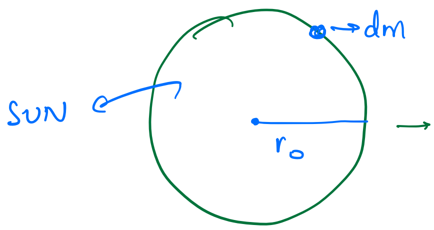
HYDROSTATIC EQUILIBRIUM AND VIRIAL THEOREM

Free fall timescale of the sun i.e. time taken for it to collapse to a point if there were no pressure support.



→ The potential Energy will be

$$dU = -\frac{G M(r_0)\, dm}{r_0}$$

→ mass interior to r_0

Now, velocity of object dm as it falls towards the center.

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = \frac{G M(r_0)}{r} - \frac{G M(r_0)}{r_0}$$

and $M(r_0)$ assumed constant,

Separation of variable and integrating,

$$\underset{\text{(free fall)}}{\tau_{ff}} = \int_0^{\tau_{ff}} dt = -\int_{r_0}^{0} \left[2GM(r_0)\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} dr$$

$$= \left(\frac{r_0^3}{2GM(r_0)}\right)^{1/2} \boxed{\int_0^1 \left(\frac{x}{1-x}\right)^{1/2} dx} \rightarrow \text{equals } \pi/2$$

mean density after factoring $4\pi/3$

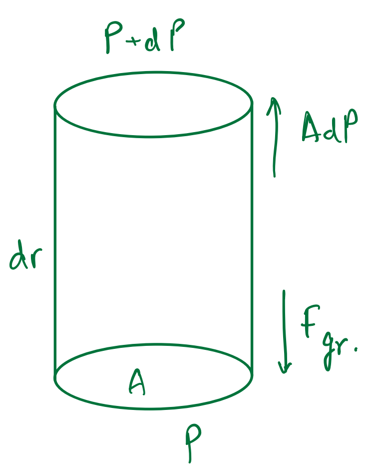
$$\therefore \quad \tau_{ff} = \left(\frac{3\pi}{32 G \bar{\rho}} \right)^{1/2}$$

and substituting the values, we get

$$\tau_{ff_\odot} = 1800 \text{ seconds}$$

→ Thus without pressure support, this is how much it will take for sun to collapse.

This does not happen because Sun is in Hydrostatic equilibrium



→ Force of gravitation on mass element of cross sectional Area A

So, balancing happens as

$$\frac{-GM(r)\,dm}{r^2} - A\,dP = 0$$

and also, $\rho(r) = \frac{dm}{A\,dr} \Rightarrow dm = \rho(r)\,A\,dr$

$$\Rightarrow \frac{-GM(r)\,\rho(r)\,A\,dr}{r^2} - A\,dP = 0$$

And some algebraic manipulation,

$$\Rightarrow \frac{-GM(r)\,\rho(r)\,A}{r^2} - A\frac{dP}{dr} = 0$$

Equation of Hydrostatic Equilibrium. $\leftarrow$ $\therefore$ $\boxed{\dfrac{dP(r)}{dr} = \dfrac{-GM(r)\,\rho(r)}{r^2}}$

The pressure gradient is negative because to counteract gravity the pressure must decrease outward (i.e. with increasing r)

Tweaking the equation a bit

$$\int_0^{r_*} 4\pi r^3 \frac{dP}{dr}\,dr = -\int_0^{r_*} \frac{GM(r)\,\rho(r)}{r^2}\,4\pi r^3\,dr$$

$$\Rightarrow \int_0^{r_*} 4\pi r^3 \frac{dP}{dr}\,dr = \boxed{-\int_0^{r_*} \frac{GM(r)\,\rho(r)}{r}\,4\pi r^2\,dr}$$

$\rightarrow$ This quantity resembles the energy that would be gained by making a star inside out from bringing a shell after shell from infinite distance.

And this is just the gravitational potential self energy of the star. $\leftarrow$ Each shell have mass,

$$dM(r) = \rho(r)\,4\pi r^2\,dr$$

Now,

$$\left[P(r)\,4\pi r^3\right]_0^{r_*} - 3\int_0^{r_*} P(r)\,4\pi r^2\,dr$$

and surface of star will be defined as the radius when Pressure goes to zero.

Mean Pressure $\boxed{\bar{P} = -\frac{1}{3}\frac{E_{gr}}{V}}$ → Virial Theorem for gravitionally bound system.

Now, if we take Thermodynamics Equations,

$PV = NKT$ → at every point in star, the gas equation is

and its thermal energy will be

$$E_{thermal} = \frac{3}{2}NKT = \frac{3}{2}PV$$

$$\therefore P = \frac{2}{3}\frac{E_{thermal}}{V}$$ → meaning local pressure equals 2/3 the local thermal energy density.

$$\bar{P}V = \frac{2}{3}\left(E_{thermal}\right)^{total}$$

and again from Virial Theorem,

$$\boxed{E^{total}_{thermal} = -\frac{E_{grav}}{2}}$$ → culmination of above formulas

The above equation says that when a star contracts and loses energy, i.e., its gravitational self energy becomes more negative, its thermal energy rises. This

Now, the third form of Virial Theorem is obtained by considering both thermal and gravitational energy.

$$E_{total} = E_{th}^{total} + E_{gr} = -E_{th}^{total} = \frac{E_{gr}}{2}$$

E_{total} being negative means that star is bound.

And as the stars radiate away their energy the gravitational energy becomes more negative i.e. the star will eventually collapse.

Now, for ρ = constant

$$E_{gr} = -\int_0^{r_*} \frac{GM(r)\,\rho(r)\,4\pi r^2 dr}{r}\,dr = -\int_0^{r_*} \frac{G\,\frac{4}{3}\pi r^3 \rho^2\, 4\pi r^2\, dr}{r}$$

$$= -\frac{3}{5}\,\frac{GM_*^2}{r_*}$$

$$\text{or } E_{gr} \sim -\frac{GM^2}{r}$$

The mean pressure inside sun is

$$\bar{P}_\odot \sim \frac{1}{3}\,\frac{GM_\odot^2}{\frac{4}{3}\pi r_\odot^3\, r_\odot} = \frac{GM_\odot^2}{4\pi r_\odot^4} \approx 10^{15}\ \text{dyne cm}^{-2} = \boxed{10^9\ \text{Atm}}$$

And to get the associated temperature with this pressure, we employ virial Theorem, to deduce Virial Temperature,

$$\frac{3}{2} N k T_{virial} \sim \frac{1}{2} \frac{G M_\odot^2}{r_\odot} = \frac{1}{2} \frac{G M_\odot N \bar{m}}{r_\odot}$$

let us assume again a classical non-relativistic ideal gas, with particles of mean mass $\bar{m}$

And considering ionized Hydrogen ion

$$\bar{m} = \frac{m_e + m_p}{2} = \frac{m_H}{2} \quad \left\{ \text{as } m_e \approx \frac{1}{2000} m_p \right\}$$

and $\boxed{m_H = 1.7 \times 10^{-24}\ g}$

Then typical thermal energy will be,

$$k T_{virial} \sim \frac{G M_\odot m_H}{6 r_\odot}$$

$$= \frac{6.67 \times 10^{-8}\ cgs \times 2 \times 10^{33}\ g \times 1.7 \times 10^{-24}\ g}{6 \times 7 \times 10^{10}\ cm}$$

$$\left\{ \begin{aligned} k &= 1.4 \times 10^{-16}\ erg\ K^{-1} \\ &= 8.6 \times 10^{-5}\ eV\ K^{-1} \end{aligned} \right.$$

$$= 5.4 \times 10^{-10}\ erg = 0.34\ keV$$

$$T_{virial} = 4 \times 10^6$$

→ and clearly nuclear reaction start to happen at this T.

This process of nuclear reactions replenishes the energy that is radiated off the star.

and consequently halting the gravitational collapse.