Statistical properties of random walk with resetting

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We study motion of a random walker in the presence of stochastic resetting. Stochastic resetting is a simple mechanism which can bring the walker from any position to a predetermined specific location with some probability. Thus, in addition to the short range diffusive motion, the walker also experiences intermittent long jumps that reset the walker back at this preferred location. In this project, we first review the spatial properties of a 1D random walker. We then extend the studies by implementing resetting which dramatically changes the statistical properties. Notably, we see resetting renders a time invariant steady state probability distribution with a saturating mean squared displacement for the walker at long times. This is in stark comparison to the reset free case. We present analytical results and corroborate with numerical simulations.

I. INTRODUCTION AND MOTIVATION

Random walk is defined as a mathematical representation of movement through random steps on a lattice at discrete times [1]. We will define a walker whose movement is contained in 1 dimension only with discrete timestamps. It is known that the reaching time of a simple symmetric random walker to a specific coordinate L starting from the origin is infinite. This is because the walker has equal probability $p = q = \frac{1}{2}$ to make right and left jumps. Thus there are stochastic trajectories which go to the opposite direction of the desired site L thus resulting in an overshooting time. This can be also understood in the following qualitative way: position distribution of a random walker concurs to a time dependent distribution which never reaches equilibrium. Thus, the walker is never confined and evolves indefinitely spreading the trajectories to explore the extremes of the phase space.

The above mentioned discussion clearly suggests that an ideal random walk is not an efficient search strategy. Thus, a long standing problem in random walk literature has been to design efficient search strategies that can make the walker to complete a search process in finite time. One such strategy is known as 'resetting' which has been a focal point of current studies in statistical physics [2, 3]. Here, in addition to short ranged nearest neighbour jumps by the walker, we also introduce another jump (of arbitrary range) process that takes the walker from any coordinate in the lattice and puts to a preferred location (say the origin) in zero time. This is like 'resetting a walker to the origin'. In this report, we study motion of such a walker. Our study is a mixture of both analytical and numerical results. The most important observation that came out from our study is the following: We find that the probability density function of the resetting-walker attains to a time independent form in large time. This function is peaked around the origin which tells us that the most probable position for the walker to be found is near the origin. Crucially, this observation hints towards the possible outcome that the walker may reach the target in a finite time compared to the non-resetting case where the walker tends to escape. This in turns suggests that there is a significant gain in terms of the search performance by the introduction of the resetting.

The remainder of the report is structured as follows. We first review the spatial properties of a simple random walker. Next, we add resetting to the walker, and write governing equations. We then show how some of these properties are being ramified in the presence of resetting.

II. NOTATIONS

For brevity, we start our discussion by introducing the notations. Jumps and steps are used interchangeably in paper. We consider p as the probability of taking $+1$ step, to right and q as the probability of taking -1 step, to left. We further denote x_n as the position of the walker after n steps. Finally, we denote $P(m, N)$ as the probability distribution function for finding the walker at point m after N steps.

III. MODEL

We consider a random walker which is at origin at $t = 0$. At each step, the walker is independent to go either left or right with probability of q or p respectively where $p + q = 1$ (Fig. 1)

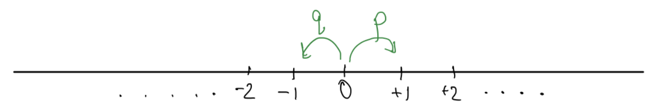

FIG. 1. A schematic of random walker defined in one dimension lattice with movement given as $+1$ step with probability p or a step of -1 with probability $1 - p$ or q.

We are interested in the following question – what is the

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probability $p(m, N)$ that the walker will be at m step point after taken N steps ? To answer this question, let us denote $N = n_1 + n_2$ as the total number of steps taken where $n_1 = \frac{1}{2}(N+m)$ are jumps to the right and $n_2 = \frac{1}{2}(N-m)$ are jumps to the left. Therefore, the net displacement , m becomes $m = n_1 - n_2$.

Since the left-right jumps are independent of each other, the probability for making n_1 jumps to right and n_2 jumps to the left is given by

$$p^{n_1} q^{n_2} = p^{\frac{1}{2}(N+m)} q^{\frac{1}{2}(N-m)} \tag{1}$$

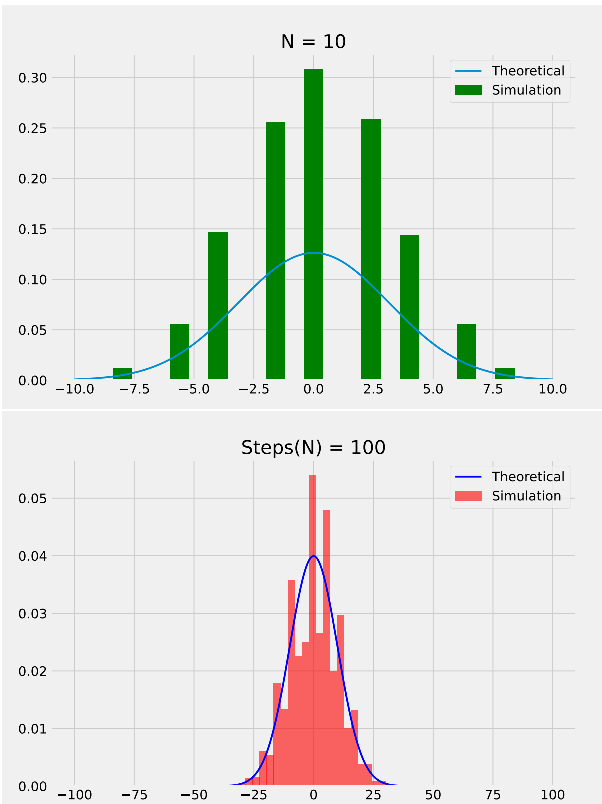

FIG. 2. Theory and Simulation. We observe how the plots get refined with increasing number of steps. Position of walker essentially, at whole depicted by the distribution in the latter figure whereas in the former figure, the distribution does not gives us the accurate answers.

Let us now compute the total number of paths made by the walker in N steps. The first object of the jumps can be chosen in N ways, later the $(N-1)$, then $(N-2)$......$(N-n_1-1)$ ways.
So, the total number of distinguishable ways to have n_1 steps to the right and n_2 to the left will be

$$\frac{N!}{n_1!} = \frac{N!}{n_1!(N-n_1)!} \tag{2}$$

So, therefore the probability of being at at position m after N jumps or notably $p(m, N)$ is

$$p(m, N) = \frac{N!}{(\frac{N+m}{2})!(\frac{N-m}{2})!} p^{\frac{1}{2}(N+m)} q^{\frac{1}{2}(N-m)} \tag{3}$$

Devising the equation for number of steps to the right ,

$$n \equiv n_1 = \frac{N+m}{2} \tag{4}$$

$$p(m, N) = p_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n} \tag{5}$$

$$P_N(n) = \binom{N}{n} p^n q^{N-n} \tag{6}$$

which is verified in Fig. 2.

A. Moments of the displacement for the random Walk

We can calculate all moments of n at any fixed time N since we know the probability distribution $p(m, N)$ from Eq. (6). For first moment, we have

$$np^n = p\frac{\partial}{\partial p}(p^n) \tag{7}$$

$$\langle n \rangle = \sum_{n=0}^{N} np^n q^{N-n}, \tag{8}$$

$$\sum_{n=0}^{N} \binom{N}{n} p\frac{\partial}{\partial p}(p^n) q^{N-n} \tag{9}$$

$$p\frac{\partial}{\partial p} \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} \tag{10}$$

$$p\frac{\partial}{\partial p}(p+q)^N = Np(p+q)^{N-1} = Np \tag{11}$$

Further for second moment,

$$\langle n^2 \rangle = \sum_{n=0}^{N} n^2 p_N(n) = \sum n \underbrace{\langle n p_N(n) \rangle}_{A}$$

and we already have derived the value to the quantity A,

$$\sum_{n=0}^{N} n^2 p_N(n) = n\frac{\partial}{\partial p} \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} p\frac{\partial}{\partial} p^n$$
$$p\frac{\partial}{\partial p}(p+q)^N = p\frac{\partial}{\partial p}\left(Np + (p+q)^{N-1}\right)$$
$$= (Np) + \Big(N-1)Np^2\Big)$$

Moments for the displacement simply reads

$$< m >= 2 < n > -N = N(p-q) \tag{12}$$

and for second moment ,

$$< m^2 >= 4 < n^2 > -4 < n > N + N^2 \tag{13}$$

Variance of displacement,

$$\sigma_m^2 = 4\sigma^2 = 4Npq, \tag{14}$$

which is verified in Fig. 3.

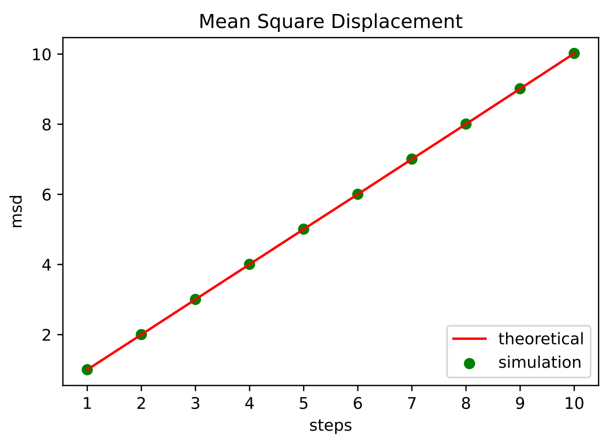

FIG. 3. Mean Square Displacement for given walker with 10 steps. As is the case for simple random walker ($p = q = 0.5$) we get a linear dependence.

We plot the MSD as a function of N in Fig. 3. We see a perfect match between the theoretical and simulation result.

B. Arguments based on the Central Limit Theorem

We observed that PDF ($p_m(N)$) for 1D walker is exactly Binomial Distribution. But when we start to increase the number of walkers, we start to observe some really interesting results. This can be seen in Fig. 2 and Fig. 4. We now will understand the math behind this Gaussian behavior. First we note that, motion of a RW(random walker) can be written as

$$x_n = x_{n-1} + \zeta_n \tag{15}$$

where ζ_n is a random variables defined as

$$\zeta_n = \begin{cases} +1 & \text{with prob p} \\ -1 & \text{with prob q} \end{cases} \tag{16}$$

equivalently writing, to note $p(\zeta)$ has the following PDF

$$p(\zeta) = p\delta(\zeta - 1) + q\delta(\zeta + 1) \tag{17}$$

Iterating the steps for $n = 0, 1, 2, ...$ we write,

$$x_1 = x_0 + \zeta_0 \tag{18}$$
$$x_2 = x_1 + \zeta_1 \tag{19}$$
$$x_2 = x_0 + \zeta_0 + \zeta_1 \tag{20}$$
$$.... \tag{21}$$
$$x_n = x_0 + \sum_{i=1}^{n-1} \zeta_i \tag{22}$$

Note that at each step, ζ is statistically identical and also independent. In other words, ζ is an $i.i.d$ random variable. From the knowledge of central limit theorem, we know that a random variable which is a sum of such random variables will attain to a Gaussian distribution as long as the mean and variance of the parent random variables are finite.

This is the reason, we see the distribution converging to a Gaussian. note that, the Gaussianity is observed within the $\sqrt{N}$ width around the mean. Refer Fig. 4. It is crucial to note that this result hold true for any random variable ζ (independent of any specific form of their PDF) as long as they remain IID with finite mean and variance.

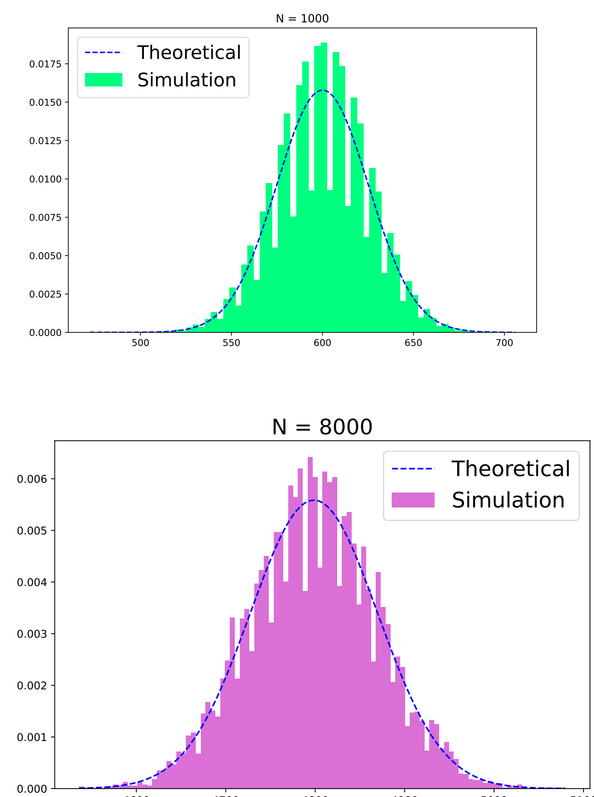

FIG. 4. Walker with $p = 0.8$ and $q = 0.2$ for 1000 and 8000 steps, we continue to get a Gaussian distribution with μ defined as $N(p - q)$. Our simulations holds good accuracy with the mathematics.

We tested the walker with $p = q = \frac{1}{2}$ and observed that mean tends at 0 i.e $\mu = 0$ which is in total agreement with the theory. We, likewise obtained a Gaussian distribution.
Then we allowed walker with $p \neq q$ and we set $p = 0.6$ and $q = 0.4$ and as we would expect the average $\mu \neq 0$ and resultant we observed a rightward shift in the plot as $p > q$

IV. RESETTING WALKER

In this section, we discuss the resetting walker.We allow the walker to restart the whole process for certain probability r. By 'resetting', we mean to relocate the walker instantly to the initial position and the process restarts from there.

To note, when we say the walker is reset the x_r, we explicitly intend that walker have no memory of past walk. Else, if such conditions are not met, we can not ignore the previous walk after each resetting. This the necessary condition for resetting.

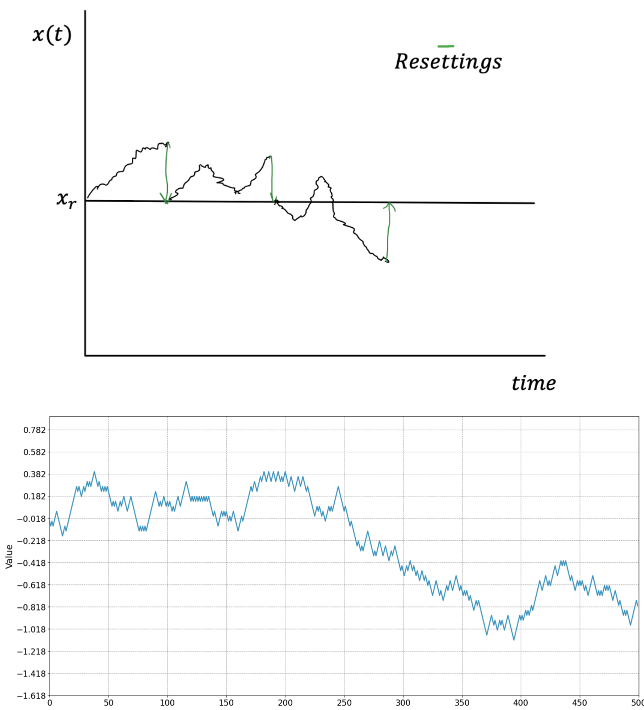

FIG. 5. A schematic of walker with resetting (above) and resetting-less (below) events.

So, for resetting walker, we can define the governing equation for the position of the walker as

$$x_{n+1} = \begin{cases} x_r & \text{with probab r} \\ x_n + \zeta_n & \text{with probab 1-r} \end{cases} \quad (23)$$

i.e., with probability r, the walker is reset to x_r independent of its current location. Otherwise, it makes the random moves.

A. Governing Equations for the PDF

To write the governing time dependent equations for the PDF, we first observe that the walker erases its memory after each resetting event. In other words, resetting mechanism makes the intervals between any two reset events independent from each other. This is a crucial observation that helps us reduce the problem to great extent. Provided that n resetting happens (note that at least 1 resetting should happen for this approach), we can have the advantage of eliminating all the $n-1$ resetting in our mind and keep track of only the last resetting event. Thus, our problem reduces to a simple random walker with one resetting event conditioned that this was the last one before the measurement.

We now define a walker with following properties. We denote the step number of last resetting as n_L, the final position of walker at present defined by x, and steps taken post last resetting defined by $N - n_L = m$. We assume that after each resetting, the walker starts from the location x_r.

PDF of the position, $P(x, N)$ can then be decomposed in the following way

$$P_r(x, N|x_0, 0; x_r) = \Big\{ \text{(resetting trajectories)} + \text{(non-resetting trajectories)} \Big\}$$

Though the probability of a resetting walker, taking a a walk without any resetting in quite less but nevertheless it is finite. There can be such walk. As we can infer, that walk will simply be a general random walk as we discussed earlier.

Now, for N steps, if a walker takes step without resetting, the probability is $1 - r$. So, for N such steps, the probability of not having a single resetting jump is simply

$$(1 - r) \cdot (1 - r) \cdot (1 - r).....N\ times = (1 - r)^N \quad (24)$$

We can formalise the PDF of position position as

$$P_r(x, N|x_0, 0; x_r) = \Big((1 - r)^N P_0(x, N|x_0, 0) + \quad (25)$$

$$\sum_{n_L=1}^{N} r(1 - r)^{N-n_L} P_0(x, N|x_r, n_L) \Big), \quad (26)$$

where $P_0(x, N|x_0, 0)$ is the PDF of the walker in the absence of resetting. Thus the PDF of the resetting walker can be written in terms of the reset-free walker. Since we know the reset-free PDF (binomial), we can just plug in that expression in above and compute the reset PDF. This was not done analytically but we have done it numerically and checked with the simulation results.

We further start to observe how the resetting affects the distribution. With increasing number of steps, we instantly observe that graph remains no longer Gaussian. Secondly, we note that after many steps, the PDF attains to a time independent form. To demonstrate this (for different time steps), we provide Fig. 7.

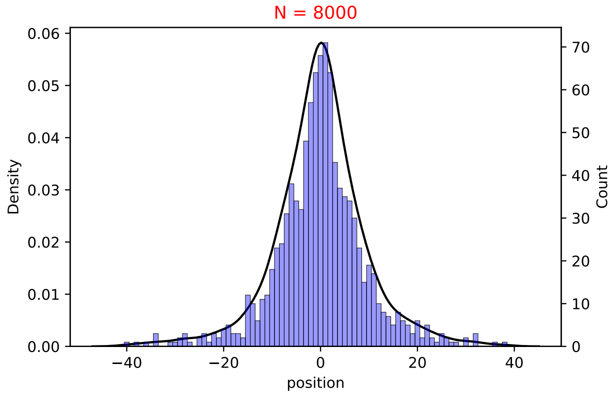

FIG. 6. Walker with resetting for $p_r = 0.3$.Defined for symmetric walker with $p = q = 0.5$. As expected, we see a spike and at peak we can observe the plot no longer remains differentiable.

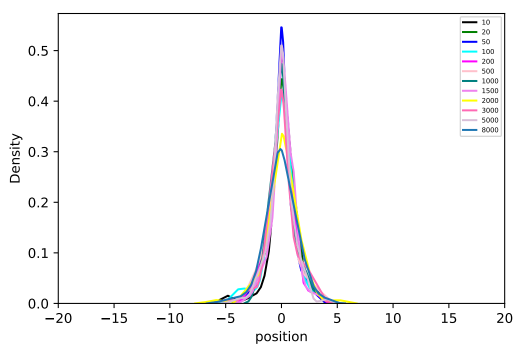

FIG. 7. Position distribution of walker for different number of steps. We observe that plots start to overlap with increment in number of steps. (p,q) and resetting are defined for probability 0.6 and 0.4 respectively and $p_r = 0.3$.

We already saw in Fig. 7 that after a certain number of steps, the position distribution continues to remain invariant under time. Since the PDF is independent of time, moments of the walker should also be. We confirm this result by observing the mean squared displacement (MSD) of the walker. Thus, for a large number of steps, we should obtain a plot that reflects to this above-mentioned fact. Fig. 8 indeed refers to this observation.

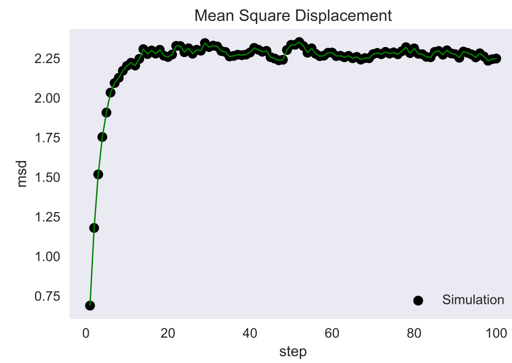

FIG. 8. MSD of a random walker in the presence of resetting as a function of steps. p and q defined for the walker are 0.6 and 0.4 respectively and $p_r = 0.3$.

V. CONCLUSION

In this report, we reviewed the problem of a simple random walker and learned that the position distribution of the walker is given by the Binomial Distribution. We discussed the results for symmetric as well as biased walkers. The mean displacement is given by $N(p - q)$ which explains centering of the peak around 0 for symmetric ($p = q$) and shift for biased one ($p \neq q$) respectively. In the second part of the report, we introduced the concept of a resetting walker. Interestingly, we observed that the position distribution attains to a time-independent form after a large number of steps. We confirmed it numerically. This was also confirmed with the fact that the MSD saturates to a threshold (plateau) value after certain steps. This naturally reinstates the fact that a saturating MSD may imply a steady state (time independent state) which is indeed the case here. As a future goal, we would like to delve deeper into the first passage time statistics and show explicitly how the knowledge gained here can be utilized to predict many statistical metrics such as the mean, variance of the search time.

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[3] Bonomo, O.L. and Pal, A., 2021. First passage under restart for discrete space and time: Application to one-dimensional confined lattice random walks. Physical Review E, 103(5), p.052129.