \\ x_2 &= \frac{1}{2} \end{aligned}$$
2. Plug into the first equation to find the x_2 variable
$$\begin{aligned} x_1 + \frac{1}{2} &= 1 \\ x_1 &= \frac{1}{2} \end{aligned}$$
3. Thus $(x_1, x_2) = (\frac{1}{2}, \frac{1}{2})$ is the only solution.

2.2 Does every system have a solution?

No! Observe the system
$$\begin{aligned} x_1 - 2x_2 &= -3 \\ 2x_1 - 4x_2 &= 8 \end{aligned}$$

Its process of solving is as follows

1. Multiply the first equation by 2 to eliminate the x_1 variable
$$2x_1 - 4x_2 = -6$$
2. Subtract the first equation from the second to cancel x_1
$$0 = 14$$
3. The equation $0 = 14$ is always false, so **no solutions** exist.

2.2.1 Example Problem

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$

How many solutions are there to this system of equations?

Solution

1. Multiply the first equation by -2 to eliminate x_1

$$-2x_1 - 2x_2 = -6$$

2. Both the first and second equation are the same. Subtracting the two in order to cancel out x_1 will result in

$$0 = 0$$

3. This means both equations have the same solutions. Therefore, the system is said to have **infinitely many solutions**.

2.3 Three Types of Linear Systems

A linear system comes in three forms. It has either **one unique solution**, **no solution**, or **infinitely many solutions**.

The **solution set** of a linear system is the set of all solutions of the linear system. Two linear systems are **equivalent** if they have the same solution set.

The general strategy is to replace one system with an equivalent system that is easier to solve.

2.3.1 Example Problem

Transform this linear system into another easier equivalent system

$$\begin{aligned} x_1 - 3x_2 &= 1 \\ -x_1 + 5x_2 &= 3 \end{aligned}$$

Solution - Add the first equation to the second equation

$$\begin{aligned} x_1 - 3x_2 &= 1 \\ 2x_2 &= 4 \end{aligned}$$

$x_2 = 2$ and $x_1 = 1 + 3(2) = 7$

3 Matrices and Linear Systems

Definition - An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.

Example Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 + \sqrt{5} \end{bmatrix} \quad \begin{bmatrix} 1 & -2 & 3 & 4 \\ 5 & -6 & 7 & 8 \\ -9 & 10 & 11 & 12 \end{bmatrix}$$

In terms of the entries of A:

$$A = \begin{bmatrix} a_1 1 & a_1 2 & \dots & a_1 n \\ a_2 1 & a_2 2 & \dots & a_2 n \\ \vdots & \vdots & \ddots & \vdots \\ a_m 1 & a_m 2 & \dots & a_m n \end{bmatrix}$$

where a_{ij} is in the ith row and jth column

Definition - For a linear system, we define the **coefficient** and **augmented** matrix as follows:

Linear System

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Coefficient Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Augmented Matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

3.1 Example Problem

Determine the coefficient matrix and augmented matrix of the linear system

$$\begin{aligned} x_1 - 3x_2 &= 1 \\ -x_1 + 5x_2 &= 3 \end{aligned}$$

Solution - The coefficient matrix would be

$$\begin{bmatrix} 1 & -3 \\ -1 & 5 \end{bmatrix}$$

and the augmented matrix would be

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ -1 & 5 & 3 \end{array}\right]$$

3.2 Elementary Row Operation

An **elementary row operation** is one of the following

- **Replacement** - add a multiple of one row to another row: $R_i \to R_i + cR_j$, where $i \neq j$.
- **Interchange** - Interchange two rows: $R_i \leftrightarrow R_j$
- **Scaling** - Multiply all entries in a row by a nonzero constant: $R_i \to cR_i$, where $c \neq 0$

3.2.1 Example Problem

Give several examples of elementary row operations

Solution

- Replacement

$$R_2 \to R_2 + 3R_1$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

- Interchange

$$R_1 \leftrightarrow R_3$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \to \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Scaling

$$R_2 \to 3R_2$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Elementary row operations can undo or **reverse** each other. For example, the elementary row operation $R_3 \to R_3 - 3R_1$ reverses the row operation of $R_3 \to R_3 + 3R_1$

$$R_3 \to R_3 + 3R_1$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Every row operation is reversible. Above showed an example of reversing the replacement operator. Similarly, the scaling operator $R_2 \to cR_2$ is reversed by the scaling operator $R_2 \to \frac{1}{c}R_2$. Row interchange $R_1 \leftrightarrow R_2$ is reversible by performing it twice.

Two matrices are **row equivalent** if one matrix can be transformed into the other matrix by a sequence of elementary row operations.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

4 Echelon forms of matrices

Definition - A matrix is in echelon form or **row echelon form** when

1. All **nonzero rows** (rows with at least one nonzero element) are above any rows of all zeroes
2. The **leading entry** (the first nonzero number from the left) of a nonzero row is always strictly to the right of the leading entry of the row above it.

4.1 Example

The following matrices achieve row echelon form

$$\begin{bmatrix} 2 & -2 & 3 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 & 2 & 0 & 5 \\ 0 & 2 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4.2 RREF - Row Reduced Echelon Form

A matrix is in **reduced row echelon form** (RREF) if it is in row echelon form **and**

- The leading entry in each nonzero row is 1
- Each leading entry is the only nonzero entry in its column

4.2.1 Examples

The following matrices are in RREF

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 2 & 5 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 1 & \frac{1}{2} & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -3 & 4 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

4.3 Theorem 1

Each matrix is row-equivalent to one and only one matrix in reduced echelon form.

Definition - We say a matrix B is the **reduced echelon form** (RREF) of a matrix if A and B are row-equivalent and B is in reduced echelon form.

4.3.1 Example Problem

Is each matrix also row-equivalent to one and only one matrix in echelon form?

Solution - No! For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ are row-equivalent and both in echelon form.

4.4 Calculating RREF

Find the rref of matrix $\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \end{bmatrix}$

Solution - to achieve RREF, the leading entry of each nonzero row needs to be 1 and each leading entry is the only nonzero entry in the column

1. $R_2 \to R_2 - R_1$
$$\begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{bmatrix}$$
2. $R_1 \to \frac{1}{3}R_1$, $R_2 \to \frac{1}{2}R_2$
$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{bmatrix}$$
3. $R_1 \to R_1 + 3R_2$
$$\begin{bmatrix} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{bmatrix}$$

4.5 Pivot Position

The position of a leading entry in an echelon form of a matrix. A **pivot column** is a column that contains a pivot position

4.5.1 Example Problem

Locate the pivot columns of the following matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution

1. $R_1 \leftrightarrow R_3$
$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

2. $R_2 \rightarrow R_2 + R_1$
$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

3. $R_3 \rightarrow R_3 + 1.5R_2$
$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

The columns of **1**, **2**, and **4** are the pivot columns of A.

4.6 Pivot Variables

Basic Variable (Pivot Variable) - A variable that corresponds to a pivot column in the coefficient matrix of a linear system. A **free variable** is a variable that is not a pivot variable.

4.6.1 Example Problem

Consider the augmented matrix and system. Determine the basic and free variables.

$$\left[\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \qquad \begin{aligned} x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 8x_4 &= 5 \\ x_5 &= 7 \end{aligned}$$

Solution - The first, third and fifth columns are pivot columns. Therefore, x_1, x_3, and x_5 are basic variables and x_2, x_4 are free variables.

5 Gaussian Elimination

The idea behind **Gaussian Elimination** is to solve linear systems for the pivot variables in terms of free variables (if any) in the equation
Specifically, Gaussian Elimination is an **algorithm** or process used to solve linear systems backed behind matrices

1. Write down the augmented matrix
2. Find the RREF (reduced row echelon form) of the matrix
3. Write down the equations corresponding to the RREF
4. Express pivot variables in terms of free variables

5.1 Example

Find the general solution of

$$\begin{aligned} 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

The solution simply involves following the Gaussian Elimination algorithm

1. Write down the augmented matrix
$$\left[\begin{array}{ccccc|c} 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$
2. Find the RREF of the matrix
$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 5 & -4 \\ 0 & 1 & -2 & 2 & 1 & -3 \end{array}\right]$$
3. Write down equations corresponding to the RREF
$$\begin{aligned} x_1 &= -2x_3 + 3x_4 + 5x_5 = -4 \\ x_2 &= -2x_3 + 2x_4 + x_5 = -3 \end{aligned}$$
4. Express pivot variables in terms of free variables
$$\begin{aligned} x_1 &= 2x_3 - 3x_4 - 5x_5 - 4 \\ x_2 &= 2x_3 - 2x_4 - x_5 - 3 \\ x_3, x_4, x_5 &= \text{free} \end{aligned}$$

5.2 Theorem 1

A linear system is **consistent** if and only if an echelon form of the augmented matrix has no row of the form $\left[\begin{array}{ccc|c} 0 & \cdots & 0 & b \end{array}\right]$, where b is nonzero.

Linear systems are consistent when

1. There is a **unique** solution (no free variables)
2. Infinitely **many** solutions (at least one free variable)

5.2.1 Example

If a linear system has an augmented matrix of $\left[\begin{array}{cc|c} 3 & 4 & -3 \\ 3 & 4 & -3 \\ 6 & 8 & -5 \end{array}\right]$, what can be inferred about the number of solutions in the system?

Solution

Convert matrix to echelon form $\left[\begin{array}{cc|c} 3 & 4 & -3 \\ 3 & 4 & -3 \\ 6 & 8 & -5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 4 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$

There is no solution because of the $\left[\begin{array}{cc|c} 0 & 0 & 1 \end{array}\right]$ row (the bottom row in the echelon form of the original matrix)

As a linear equation, this row is equivalent to $0x_1 + 0x_2 = 1$, which is an equation that has no solution!

6 Linear Combinations

Consider the $m \times n$ matrices $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$, and $B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \ddots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix}$

6.1 Sum and Scalar Product

The **sum** of $A + B$ would be

$$\begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

Similarly, the **product** cA for a scalar c is

$$\begin{bmatrix} ca_{11} & ca_{12} & \dots & ca_{1n} \\ ca_{21} & ca_{22} & \dots & ca_{2n} \\ \dots & \dots & \ddots & \dots \\ ca_{m1} & ca_{m2} & \dots & ca_{mn} \end{bmatrix}$$

6.1.1 Adding Matrices Example

$$\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1+2 & 0+3 \\ 5+3 & 2+1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 8 & 3 \end{bmatrix}$$

6.1.2 Scalar Multiplying Matrices Example

$$5\begin{bmatrix}2 & 1 & 0\\ 3 & 1 & -1\end{bmatrix} = \begin{bmatrix}5\cdot 2 & 5\cdot 1 & 5\cdot 0\\ 5\cdot 3 & 5\cdot 1 & 5\cdot -1\end{bmatrix} = \begin{bmatrix}10 & 5 & 0\\ 15 & 5 & -5\end{bmatrix}$$

6.2 Row and Column Vectors

Column Vectors are $m \times 1$-matrices, while **Row Vectors** are $1 \times n$-matrices

6.2.1 Examples of Row and Column Vectors

$$\begin{bmatrix}1\\ 2\end{bmatrix} \quad \begin{bmatrix}1 & 2\end{bmatrix} \quad \begin{bmatrix}0\\ -2\\ 3\end{bmatrix} \quad \begin{bmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\end{bmatrix}$$

6.3 Transpose

If A is $m \times n$, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A. In terms of matrix elements: $(A^T)_{ij} = A_{ji}$

6.3.1 Figuring out the Transpose of a Matrix

What is the transpose of $A = \begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$?

In a transpose, the rows of the original matrix simply become the columns of the transposed matrix

$$A^T = \begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$$

6.4 Span

The **linear combination** of $m \times n$-matrices $A_1, A_2, \ldots, A_p$ with **coefficients** $c_1, c_2, \ldots, c_p$ is defined as

$$c_1A_1 + c_2A_2 + \cdots + c_pA_p$$

For example, for $m \times n$-matrices A_1 and A_2, some examples of linear combinations of these two matrices are

$$3A_1 + 2A_2 \quad A_1 - 2A_2 \quad \frac{1}{3}A_1 = \frac{1}{3}A_1 + 0A_2$$

The **span** $(A_1, \dots, A_p)$ is defined as the set of all linear combinations of $A_1, \dots, A_p$, or simply

$$\text{span}(A_1, \dots, A_p) := \{c_1 A_1 + c_2 A_2 + \cdots + c_p A_p : c_1, \dots, c_p \text{ scalars}\}$$

The **set** of all column vectors of length m is represented as $\mathbb{R}^m$

For example, let $a_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, a_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, and $b = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$. Is b a linear combination of a_1, a_2?

Find x_1, x_2 such that

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

1. Form the system of equations

$$\begin{aligned} &x_1 + 4x_2 = -1 \\ &0x_1 + 2x_2 = 8 \\ &3x_1 + 14x_2 = -5 \end{aligned}$$

2. Form the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 2 & 8 \\ 3 & 14 & -5 \end{array}\right]$$

3. Compute the echelon form

$$\left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 2 & 8 \\ 0 & 2 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & -1 \\ 0 & 2 & 8 \\ 0 & 0 & -10 \end{array}\right]$$

With this echelon form matrix, it can be concluded that b is not a linear combination of a_1 and a_2 because the system is inconsistent

This means that there is no solution that exists for the bottom row linear equation

$$0x_1 + 0x_2 = -10$$

Therefore, this means that there is no solution that exists for the derived linear system **and** the original vector equation, $x_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$

Geometrically speaking, this means that b is not in the span of a_1 and a_2

6.5 TTK - Things to Know

Solving linear systems is the same as finding linear combinations!

Theorem 1 - A vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

has the same solution set as the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} a_1 & a_2 & \cdots & a_n & b \end{array}\right]$$

In particular, b can be generated by a linear combination of $a_1, a_2, \cdots, a_n$ if and only if there is a solution to the linear system corresponding to the augmented matrix

A matrix is defined in terms of its **colums** or **rows**

$$A := \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \quad \text{or} A := \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}$$

7 Matrix Vector Multiplication

Suppose x is a vector in $\mathbb{R}^m$ and $A = \begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}$ an $m \times n$-matrix. The product Ax is defined by

$$Ax = x_1a_1 + x_2a_2 + \cdots + x_na_n$$

- Ax is a linear combination of the columns of A using the entries in x as coefficients.
- Ax is only defined if the number of entries of x is equal to the number of columns of A

7.1 Example Problem

If $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \end{bmatrix}$, and $x = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, determine Ax and Bx

$$Ax = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$Bx = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 21 \end{bmatrix}$$

7.2 Example Problem 2

Consider the vector equation

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Find a 2×2 matrix A such that (x_1, x_2) is a solution to the above equation if and only if

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}?$$

Take $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Form a linear equation with A using a linear combination Ax, where x represents the column vector of unknown variables

$$Ax = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is the vector expression that matches the problem statement and we know that it is equivalent to the column vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is also equivalent to to the column vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

Therefore one possible 2×2 matrix is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

7.3 Theorems and Vocab

Let $A = \begin{bmatrix} a_1, \cdots, a_n \end{bmatrix}$ be an $m \times n$-matrix and b in $\mathbb{R}^m$. The the following are equivalent

- $(x_1, x_2, \ldots, x_n)$ is a solution of the vector equation, $x_1 a_1 + x_2 a_2 + \cdots x_n a_n = b$
- $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a solution to the matrix equation, $Ax = b$
- $(x_1, x_2, \ldots, x_n)$ is a solution of the system with augmented matrix, $\left[\begin{array}{c|c} A & b \end{array}\right]$

The notation for the system of equations with augmented matrix $\begin{bmatrix} A & b \end{bmatrix}$ will be written as $Ax = b$

7.4 Matrices as Machines

Let A be a $m \times n$ matrix.

1. Input: n-component vector $x \in \mathbb{R}^m$
2. Output: m-component vector $b = Ax \in \mathbb{R}^m$

For example, consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. What does this machine do?

Solution

1. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be our input

$$Ax = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

2. Therefore, the machine A switches the entries of the vector x

Geometrically speaking, this machine reflects across the $x_1 = x_2$-line

7.5 Composition of Machines

Let A be an $m \times n$ matrix and B be an $k \times l$ matrix. Now we can compose the two machines

However, this composition only works for some k, l, m, n. For which?

Solution

- If A is an $m \times n$-matrix and x in $\mathbb{R}^n$, then Ax is in $\mathbb{R}^m$
- In order to calculate $B(Ax)$ when then need B to have m columns.
- So we need $l = m$. Both n and k can be arbitrary.

7.5.1 Worked Example

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ be as before. Is $A(Bx) = B(Ax)$?

Solution No, projection and reflection do not commute!

$$A(B \begin{bmatrix} 1 \\ 2 \end{bmatrix}) = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad B(A \begin{bmatrix} 1 \\ 2 \end{bmatrix}) = B \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

8 Matrix Multiplication

Let A be an $m \times n$ matrix and let $B = \begin{bmatrix} b_1 \dots b_p \end{bmatrix}$ be an $n \times p$-matrix. We define

$$AB := \begin{bmatrix} Ab_1 & Ab_2 & \dots & Ab_p \end{bmatrix}$$

Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$

Solution

$$Ab_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -7 \end{bmatrix} = -3 \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix} - 7 \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

These are the columns of the product matrix, where $AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$

Note that Ab_1 and Ab_2 are linear combinations of the columns of A. This means that each column of AB is a linear combination of the columns of A using coefficients from the corresponding columns of B

8.1 Worked Example

Let A be an $m \times n$ matrix and let B be an $n \times p$-matrix. We define

$$AB := \begin{bmatrix} Ab_1 & Ab_2 & \dots & Ab_p \end{bmatrix}$$

If C is a 4×3 and D is a 3×2, are CD and DC defined? What are their sizes or dimensions?

Solution

1. The product AB can only be defined if B has as many rows as A has columns
2. If this is the case, then AB has as many rows as A and as many columns as B
3. Therefore, CD is defined and has a 4×2 dimension, while DC is not defined

Recall that matrices can be thought of as machines

- Let B be $n \times p$: input $x \in \mathbb{R}^p$, output $c = Bx \in \mathbb{R}^n$

- Let A be $m \times n$: input $y \in \mathbb{R}^n$, output $b = Ay \in \mathbb{R}^m$

Compute $(AB)x$ and $A(B(x))$. Are these the same?

Solution

$$Bx = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$$

$$A(Bx) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix} = (x_1 + 2x_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (x_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

$$(AB)x = \begin{bmatrix} Ab_1 & Ab_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 Ab_1 + x_2 Ab_2 = x_1 (1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}) + x_2 (2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix})$$

$$= x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then for every $x \in \mathbb{R}^p$

$$A(Bx) = (AB)x$$

8.2 Row Column Rule

Let A be $m \times n$ and B be $n \times p$ such that

$$A = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix}, \text{ and } B = \begin{bmatrix} C_1 & \cdots & C_p \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} R_1 C_1 & \cdots & R_1 C_p \\ R_2 C_1 & \cdots & R_2 C_p \\ R_m C_1 & \cdots & R_m C_p \end{bmatrix} \text{ and } (AB)_{ij} = R_i C_j = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

8.2.1 Example Problem

Let $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined

$$AB = \begin{bmatrix} 2 \cdot 2 + 3 \cdot 0 + 6 \cdot 4 & 2 \cdot (-3) + 3 \cdot 1 + 6 \cdot (-7) \\ -1 \cdot 2 + 0 \cdot 0 + 1 \cdot 4 & -1 \cdot (-3) + 0 \cdot 1 + 1 \cdot (-7) \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$$

8.3 Outer Product Rule

Let A be $m \times n$ and B be $n \times p$ such that

$$A = \begin{bmatrix} C_1 \cdots C_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$$

Then

$$AB = C_1R_1 + \cdots + \cdots + C_nR_n$$

8.3.1 Example Problem

Let $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined

Solution

$$\begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 24 & -42 \\ 4 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$$

9 Properties of Matrix Multiplication

The **identity matrix** I_n of size n is defined as

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Let A be an $m \times n$ matrix and B and C be matrices for which the indicated sums nad products are defined

1. $A(BC) = (AB)C$ (associative law of multiplication)
2. $A(B + C) = AB + AC, (B + C)A = BA + CA$ (distributive laws)
3. $r(AB) = (rA)B = A(rB)$ for every scalar r
4. $A(rB + sC) = rAB + sAC$ for every scalars r, s (linearity of matrix multiplication)
5. $I_mA = A = AI_n$ (identity for matrix multiplication)

9.1 Matrix Identities vs Real Number Identities

While matrix multiplication properties are analogous to that of real numbers, not all properties of real numbers hold for matrices

For Example, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Determine AB and BA. Are these matrices the same?

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

No. $AB \neq BA$. These matrices are not the same. Matrix multiplication is not commutative!

9.2 Transpose Property of Matrices

Have $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. What is $(AB)^T$? What about $A^T B^T$ and $B^T A^T$?

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 10 \end{bmatrix}$$
$$(AB)^T = \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$
$$A^T B^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$
$$B^T A^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 10 \end{bmatrix}$$

The transpose of a product is the product of transposes in **opposite order**

$$(AB)^T = B^T A^T$$

9.3 Powers of Matrices

Let $A^k = A \cdots A$ for k-times; that is A^k is obtained by multiplying A $k - times$ with itself

For which matrices A does A^k make sense? If A is $m \times n$ what can m and n be?

To be able to multiply A by any $m \times n$-matrix, we need that $m = n$

Determine $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3$

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix}$$

Higher powers of matrices are more difficult to calculate using this method!

10 Elementary Matrices

Let A be a 3×3-matrix. What happens to A if you multiply it by one of E_1, E_2, and E_3?

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + 2a_{11} & a_{32} + 2a_{12} & a_{33} + 2a_{13} \end{bmatrix}$$

If an elementary row operation is performed on an $m \times n$-matrix A, the resulting matrix can be written as EA, where the $m \times m$-matrix E is created by performing the same row operations on I_m

An **elementary matrix** is one that an elementary row operation can be performed upon the identity matrix

Let A, B be two $m \times n$ matrices and row-equivalent. Then there is a sequence $m \times m$-elementary matrices $E_1, \dots, E_l$ such that

$$E_l \dots E_1 A = B$$

Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find two elementary matrices E_1, E_2 such that $E_2 E_1 A = B$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B$$

Set $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$$E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B$$

Recall that

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Find elementary matrices E_1^{-1} and E_2^{-1} such that $A = E_1^{-1} E_2^{-1} = B$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B$$

Set $E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Notice how the row operations were reversed to go from matrix B to A

11 Inverse of a Matrix

The inverse of a real number a is denoted as a^{-1}. For example, $7^{-1} = \frac{1}{7}$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

Not all real numbers have inverse. 0^-1 is not well defined, since there is no real number b such that $0 \cdot b = 1$

Recall that the identity matrix I_n is the $n \times n$-matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the inverse of A

11.1 Example Problem

What is the inverse of $\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

Elementary matrices are invertible because row operations are reversible. So the inverse matrix is the elementary matrix corresponding to $R_2 \to R_2 - 5R_1$: $\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Let A be an invertible matrix, then its inverse C is unique

Assume B and C are both inverses of A. Then

$$BA = AB = I_n$$
$$CA = AC = I_n$$

Thus, $B = BI_n = BAC = I_nC = C$

11.2 Inverse Properties

Suppose A and B are invertible. Then

1. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1})
2. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. A^T is invertible and $(A^T)^{-1}1\ (A^{-1})^T$

Proofs

$$AA^{-1} = I = A^{-1}A$$

$$(B^{-1}A^{-1})(AB) = B^{-1}IB = B^{-1}B$$
$$(AB)(B^{-1}A^{-1}) = AI^{-1} = AA^{-1} = I$$

$$A^T(A^{-1})^T = (A^-1A)^T = I^T = I$$

11.3 Multiplying the inverse of A

A^{-1} is denoted as the inverse of A. Multiplying by A^{-1} is like "dividing by A"

- Writing $\frac{A}{B}$ is unclear whether this means AB^{-1} or $B^{-1}A$, and these two matrices are completely different

If $AB = I$, then $A^{-1} = B$ and so $BA = I$

Similarly, not all $n \times n$ matrices are invertible. For example, the 2×2 matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is not invertible

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \neq I_2$$

Recall that the identity 2×2 matrix (I_2) is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The second row of matrix $\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \end{bmatrix}$, the second row of the identity matrix. Therefore, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ can't be invertible

Suppose that A is an invertible $n \times n$ matrix. Then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$

Proof

The vector $A^{-1}b$ is a solution, because

$$A(A^{-1}b) = (AA^{-1})b = I_n b = b$$

Suppose there is another solution w, then $Aw = b$. Thus

$$w = I_n w = A^{-1}Aw = A^{-1}b$$

Additionally, A must have n pivots because otherwise $Ax = b$ would not have a solution of each b

12 Computing the Inverse

A 1×1 matrix $\begin{bmatrix} a \end{bmatrix}$ is invertible when $a \neq 0$ and its inverse is $\begin{bmatrix} \frac{1}{a} \end{bmatrix}$

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible

Proof

$$\begin{aligned} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \frac{1}{ad - bc} \begin{bmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Let A be an $n \times n$. The following are equivalent:

- A is invertible
- the RREF of A is I_n

Proof

Suppose A can be row-reduced to the identity matrix

$$A = A_0 \rightsquigarrow A_1 \rightsquigarrow \cdots \rightsquigarrow A_m = I_n$$

Thus there are elementary matrices $E_1, \cdots, E_m$ such that

$$E_m E_{m-1} \cdots E_1 A = I_n$$

Thus

$$A^{-1} = E_m E_{m-1} \cdots E_1 = E_m E_{m-1} \cdots E_1 I_n$$

This boils down to the ideaw where we suppose A is invertible. Every sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}

12.1 Algorithm

Place A and I side by side to form an augmented matrix of $\begin{bmatrix} A \mid I \end{bmatrix}$

This becomes a $n \times 2n$ matrix (Big Augmented Matrix), instead of $n \times (n+1)$

Perform row operations on this matrix (which will produce identical operations on A and I).

By Theorem: $\begin{bmatrix} A \mid I \end{bmatrix}$ will row reduce to $\begin{bmatrix} I \mid A^{-1} \end{bmatrix}$

12.1.1 Example

Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists

$$\begin{bmatrix} A \mid I \end{bmatrix} = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{array}\right]$$