The Review of Financial Studies

Back-Running: Seeking and Hiding Fundamental Information in Order Flows*

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We model the strategic interaction between fundamental investors and "back-runners," whose only information is about the past order flow of fundamental investors. Back-runners partly infer fundamental investors' information from their order flow and exploit it in subsequent trading. Fundamental investors counteract back-runners by randomizing their orders, unless back-runners' signals are too imprecise. Surprisingly, a higher accuracy of back-runners' order flow information can harm back-runners and benefit fundamental investors. As an application of the model, the common practice of payment for (retail) order flow reveals information about institutional order flow and enables back-runners to earn large profits. (*JEL* G14, G18)

Received July 4, 2017; editorial decision February 12, 2019 by Editor Andrew Karolyi.

This paper studies the strategic interaction between fundamental informed trading and order flow informed trading, as well as its implications for market equilibrium outcomes. By order flow informed trading, we mean strategies that begin with no innate trading motives—be it fundamental information or liquidity needs—but instead learn about other investors' order flows and then act accordingly. A primary example of order flow informed trading is "order anticipation" strategies. According to the Securities and Exchange Commission (2010, pp. 54–55), order anticipation "involves any means to ascertain the

*For helpful discussions and comments, we thank Andrew Karolyi (editor); two anonymous referees; Kerry Back, Dan Bernhardt, Bradyn Breon-Drish, Mike Burkart, Henry Cao, David Easley, Itay Goldstein, Jungsuk Han, Terry Hendershott, Eric Hughson, Andrew Karolyi, Leonid Kogan, Pete Kyle, Doron Levit, Wei Li, Albert Menkveld, Maureen O'Hara, Christine Parlour, Uday Rajan, Ryan Riordan, Ioanid Rosu, Gideon Saar, Francesco Sangiorgi, Duane Seppi, Matthew Spiegel, Gunter Strobl, Laura Veldkamp, Xavier Vives, Yajun Wang, Bin Wei, Mao Ye, Jerry Yu, and Bart Zhou Yueshen; seminar and conference participants at the 2015 Finance Theory Group Summer School, the 2015 Northern Finance Association annual meeting, the 11th Central Bank Workshop on the Microstructure of Financial Markets, the 26th Annual Conference on Financial Economics and Accounting, the 2016 ASSA annual meetings, the 2016 China International Conference in Finance, the 2017 SFS Cavalcade, the 2017 WFA; and finance seminars at DePaul University, Gothenburg University, Lund University, Stockholm School of Economics, and Wisconsin Business School. We especially thank Hyungjune Kang for providing excellent research assistance. Send correspondence to Haoxiang Zhu, MIT Sloan School of Management, 100 Main Street E62-623, Cambridge, MA 02142; telephone: +1-617-253-2478. E-mail: zhuh@mit.edu.

The Review of Financial Studies 33 (2020) 1484–1533

doi:10.1093/rfs/hhz070 Advance Access publication July 2, 2019

existence of a large buyer (seller) that does not involve violation of a duty, misappropriation of information, or other misconduct. Examples include the employment of sophisticated pattern recognition software to ascertain from *publicly available information* the existence of a large buyer (seller), or the sophisticated use of orders to 'ping' different market centers in an attempt to locate and trade in front of large buyers and sellers [emphasis added]."

Order anticipation strategies have always been controversial and recently generated heated debates in the context of high-frequency traders (HFTs), especially following the publication of Lewis (2014).[1] Although most (reluctantly) agree that such strategies are legal in today's regulatory framework, many investors and regulators have expressed severe concerns that they could harm market quality and long-term investors. For example, in its influential Concept Release on Equity Market Structure, Securities and Exchange Commission (2010, p. 56) asks: "Do commenters believe that order anticipation significantly detracts from market quality and harms institutional investors?"

Motivated by such regulatory concerns, this paper proposes a simple model to formally analyze the strategic interaction between institutional investors and strategic traders that use order anticipation strategies based on past order flows, which we refer to as "back-running." We also examine the effect of back-running on institutional and retail investors as well as market quality.

Our model adds back-runners to an otherwise standard two-period Kyle (1985) model. There are $I \geq 1$ fundamental investors, $J \geq 1$ back-runners, noise traders, and a competitive market maker. In the first period, each fundamental investor observes a component f_i of the true asset value v, where $v = p_0 + \sum_i f_i$ and the components $\{f_i\}$ are independent. Fundamental investors and noise traders submit market orders, which are executed by the competitive market maker at the conditional expected value of the asset given the total order flows. Although back-runners start with no fundamental information or liquidity needs, each receives a noisy signal of the fundamental investors' total order flow in period 1, denoted X_1, after that order is executed by the market maker. This information allows the back-runners to partly infer the fundamental investors' private information. In the second period, back-runners join fundamental investors and noise traders in trading in the market, and their aggregate order flow is again filled by the competitive market maker. We characterize a linear equilibrium in which each fundamental investor and each back-runner maximizes her profit, taking everyone else's strategy as given.

The presence of back-runners substantially changes the strategies of fundamental investors. In particular, we show that if the back-runners' signals about order flows are sufficiently precise, their optimal trading strategies involve randomization, that is, mixed strategies. To see why a pure strategy is not

[1] For example, Harris (2003, p. 245) writes "(o)rder anticipators are *parasitic traders*. They profit only when they can prey on other traders" [emphasis in original].

optimal, consider the extreme case in which the back-runners' signals are perfect. In this case, a pure strategy by fundamental investors completely reveals their private information to the back-runners and reduces their profits through competition. Instead, fundamental investors' optimal strategy is to add endogenous, normally distributed noise into their period-1 orders. By continuity, randomization remains optimal if back-runners' signals are sufficiently precise, that is, if the standard deviation of the noise in signal is below a certain threshold that we can compute. A mixed strategy equilibrium of this nature is first shown by Huddart, Hughes, and Levine (2001) in a model with a monopolist insider, whose trades are publicly disclosed ex post. The mixed strategy also nicely echoes Stiglitz's (2014, p. 8) remark on high-frequency trading: "[T]he informed, knowing that there are those who are trying to extract information from observing (directly or indirectly) their actions, will go to great lengths to make it difficult for others to extract such information."

A major contribution of our analysis relative to Huddart, Hughes, and Levine (2001) is that randomization is not only possible but also likely. In particular, we show that a moderate number of back-runners sufficiently widens the parametric region for randomization so that a mixed strategy equilibrium is likely to obtain. For example, for ten back-runners, a choice motivated by van Kervel and Menkveld (2019), the noise threshold in a back-runner's signal is comparable to the amount of noise trading in the entire market. This is not a stringent condition on the accuracy of back-runners' order flow information. Intuitively, as more back-runners join the market, information leakage becomes more costly, and the fundamental investors have stronger incentives to randomize.

Furthermore, the model generates new theoretical results that we did not expect ex ante. For example, in the mixed strategy region, for sufficiently many back-runners, increasing the accuracy of back-runners' signals can actually reduce back-runners' profits and increase fundamental investors' profits. This result is analytically proven for the special case of a single fundamental investor and holds in numerical calculations for a general number of fundamental investors. In addition, in the pure strategy region, the total profit of back-runners increases in the number J of back-runners if $J \leq 3$ and decreases in J if $J > 3$. This result suggests that the industry structure of back-runners is likely to gravitate toward a tight oligopoly of three or four firms, but not a duopoly or a monopoly.

The most direct empirical prediction from our theory is that certain HFTs that can detect institutional orders are also able to anticipate their future paths and then trade in the *same direction* for profits. Our prediction contrasts with the market-making view about HFTs, which should trade in the opposite direction of institutional investors. To test this prediction, the relevant data should identify institutional investors and HFTs. To the best of our knowledge, three empirical studies of HFTs have used data of this granularity: van Kervel and Menkveld (2019) in the Swedish equity market, Korajczyk and Murphy (2019) in the Canadian equity market, and Tong (2015) in the U.S. equity market.

van Kervel and Menkveld (2019) and Korajczyk and Murphy (2019) directly test and support our prediction about HFT behavior. van Kervel and Menkveld (2019) find that certain HFTs initially provide liquidity when institutional investors start executing their orders, but if such order execution takes a long time, the HFTs eventually reverse course and trade in the same direction as the institutions. Moreover, HFTs' same-direction trading is associated with higher permanent price impact than opposite-direction trading. Korajczyk and Murphy (2019) find that "there is a significant increase in same-direction abnormal trading activity by HFTs relative to their opposite-direction abnormal activity when an institutional trade is being executed." They also find that institutional trading costs dropped after an exogenous reduction in HFT activity, caused by a regulatory change that increases the cost of sending electronic messages to Canadian exchanges. Tong's (2015) empirical strategy is not a direct mapping to our theory, but her evidence—an increase in HFT activity is associated with a higher implementation-shortfall cost of institutions—is consistent with our prediction.[2]

Beyond the HFT context, back-running also provides a useful framework to interpret the behavior of financial intermediaries. Using transaction-level data from a group of institutional investors, Di Maggio et al. (2019) find evidence that brokers sometimes leak informed order-flow information to their best clients, who then earn excess returns by trading in the same direction as those informed orders. In a follow-up study, Barbon et al. (Forthcoming) present similar evidence about fire sales, although in this case brokers leak information about liquidity shocks, not fundamental values.

Our results provide a natural theoretical foundation for the use of randomization in the execution of large institutional orders. While randomization in our two-period model boils down to adding a mean-zero perturbation, the practical implementation could involve using irregular execution size, time intervals, or speed, among others. For example, Sağlam (2018) finds that the implementation shortfall of a client order is higher if it is split and executed in a regular manner, in the sense of more regular trade sizes, more regular time intervals between trades, or a more regular rate of execution. Sağlam's interpretation, as well as ours, is that any such regularity makes the large client order detectable and makes back-running easier.

As an application of the model, we estimate the value of retail order flow information in U.S. equity markets through the lens of back-running. Under the interpretation that retail order flows are proxies for noise trading and institutional order flows are proxies for informed trading, information about retail order flows is equivalent to information about institutional order flows, by market clearing. Numerical solutions of the equilibrium under reasonable

[2] Somewhat relatedly, Hirschey (2018) finds that HFTs' aggressive orders lead those of other investors, and these patterns are stronger in situations when the non-HFTs are less concerned with hiding their order flows. His data do not identify institutional orders.

parameters suggest that back-runners' profits are in the order of 5–30 bps of retail dollar volume, or billions of dollars per year, whereas institutional investors' profits are in the order of 70–80 bps of retail dollar volume, or tens of billions of dollars per year. The order of magnitude of these estimates suggests that the common practice of payment for (retail) order flow in U.S. equity markets could be an important yet overlooked channel of the back-running of institutional orders.

Finally, let us caution that our model is inherently stylized, with only two periods and exogenous noise trading, among other assumptions. A full dynamic model would be more realistic and a better match to the data, but we have not found a way to solve it. In addition, our theory is meant to capture only one aspect of HFTs, namely their back-running strategies. The other side of the coin, namely, the market-making strategies of HFTs, also receives strong empirical support, as surveyed by Jones (2013) and Menkveld (2016). A useful future research direction is to incorporate multiple dimensions of HFT strategies in a coherent theoretical framework and use it for better understanding of the data and implications for policy. The ambiguous and nonmonotone theoretical predictions on market quality and various traders' profits in our current model already suggest that the qualitative results will likely remain ambiguous in a more comprehensive model. Yet future research may find useful empirical proxies or structural methods to bound parameter values to a narrower range, where model implications can be directly assessed quantitatively.

At a technical level, the model of our paper is closest to that of Huddart, Hughes, and Levine (2001), which is an extension of Kyle (1985). Motivated by the mandatory disclosure of trades by firm insiders, they assume that the insider's orders are disclosed publicly and perfectly after being filled. They show that the only equilibrium in their setting is a mixed strategy one. In their model the mandatory public disclosure unambiguously improves price discovery and market liquidity in each period. Buffa (2013) studies disclosure of insider trades when the insider is risk averse. His equilibrium with disclosure also features mixed strategies. In contrast to Huddart, Hughes, and Levine (2001), however, he shows that disclosing insider trades can harm price discovery by making the risk-averse insider trade less aggressively. Besides these two most closely related papers, several other papers with mixed strategy equilibria can be found in the literature, but they are quite different in terms of the economic questions or modeling approaches.[3]

Our results differ from those of Huddart, Hughes, and Levine (2001) and Buffa (2013) along a number of important dimensions. First, our model is

[3] In a continuous-time extension of Glosten and Milgrom (1985) model, Back and Baruch (2004) show that there is a mixed strategy equilibrium in which the informed trader's strategy is a point process with stochastic intensity. Baruch and Glosten (2013) show that "flickering quotes" and "fleeting orders" can arise from a mixed strategy equilibrium in which quote providers repeatedly undercut each other. Yueshen (2015) shows that if market makers are not perfectly competitive and the number of market makers is uncertain, then market makers who are present use a mixed pricing strategy. These papers do not explore the question of trading on order flow information or a switch between pure and mixed strategy equilibria.

more general in allowing an arbitrary number of fundamental investors and back-runners. This setup reveals some new theoretical results that one would not expect ex ante. For example, as mentioned earlier, an increase in the accuracy of back-runners' signals may reduce their profits if there are sufficiently many of them. Second, equipped with the general model, we show that the mixed strategy equilibrium is not only possible but also likely, under reasonable parameters. This broadens the applicability of this class of models. Third, while their models apply to public disclosure of insider trades, our model is much more suitable to analyze the *private* learning of order flow information by proprietary firms such as HFTs. Fourth, and finally, while our model remains stylized, it is more general than earlier models so that we have more confidence in calibrating it. The model-implied profits of institutional investors are fairly close to those from mutual fund studies, which, in turn, suggests that our estimation of back-runners' profits are not entirely off target.

Also related to our paper, Madrigal (1996) considers a two-period Kyle (1985) model with an insider and a "(non-fundamental) speculator." Madrigal's equilibrium analysis contains some errors and misses the mixed strategy equilibrium, so we refer readers to Yang and Zhu (2017) for a full discussion.

An earlier literature explores information about liquidity-driven order flows, including Brunnermeier and Pedersen (2005), Attari, Mello, and Ruckes (2005), Cao, Evans, and Lyons (2006), Carlin, Lobo, and Viswanathan (2007), and Bernhardt and Taub (2008). Our model differs from them in two ways: (1) the relevant information is about asset fundamentals, not liquidity needs, and (2) order flow information is learned over time, not endowed instantly. More recently, in partial equilibrium setting, Boulatov, Bernhardt, and Larionov (2016) study the strategic interaction among a large liquidity trader and several "parasitic traders" when the price impact is exogenously given.

In terms of its applications, our paper is most related to the recent theoretical literature on HFTs. Biais, Foucault, and Moinas (2015) model HFTs as agents who have a higher probability of finding a trade and who have information about the asset's fundamental value. Foucault, Hombert, and Roşu (2016) model HFTs who continually receive proprietary information about the innovations (or news) in the asset's fundamental value. Roşu (2018) develops a model in which traders differ in their information processing speed and shows that the order flow of fast traders predicts the order flow of slow traders (anticipatory trading). Jovanovic and Menkveld (2012) show that informed HFTs can alleviate adverse selection and restore trades. Hoffmann (2014) show that HFTs' ability to react to information fast reduce their own risk of being picked off but may have the opposite effect on slow traders. Budish, Cramton, and Shim (2015) argue that HFTs' ability to "snipe" stale quote is a major concern for the design of exchanges. Baldauf and Mollner (2018) show that liquidity-providing HFTs and quote-sniping HFTs respond differently to the same publicly observed transaction. Cespa and Vives (2016) model a market in which agents' heterogeneous speeds creates market instability.

Li (2014) models the "front-running" behavior of multiple HFTs who observe the aggregate order flow ex ante with noise. Relative to these studies and many others (see Menkveld 2016 for a comprehensive survey), back-runners in our model are not as informed as fundamental investors, but back-runners can collect information from fundamental investors' trading behavior. It is the separation between fundamental information and order flow information that gives rise to the interesting interactions and implications derived from our model.

1. A Model of Back-Running

We consider a variant of the two-period Kyle (1985) model with one risky asset. The risky asset has a liquidation value given by a random variable

$$v = p_0 + \sum_{i=1}^{I} f_i = p_0 + f_1 + ... + f_I, \tag{1}$$

where p_0 is a commonly known constant, each f_i is normally distributed with mean 0 and variance $\sigma_f^2 > 0$, and $\{f_1, ..., f_I\}$ are mutually independent. Such an additive payoff structure is used in, for example, Back, Cao, and Willard (2000), Bernhardt and Miao (2004), Brunnermeier (2005), Goldman (2005), Yuan (2005), Kondor (2012), and Goldstein and Yang (2015), among others.[4] For convenience, we write $\Sigma_0 \equiv Var(v) = I\sigma_f^2$.

The market is populated by four types of players: $I \geq 1$ fundamental investors, $J \geq 1$ back-runners, a representative competitive market-maker, and noise traders. Everyone is risk neutral.

At the beginning of period 1, fundamental investor i observes the private signal f_i. She places market orders $x_{1,i}$ and $x_{2,i}$ in periods 1 and 2, respectively. Let us denote the aggregate period-1 order flow of all fundamental investors as

$$X_1 \equiv \sum_{i=1}^{I} x_{1,i}. \tag{2}$$

At the beginning of period 2, back-runner j observes a private signal of the collective trade of fundamental investors in period 1:

$$s_j = X_1 + \varepsilon_j, \tag{3}$$

where

$$\varepsilon_j \sim N\left(0, \sigma_\varepsilon^2\right) \text{ with } \sigma_\varepsilon \in [0, \infty), \tag{4}$$

where ε_j is independent of each other and of all other random variables. The parameter σ_ε captures the accuracy of back-runners' signals (a smaller σ_ε means

4 As pointed out by Paul (1993, p. 1477), this additive structure of information "is in the spirit of Hayek's view that one of the most important functions of the price system is the decentralized aggregation of information and that no one person or institution can process all information relevant to pricing." In practical terms, the additive structure of information captures that different institutions have expertise in different styles or industries (see Goldstein and Yang 2015 for more discussions).

more accurate signals). We allow the degenerate case of $\sigma_\varepsilon = 0$, that is, the back-runners observe X_1 perfectly. After observing the private signal s_j and period-1 price p_1, back-runner j places a market order $d_{2,j}$ in period 2.

The total order flows from noise traders in period 1 and period 2 are, respectively,

$$u_1 \sim N\left(0, \sigma_u^2\right), \text{ with } \sigma_u > 0, \tag{5}$$

$$U_2 \sim N\left(0, \sigma_U^2\right), \text{ with } \sigma_U > 0. \tag{6}$$

The potential heterogeneity in the noise traders' variances in the two periods is a reduced-form way to model the relative length of the two periods. For example, if period 2 is much longer than period 1, we would expect $\sigma_U >> \sigma_u$.

The aggregate order flows in the two periods are, respectively,

$$y_1 = \sum_{i=1}^{I} x_{1,i} + u_1, \tag{7}$$

$$y_2 = \sum_{i=1}^{I} x_{2,i} + \sum_{j=1}^{J} d_{2,j} + U_2. \tag{8}$$

At the end of period t, after observing the total order flow y_t for that period, the market-maker sets price p_t according to the weak-efficiency rule, that is,

$$p_1 = E(v|y_1) \text{ and } p_2 = E(v|y_1, y_2). \tag{9}$$

2. Equilibrium

We look for a perfect Bayesian equilibrium, in which the I fundamental investors and the J back-runners choose their trading strategies to maximize expected profits. The market maker's strategy is pinned down by the weak-efficiency rule.

We conjecture the following symmetric linear strategies:

$$x_{1,i} = \beta_1 f_i + z_i, \text{ with } z_i \sim N\left(0, \sigma_z^2\right), \tag{10}$$

$$x_{2,i} = \beta_2 [f_i - E(f_i|y_1)] - \beta_x \left[x_{1,i} - E\left(x_{1,i}|y_1\right)\right], \tag{11}$$

$$d_{2,j} = \delta\left[s_j - E\left(s_j|y_1\right)\right], \tag{12}$$

$$p_1 = p_0 + \lambda_1 y_1, \tag{13}$$

$$p_2 = p_1 + \lambda_2 y_2. \tag{14}$$

These strategies are symmetric in the sense that the I fundamental investors choose identical strategies and the J back-runners choose identical strategies. That is, the parameters β_1, β_2, β_x, and σ_z do not depend on i in (10) and (11), and the parameter δ does not depend on j in (12). The form of Equations (10)–(12) is motivated by Bernhardt and Miao (2004), who specify that the trading strategy of an informed agent is a linear function of each piece of his unrevealed private information, that is, the difference between each signal of an informed agent and the expectation of that signal given public information. In Equation (10), we allow fundamental investors to play a mixed strategy in period 1 by adding a random noise term z_i. We can show that fundamental investors and back-runners always play pure strategies in period 2 in a symmetric linear equilibrium and, thus, we do not include a random noise term in Equations (11) and (12).

We have followed Huddart, Hughes, and Levine (2001) and restricted attention to normally distributed z_i in order to maintain tractability. If $\sigma_z=0$, fundamental investors play a pure strategy in period 1, and we refer to the resultant linear equilibrium as a *pure strategy equilibrium*. If $\sigma_z>0$, fundamental investors play a mixed strategy in period 1, and we refer to the resultant linear equilibrium as a *mixed strategy equilibrium*. As we will show shortly, by adding noise into their orders, the fundamental investors limit the back-runners' ability to infer their private information about v. To an outside observer, the endogenously added noise z_i may look like exogenous noise trading.

As usual, Equations (13) and (14) simply say that the equilibrium pricing rule is a linear function of net order flows.

2.1 Main derivation steps

2.1.1 Market maker's decisions. In period 1, the market maker sees the aggregate order flow y_1 and sets $p_1=E(v|y_1)$. Using the conjectured trading strategies and the projection theorem, we can compute

$$\lambda_1=\frac{Cov(v,y_1)}{Var(y_1)}=\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}. \tag{15}$$

Similarly, in period 2, the market maker sees $\{y_1,y_2\}$ and sets $p_2=E(v|y_1,y_2)$, which implies that $\lambda_2=\frac{Cov(v,y_2|y_1)}{Var(y_2|y_1)}$. By the conjectured trading strategies and applying the projection theorem, we have

$$\lambda_2=\frac{(\beta_2-\beta_x\beta_1+\delta J\beta_1)\Sigma_0-\frac{\beta_1\Sigma_0\left[(\beta_2-\beta_x\beta_1+\delta J\beta_1)\beta_1\Sigma_0+(\delta J-\beta_x)I\sigma_z^2\right]}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}}{\left[\begin{array}{c}(\beta_2-\beta_x\beta_1+\delta J\beta_1)^2\Sigma_0+(\delta J-\beta_x)^2I\sigma_z^2+\delta^2J\sigma_\varepsilon^2+\sigma_U^2\\-\frac{\left[(\beta_2-\beta_x\beta_1+\delta J\beta_1)\beta_1\Sigma_0+(\delta J-\beta_x)I\sigma_z^2\right]^2}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}\end{array}\right]}. \tag{16}$$

2.1.2 Back-runners' decisions. $\pi_{2,j}^B=d_{2,j}(v-p_2)$ denotes back-runner j's profit that comes from her trade $d_{2,j}$ in period 2. Back-runner j observes $\{s_j,p_1\}$ and chooses $d_{2,j}$ to maximize $E(\pi_{2,j}^B|s_j,p_1)$. Using the conjectured trading strategies and the pricing function (14), we can compute the first-order condition (FOC), which delivers

$$d_{2,j}=\frac{E\left(v-p_1|s_j,p_1\right)}{2\lambda_2}-\frac{E\left(\sum_i x_{2,i}+\sum_{j'\neq j}d_{2,j'}\,|\,s_j,p_1\right)}{2}. \tag{17}$$

The second-order condition (SOC) is

$$\lambda_2>0. \tag{18}$$

Note that by (13), the information set $\{s_j,p_1\}$ is equivalent to $\{s_j,y_1\}$. Using this fact and the conjectured trading strategies, we apply the projection theorem to show that both $E\left(v-p_1|s_j,p_1\right)$ and $E\left(\sum_i x_{2,i}+\sum_{j'\neq j}d_{2,j'}|s_j,p_1\right)$ are linear functions of $s_j-E\left(s_j|y_1\right)$. Inserting these linear functions into (17), we express $d_{2,j}$ as a linear function of $s_j-E\left(s_j|y_1\right)$. Finally, we compare this expression with the conjectured strategy (12) to arrive at the following equation:

$$\delta=\frac{1}{2}\left[\begin{array}{c}\frac{1}{\lambda_2}\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2}\\ -\left(\beta_2\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2}-\beta_x\right)-(J-1)\delta\end{array}\right]\frac{\sigma_\varepsilon^{-2}}{\left(\beta_1^2\Sigma_0+I\sigma_z^2\right)^{-1}+\sigma_\varepsilon^{-2}+\sigma_u^{-2}}. \tag{19}$$

2.1.3 Fundamental investors' problems. Fundamental investors trade in both periods. We solve their problems by backward induction. Let $\pi^F_{t,i}=x_{t,i}(v-p_t)$ denote fundamental investor i's profit that comes from her period-t trade $x_{t,i}$. In period 2, fundamental investor i chooses $x_{2,i}$ to maximize $E(\pi^F_{2,i}|f_i,p_1,x_{1,i})$. Taking the FOC results in the following solution:

$$x_{2,i}=\frac{E\left(v-p_1|f_i,p_1,x_{1,i}\right)}{2\lambda_2}-\frac{E\left(\sum_{i'\neq i}x_{2,i'}+\sum_j d_{2,j}|f_i,p_1,x_{1,i}\right)}{2}. \tag{20}$$

The SOC is still $\lambda_2>0$, as given by (18). Applying the projection theorem, we can express $E\left(v-p_1|f_i,p_1,x_{1,i}\right)$ and $E\left[\sum_{i'\neq i}x_{2,i'}+\sum_j d_{2,j}|f_i,p_1,x_{1,i}\right]$ as linear functions of $f_i-E(f_i|y_1)$ and $x_{1,i}-E\left(x_{1,i}|y_1\right)$. Inserting these expressions into (20) and comparing with the conjectured strategy (11), we have

$$\beta_2 = \frac{1}{2\lambda_2}, \tag{21}$$

$$\beta_x = \frac{1}{2}\left[\begin{array}{c}\frac{1}{\lambda_2}\frac{\beta_1\frac{I-1}{I}\Sigma_0}{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}\\ +(I-1)\frac{-\beta_2\beta_1\frac{\Sigma_0}{I}+\beta_x\left(\beta_1^2\frac{\Sigma_0}{I}+\sigma_z^2\right)}{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}+J\delta\frac{\sigma_u^2}{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}\end{array}\right]. \tag{22}$$

In period 1, fundamental investor i observes f_i and chooses $x_{1,i}$ to maximize $E(\pi^F_{1,i}+\pi^F_{2,i}|f_i)$, where the second-period profit $\pi^F_{2,i}$ is generated from the optimal trading strategy (11). Direct computation shows

$$E\left(\pi_{F,1}+\pi_{F,2}|f_i\right)$$
$$=-\left[\lambda_1-\lambda_2\left(\beta_x\frac{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}\right)^2\right]x_{1,i}^2$$
$$+\left[1-2\lambda_2\beta_2\left(\beta_x\frac{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}\right)\right]f_i x_{1,i}$$
$$+\lambda_2\left[\beta_2^2 f_i^2+\left(\begin{array}{c}\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}\\ -\beta_x\frac{\beta_1^2\frac{1}{I}\Sigma_0+\sigma_z^2}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}\end{array}\right)^2\left(\begin{array}{c}\beta_1^2\frac{I-1}{I}\Sigma_0\\ +(I-1)\sigma_z^2+\sigma_u^2\end{array}\right)\right]. \quad (23)$$

Depending on whether fundamental investors play a mixed or a pure strategy (i.e., whether σ_z is equal to 0), we have two cases:

Case 1. Mixed Strategy ($\sigma_z > 0$)

For a mixed strategy to sustain in equilibrium, fundamental investors have to be indifferent among all realizations of their order flows. This, in turn, means that coefficients on $x_{1,i}^2$ and $x_{1,i}$ in (23) are equal to zero, that is,

$$\lambda_1-\lambda_2\left(\beta_x\frac{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}\right)^2=0, \quad (24)$$

$$1-2\lambda_2\beta_2\left(\beta_x\frac{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}\right)=0. \quad (25)$$

Case 2. Pure Strategy ($\sigma_z = 0$)

If fundamental investors play a pure strategy, $z_i = 0$ (and $\sigma_z = 0$) in the conjectured strategy, and thus (10) degenerates to $x_{1,i} = \beta_1 f_i$. We take the FOC of (23) and solve $x_{1,i}$ as a linear function of f_i, which, compared with the conjectured pure strategy $x_{1,i} = \beta_1 f_i$, implies

$$\beta_1=\frac{1-2\lambda_2\beta_2\left(\beta_x\frac{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}\right)}{2\left[\lambda_1-\lambda_2\left(\beta_x\frac{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}\right)^2\right]}. \quad (26)$$

The SOC is

$$\lambda_1-\lambda_2\left(\beta_x\frac{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}\right)^2>0. \quad (27)$$

In sum, a mixed strategy equilibrium is defined in terms of seven unknowns: β_1, σ_z, β_2, β_x, δ, λ_1, and λ_2. They are characterized by seven equations, (15), (16), (19), (21), (22), (24), and (25), together with one SOC, (18).

In a pure strategy equilibrium, we have $\sigma_z=0$; thus, it is defined in terms of six unknowns: $\beta_1,\beta_2,\beta_x,\delta,\lambda_1$, and λ_2. These six unknowns are determined by six equations, (15), (16), (19), (21), (22), and (26), together with two SOCs, (18) and (27).

2.2 Mixed strategy and pure strategy equilibria

After further simplification, we can characterize a mixed strategy equilibrium and a pure strategy equilibrium in more parsimonious forms. That said, the mathematical expressions are still involved and may not appear intuitive at first sight. We will postpone the discussion of the intuition to Section 3 (the special case of $I=1$).

Proposition 1 (Mixed strategy equilibrium). A mixed strategy equilibrium is characterized by the following system of equations in three unknowns $(\delta,\beta_1,\sigma_z)$:

$$\delta=\frac{\left(\beta_1^2\Sigma_0 I-\beta_1^2\Sigma_0+I^2\sigma_z^2-I\sigma_z^2+3I\sigma_u^2-\sigma_u^2\right)\left(\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2\right)}{2J\sigma_u^2\left(\beta_1^2\Sigma_0 I-\beta_1^2\Sigma_0+I^2\sigma_z^2-I\sigma_z^2+I\sigma_u^2\right)}, \tag{28}$$

$$\delta=\frac{\frac{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}{2\left(\beta_1^2\Sigma_0+I\sigma_z^2\right)}+\frac{I-1}{2I}\frac{\left(\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2\right)}{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+2\sigma_u^2}}{2\frac{\left(\beta_1^2\Sigma_0+I\sigma_z^2\right)^{-1}+\sigma_\varepsilon^{-2}+\sigma_u^{-2}}{\sigma_\varepsilon^{-2}}-\frac{J\sigma_u^2}{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+2\sigma_u^2}+(J-1)}, \tag{29}$$

$$\delta^2 J\sigma_\varepsilon^2+\sigma_U^2=\frac{\left(\sigma_u^2-\beta_1^2\Sigma_0\right)\left(\sigma_u^2+\beta_1^2\Sigma_0+\sigma_z^2 I\right)^2}{4\beta_1^2\Sigma_0\sigma_u^2}, \tag{30}$$

where $\beta_1\in\left(0,\frac{\sigma_u}{\sqrt{\Sigma_0}}\right)$.

After we solve $(\delta,\beta_1,\sigma_z)$, the other variables are given by

$$\lambda_1 = \lambda_2=\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}, \tag{31}$$

$$\beta_x = \frac{\frac{1}{2\lambda_2}\beta_1\frac{I-1}{I}\Sigma_0+J\delta\sigma_u^2}{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+2\sigma_u^2}, \tag{32}$$

$$\beta_2 = \frac{1}{2\lambda_2}. \tag{33}$$

The price discovery variables are

$$\Sigma_1=Var(v|y_1)=\frac{\left(I\sigma_z^2+\sigma_u^2\right)\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}, \tag{34}$$

$$
\begin{aligned}
\Sigma_2 &= Var(v|y_1, y_2) \\
&= \Sigma_0\left[\left(IJ^2\delta^2\sigma_u^2 + IJ\delta^2\sigma_\varepsilon^2 - 2IJ\delta\sigma_u^2\beta_x + I\sigma_U^2 + I\sigma_u^2\beta_x^2\right)\sigma_z^2\right. \\
&\quad \left. + \left(J\delta^2\sigma_u^2\sigma_\varepsilon^2 + \sigma_U^2\sigma_u^2\right)\right] \\
&\quad \times \left[\left(IJ^2\delta^2\sigma_u^2 + IJ\delta^2\sigma_\varepsilon^2 - 2IJ\delta\sigma_u^2\beta_x + \Sigma_0 I\beta_2^2 + I\sigma_U^2 + I\sigma_u^2\beta_x^2\right)\sigma_z^2\right. \\
&\quad + \Sigma_0 J^2\delta^2\beta_1^2\sigma_u^2 + \Sigma_0 J\delta^2\beta_1^2\sigma_\varepsilon^2 + J\delta^2\sigma_u^2\sigma_\varepsilon^2 \\
&\quad - 2\Sigma_0 J\delta\beta_1^2\sigma_u^2\beta_x + 2\Sigma_0 J\delta\beta_1\beta_2\sigma_u^2 + \Sigma_0\beta_1^2\sigma_U^2 \\
&\quad \left. + \Sigma_0\beta_1^2\sigma_u^2\beta_x^2 - 2\Sigma_0\beta_1\beta_2\sigma_u^2\beta_x + \Sigma_0\beta_2^2\sigma_u^2 + \sigma_U^2\sigma_u^2\right]^{-1}.
\end{aligned} \tag{35}
$$

The expected profits of each fundamental investor and each back-runner are, respectively,

$$
E(\Pi_i^F) = E(\pi_{1,i}^F + \pi_{2,i}^F) \tag{36}
$$

$$
= \lambda_2\left[\beta_2^2\frac{\Sigma_0}{I} + \begin{pmatrix} \beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0 + I\sigma_z^2 + \sigma_u^2} \\ -\beta_x\frac{\beta_1^2\frac{1}{I}\Sigma_0 + \sigma_z^2}{\beta_1^2\Sigma_0 + I\sigma_z^2 + \sigma_u^2}\end{pmatrix}^2\left(\beta_1^2\frac{I-1}{I}\Sigma_0 + (I-1)\sigma_z^2 + \sigma_u^2\right)\right], \tag{37}
$$

$$
E\left(\pi_{2,j}^B\right) = \lambda_2\delta^2\left(\sigma_\varepsilon^2 + \frac{\Sigma_X\sigma_u^2}{\Sigma_X + \sigma_u^2}\right), \text{ with } \Sigma_X = \beta_1^2\Sigma_0 + I\sigma_z^2, \tag{38}
$$

and the expected loss of noise traders is

$$
\lambda_1\sigma_u^2 + \lambda_2\sigma_U^2. \tag{39}
$$

Proposition 2 (Pure strategy equilibrium). A pure strategy equilibrium is characterized by the following system in two unknowns $(\beta_1, \lambda_2) \in \mathbb{R}_{++}^2$:

$$
\lambda_2 = \frac{(\beta_2 - \beta_x\beta_1 + \delta J\beta_1)\frac{\Sigma_0\sigma_u^2}{\beta_1^2\Sigma_0 + \sigma_u^2}}{(\beta_2 - \beta_x\beta_1 + \delta J\beta_1)^2\frac{\Sigma_0\sigma_u^2}{\beta_1^2\Sigma_0 + \sigma_u^2} + \delta^2 J\sigma_\varepsilon^2 + \sigma_U^2}, \tag{40}
$$

$$\beta_1 = \frac{1-\left(\beta_x \frac{\beta_1^2 \frac{I-1}{I}\Sigma_0+\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)}{2\left[\lambda_1-\lambda_2\left(\beta_x \frac{\beta_1^2 \frac{I-1}{I}\Sigma_0+\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)^2\right]}, \tag{41}$$

where

$$\beta_2 = \frac{1}{2\lambda_2}, \tag{42}$$

$$\delta = \frac{\frac{1}{2\lambda_2}\left(\frac{1}{\beta_1}+\frac{\beta_1\frac{I-1}{I}\Sigma_0}{\beta_1^2\frac{I-1}{I}\Sigma_0+2\sigma_u^2}\right)}{2\frac{\left(\beta_1^2\Sigma_0\right)^{-1}+\sigma_\varepsilon^{-2}+\sigma_u^{-2}}{\sigma_\varepsilon^{-2}}-\frac{J\sigma_u^2}{\beta_1^2\frac{I-1}{I}\Sigma_0+2\sigma_u^2}+(J-1)}, \tag{43}$$

$$\beta_x = \frac{\frac{1}{2\lambda_2}\beta_1\frac{I-1}{I}\Sigma_0+J\delta\sigma_u^2}{\beta_1^2\frac{I-1}{I}\Sigma_0+2\sigma_u^2}, \tag{44}$$

$$\lambda_1 = \frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}, \tag{45}$$

and one SOC:

$$\lambda_1-\lambda_2\left(\beta_x \frac{\beta_1^2 \frac{I-1}{I}\Sigma_0+\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)^2>0. \tag{46}$$

The price discovery variables are

$$\Sigma_1 = Var(v|y_1)=\frac{\sigma_u^2\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}, \tag{47}$$

$$\begin{aligned}\Sigma_2 &= Var(v|y_1,y_2)\\ &= \Sigma_0\frac{J\delta^2\sigma_u^2\sigma_\varepsilon^2+\sigma_U^2\sigma_u^2}{\left[\begin{array}{c}\Sigma_0J^2\delta^2\beta_1^2\sigma_u^2+\Sigma_0J\delta^2\beta_1^2\sigma_\varepsilon^2+J\delta^2\sigma_u^2\sigma_\varepsilon^2\\ -2\Sigma_0J\delta\beta_1^2\sigma_u^2\beta_x+2\Sigma_0J\delta\beta_1\beta_2\sigma_u^2+\Sigma_0\beta_1^2\sigma_U^2\\ +\Sigma_0\beta_1^2\sigma_u^2\beta_x^2-2\Sigma_0\beta_1\beta_2\sigma_u^2\beta_x+\Sigma_0\beta_2^2\sigma_u^2+\sigma_U^2\sigma_u^2\end{array}\right]}.\end{aligned} \tag{48}$$

The expected profits of each fundamental investor and each back-runner are, respectively,

$$\begin{aligned}E\left(\Pi_i^F\right)&=E(\pi_{1,i}^F+\pi_{2,i}^F)\\ &=\left[\lambda_1-\lambda_2\left(\beta_x \frac{\beta_1^2 \frac{I-1}{I}\Sigma_0+\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)^2\right]\beta_1^2\frac{\Sigma_0}{I}\end{aligned} \tag{49}$$

$$+\lambda_2\left[\beta_2^2\frac{\Sigma_0}{I}+\left(\beta_2\frac{\beta_1\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}-\beta_x\frac{\beta_1^2\frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)^2\left(\beta_1^2\frac{I-1}{I}\Sigma_0+\sigma_u^2\right)\right],$$

$$E\left(\pi_{2,j}^B\right)=\lambda_2\delta^2\left(\sigma_\varepsilon^2+\frac{\Sigma_X\sigma_u^2}{\Sigma_X+\sigma_u^2}\right),\text{ with }\Sigma_X=\beta_1^2\Sigma_0, \tag{50}$$

and the total expected loss of noise traders is

$$\lambda_1\sigma_u^2+\lambda_2\sigma_U^2. \tag{51}$$

2.3 Baseline parameters for numerical comparative statics

The model can be numerically solved for any given set of parameters: I, J, p_0, $\Sigma_0, \sigma_u, \sigma_U$, and σ_ε. To help intuition, we interpret the full model as a trading day, with each period corresponding to half a day. The traded asset is interpreted as a typical stock in the U.S. equity market. In numerically calculating the equilibrium, we set the baseline parameters as follows:

- p_0 and Σ_0. Because $\Sigma_0=Var[v]=p_0^2Var[v/p_0]$, we normalize $p_0=1$ and set Σ_0 to be the daily stock return variance. According to the CBOE,[5] the VIX index has a daily average of about 18.5 from January 2004 to February 2018, corresponding to an annualized volatility of 18.5% for the S&P 500 index. Because a typical stock is more volatile than a stock index, we set the baseline parameter to be an annualized return volatility of 30%. Hence, we set

$$\Sigma_0=\frac{(30\%)^2}{252}=0.00036, \tag{52}$$

corresponding to the volatility of $\sqrt{0.00036}=1.9\%$ per day, very close to the calibration of Kyle and Obizhaeva (2016) that a typical stock has a daily return volatility of 2%.
- σ_u and σ_U. We interpret the noise traders in the model as retail investors, and set $\sigma_u=\sigma_U=1$ million shares. Given the normalized price of $p_0=1$, this implies a daily dollar volume of retail investors of

$$E(|u_1p_1|+|U_2p_2|)\approx E(|u_1|+|U_2|)p_0$$
$$=\sqrt{\frac{2}{\pi}}\times 2\times 1=1.6 \text{ millions of dollars}, \tag{53}$$

where the first step ("$\approx$") relies on the approximation that p_1 and p_2 are close to p_0 because the daily stock volatility is small. The implied daily

[5] See http://www.cboe.com/products/vix-index-volatility/vix-options-and-futures/vix-index/vix-historical-data.

retail volume of \$1.6 million corresponds to a typical medium-cap stock in U.S. equity markets.[6]

- I and J. van Kervel and Menkveld (2019) investigate the behavior of high-frequency traders around large institutional investors' order execution on Nasdaq OMX. Their data contain 4 institutions and 10 high-frequency traders. Because they observe the activities of all HFTs on trade reports, setting $J=10$ seems reasonable. However, the actual number of institutional investors is likely to be higher than 4. We thus consider a few possibilities, $I=4$, $I=100$, and $I=10,000$. In our symmetric model, each fundamental investor's information is $1/I$th of the total private information in terms of variance.

  One way to interpret the choice of I is to map it to the predictability of v based on f_i. In our model, the correlation between each fundamental investor's signal f_i and the fundamental value v is

$$Corr(f_i,v)=\frac{Cov(f_i,v)}{\sqrt{Var[f_i]Var[v]}}=\frac{1}{\sqrt{I}}. \tag{54}$$

  Under the assumption of $I=4$, $I=100$, and $I=10,000$, the correlation is 0.5, 0.1, and 0.01, respectively, corresponding to an R^2 in a univariate regression of v on f_i of 0.25, 0.01, and 0.0001 at the daily frequency. Thus, $I=4$ seems too low, and $I=100$ or $I=10,000$ seems more seasonable.

  We can also judge how reasonable the baseline calibrations of I are by the "alpha" (or risk-adjusted return) generated by fundamental investors. Each individual fundamental investor's signal has a daily volatility of $\sqrt{\Sigma_0/I}$, and the daily alpha generated should be in the same order of magnitude.[7] Under $\sqrt{\Sigma}=1.9\%$ and $I=10,000$, the alpha is roughly 1.9 bps per day, or 4.8% per year, assuming that each fundamental investor trades every day of the year. Alternatively, if $I=100$ and each fundamental investor trades only once per month, the annual alpha is in the order of $1.9\%/10\times 12\approx 2.3\%$ per year, which is also reasonable.
- The only remaining parameter, σ_ε, is difficult to observe and is a key variable that determines the nature of the equilibrium. Thus, we primarily explore the variation in σ_ε in subsequent analysis.

Figure 1 shows equilibrium outcomes as functions of σ_ε (equivalently, as functions of $\sigma_\varepsilon/\sigma_u$, because $\sigma_u=1$), for various values of I. Looking across

[6] Using 1 month of disaggregated data in various type of trading venues in U.S. equity markets, Menkveld, Yueshen, and Zhu (2017) report that the average retail trading volume of large-, medium-, and small-cap stocks is 3,783 shares, 133 shares, and 70 shares per minute. Converted to daily frequency, the three categories have daily share volume of 145.78 million shares, 51,870 shares, and 27,300 shares, respectively. If a typical stock has a price of \$30, then a typical medium-cap stock would have a daily dollar volume of \$1.56 million.

[7] A fully informed investor who buys one share if $v-p_0>0$ and sells one share otherwise makes a return proportional to the volatility of v, that is, $E(|v-p_0|/p_0)\propto\sqrt{Var(v/p_0)}$, which is just $\sqrt{Var(v)}$ under the normalization of $p_0=1$.

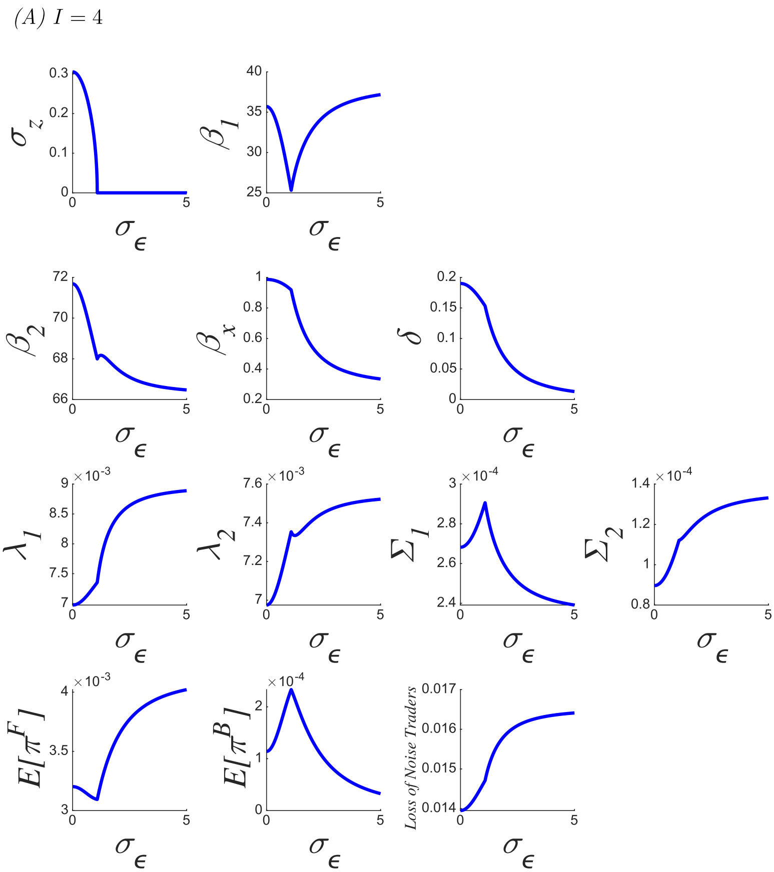

Figure 1
Comparative statics of the model
Parameters: $\Sigma_0 = 0.3^2/252$, $\sigma_u = \sigma_U = 1$, and $J = 10$. The three panels show $I = 4$, $I = 100$, and $I = 10{,}000$, respectively.

the three panels (as well as many others that we have calculated, but have not reported), we observe that the shape of equilibrium does not depend critically on I. Moreover, the other equilibrium outcome variables are also relatively insensitive to I, with the exception of σ_z and $E(\Pi^F)$. As I increases, σ_z seems to decrease at the rate $\sqrt{1/I}$, whereas $E(\Pi^F)$ seems to decrease at the rate $1/I$. The insensitivity to I of the equilibrium probably has to do with the model setup that each fundamental investor observes an *iid* piece of information and acts effectively as a monopolist on her piece of private information.[8]

[8] As a comparison, Holden and Subrahmanyam (1992) show that if multiple informed traders have *identical* information about the fundamental value, the competition produces very aggressive trading that the information is revealed very quickly. This does not happen in our setting, because, again, each fundamental investor has

(B) $I = 100$

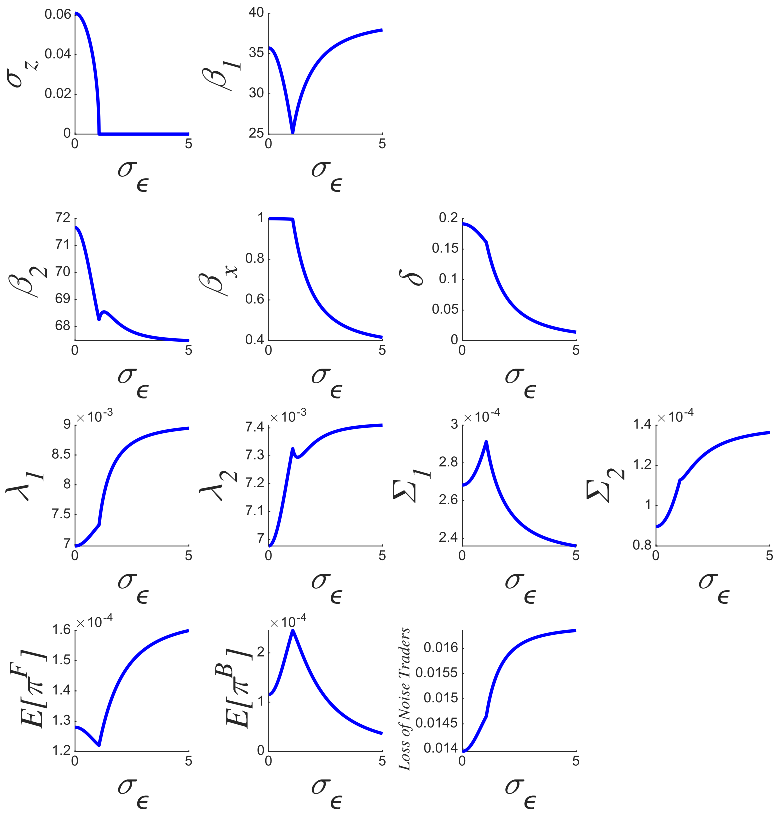

Figure 1
(*Continued*)

In all panels of Figure 1, a mixed strategy equilibrium obtains if σ_ε is below about 1 (million shares). This condition implies a fairly inaccurate signal of back-runners because the total volume of noise traders is about 1.6 (million shares per day). In other words, the condition for the existence of a mixed strategy equilibrium is not too stringent.

Further analytical results and sharper intuition can be obtained in the special case of a monopolist fundamental investor, that is, $I = 1$. We turn to this case in

---

a unique piece of information. Back, Cao, and Willard (2000) show that if multiple informed traders receive imperfectly correlated signals about the fundamental value, then the "common" part of the signals is revealed very quickly (like in Holden and Subrahmanyam 1992), but the remaining (idiosyncratic) parts are revealed very slowly. Because of the analytical challenge, we have not solved the case with correlated signals (i.e., f_i and f_j are correlated), but the insight from Back, Cao, and Willard (2000) suggests that our results on mixed strategies would still apply whenever fundamental investors are trading on their unique pieces of information, after revealing the common part, if any.

(C) $I = 10{,}000$

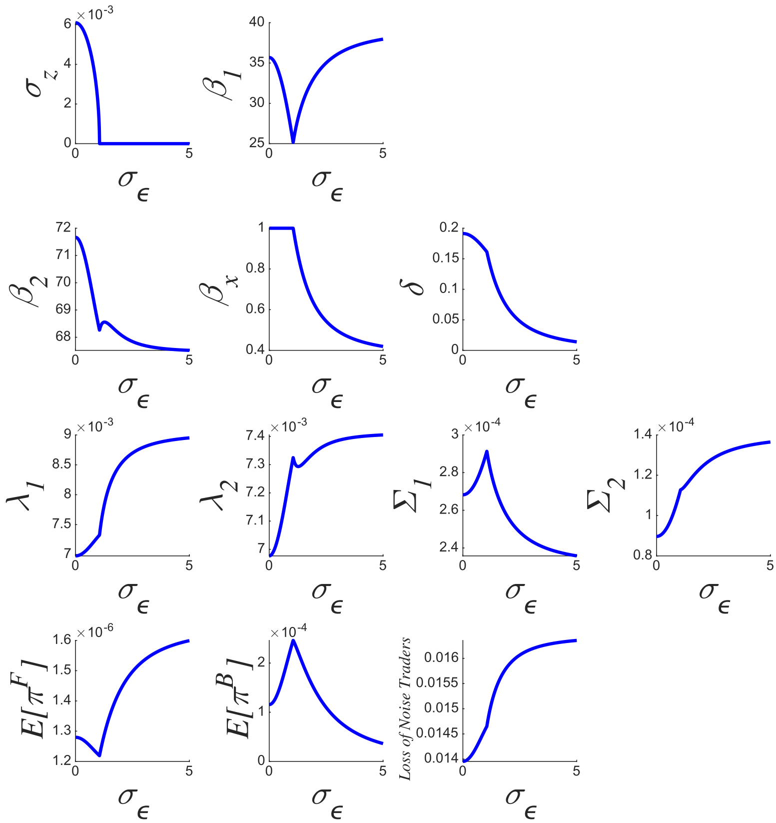

Figure 1
(*Continued*)

the next section. Again, in light of Figure 1, the special case of $I=1$ does not appear particularly restrictive for the vast majority of equilibrium outcomes. (As we will show later, the equilibrium outcomes are more sensitive to J.)

3. Special Case of a Monopolist Fundamental Investor

The equilibrium characterization is not in closed form in the previous section. To gain further intuition, in this section we consider the special case of $I=1$, that is, a monopolist fundamental investor. In addition, our comparative statics analysis reveals that it is the number J of back-runners rather than the number I of fundamental investors that drives the patterns of variables. Thus, focusing on the special case of $I=1$ allows us to deliver our qualitative results most parsimoniously.

Under $I = 1$, we can show that the equilibrium is ultimately characterized by three parameters:

$$J \quad : \quad \text{number of back-runners} \tag{55}$$

$$\theta \equiv \frac{\sigma_\varepsilon^2}{\sigma_u^2} : \text{accuracy of back-runners' information} \tag{56}$$

$$\eta \equiv \frac{\sigma_U^2}{\sigma_u^2} : \text{relative size of markets over the two periods} \tag{57}$$

3.1 Mixed strategy equilibrium

Proposition 3. Suppose $I = 1$. A linear mixed strategy equilibrium exists if and only if

$$(J - 4\theta)\left[\frac{4\theta}{J} + \eta\left(\frac{J+2+4\theta}{J+1}\right)^2\right] > 2(1+4\theta), \tag{58}$$

where $\theta = \sigma_\varepsilon^2/\sigma_u^2$ and $\eta = \sigma_U^2/\sigma_u^2$. If a mixed strategy equilibrium exists, it is the unique linear mixed strategy equilibrium, which is specified by equations (10)–(14) with

$$\sigma_z^2 = \left[\frac{J-4\theta}{J+2+4\theta} - \frac{1}{1+\frac{4\theta}{J}+\eta\left(\frac{J+2+4\theta}{J+1}\right)^2}\right]\sigma_u^2, \tag{59}$$

$$\beta_1 = \frac{1}{\sqrt{1+\frac{4\theta}{J}+\eta\left(\frac{J+2+4\theta}{J+1}\right)^2}}\frac{\sigma_u}{\sqrt{\Sigma_0}}, \tag{60}$$

$$\beta_2 = \frac{J+1}{J+2+4\theta}\sqrt{1+\frac{4\theta}{J}+\eta\left(\frac{J+2+4\theta}{J+1}\right)^2}\frac{\sigma_u}{\sqrt{\Sigma_0}}, \tag{61}$$

$$\beta_x = \frac{J+1}{J+2+4\theta}, \tag{62}$$

$$\delta = \frac{2(J+1)}{J(J+2+4\theta)}, \tag{63}$$

$$\lambda_1 = \lambda_2 = \frac{J+2+4\theta}{2(J+1)\sqrt{1+\frac{4\theta}{J}+\eta\left(\frac{J+2+4\theta}{J+1}\right)^2}}\frac{\sqrt{\Sigma_0}}{\sigma_u}. \tag{64}$$

The conditions under which the mixed strategy equilibrium exists can be spelled out more explicitly.

Corollary 1. Suppose $I = 1$. Then

- If J is sufficiently large, there always exists a unique linear mixed strategy equilibrium.
- For a fixed and sufficiently small σ_ε^2, as $\eta \equiv \frac{\sigma_U^2}{\sigma_u^2}$ becomes sufficiently large, there always exists a unique linear mixed strategy equilibrium.
- If $\sigma_\varepsilon = 0$, then there exists a linear mixed strategy equilibrium if and only if

$$J\left(\frac{J+2}{J+1}\right)^2 \eta > 2. \tag{65}$$

If a mixed strategy equilibrium exists, it is the unique linear mixed strategy equilibrium.

This corollary essentially says that a mixed strategy equilibrium is more likely to obtain if J or σ_U^2/σ_u^2 is sufficiently large, or if σ_ε^2 is sufficiently small. The intuition could be seen by considering the potential loss of the fundamental investor due to information leakage. If information is potentially leaked to more back-runners (larger J), the fundamental investor should be more cautious and uses a mixed strategy to reduce information leakage. Likewise, if period 2 of the economy has more noise traders, implying a higher potential profit from information, then the fundamental investor should be more cautious and use a mixed strategy. Finally, a small σ_ε^2 implies more precise order-flow information, which again encourages mixing by the fundamental investor.

Figure 2 illustrates the existence of a mixed strategy equilibrium for various values of J and $\theta \equiv \sigma_\varepsilon^2/\sigma_u^2$, fixing $I=1$ and $\sigma_U/\sigma_u = 1$. In general, a mixed strategy equilibrium exists if and only if the back-runners' signals are precise enough (small enough σ_ε). And this threshold is more likely to be satisfied if there are more back-runners. Unless J is very small, the condition in $\sigma_\varepsilon/\sigma_u$ for the existence of a mixed strategy equilibrium is not too stringent.

3.2 Pure strategy equilibrium

Likewise, we can characterize the pure strategy equilibrium as follows.

Proposition 4. Suppose $I=1$ and define $k \equiv \frac{\beta_1^2 \Sigma_0}{\sigma_u^2}$. A linear pure strategy equilibrium exists if and only if the following two conditions defined over $k \in (0,1]$ are satisfied:

- k solves a seventh-order polynomial:

$$f(x) = A_7 k^7 + A_6 k^6 + A_5 k^5 + A_4 k^4 + A_3 k^3 + A_2 k^2 + A_1 k_1 + A_0, \tag{66}$$

where the $A_0, A_1, \ldots, A_7$ coefficients are given by equations (A22)–(A29) in the appendix.

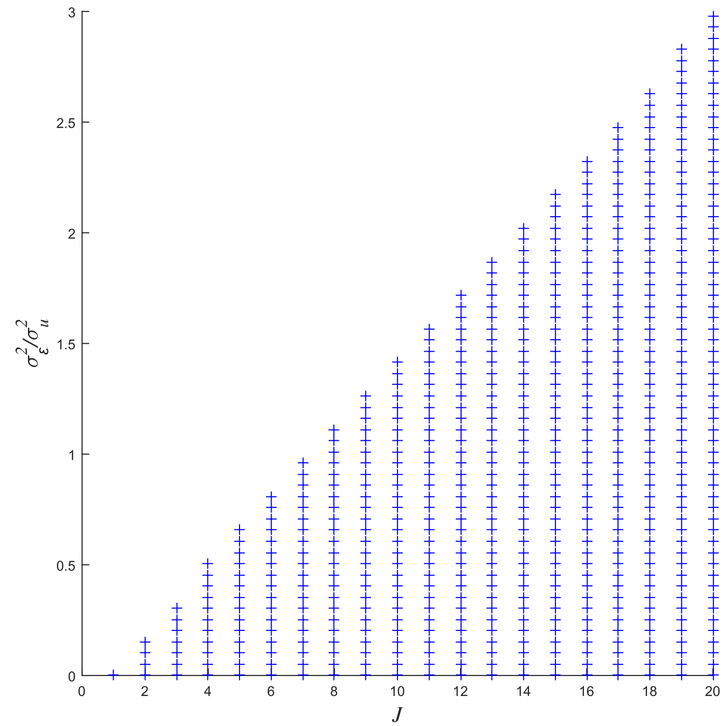

Figure 2
Mixed strategy region, denoted by "+"
Parameters: $I=1$ and $\sigma_U/\sigma_u=1$.

- The following SOC is satisfied:

$$\lambda_1-\lambda_2\left(\beta_x\frac{\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\beta_2\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)^2>0, \tag{67}$$

where

$$\beta_1 = \frac{\sqrt{k}\sigma_u}{\sqrt{\Sigma_0}}, \tag{68}$$

$$\lambda_1 = \frac{\sqrt{\Sigma_0}}{\sigma_u}\frac{\sqrt{k}}{1+k}, \tag{69}$$

$$\lambda_2 = \frac{\sqrt{\Sigma_0}}{\sigma_u}\sqrt{\frac{\left[\begin{array}{c}4k^2\theta^2+4k\theta+4\theta^2+Jk^2+8k\theta^2\\+4k^2\theta+k^2+Jk\theta+Jk^2\theta\end{array}\right]}{\eta(k+1)(2k+4\theta+4k\theta+Jk)^2}}, \tag{70}$$

$$\beta_2 = \frac{1}{2\lambda_2}, \tag{71}$$

$$\delta = \frac{1}{2\lambda_2} \frac{1}{2\left[\theta\left(\frac{1}{k}+1\right)+1\right]+\frac{J}{2}-1} \frac{\sqrt{\Sigma_0}}{\sqrt{k}\sigma_u}, \tag{72}$$

$$\beta_x = \frac{J}{2}\delta, \tag{73}$$

with $\theta = \sigma_\varepsilon^2/\sigma_u^2$ and $\eta = \sigma_U^2/\sigma_u^2$.

3.3 Switch of equilibrium

Proposition 5 (Mixed versus pure strategy equilibrium). Suppose $I=1$.

1. Fix $(\Sigma_0, \sigma_u, \sigma_U, J)$. Then
   (a). If σ_ε is sufficiently large, then there is no linear mixed strategy equilibrium, and there is a unique linear pure strategy equilibrium.
   (b). If σ_ε is sufficiently small and if $(J+1)\eta \geq 2$, then there is no linear pure strategy equilibrium, and there is a unique linear mixed strategy equilibrium.
2. Fix (Σ_0, σ_u) and set $\sigma_\varepsilon = 0$. Then
   (a). If both J and η are sufficiently small, then there is no linear mixed strategy equilibrium, and there is a unique linear pure strategy equilibrium.
   (b). If either J or η is sufficiently large, then there is no linear pure strategy equilibrium, and there is a unique linear mixed strategy equilibrium.

The intuition for Part 1(a) can be obtained by considering the extreme case of $\sigma_\varepsilon \to \infty$. This case degenerates to the familiar Kyle (1985) setting, in which mixing is not an optimal choice for the fundamental investor. The parametric condition in Part 1(b), $(J+1)\eta \geq 2$, is marginally more stringent than the condition in Corollary 1, $J\left(\frac{J+2}{J+1}\right)^2 \eta > 2$. Note that the parametric condition $(J+1)\eta \geq 2$ is satisfied by any J if $\sigma_U^2/\sigma_u^2 \geq 1$. In that case, as σ_ε approaches zero, the only possible equilibrium has mixed strategies.

The intuition for Part 2 of Proposition 5 could be obtained by asking how much profit the fundamental investor loses to back-runners if she uses a pure strategy. If J is large, leaking information to more back-runners is more costly. If η is large, then the profit in period 2 is too large to be compromised by information leakage. Either effect encourages the fundamental investor to add noise, leading to a mixed strategy equilibrium, like in part 2(b). Conversely, a pure strategy equilibrium obtains if both J and η are relatively small, like in part 2(a).

The conditions in Proposition 5 are mostly about very large or very small values of the parameters. It would be desirable to further tighten the parameter range for which only one case of equilibrium is obtained, but we have not been able to do so. Likewise, we have been unable to analytically prove that a pure strategy equilibrium and a mixed strategy equilibrium do not coexist for a given set of parameters, although numerically we have always obtained a unique equilibrium.

3.4 Comparative statics with respect to σ_ε

While the model can be characterized in closed form, the comparative statics are not as tractable. We can, however, characterize the comparative statics in closed form for the two special cases: $\sigma_\varepsilon \to 0$ and $\sigma_\varepsilon \to \infty$.

3.4.1 Mixed strategy equilibrium (if σ_ε is sufficiently small).

Proposition 6. Suppose $I=1$, $(J+1)\eta \geq 2$, and σ_ε is sufficiently small, so that the unique linear equilibrium is a mixed strategy equilibrium. In this case $\lambda_1=\lambda_2$ and we denote both by λ. Then

$$\frac{\partial \sigma_z^2}{\partial \sigma_\varepsilon}<0, \quad \frac{\partial \beta_1}{\partial \sigma_\varepsilon}<0, \quad \frac{\partial \beta_x}{\partial \sigma_\varepsilon}<0, \quad \frac{\partial \delta}{\partial \sigma_\varepsilon}<0; \tag{74}$$

$$\frac{\partial \Sigma_1}{\partial \sigma_\varepsilon}>0, \quad \frac{\partial \Sigma_2}{\partial \sigma_\varepsilon}>0; \tag{75}$$

$$\frac{\partial \beta_2}{\partial \sigma_\varepsilon} < 0 \quad \text{if and only if} \quad J>2; \tag{76}$$

$$\frac{\partial \lambda}{\partial \sigma_\varepsilon} > 0 \quad \text{if and only if} \quad J>2; \tag{77}$$

$$\frac{\partial \left(\lambda \sigma_u^2+\lambda \sigma_U^2\right)}{\partial \sigma_\varepsilon} > 0 \quad \text{if and only if} \quad J>2; \tag{78}$$

$$\frac{\partial E\left[\Pi^F\right]}{\partial \sigma_\varepsilon}<0 \quad \text{if and only if}$$

$$(\eta+1)J^4-(\eta+3)J^3-6(3\eta+2)J^2-4(7\eta+2)J-8\eta>0; \text{ and} \tag{79}$$

$$\frac{\partial E\left[\pi_{2,j}^B\right]}{\partial \sigma_\varepsilon}>0 \quad \text{if and only if}$$

$$(\eta+1)J^4+\eta J^3-(10\eta+7)J^2-10(2\eta+1)J-4(2\eta+1)>0. \tag{80}$$

We begin with variables that describe trading strategies in the first two lines of Proposition 6. An increase in σ_ε means that the back-runners' information becomes less precise, which also reduces the back-runners' trading intensity (smaller δ). Worrying less about information leakage, the fundamental investor adds less noise (smaller σ_z^2) and relies on less aggressive trading (smaller β_1) in period 1 to defend herself. For the same reason, in period 2, the fundamental investor also trades less aggressively against the noise in period-1 order flow (smaller β_x). Less information corresponds to less price discovery, shown in larger conditional variances Σ_1 and Σ_2.

The last five items of Proposition 6 shows how a reduction in back-runners' information accuracy affects the fundamental investor's trading intensity in period 2, market liquidity (equal in the two periods), and the expected profits of various agents. All these relations depend on J.

In particular, if J is sufficiently large, lowering the accuracy in back-runners' signals increases their own profits and decreases the fundamental investor's profits. This is quite surprising. The bottom row of panel A of Figure 3 illustrates these surprising patterns for $J=10$. If J is small, the opposite pattern is true, as shown in the bottom row of panel B of Figure 3 for $J=2$.

Our intuition for the profit patterns is the following trade-off. It is slightly easier to describe it for a reduction of σ_ε around 0, that is, as back-runners' information improves.

- Competition: As σ_ε decreases, in period 2, the fundamental investor faces more competition from back-runners, who are now endowed with more accurate information. The fundamental investor thus suffers from more competition.
- Endogenous noise: As σ_ε decreases, a mixed strategy with a larger σ_z is supported as an equilibrium. Because only the fundamental investor knows her own added noise z, this information gives her an information advantage about the period-2 price relative to the back-runners. Moreover, because back-runners receive different noisy signals of X_1, this noise generates additional uninformed order flows from the back-runners that the fundamental investor can exploit. In this dimension, the fundamental investor can benefit.

Proposition 6 shows that, in terms of the fundamental investor's profit, the competition effect dominates for a small J but the endogenous-noise effect dominates for a large J. The reason that J matters in this trade-off could be further understood as follows. The effect of competition is likely concave: the first competitor reduces profits by a large margin, but each additional competitor reduces the profit by a smaller amount. The endogenous noise channel, however, is more likely linear, in the sense that each additional back-runner's signal is contaminated by the endogenous noise z and contains a fresh idiosyncratic noise ε (because $\sigma_\varepsilon > 0$). Both noises can be exploited by the fundamental investor in period 2. Once J is sufficiently large, the fundamental investor's benefit of exploiting these noises in back-runners' order flows dominates the cost of competition.

This intuition could also explain the comparative statics with respect to β_2 and λ. For a large J, the intuition above suggests that the back-runners inject

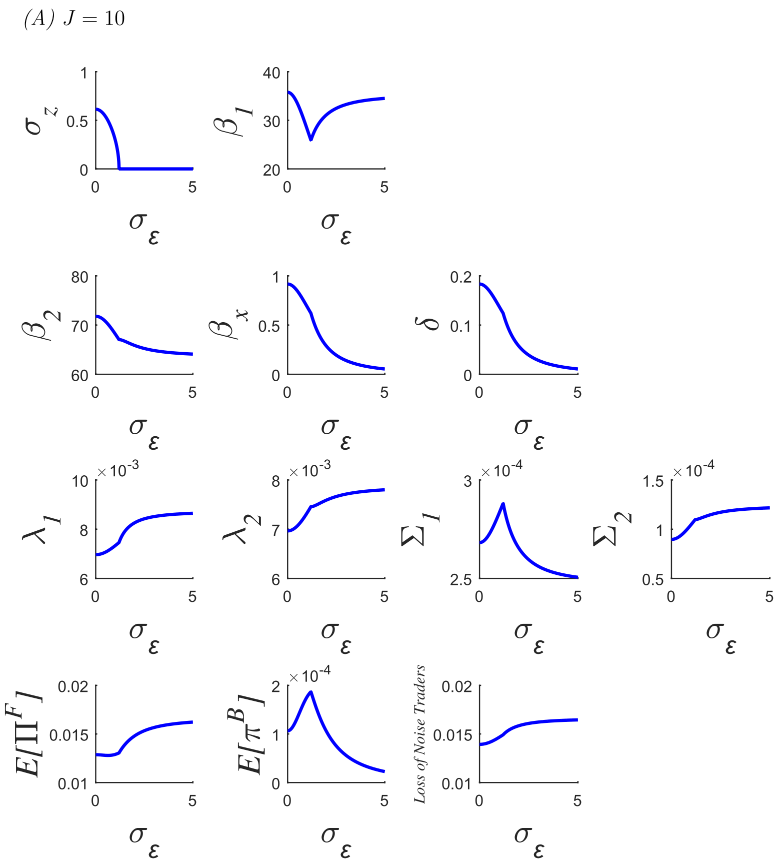

Figure 3
Comparative statics of the model for a monopolist fundamental investor
Parameters: $\Sigma_0 = 0.3^2/252$, $\sigma_u = \sigma_U = 1$, and $I = 1$. The two panels show $J = 10$ and $J = 2$, respectively.

"more" noise $\{\varepsilon_j\}$ in the market than information. Thus, by increasing the back-runners' trading intensity, a reduction in σ_ε leads to better market liquidity (smaller λ) and more aggressive trading by the fundamental investor in period 2 (larger β_2). The opposite is true if J is small.

The thresholds for J in the last five equations of Proposition 6 are not all identical, suggesting that the trade-off mentioned above differentially applies to different variables. Beyond the "large J versus small J" intuition and the analytical proof, we have not been able to find the exact economic intuition for the difference in these if-and-only-if conditions.

3.4.2 Pure strategy equilibrium (if σ_ε is sufficiently large).

Proposition 7. Suppose $I = 1$ and σ_ε is sufficiently large, so that the unique linear equilibrium is a pure strategy equilibrium. Then

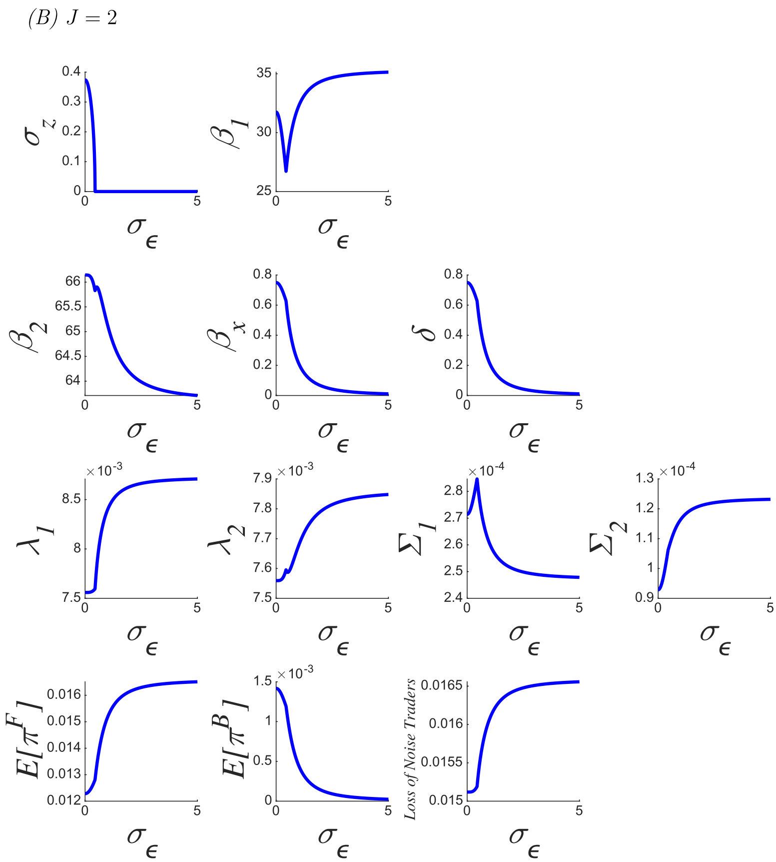

Figure 3
Continued

$$\frac{\partial \beta_1}{\partial \sigma_\varepsilon} > 0, \quad \frac{\partial \beta_2}{\partial \sigma_\varepsilon} < 0, \quad \frac{\partial \beta_x}{\partial \sigma_\varepsilon} < 0, \quad \frac{\partial \delta}{\partial \sigma_\varepsilon} < 0; \tag{81}$$

$$\frac{\partial \lambda_1}{\partial \sigma_\varepsilon} > 0, \quad \frac{\partial \lambda_2}{\partial \sigma_\varepsilon} > 0, \quad \frac{\partial \Sigma_1}{\partial \sigma_\varepsilon} < 0; \tag{82}$$

$$\frac{\partial \Sigma_2}{\partial \sigma_\varepsilon} > 0 \quad \text{if and only if} \quad \eta < \hat{\eta} \approx 3.06; \tag{83}$$

$$\frac{\partial E\left[\Pi^F\right]}{\partial \sigma_\varepsilon} > 0, \quad \frac{\partial E\left[\pi_{2,j}^B\right]}{\partial \sigma_\varepsilon} < 0, \quad \text{and} \quad \frac{\partial\left(\lambda_1 \sigma_u^2 + \lambda_2 \sigma_U^2\right)}{\partial \sigma_\varepsilon} > 0. \tag{84}$$

For a sufficiently large σ_ε, a pure strategy equilibrium obtains. As the back-runners' information precision decreases (higher σ_ε), they naturally trade less aggressively (smaller δ) and make lower profits (smaller $E[\pi^B_{2,j}]$). Worrying less about information leakage, the fundamental investor shifts more trading activity to the first period (larger β_1 and smaller β_2 and β_x) and makes more profits (larger $E[\Pi^F]$). Because of this shift of informed activity to the first period, price impact increases and price discovery improves in period 1. Price impact also increases in period 2, presumably because the fundamental investor smooths trading intensity across the two periods.

The period-2 price discovery, however, depends on $\eta \equiv \sigma^2_U/\sigma^2_u$. If η is sufficiently low (in this case lower than 3.06), namely, if period 2's noise trading is not too high relative to period 1's, then period-2 price discovery improves as σ_ε becomes smaller. Intuitively, as long as period 2's noise trading is not too high, the fundamental investor would still trade nontrivial amount in period 1, leading to potential information leakage. This information leakage naturally improves price discovery in period 2. This qualitative intuition seems generic, although the specific numerical threshold of 3.06 is likely specific to the two-period setting.

3.5 Comparative statics with respect to J

Now, we turn to comparative statics with respect to J, the number of back-runners. It would be desirable to prove them for any value of σ_ε, but, because of analytical difficulty, we can only sign the comparative statics for $\sigma_\varepsilon = 0$, that is, perfect order-flow information. (Of course, by continuity, the same results obtain if $\sigma_\varepsilon = 0$ is sufficiently small.)

3.5.1 Mixed strategy equilibrium (if J is sufficiently large). Recall from Proposition 5 that a large J leads to a mixed strategy equilibrium.

Proposition 8. Suppose $I = 1$, $\sigma_\varepsilon = 0$, and J is sufficiently large, so that the unique linear equilibrium is a mixed strategy equilibrium. In this case $\lambda_1 = \lambda_2$ and we denote both by λ. Then

$$\frac{\partial \sigma^2_z}{\partial J} > 0, \quad \frac{\partial \beta_1}{\partial J} > 0, \quad \frac{\partial \beta_2}{\partial J} > 0, \quad \frac{\partial \beta_x}{\partial J} > 0, \quad \frac{\partial \delta}{\partial J} < 0, \quad \frac{\partial \lambda}{\partial J} < 0; \tag{85}$$

$$\frac{\partial \Sigma_1}{\partial J} < 0 \quad \text{if and only if} \quad \eta > 1; \tag{86}$$

$$\frac{\partial \Sigma_2}{\partial J} < 0 \quad \text{if and only if} \quad \eta > 1; \tag{87}$$

$$\frac{\partial E\left(\Pi^F\right)}{\partial J} > 0, \quad \frac{\partial E\left(\pi^B_{2,j}\right)}{\partial J} < 0, \quad \frac{\partial \left[J \times E\left(\pi^B_{2,j}\right)\right]}{\partial J} < 0,$$
$$\text{and} \quad \frac{\partial \left(\lambda \sigma^2_u + \lambda \sigma^2_U\right)}{\partial J} < 0. \tag{88}$$

These comparative statics are quite natural. If there are more back-runners (larger J), the fundamental investor front-loads her trades more (larger β_1) and simultaneously adds more noise in her order flow (larger σ_z^2) in period 1. The net effect is that the price impact of trades drops (smaller λ, same for both periods). The smaller price impact of trades encourages the fundamental investors to trade more aggressively in period 2 (large β_2). On the other hand, the endogenous noise injected by fundamental investors obscures back-runners' order flow information, so back-runners trade less aggressively in period 2 (smaller δ), reducing their profits. Conversely, fundamental investors' profits increase, and noise traders' losses decrease in J.

We have expected the price discovery variables Σ_1 and Σ_2 to decrease as more back-runners are added, but this is true if and only if η is sufficiently large (in this case, if $\eta > 1$), that is, if the fundamental investor has large enough profits to protect from back-running.

Theoretical ambiguity may help reconcile seemingly conflicting evidence in the empirical literature. For example, Brogaard, Hendershott, and Riordan (2014) find that, on average, HFTs contribute to price discovery by trading in the direction of permanent price movement. At the same time, Weller (2018) finds that the more active is algorithmic trading, the more delayed is the price jump near earnings announcement, suggesting delayed price discovery.

3.5.2 Pure strategy equilibrium (if both J and η are small). Recall from Proposition 5 that a sufficiently small J combined with a sufficiently small η leads to a pure strategy equilibrium.

Proposition 9. Suppose $I=1$, $\sigma_\varepsilon=0$, and both J and η are small, so that the unique linear equilibrium is a pure strategy equilibrium. Then

$$\frac{\partial\beta_1}{\partial J}>0,\quad \frac{\partial\beta_2}{\partial J}>0,\quad \frac{\partial\beta_x}{\partial J}<0,\quad \frac{\partial\delta}{\partial J}<0,\quad \frac{\partial\lambda_1}{\partial J}>0,\quad \frac{\partial\lambda_2}{\partial J}<0; \tag{89}$$

$$\frac{\partial\Sigma_1}{\partial J}<0,\quad \frac{\partial\Sigma_2}{\partial J}<0; \tag{90}$$

$$\frac{\partial E\left(\Pi^F\right)}{\partial J} < 0,\quad \frac{\partial E\left(\pi_{2,j}^B\right)}{\partial J}<0,\quad \frac{\partial\left(\lambda_1\sigma_u^2+\lambda_2\sigma_U^2\right)}{\partial J}<0; \tag{91}$$

$$\frac{\partial E\left(J\times\pi_{2,j}^B\right)}{\partial J} < 0 \quad \text{if and only if} \quad J>3. \tag{92}$$

It is informative to compare the comparative statics in Proposition 9 to those in Proposition 8. The difference reveals the economic difference between the mixed strategy equilibrium and the usual pure strategy equilibrium.

A major difference here is that in the pure strategy equilibrium, the fundamental investor cannot adjust the size of the endogenous noise (it is zero). As J increases, the fundamental investor's adjustment has to come from trading intensity, and this part goes in the same direction as the mixed strategy equilibrium (β_1 and β_2 increase in J). Naturally, the price impact of trade increases in J in period 1. Competition among more back-runners causes each of them to trade less aggressively (δ decreases), and the price impact goes down in period 2 (λ_2 decreases). Price discovery improves in both periods as J increases, presumably because, again, the only way for the fundamental investor to reduce information leakage is to front-load her trades.

The lack of endogenous noise injected by the fundamental investor in the pure strategy equilibrium also shows up in the profits. As J increases, the fundamental investor's profit declines in the pure strategy equilibrium, opposite to the pattern in the mixed strategy equilibrium. Interestingly, the total profit of the back-runners decreases in J if and only if $J > 3$. That is, back-runners receive the maximal profits at $J = 3$ or $J = 4$, which suggests that the "optimal" industry structure for back-runners is a tight oligopoly of three or four firms, but not a monopoly or duopoly.

4. Application: Value of Retail Order Flows

We conclude this paper with an application: the value of retail order flows through the lens of back-running. This question is highly relevant because of the widespread practice of payment for order flow, whereby retail brokers aggregate customer orders and route them to a third party for execution, in return for a fee. From some of the largest retail brokerage firms' public disclosures, SEC (2016) finds that payments for order flow to these firms range from \$92 million to \$304 million in 2014. The current discussion about payment for order flow focuses on its distorted routing incentives, namely, retail brokers may choose the destination of these retail orders to maximize fees or rebates, which are kept by the retail brokers.[9]

In this section, we illustrate an orthogonal channel through which handling retail orders gives retail brokers an economic benefit. To make this point most clearly, we rewrite each back-runner's signal s_j as $s_j = X_1 + \varepsilon_j = y_1 - u_1 + \varepsilon_j$, where y_1 is publicly observable. Thus, each back-runner effectively receives a signal about the total noise traders' past order flow u_1, with the same perturbation ε_j. Under a common interpretation that retail orders are proxies for noise trading, we could view back-runners as, say, proprietary trading firms

[9] Battalio, Corwin, and Jennings (2016) find that for 4 of the 10 retail brokers they examine, rebates seem to be a significant determinant of their routing decisions. The SEC (2016) discusses potential issues raised by payment for order flow, particularly the distorted routing incentives, as well as various options of addressing them, ranging from an outright ban to passing on the fees and rebates to customers. In 2012, the U.K. FSA (FCA's precursor) effectively banned payment for order flow in the United Kingdom, citing the conflict of interest created between clients and brokers by this practice.

that directly handle and fill retail orders, which are routed by retail brokers (e.g., Charles Schwab, E*Trade, or TD Ameritrade). According to *Bloomberg*, Citadel and KCG dominate the handling of retail orders in the United States (see Massa 2017). Our model does not have a concern of "front-running," suboptimal order routing decisions, or intercepting marketable retail investors, all of which are the focus of previous literature. Instead, we show that the knowledge about retail investors' trades can lead to back-running, and the profits from back-running would be an estimate of the economic value of observing or handling retail orders.

We now quantitatively explore the value of retail order flow information through the lens of back-running. As before, the full model is interpreted as a trading day, with each period being half a day. For ease of interpretation, we express the profits of fundamental investors and back-runners in multiples of the dollar volume of noise traders:

$$Q^F = \frac{I \times E\left(\Pi_i^F\right)}{E\left[\left(|u_1|+|U_2|\right)p_0\right]}, \tag{93}$$

$$Q^B = \frac{J \times E\left(\pi_{2,j}^B\right)}{E\left[\left(|u_1|+|U_2|\right)p_0\right]}, \tag{94}$$

where the denominator uses on the approximation that p_1 and p_2 are close to p_0.

We can show that Q^F and Q^B only depend on five variables: I, J, $\frac{\sqrt{\Sigma_0}}{p_0}$, $\frac{\sigma_U}{\sigma_u}$, and $\frac{\sigma_\varepsilon}{\sigma_u}$. That is:

$$Q^F = Q^F\left(I, J, \frac{\sqrt{\Sigma_0}}{p_0}, \frac{\sigma_U}{\sigma_u}, \frac{\sigma_\varepsilon}{\sigma_u}\right) \text{ and } Q^B = Q^B\left(I, J, \frac{\sqrt{\Sigma_0}}{p_0}, \frac{\sigma_U}{\sigma_u}, \frac{\sigma_\varepsilon}{\sigma_u}\right). \tag{95}$$

Figure 1 and panel A of Figure 3 reveal that the profits of back-runners could be nonmonotone in σ_ε. Thus, we consider two possible values of σ_ε:

- $\sigma_\varepsilon^2 = 0$. This corresponds to perfect information about the entire retail order flow in the first half of a trading day.
- σ_ε is chosen so that it maximizes the total profits of back runners, that is, for a given J, we define $\sigma_\varepsilon^{opt} \equiv \arg\max_{\sigma_\varepsilon} Q^B\left(I, J, \frac{\sqrt{\Sigma_0}}{p_0}, \frac{\sigma_U}{\sigma_u}, \frac{\sigma_\varepsilon^2}{\sigma_u^2}\right)$. This choice allows the possibility that back-runners also optimize their information technology.

Although these choices of σ_ε may appear to be low, we believe they are in fact reasonable choices because two firms dominate the handling of retail orders in U.S. equity markets. Alternatively, one can interpret our calibration results as upper bounds for the value of retail order flow information. The ratio σ_U/σ_u is set to be 1.

Figure 4 plots $\sigma_\varepsilon^{opt}/\sigma_u$, σ_z, Q^F and Q^B, all as functions of J. The annual stock return volatility is kept at 30%. The only difference in the four panels is

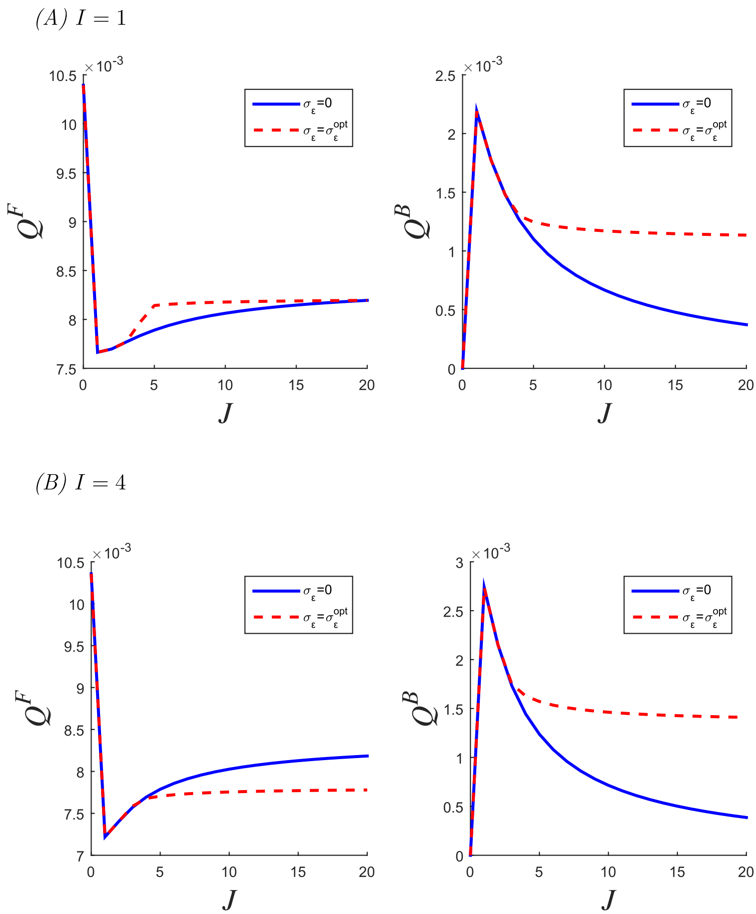

Figure 4
Profits of fundamental investors and back-runners with various *I*
Parameters: $\Sigma_0=0.3^2/252$, and $\sigma_u=\sigma_U=1$. The four panels show $I=1$, $I=4$, $I=100$, and $I=10,000$, respectively.

I, which takes the value of 1, 4, 100, or 10,000. As before, the general shape and magnitude of these variables are insensitive to I.

Figure 4 shows that the value of order-flow information is rather similar for both choices of σ_ε. Conditional on nontrivial back-running ($J\geq 1$), the institutional investors' (fundamental investors') total profits are about 70–80 bps of retail trading volume (noise traders' volume), and the back-runners' profits are about 5–30 bps of retail trading volume. Moreover, conditional on $J\geq 1$, fundamental investors' total profits are generally increasing in J, but back-runners' total profits are generally decreasing in J.

Figure 5 plots Q^F and Q^B under larger or smaller return volatilities, $\Sigma_0=(50\%)^2/252$ or $\Sigma_0=(10\%)^2/252$, but fixing $I=100$. We observe that the shapes

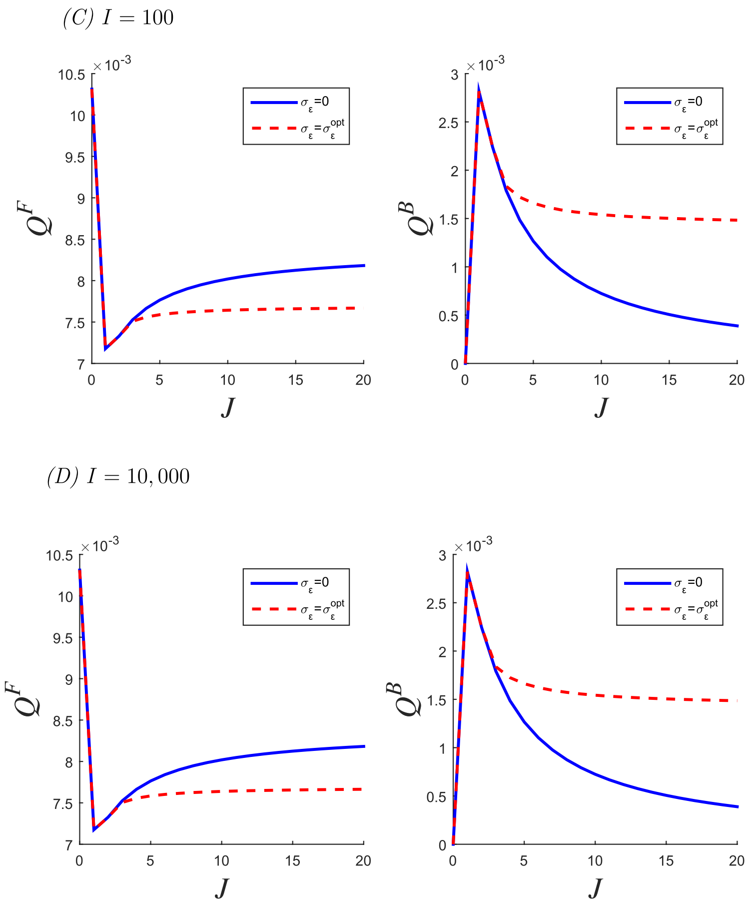

Figure 4
(*Continued*)

of Q^F and Q^B are identical to the case of $\Sigma_0 = (30\%)^2/252$, but the magnitude is simply scaled by $\sqrt{\Sigma_0}$. This means that inaccurate assumptions about return volatility do not change the order of magnitude of these estimated profits.

The magnitude of these numbers suggests that retail order flows are highly valuable for back-runners. A typical daily dollar volume in the U.S. equity market is about \$200 billion and retail volume is about 10% of the total. Under the assumption of 30% annualized return volatility for a typical stock, back-runners' profits are, at the high end of $Q^B = 30$ bps, in the order of \$200 billion $\times 10\% \times 0.003\% =$ \$60 million per day, or about \$15 billion per year. At the low end of $Q^B = 5$ bps, back-runners' profits from retail order flow information are in the order of \$2.5 billion. The magnitude is economically large.

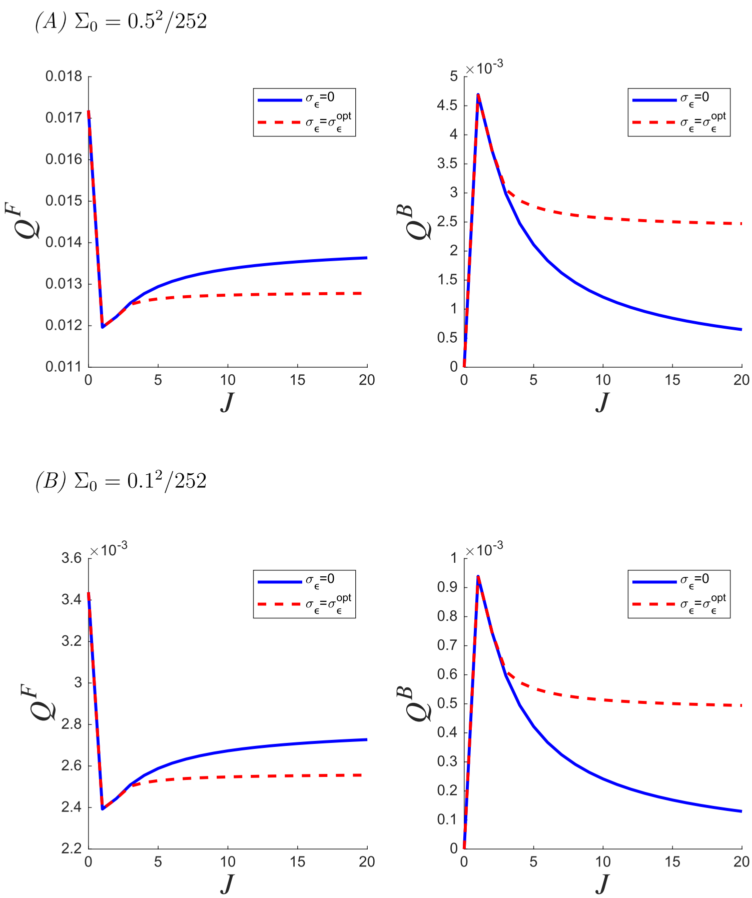

Figure 5
Profits of fundamental investors and back-runners with various Σ_0
Parameters: $\sigma_u = \sigma_U = 1$ and $I = 100$. The two panels show $\Sigma_0 = 0.5^2/252$ and $\Sigma_0 = 0.1^2/252$, respectively.

By analogous calculations, using $Q^F \approx 75$ bps, institutional investors' daily profits in the U.S. equity market is in the order of \$150 million per day, or about \$37.5 billion per year. As a comparison, Berk and van Binsbergen (2015) estimate that an average U.S. mutual fund generates a value of about \$3.2 million per year, or about \$19.2 billion per year in aggregate for their sample of 6,000 funds. Our estimate from an extremely stylized model is in the same order of magnitude.

5. Conclusion

This paper presents a theory of back-running and its implications. Back-runners start with no innate trading motive but observe past order flow information of fundamental investors (or equivalently, noise traders). Order flow information allows back-runners to partly infer the information of fundamental investors and exploit it in subsequent trading. We characterize conditions under which the resultant equilibrium involves mixed strategies or pure strategies. Various market outcomes depend on the number of back-runners more than on the number of fundamental investors. When the number of back-runners is sufficiently large, some usual intuition flips. For instance, more accurate order flow information may reduce back-runners' profits and increase fundamental investors' profits. A straightforward application of the model reveals a high value for past order flow information from retail investors, which effectively gives a signal about past institutional order flows. Under reasonable parameters, the total value earned by back-runners is 5–30 bps of retail dollar volume and the total value earned by institutional investors is 70–80 bps of retail dollar volume.

Appendix: Proofs

Proof of Proposition 1

A mixed strategy equilibrium is defined by seven unknowns, $(\beta_1,\sigma_z,\beta_2,\beta_x,\delta,\lambda_1,\lambda_2)$, which are characterized by seven equations, (15), (16), (19), (21), (22), (24), and (25), together with one SOC (18). By (21), (24), and (25), we show

$$\lambda_1 = \lambda_2=\lambda, \qquad \text{(A1)}$$

$$\beta_x \frac{\beta_1^2 \frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+\sigma_u^2}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}+\beta_2\frac{\beta_1 \frac{1}{I}\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2} = 1. \qquad \text{(A2)}$$

We are thus left with six unknowns, $(\beta_1,\sigma_z,\beta_2,\beta_x,\delta,\lambda)$. We then express $(\lambda,\beta_x,\beta_2)$ as functions of $(\beta_1,\sigma_z,\delta)$ and further simplify the system in terms of three unknowns $(\beta_1,\sigma_z,\delta)$, characterized by Equations (28)–(30) in Proposition 1.

By (15) and (A1), we have

$$\lambda_1=\lambda_2=\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}. \qquad \text{(A3)}$$

Using (21) and (A3), we can compute

$$\beta_2=\frac{\beta_1^2\Sigma_0+I\sigma_z^2+\sigma_u^2}{2\beta_1\Sigma_0}. \qquad \text{(A4)}$$

By (21) and (22), we obtain the expression of β_x in terms of (β_1,σ_z), which is given by Equation (32) in Proposition 1. Then, using (21), (A1), (A2), and (32), we can express δ in terms of (β_1,σ_z), arriving at Equation (28) in Proposition 1.

Inserting (21) and (32) into (19) leads to

$$\delta=\frac{\frac{1}{2\lambda_2}\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+I\sigma_z^2}+\frac{1}{2\lambda_2}\frac{\beta_1\frac{I-1}{I}\Sigma_0}{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+2\sigma_u^2}}{2\frac{\left(\beta_1^2\Sigma_0+I\sigma_z^2\right)^{-1}+\sigma_\varepsilon^{-2}+\sigma_u^{-2}}{\sigma_\varepsilon^{-2}}-\frac{J\sigma_u^2}{\beta_1^2\frac{I-1}{I}\Sigma_0+(I-1)\sigma_z^2+2\sigma_u^2}+(J-1)}. \tag{A5}$$

Combining the above equation with (A3), we obtain Equation (29) in Proposition 1.

Equation (30) in Proposition 1 is simplified from Equation (16). Specifically, by (32) and (A3), we show

$$\beta_x=\left(I-\frac{1}{2}\right)\frac{\sigma_u^2+\beta_1^2\Sigma_0+\sigma_z^2 I}{\sigma_z^2 I^2-\beta_1^2\Sigma_0+\sigma_u^2 I-\sigma_z^2 I+\beta_1^2\Sigma_0 I}. \tag{A6}$$

Using (28), (A4), and (A6), we obtain

$$\lambda_2=\frac{\frac{\sigma_u^2+\beta_1^2\Sigma_0+\sigma_z^2 I}{2\beta_1}}{\frac{1}{4}\left(\sigma_u^2+\beta_1^2\Sigma_0\right)\frac{\left(\sigma_u^2+\beta_1^2\Sigma_0+\sigma_z^2 I\right)^2}{\beta_1^2\Sigma_0\sigma_u^2}+\delta^2 J\sigma_\varepsilon^2+\sigma_U^2}, \tag{A7}$$

which, combined with (A3), leads to Equation (30) in Proposition 1.

Equations (28)–(30) in Proposition 1 form a system of three equations in terms of three unknowns $(\delta,\beta_1,\sigma_z)$. For the range of $\beta_1\in\left(0,\frac{\sigma_u}{\sqrt{\Sigma_0}}\right)$, the lower bound comes from (A3) and the SOC (18), and the upper bound is the result of Equation (30). The price discovery and profit variables come from direct computations.

Proof of Proposition 2

A pure strategy equilibrium is defined in terms of six unknowns $(\beta_1,\beta_2,\beta_x,\delta,\lambda_1,\lambda_2)$, which are characterized by six equations, (15), (16), (19), (21), (22), and (26), together with two SOCs, (18) and (27). We prove Proposition 2 by expressing $(\beta_2,\beta_x,\delta,\lambda_1)$ as functions of $(\beta_1\lambda_2)$, which simplifies the system in terms of two unknowns $(\beta_1\lambda_2)$.

Equation (42) is simply Equation (21), which expresses β_2 as a function of λ_2. Equations (44) and (45) express (β_x,λ_1) as functions of (β_1,λ_2) and they are obtained from Equations (32) and (15), respectively, replaced with $\sigma_z=0$. Inserting Equations (21) and (44) into Equation (19) and noting $\sigma_z=0$, we obtain Equation (43), which expresses δ as a function of (β_1,λ_2). Using (42) and $\sigma_z=0$, we can rewrite (16), (26), and (27), respectively, as (40), (41), and (46).

The requirement $\lambda_2>0$ comes from the SOC, (18). The requirement $\beta_1>0$ is implied jointly by (45) and (46). The price discovery and profit variables arise from direct computations.

Proof of Proposition 3

Using $I=1$, we simplify Equations (28)–(30) as the following three equations:

$$\delta = \frac{\sigma_u^2+\beta_1^2\Sigma_0+\sigma_z^2}{J\sigma_u^2}, \tag{A8}$$

$$\delta = \frac{\frac{\beta_1^2\Sigma_0+\sigma_z^2+\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_z^2}}{4\frac{\left(\beta_1^2\Sigma_0+\sigma_z^2\right)^{-1}+\sigma_\varepsilon^{-2}+\sigma_u^{-2}}{\sigma_\varepsilon^{-2}}+J-2}, \tag{A9}$$

$$\delta^2 J\sigma_\varepsilon^2+\sigma_U^2 = \frac{1}{4}\left(\sigma_u^2-\beta_1^2\Sigma_0\right)\frac{\left(\sigma_u^2+\beta_1^2\Sigma_0+\sigma_z^2\right)^2}{\beta_1^2\Sigma_0\sigma_u^2}. \tag{A10}$$

We now solve δ, β_1, and σ_z sequentially. First, by (A8), we have

$$\beta_1^2\Sigma_0+\sigma_z^2=(J\delta-1)\sigma_u^2, \tag{A11}$$

which is inserted into (A9), leading to the value of δ given by (63) in Proposition 3. Second, plugging (63) and (A11) into (A10) and with some algebra, we have

$$4\left(\frac{J+1}{J+2+\frac{4\sigma_\varepsilon^2}{\sigma_u^2}}\right)^2\frac{1}{J}\frac{\sigma_\varepsilon^2}{\sigma_u^2}+\frac{\sigma_U^2}{\sigma_u^2}=\left(1-\frac{\beta_1^2\Sigma_0}{\sigma_u^2}\right)\left(\frac{J+1}{J+2+\frac{4\sigma_\varepsilon^2}{\sigma_u^2}}\right)^2\frac{\sigma_u^2}{\beta_1^2\Sigma_0}, \tag{A12}$$

which gives the solution to β_1 in (60). Last, using (63), (60), and (A8), we compute the value of σ_z^2 given by (59) in Proposition 3.

Once $(\delta,\beta_1,\sigma_z)$ are solved, the other variables are given by Equations (31)–(33) with $I=1$.

By definition, a mixed strategy equilibrium exists if and only if $\sigma_z>0$. By the expression of σ_z in (59), we can show that $\sigma_z>0$ if and only condition (58) holds.

Proof of Corollary 1

Corollary 1 directly follows from condition (58).

Proof of Proposition 4

Inserting $I=1$ into Equations (41), (43), and (44), we have

$$\beta_1 = \frac{1-\left(\beta_x\frac{\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\beta_2\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)}{2\left[\lambda_1-\lambda_2\left(\beta_x\frac{\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\beta_2\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)^2\right]}, \tag{A13}$$

$$\delta = \frac{\frac{1}{2\lambda_2}\frac{1}{\beta_1}}{2\frac{\left(\beta_1^2\Sigma_0\right)^{-1}+\sigma_\varepsilon^{-2}+\sigma_u^{-2}}{\sigma_\varepsilon^{-2}}+\frac{J}{2}-1}, \tag{A14}$$

$$\beta_x = \frac{J\delta}{2}. \tag{A15}$$

Using (21), (45), and (A15), we can further simplify Equation (A13) as follows:

$$\begin{aligned} & 2\lambda_2\left(\frac{2\beta_1^2\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}-1\right) \\ &= \left(\frac{J\delta\lambda_2\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)\left[\beta_1\left(\frac{J\delta\lambda_2\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)-1\right]. \end{aligned} \tag{A16}$$

To arrive at the polynomial (66), we want to express $\delta\lambda_2$ and λ_2^2 as functions of β_1 and then insert these expressions into the squared Equation (A16). By Equation (A14),

$$\lambda_2\delta = \frac{\beta_1\sigma_u^2\Sigma_0}{4\sigma_u^2\sigma_\varepsilon^2+2\beta_1^2\Sigma_0\sigma_u^2+4\beta_1^2\Sigma_0\sigma_\varepsilon^2+J\beta_1^2\Sigma_0\sigma_u^2}. \tag{A17}$$

Inserting (21) and (A15) into (40), we have

$$\sigma_U^2\lambda_2^2 = \left[\frac{1+J\beta_1\delta\lambda_2}{2}-\left(\frac{1+J\beta_1\delta\lambda_2}{2}\right)^2\right]\frac{\Sigma_0\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}-J\sigma_\varepsilon^2(\delta\lambda_2)^2. \tag{A18}$$

In the above equation, we replace $\lambda_2\delta$ given by (A17) to derive

$$\lambda_2^2 = \frac{\sigma_u^2}{\sigma_U^2}\frac{\Sigma_0\left[\begin{array}{c} 4\sigma_u^4\sigma_\varepsilon^4+\beta_1^4\Sigma_0^2\sigma_u^4+4\beta_1^4\Sigma_0^2\sigma_\varepsilon^4+8\beta_1^2\Sigma_0\sigma_u^2\sigma_\varepsilon^4+4\beta_1^2\Sigma_0\sigma_u^4\sigma_\varepsilon^2 \\ +4\beta_1^4\Sigma_0^2\sigma_u^2\sigma_\varepsilon^2+J\beta_1^4\Sigma_0^2\sigma_u^4+J\beta_1^2\Sigma_0\sigma_u^4\sigma_\varepsilon^2+J\beta_1^4\Sigma_0^2\sigma_u^2\sigma_\varepsilon^2 \end{array}\right]}{\left(\sigma_u^2+\beta_1^2\Sigma_0\right)\left(4\sigma_u^2\sigma_\varepsilon^2+2\beta_1^2\Sigma_0\sigma_u^2+4\beta_1^2\Sigma_0\sigma_\varepsilon^2+J\beta_1^2\Sigma_0\sigma_u^2\right)^2}. \tag{A19}$$

We then square the both sides of (A16) in order to use (A17) and (A19) to substitute $\lambda_2\delta$ and λ_2^2. Doing so requires that the terms $\frac{2\beta_1^2\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}-1$ and $\beta_1\left(J\delta\lambda_2\frac{\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)-1$ have the same sign, that is,

$$\left(\frac{2\beta_1^2\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}-1\right)\left[\beta_1\left(\frac{J\delta\lambda_2\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2}+\frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)-1\right]\geq 0. \tag{A20}$$

Inserting the expression of $\lambda_2\delta$ (A17) into the above condition, we find that the above inequality is equivalent to requiring

$$\beta_1^2 \leq \frac{\sigma_u^2}{\Sigma_0}, \tag{A21}$$

which must be true in a pure strategy equilibrium, as stated in Proposition 2. Thus, we take square of (A16), insert (A17) and (A19) to substitute $\lambda_2\delta$ and

λ_2^2, and define $k \equiv \frac{\beta_1^2 \Sigma_0}{\sigma_u^2} \in (0,1]$, which yields the seventh-order polynomial of k in (66), where

$$A_7 = \left(J + 4\theta + J\theta + 4\theta^2 + 1\right)(J + 4\theta + 2)^2, \tag{A22}$$

$$A_6 = (J + 4\theta + 2)\left(-3J + 4\theta + 4J\theta + 40\theta^2 + 48\theta^3 + 12J\theta^2 - J^2 - 2\right), \tag{A23}$$

$$\begin{aligned} A_5 = {} & 64\theta^4 - 64\theta - 4\eta - 24J^2\theta^2 - 88J\theta - 4J\eta - 192\theta^2 - 128\theta^3 \\ & - 8J - 32\theta\eta - 160J\theta^2 \\ & - 32J^2\theta - 48J\theta^3 - 2J^3\theta - J^2\eta - 96\theta^2\eta - 128\theta^3\eta \\ & - 64\theta^4\eta - 5J^2 - J^3 - 4J^2\theta^2\eta \\ & - 24J\theta\eta - 48J\theta^2\eta - 4J^2\theta\eta - 32J\theta^3\eta - 4, \end{aligned} \tag{A24}$$

$$\begin{aligned} A_4 = {} & 8J - 24J^2\theta^2 - 4J\eta - 192\theta^2 - 512\theta^3 - 320\theta^4 - 32\theta\eta \\ & - 160J\theta^2 - 192J\theta^3 - 2J^2\eta - 192\theta^2\eta \\ & - 384\theta^3\eta - 256\theta^4\eta + 5J^2 + J^3 - 16J^2\theta^2\eta - 48J\theta\eta \\ & - 144J\theta^2\eta - 12J^2\theta\eta - 128J\theta^3\eta + 4, \end{aligned} \tag{A25}$$

$$\begin{aligned} A_3 = {} & 32\theta + 12J^2\theta^2 + 44J\theta + 96\theta^2 - 128\theta^3 - 320\theta^4 \\ & + 80J\theta^2 + 16J^2\theta - 48J\theta^3 + J^3\theta - J^2\eta \\ & - 96\theta^2\eta - 384\theta^3\eta - 384\theta^4\eta - 24J^2\theta^2\eta - 24J\theta\eta \\ & - 144J\theta^2\eta - 12J^2\theta\eta - 192J\theta^3\eta, \end{aligned} \tag{A26}$$

$$\begin{aligned} A_2 = -4\theta\Big(& -24\theta - 20J\theta - 64\theta^2 - 16\theta^3 - 24J\theta^2 - 3J^2\theta + J^2\eta + 32\theta^2\eta + 64\theta^3\eta \\ & + 12J\theta\eta + 32J\theta^2\eta + 4J^2\theta\eta\Big), \end{aligned} \tag{A27}$$

$$A_1 = -4\theta^2\left(-32\theta - 12J\theta - 48\theta^2 + J^2\eta + 16\theta^2\eta + 8J\theta\eta\right), \tag{A28}$$

$$A_0 = 64\theta^4. \tag{A29}$$

The SOC (67) is obtained from (46) with $I=1$. The expressions of the other variables in the proposition simply substitute $k\equiv\frac{\beta_1^2\Sigma_0}{\sigma_u^2}$, $\theta\equiv\frac{\sigma_\varepsilon^2}{\sigma_u^2}$, and $\eta\equiv\frac{\sigma_U^2}{\sigma_u^2}$.

Proof of Proposition 5
Part (a): Switch based on σ_ε

When σ_ε is large: For sufficiently large σ_ε, condition (58) in Proposition 3 is violated and thus there exists no linear mixed strategy equilibrium. We now show that there exists a unique linear pure strategy equilibrium. The polynomial (66) in Proposition 4 is equivalent to the following equation:

$$\begin{aligned} & f(k;\theta,\eta) \\ \equiv\; & \log(k+1)+2\log(2k+4\theta+4k\theta+Jk)+2\log(1-k) \\ & +\log\left(4k^2\theta^2+4k\theta+4\theta^2+Jk^2+8k\theta^2+4k^2\theta+k^2+Jk\theta+Jk^2\theta\right) \\ & -\log k\eta-2\log(k+2\theta+2k\theta)-2\log(J+2k+4\theta+4k\theta+Jk) \\ =\; & 0. \end{aligned} \tag{A30}$$

We can easily show $f(0;\theta,\eta)>0$ and $f(1;\theta,\eta)<0$, and, thus, by the intermediate-value theorem, there exists a solution to k for any value of $\theta=\frac{\sigma_\varepsilon^2}{\sigma_u^2}$.

Note that as $\sigma_\varepsilon\to\infty$, we have $\theta=\frac{\sigma_\varepsilon^2}{\sigma_u^2}\to\infty$. We can show that

$$\left.\frac{\partial f(k;\theta,\eta)}{\partial k}\right|_{\theta\to\infty}\propto-\frac{2k^2+k+1}{k\left(1-k^2\right)}<0. \tag{A31}$$

Thus, for sufficiently large σ_ε, $f(\cdot;\theta,\eta)$ is downward sloping, and, hence, the solution to (A30) is unique.

To prove that this solution forms a linear pure strategy equilibrium, we remain to show that the SOC (67) is satisfied as well. Using the expressions of $\lambda_1,\lambda_2,\beta_1,\beta_2$, and β_x in Proposition 4, we can show that as $\theta\to\infty$, condition (67) is equivalent to

$$4+4k-k\eta>0. \tag{A32}$$

Note that in a linear pure strategy equilibrium, $k\eta$ is given by (A30) as follows:

$$\begin{aligned} k\eta= & \frac{(k+1)(2k+4\theta+4k\theta+Jk)^2(1-k)^2\begin{pmatrix}4k^2\theta^2+4k\theta+4\theta^2+Jk^2\\+8k\theta^2+4k^2\theta+k^2+Jk\theta+Jk^2\theta\end{pmatrix}}{(k+2\theta+2k\theta)^2(J+2k+4\theta+4k\theta+Jk)^2} \\ & \to(1-k)^2(1+k),\text{ as }\theta\to\infty. \end{aligned} \tag{A33}$$

Hence, as $\theta\to\infty$, condition (A32) is satisfied:

$$4+4k-k\eta\to(3-k)(k+1)^2>0\text{ as }\theta\to\infty. \tag{A34}$$

When σ_ε is small: By Corollary 1, for sufficiently small σ_ε, a mixed strategy equilibrium exists if and only if (65) is satisfied. Note that

$$(J+1)\eta - J\eta\left(\frac{J+2}{J+1}\right)^2 = -\frac{\left(J^2+J-1\right)\eta}{(J+1)^2} < 0 \Rightarrow \tag{A35}$$

$$J\eta\left(\frac{J+2}{J+1}\right)^2 > (J+1)\eta > 2. \tag{A36}$$

Thus, if $(J+1)\eta > 2$, then condition (65) holds, and there exists a unique mixed strategy equilibrium.

Next, we prove that there is no linear pure strategy equilibrium under the condition $(J+1)\eta > 2$. Note that in a pure strategy equilibrium, we have $\beta_1 > 0$. By (A13), we have

$$1-\left(\beta_x \frac{\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2} + \beta_2 \frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right) > 0. \tag{A37}$$

Using (21) and (A15), we can rewrite the above condition as

$$4\lambda_2^2 > \left(J\delta\lambda_2 \frac{\sigma_u^2}{\beta_1^2\Sigma_0+\sigma_u^2} + \frac{\beta_1\Sigma_0}{\beta_1^2\Sigma_0+\sigma_u^2}\right)^2. \tag{A38}$$

We employ the expressions of λ_2 and $\delta\lambda_2$ in (70) and (A17), together with the definitions $k \equiv \frac{\beta_1^2\Sigma_0}{\sigma_u^2}$ and $\theta \equiv \frac{\sigma_\varepsilon^2}{\sigma_u^2}$, we can show that condition (A38) is equivalent to the following:

$$\eta < 4(k+1)\frac{4k^2\theta^2+4k\theta+4\theta^2+Jk^2+8k\theta^2+4k^2\theta+k^2+Jk\theta+Jk^2\theta}{k(J+2k+4\theta+4k\theta+Jk)^2}. \tag{A39}$$

Setting $\theta = 0$, the right-hand side (RHS) of the above condition degenerates to $\frac{4(J+1)k(k+1)}{(J+2k+Jk)^2}$. Thus, for sufficiently small σ_ε (and hence θ), in a pure strategy equilibrium, we must have:

$$\eta < \frac{4(J+1)k(k+1)}{(J+2k+Jk)^2}. \tag{A40}$$

Taking derivative shows that $\frac{4(J+1)k(k+1)}{(J+2k+Jk)^2}$ is increasing in $k \in (0,1]$. Thus, its maximum is achieved at $k=1$; that is, $\max_{k\in(0,1]} \frac{4(J+1)k(k+1)}{(J+2k+Jk)^2} = \frac{4(J+1)2}{(J+2+J)^2} = \frac{2}{J+1}$. Hence, if $(J+1)\eta > 2$, then condition (A40) is never satisfied and thus, there exists no linear pure strategy equilibrium.

Part (b): Switch based on J and η

When both J and η are small:. Now, we fix J and consider a process of $\eta \to 0$. By the third part of Corollary 1, there is no mixed strategy equilibrium because $J\left(\frac{J+2}{J+1}\right)^2\eta < 2$ for sufficiently small η. Now let us show that there exists a unique pure strategy equilibrium.

At $\sigma_\varepsilon = 0$ (and hence $\theta = \frac{\sigma_\varepsilon^2}{\sigma_u^2} = 0$), the polynomial (A30) characterizing the pure strategy equilibrium becomes

$$(J+1)(J+2)^2 k(1-k)^2(k+1) = \eta(J+2k+Jk)^2. \tag{A41}$$

As $\eta \to 0$, the solution of k must either converge to 0 or 1. Now, we show that $k=0$ violates the SOC, whereas $k=1$ does not, so that the unique pure strategy equilibrium features $k \to 1$ (as $\eta \to 0$ and J is fixed).

Inserting $\theta = 0$ into the expressions of δ, β's, and λ's in Proposition 4, the SOC (67) degenerates to

$$k\sqrt{k} - \frac{\sqrt{\eta}}{\sqrt{k+1}} \frac{J+2}{4\sqrt{J+1}} \left(\frac{J}{J+2} + k\right)^2 > 0. \tag{A42}$$

As $\eta \to 0$ and $k \to 1$, the left-hand side (LHS) goes to 1, and, thus, the SOC is satisfied. In contrast, suppose that $k \to 0$ as $\eta \to 0$. By (A41), we must have

$$k \propto \eta \frac{J^2}{(J+1)(J+2)^2}. \tag{A43}$$

Inserting the above expression into the LHS of (A42), we have

$$k\sqrt{k} - \frac{\sqrt{\eta}}{\sqrt{k+1}} \frac{J+2}{4\sqrt{J+1}} \left(\frac{J}{J+2} + k\right)^2 \tag{A44}$$

$$\propto \sqrt{\eta}\left[\eta \frac{J^2}{(J+1)(J+2)^2}\sqrt{\frac{J^2}{(J+1)(J+2)^2}} - \frac{J+2}{4\sqrt{J+1}}\left(\frac{J}{J+2}\right)^2\right], \tag{A45}$$

which is negative as $\eta \to 0$. Thus, the SOC is violated.

When J or η is large:. First, fix η and let J diverge to ∞. The condition in the third part of Corollary 1 is satisfied, and, thus, there is a unique mixed strategy equilibrium. Now, we show that there exists no pure strategy equilibrium. As $J \to \infty$, the LHS of (A42) is

$$k\sqrt{k} - \frac{\sqrt{\eta}}{\sqrt{k+1}} \frac{J+2}{4\sqrt{J+1}} \left(\frac{J}{J+2} + k\right)^2 \tag{A46}$$

$$< \quad 1 - \frac{\sqrt{\eta}}{\sqrt{1+1}} \frac{J+2}{4\sqrt{J+1}} \left(\frac{J}{J+2} + 0\right)^2 \tag{A47}$$

$$\propto \quad -\frac{\sqrt{\eta}}{4\sqrt{2}}\frac{J}{J+2}\frac{J}{\sqrt{J+1}} \propto -\frac{\sqrt{\eta}}{4\sqrt{2}}\sqrt{J} < 0. \tag{A48}$$

Thus, the SOC for a pure strategy equilibrium is violated no matter the value of $k \in [0,1]$.

Second, fix J and let η diverge to ∞. Again, by the third part of Corollary 1, there exists a unique mixed strategy equilibrium. Also, the LHS of (A42) diverges to $-\infty$ and hence the SOC for a pure strategy equilibrium cannot be satisfied.

Proof of Proposition 6

We first examine the derivatives of the strategy variables. By the expressions of β_1, β_x, and δ in Proposition 3, direct computations yield

$$\frac{\partial \beta_1}{\partial \sigma_\varepsilon} < 0, \frac{\partial \beta_x}{\partial \sigma_\varepsilon} < 0, \text{ and } \frac{\partial \delta}{\partial \sigma_\varepsilon} < 0. \tag{A49}$$

By (59), we can show that $\left.\frac{\partial \sigma_z^2}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon = 0}$ has the opposite sign as

$$2J\eta(J+2)^2\left[\eta(J+2)^2 + J(J+1)\right] + \left(J^2 - 2J - 4\right)(J+1)^3, \tag{A50}$$

which is positive under the assumption of $(J+1)\eta \geq 2$. Hence, $\left.\frac{\partial \sigma_z^2}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon = 0} < 0$. By (61) and (64), we can show that

$$\left.\frac{\partial \beta_2}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon = 0} < 0 \Longleftrightarrow \left.\frac{\partial \lambda}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon = 0} > 0 \Longleftrightarrow J > 2. \tag{A51}$$

Next, we examine the price discovery variables, Σ_1 and Σ_2. Inserting the expressions of σ_z, δ, and $\beta's$ into the Σ variables in Proposition 1, we obtain

$$\Sigma_1 \quad = \quad \Sigma_0\left[1 - \frac{1}{1+\frac{4\theta}{J}+\eta\left(\frac{J+2+4\theta}{J+1}\right)^2}\frac{J+2+4\theta}{2(J+1)}\right], \tag{A52}$$

$$\Sigma_2 \quad = \quad \frac{\Sigma_0}{2}\frac{\begin{bmatrix} 16J\eta\theta^2 + \left(4J+16J\eta+8J^2\eta+4\right)\theta \\ +\left(4J\eta - J + 4J^2\eta + J^3\eta - J^2\right) \end{bmatrix}}{\begin{bmatrix} 16J\eta\theta^2 + \left(8J+16J\eta+8J^2\eta+4J^2+4\right)\theta \\ +\left(J+4J\eta+4J^2\eta+J^3\eta+2J^2+J^3\right) \end{bmatrix}}. \tag{A53}$$

Taking derivative of (A52) with respect to σ_ε shows $\frac{\partial \log \Sigma_1}{\partial \sigma_\varepsilon} > 0$. Taking derivative of (A53), we find that $\frac{\partial \log \Sigma_2}{\partial \sigma_\varepsilon}$ has the same sign as

$$J(J+2)^2\left[\eta - \frac{J+1}{(J+2)^2}\right] + 4\theta\left(J+4J\eta+2J^2\eta+4J\theta\eta+1\right). \tag{A54}$$

Under the assumption of $(J+1)\eta \geq 2$, we have $\eta - \frac{J+1}{(J+2)^2} \geq \frac{J^2+6J+7}{(J+1)(J+2)^2} > 0$ and thus, $\frac{\partial \log \Sigma_2}{\partial \sigma_\varepsilon} > 0$.

Finally, let us examine the profit variables. We insert the expressions of δ, β_1, and σ_z into the profit variables in Proposition 1 to show that the profits of the fundamental investor and each back-runner are, respectively,

$$E\left(\Pi^F\right)=\frac{\sigma_u\sqrt{\Sigma_0}}{2(J+1)}\frac{(J+1)^2\left[1+\frac{4\theta}{J}+\eta\left(\frac{J+2+4\theta}{J+1}\right)^2\right]+(1+4\theta)^2}{(J+2+4\theta)\sqrt{1+\frac{4\theta}{J}+\eta\left(\frac{J+2+4\theta}{J+1}\right)^2}}, \tag{A55}$$

$$E\left(\pi^B_{2,j}\right)=\frac{\sigma_u\sqrt{\Sigma_0}}{J^2}\frac{J+\left(\frac{J-1}{2}\right)4\theta}{(J+2+4\theta)\sqrt{1+\frac{4\theta}{J}+\eta\left(\frac{J+2+4\theta}{J+1}\right)^2}}. \tag{A56}$$

Taking derivative of the above profit expressions of profits with respect to σ_ε and setting $\sigma_\varepsilon=0$ lead to

$$\left.\frac{\partial E\left[\Pi^F\right]}{\partial\sigma_\varepsilon}\right|_{\sigma_\varepsilon=0}<0 \quad \text{if and only if}$$

$$(\eta+1)J^4-(\eta+3)J^3-6(3\eta+2)J^2-4(7\eta+2)J-8\eta>0;\ \text{and} \tag{A57}$$

$$\left.\frac{\partial E\left[\pi^B_{2,j}\right]}{\partial\sigma_\varepsilon}\right|_{\sigma_\varepsilon=0}>0 \quad \text{if and only if}$$

$$(\eta+1)J^4+\eta J^3-(10\eta+7)J^2-10(2\eta+1)J-4(2\eta+1)>0. \tag{A58}$$

Proof of Proposition 7

Note that as $\sigma_\varepsilon\to\infty$, we have $\theta=\frac{\sigma_\varepsilon^2}{\sigma_u^2}\to\infty$. So, we examine the derivatives evaluated at the limit $\theta\to\infty$. Applying the implicit function theorem to (A30), we have

$$\frac{\partial k}{\partial\theta}=-\frac{\partial f(k;\theta,\eta)/\partial\theta}{\partial f(k;\theta,\eta)/\partial k}. \tag{A59}$$

By (A31), $\left.\frac{\partial f(k;\theta,\eta)}{\partial k}\right|_{\theta\to\infty}<0$, and thus $\left.\frac{\partial k}{\partial\theta}\right|_{\theta\to\infty}$ has the same sign as $\left.\frac{\partial f(k;\theta,\eta)}{\partial\theta}\right|_{\theta\to\infty}$. Direct computation shows

$$\left.\frac{\partial f(k;\theta,\eta)}{\partial\theta}\right|_{\theta\to\infty}\propto\frac{J(2-k)}{4(k+1)\theta^2}>0. \tag{A60}$$

Hence, $\left.\frac{\partial k}{\partial\theta}\right|_{\theta\to\infty}>0$.

By the expressions of β_1 and λ_1 in Proposition 4, it is straightforward to show that $\left.\frac{\partial\beta_1}{\partial\sigma_\varepsilon}\right|_{\sigma_\varepsilon\to\infty}\propto\frac{1}{2k}\left.\frac{\partial k}{\partial\theta}\right|_{\theta\to\infty}>0$ and $\left.\frac{\partial\lambda_1}{\partial\sigma_\varepsilon}\right|_{\sigma_\varepsilon\to\infty}\propto\frac{1-k}{2k(k+1)}\left.\frac{\partial k}{\partial\theta}\right|_{\theta\to\infty}>0$.

By the expression of λ_2 in Proposition 4, we can compute

$$\frac{\partial\log\lambda_2}{\partial\theta}=\frac{Jk(k+1)4\theta(k+1)+k(J-2)}{2(2k+4\theta+4k\theta+Jk)\begin{pmatrix}4k^2\theta^2+4k\theta+4\theta^2+Jk^2+8k\theta^2\\+4k^2\theta+k^2+Jk\theta+Jk^2\theta\end{pmatrix}}$$

$$+\frac{1}{2}\left[\begin{array}{c}\frac{8k\theta^2+4\theta+2Jk+8\theta^2+8k\theta+2k+J\theta+2Jk\theta}{4k^2\theta^2+4k\theta+4\theta^2+Jk^2+8k\theta^2+4k^2\theta+k^2+Jk\theta+Jk^2\theta}\\ -\frac{1}{k+1}+\frac{2(2+4\theta+J)}{(2k+4\theta+4k\theta+Jk)}\end{array}\right]\frac{\partial k}{\partial \theta}. \tag{A61}$$

As $\theta \to \infty$, both the second term of the above expression is positive. Thus, $\left.\frac{\partial \lambda_2}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty} > 0$. Because $\beta_2 = \frac{1}{2\lambda_2}$, we have $\left.\frac{\partial \beta_2}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty} < 0$.

By the expression of δ in Proposition 4, we have

$$\frac{\partial \log \delta}{\partial \theta} = -\frac{\partial \log \lambda_2}{\partial \theta} - \frac{2\left(\frac{1}{k}+1\right)}{2\left(\theta\left(\frac{1}{k}+1\right)+1\right)+\frac{J}{2}-1} \tag{A62}$$

$$-\left[\frac{2\theta\left(-\frac{1}{k^2}\right)}{2\left(\theta\left(\frac{1}{k}+1\right)+1\right)+\frac{J}{2}-1}+\frac{1}{2}\frac{1}{k}\right]\frac{\partial k}{\partial \theta}. \tag{A63}$$

Inserting the expression of $\frac{\partial \log \lambda_2}{\partial \theta}$ in (A61) and the expression of $\frac{\partial k}{\partial \theta}$ in (A59) into the above expression, we can show

$$\left.\frac{\partial \delta}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty} \propto -\frac{4(k+1)}{2k+4\theta+4k\theta+Jk} < 0. \tag{A64}$$

Given $\beta_x = \frac{J}{2}\delta$, we also have $\left.\frac{\partial \beta_x}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty} < 0$.

Inserting $I=1$ and the expressions of δ and β's in Proposition 4 into the expressions of Σ's in Proposition 2, we have

$$\Sigma_1 = \frac{\Sigma_0}{k^2+1}, \tag{A65}$$

$$\Sigma_2 = \Sigma_0 \frac{k+2\theta+2k\theta}{(k+1)(2k+4\theta+4k\theta+Jk)}. \tag{A66}$$

Taking derivative of Σ_1 shows

$$\left.\frac{\partial \Sigma_1}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty} = -\frac{2k\Sigma_0}{\left(k^2+1\right)^2}\left.\frac{\partial k}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty} < 0 \tag{A67}$$

by $\left.\frac{\partial k}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty} > 0$. Taking derivative of Σ_2 with respect to θ and using the expression of $\frac{\partial k}{\partial \theta}$ in (A59), we can compute

$$\left.\frac{\partial \log \Sigma_2}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty} \propto \frac{Jk\theta(k+1)^2\left(k+2k^2+1\right)}{\left(4k+k^2-1\right)(2k+4\theta+4k\theta+Jk)}. \tag{A68}$$

Thus the sign of $\left.\frac{\partial \Sigma_2}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty}$ is determined by the sign of

$$g(k) \equiv 4k+k^2-1. \tag{A69}$$

Direct computation shows

$$g(k)>0 \Longleftrightarrow k>0.23607. \tag{A70}$$

Note that as $\theta \to \infty$, the value of k is determined by (A33), that is,

$$k\eta \approx (1-k)^2(1+k). \tag{A71}$$

It is easy to show that k as a function of η, which is implicitly determined by (A71), is decreasing in η. In addition, $k \to 1$ as $\eta \to 0$, and $k \propto \frac{1}{\eta}$ as $\eta \to \infty$. Thus, as η gradually increases from 0 toward ∞, k must gradually decrease from 1 toward 0. At the threshold of $k=0.23607$ in (A70), the corresponding value of η is $\hat{\eta}=3.0557$. Taken together, we have

$$\left.\frac{\partial \Sigma_2}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty} \Longleftrightarrow k>0.23607 \Longleftrightarrow \eta<\hat{\eta}\approx 3.06. \tag{A72}$$

Finally, let us examine the profit variables. Inserting $I=1$ and the expressions of δ, β's and λ's in Proposition 4 into the expressions of $E\left(\pi_{2,j}^B\right)$ and $E\left(\Pi^F\right)$ in Proposition 2, we can obtain

$$E\left(\pi_{2,j}^B\right)=\frac{\sqrt{\eta(k+1)(2k+4\theta+4k\theta+Jk)^2}\sigma_u\sqrt{\Sigma_0}}{4k\left[\begin{array}{c}2\left(\theta\left(\frac{1}{k}+1\right)+1\right)\\+\frac{J}{2}-1\end{array}\right]^2\sqrt{\left[\begin{array}{c}4k^2\theta^2+4k\theta+4\theta^2+Jk^2+8k\theta^2\\+4k^2\theta+k^2+Jk\theta+Jk^2\theta\end{array}\right]}}, \tag{A73}$$

$$E\left(\Pi^F\right)=\frac{k\sqrt{k}}{1+k}\sqrt{\Sigma_0}\sigma_u + \frac{\sqrt{\eta}(k+2\theta+2k\theta)\frac{k+2\theta+8k\theta+Jk^2+6k^2\theta+Jk+3k^2}{(k+1)\sqrt{k+1}(2k+4\theta+4k\theta+Jk)}}{\sqrt{\begin{array}{c}4k^2\theta^2+4k\theta+4\theta^2+Jk^2+8k\theta^2\\+4k^2\theta+k^2+Jk\theta+Jk^2\theta\end{array}}}\sqrt{\Sigma_0}\sigma_u. \tag{A74}$$

Direct computations show $\left.\frac{\partial E\left(\pi_{2,j}^B\right)}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty}<0$ and $\left.\frac{\partial E\left(\Pi^F\right)}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty}>0$.

The result $\left.\frac{\partial\left(\lambda_1\sigma_u^2+\lambda_2\sigma_U^2\right)}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty}>0$ directly follows from $\left.\frac{\partial \lambda_1}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty}>0$ and $\left.\frac{\partial \lambda_1}{\partial \sigma_\varepsilon}\right|_{\sigma_\varepsilon \to \infty}>0$.

Proof of Proposition 8

Now set $I=1$ and $\sigma_\varepsilon=0$ (and hence $\theta=\sigma_\varepsilon^2/\sigma_u^2=0$). Fix the values of $(\Sigma_0,\sigma_u,\sigma_U)$ and let J diverge to ∞. By part (b) of the proof of Proposition 5, the unique linear equilibrium is a mixed strategy equilibrium.

Inserting $\theta = 0$ into the expressions of δ, β's, and λ's in Proposition 3 and taking derivative with respect to J, we can sign the derivatives of these parameters directly.

Inserting $\theta = 0$ in (A52) and taking derivative, we find that

$$\frac{\partial \Sigma_1}{\partial J} < 0 \Longleftrightarrow (\eta - 1) J^2 + (4\eta - 2) J + (4\eta - 1) > 0. \tag{A75}$$

Thus, for sufficiently large J, $\frac{\partial \Sigma_1}{\partial J} < 0 \Longleftrightarrow \eta > 1$. Similarly, inserting $\theta = 0$ in (A53) and taking derivative, we can show

$$\frac{\partial \Sigma_2}{\partial J} < 0 \Longleftrightarrow \frac{(\eta - 1) J^2 + (4\eta - 2) J + (4\eta - 1)}{\eta J^2 + (4\eta - 1) J + (4\eta - 1)} > 0. \tag{A76}$$

Therefore, $\frac{\partial \Sigma_2}{\partial J} < 0 \Longleftrightarrow \eta > 1$ for sufficiently large J.

Inserting $\theta = 0$ into the profit expressions in (A55) and (A56) and taking derivatives, we can obtain

$$\frac{\partial E\left(\Pi^F\right)}{\partial J} > 0, \frac{\partial E\left(\pi_{2,j}^B\right)}{\partial J} < 0, \text{ and } \frac{\partial \left[J \times E\left(\pi_{2,j}^B\right)\right]}{\partial J} < 0, \tag{A77}$$

for sufficiently large J. The result of $\frac{\partial\left(\lambda\sigma_u^2 + \lambda\sigma_U^2\right)}{\partial J} < 0$ directly follows from $\frac{\partial \lambda}{\partial J} < 0$.

Proof of Proposition 9

Like in the proof of part (b) of Proposition 5, we still fix J and let $\eta \to 0$. From the proof of Proposition 5, we know that $k \to 1$, and its value is implicitly determined by Equation (A41). Formally, using (A41), we can compute

$$k \propto 1 - \sqrt{\eta} \frac{\sqrt{2(J+1)}}{J+2}. \tag{A78}$$

Applying the implicit function theorem to (A41), we have

$$\begin{aligned} \frac{\partial k}{\partial J} &= \frac{(J + 2k + Jk - 4)k\left(1 - k^2\right)}{(J+2)\left(\left(2k^3 + 5k^2 + 2k - 1\right)J + \left(4k^3 + 2k^2 + 2k\right)\right)} \\ &\propto \frac{J-1}{\sqrt{2(J+1)}(J+2)^2}\sqrt{\eta}, \end{aligned} \tag{A79}$$

where the second equation follows from (A78). Using (A78), direct computation shows the value of k is higher at $J = 2$ than at $J = 1$. Combining with (A79), we know k is increasing in J, and thus,

$$\frac{\partial k}{\partial J} > 0 \Rightarrow \frac{\partial \beta_1}{\partial J} > 0, \tag{A80}$$

by the expression of β_1 in Proposition 4.

Similarly, using the expressions of $\beta_2, \beta_x, \delta, \lambda_1$, and λ_2 in Proposition 4, together with $\theta = 0$, we can compute

$$\frac{\partial \beta_2}{\partial J} > 0, \frac{\partial \beta_x}{\partial J} < 0, \frac{\partial \delta}{\partial J} < 0, \frac{\partial \lambda_1}{\partial J} > 0, \frac{\partial \lambda_2}{\partial J} < 0. \tag{A81}$$

Inserting $\theta = 0$ in (A65) and (A66), taking derivative and combining with (A79), we can show

$$\frac{\partial \Sigma_1}{\partial J} < 0 \text{ and } \frac{\partial \Sigma_2}{\partial J} < 0. \tag{A82}$$

Inserting $\theta = 0$ in (A74) and (A73), taking derivative and combining with (A78) and (A79), we have

$$\frac{\partial \log E\left(\Pi^F\right)}{\partial J} \propto -\frac{1}{4}\frac{4J+5}{(J+2)^2(J+1)^2}\sqrt{2(J+1)}\sqrt{\eta} < 0, \tag{A83}$$

$$\frac{\partial \log E\left(\pi_{2,j}^B\right)}{\partial J} \propto -\frac{1}{2}\frac{3J+4}{(J+1)(J+2)} < 0, \tag{A84}$$

$$\frac{\partial(\lambda_1 + \lambda_2 \eta)}{\partial J} \propto -\frac{J}{2\sqrt{2(J+1)}(J+2)^2}\sqrt{\eta} < 0, \tag{A85}$$

and

$$\frac{\partial \log\left[J \times E\left(\pi_{2,j}^B\right)\right]}{\partial J} \propto -\frac{1}{2}\frac{J^2 - 2J - 4}{J(J+1)(J+2)}, \tag{A86}$$

which implies

$$\frac{\partial\left[J \times E\left(\pi_{2,j}^B\right)\right]}{\partial J} < 0 \Longleftrightarrow J^2 - 2J - 4 > 0 \Longleftrightarrow J > 3. \tag{A87}$$

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