PROOF OF $E = mc^2$

We already have, $F = \frac{dp}{dt}$ (partical at rest)

Then the particle's final Kinetic Energy,

$$K = \int_{x_i}^{x_f} F\,dx = \int_{x_i}^{x_f} \frac{dp}{dt}\,dx = \int_{p_i}^{p_f} \frac{dx}{dt}\,dp = \int_{p_i}^{p_f} v\,dp$$

Integrating by parts and using initial condition, $p_i = 0$

$$K = p_f v_f - \int_0^{v_f} p\,dv$$

$$= \frac{m v_f^2}{\sqrt{1 - \frac{v_f^2}{c^2}}} - \int_0^{v_f} \frac{mv}{\sqrt{1 - v^2/c^2}}\,dv$$

$$= \frac{m v_f^2}{\sqrt{1 - \frac{v_f^2}{c^2}}} - mc^2\left(\sqrt{1 - v^2/c^2} - 1\right)$$

and eventually the expression becomes,

$$K = mc^2\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right) = mc^2(\gamma - 1)$$

and consequently, the above formula reduces to

$K = \frac{1}{2}mu^2$ or $K = \frac{p^2}{2m}$ in low speed limit.