DERIVATION OF KEPLAR 1st LAW

Using effect of gravitation and Newton's second law

$$L = r \times p$$

So,

$$\frac{dL}{dt} = \frac{dr}{dt} \times p + r \times \frac{dp}{dt}$$

$$= v \times p + r \times F$$

Now, $v \times p \rightarrow 0$

and $r \times F \rightarrow F$ is a central force directed inward along r, therefore $r \times F \rightarrow 0$

$\therefore \frac{dL}{dt} = 0$ } angular momentum of a system is a constant for central force law.

Now, alternatively,

$$L = \mu r \times v \qquad (\mu = \text{reduced mass})$$

$$\left(\mu r \hat{r} \times \frac{d}{dt} r\hat{r} \right)$$

$$= \mu r \hat{r} \times \left(\frac{dr}{dt} \hat{r} + r \frac{d\hat{r}}{dt} \right)$$

$$= \mu r^2 \hat{r} \times \frac{d}{dt} \hat{r} \quad \left(\text{as } \hat{r} \times \hat{r} = 0 \right)$$

In other word, the acceleration of the reduced mass due to gravitational force exerted by M is

$$a = -\frac{GM}{r^2} \hat{r}$$

Now, taking vector cross product of accelerations of μ with its own orbital angular momentum,

$$a \times L = -\frac{GM}{r^2} \hat{r} \times \left(\mu r^2 \hat{r} \times \frac{d}{dt} \hat{r} \right)$$

$$= -GM\mu \, \hat{r} \times \left(\hat{r} \times \frac{d}{dt} \hat{r} \right)$$

and $A \times (B \times C) = (A \cdot C) B - (A \cdot B) C$ gives,

$$a \times L = -GM\mu \left[\left(\hat{r} \cdot \frac{d}{dt} \hat{r} \right) \hat{r} - \left(\hat{r} \cdot \hat{r} \right) \frac{d}{dt} \hat{r} \right]$$

and as $\hat{r}$ is a unit vector, and $\hat{r} \cdot \hat{r} = 1$

$$a \times L = GM\mu \frac{d}{dt} \hat{r}$$

$$\frac{d}{dt} (v \times L) = \frac{d}{dt} (GM\mu \hat{r})$$

and integrating wrt

$$V \times L = GM\mu \hat{r} + D$$

$D \rightarrow$ constant vector

and as v is max at perihelion, $v \times L \rightarrow$ max.
and magnitude of D determines the eccentricity of orbit
So,

$$r \cdot (V \times L) = GM\mu r \hat{r} \cdot \hat{r} + r \cdot D$$

and using identity

$$A \cdot (B \times C) = (A \times B) \cdot C \text{ gives}$$

$$(r \times V) \cdot L = GM\mu r + rD\cos\theta$$

and recalling angular momentum,

$$\frac{L^2}{\mu} = GM\mu r \left(1 + \frac{D\cos\theta}{GM\mu}\right)$$

and $e \equiv \dfrac{D}{GM\mu}$

which gives

$$\boxed{r = \frac{L^2}{\mu^2 GM(1 + e\cos\theta)}}$$

$\rightarrow$ equation of ellipse.