Stat 602 Homework 2

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Due 3/4/2019

Problem 1

Problem 4.3

Problem 4.4

Part a

```
## k-Nearest Neighbors
##
## 146 samples
##   9 predictor
##   2 classes: '1', '2'
##
## Pre-processing: centered (9), scaled (9)
## Resampling: Cross-Validated (10 fold)
## Summary of sample sizes: 131, 132, 132, 131, 132, 131, ...
## Resampling results across tuning parameters:
##
##   k   Accuracy   Kappa
##    1  0.8019048  0.6040763
##    2  0.8371429  0.6751192
##    3  0.8033333  0.6068505
##    4  0.8104762  0.6246800
##    5  0.7547619  0.5119849
##    6  0.7552381  0.5139579
##    7  0.7409524  0.4842320
##    8  0.7480952  0.5012970
##    9  0.7476190  0.5023240
##   10  0.7271429  0.4612502
##   11  0.7609524  0.5273293
##   12  0.7680952  0.5406763
##   13  0.7542857  0.5131045
##   14  0.7466667  0.4982815
##   15  0.7609524  0.5268529
##   16  0.7604762  0.5249862
##   17  0.7676190  0.5388204
##   18  0.7333333  0.4715346
##   19  0.7542857  0.5129146
##   20  0.7542857  0.5115648
##
## Accuracy was used to select the optimal model using the largest value.
## The final value used for the model was k = 2.
```

Based on 10-fold cross-validation, the accuracy is estimated to be highest when $k = 4$.

Part b

The classification rule $\hat{y} = 2I[t(x) \geq k/2] + I[t(x) < k/2]$ is equivalent to $\hat{y} = 2I[t(x)/k \geq 1/2] + I[t(x)/k < 1/2]$. Here, $t(x)/k$ is an estimate of $E[\text{type} = 2|x] = P(\text{type} = 2|x)$. Thus, this is directly an estimate of a posterior class probability which accounts for the prior class probabilities. This means that no modification of kNN needs to be made to account for differing prior probabilities.

Problem 5.1

The following are decompositions using the matrix

$$X = \begin{bmatrix} 2 & 4 & 7 & 2 \\ 4 & 3 & 5 & 5 \\ 3 & 4 & 6 & 1 \\ 5 & 2 & 4 & 2 \\ 1 & 3 & 4 & 4 \end{bmatrix}$$

Part a

The QR decomposition of X is:

$$Q = \begin{bmatrix} -0.27 & 0.57 & 0.77 & 0.04 \\ -0.54 & -0.07 & -0.13 & 0.56 \\ -0.4 & 0.37 & -0.35 & -0.71 \\ -0.67 & -0.5 & 0.1 & -0.13 \\ -0.13 & 0.53 & -0.5 & 0.4 \end{bmatrix}, \quad R = \begin{bmatrix} -7.42 & -6.07 & -10.25 & -5.53 \\ 0 & 4.15 & 5.99 & 2.28 \\ 0 & 0 & 1.06 & -1.23 \\ 0 & 0 & 0 & 3.57 \end{bmatrix}$$

Thus, the basis for $C(X)$ will be the columns of the matrix Q.

The Singular Value Decomposition of X is:

$$U = \begin{bmatrix} -0.5 & 0.53 & 0.16 & -0.66 \\ -0.5 & -0.59 & 0.13 & -0.11 \\ -0.46 & 0.49 & -0.25 & 0.65 \\ -0.39 & -0.33 & -0.68 & -0.09 \\ -0.37 & -0.17 & 0.65 & 0.35 \end{bmatrix}, \quad D = \begin{bmatrix} 16.58 & 0 & 0 & 0 \\ 0 & 3.78 & 0 & 0 \\ 0 & 0 & 3.38 & 0 \\ 0 & 0 & 0 & 0.55 \end{bmatrix},$$

$$V = \begin{bmatrix} -0.4 & -0.43 & -0.8 & 0.12 \\ -0.44 & 0.3 & 0.18 & 0.83 \\ -0.71 & 0.45 & 0.04 & -0.54 \\ -0.38 & -0.72 & 0.57 & -0.06 \end{bmatrix}$$

Thus, the basis for $C(X)$ will be the columns of the matrix U.

Part b

We can find the eigen (spectral) decomposition of $X'X$ by calculating the eigenvalues, D^2, and the eigenvectors, which will be the columns of V:

$$D^2 = \begin{bmatrix} 274.94 \\ 14.32 \\ 11.44 \\ 0.3 \end{bmatrix}, \quad V = \begin{bmatrix} -0.4 & -0.43 & -0.8 & 0.12 \\ -0.44 & 0.3 & 0.18 & 0.83 \\ -0.71 & 0.45 & 0.04 & -0.54 \\ -0.38 & -0.72 & 0.57 & -0.06 \end{bmatrix}$$

We can find the eigen (spectral) decomposition of XX' by calculating the eigenvalues, D^2, and the eigenvectors, which will be the columns of U (rather than V as for $X'X$):

$$D^2 = \begin{bmatrix} 274.94 \\ 14.32 \\ 11.44 \\ 0.3 \end{bmatrix}, \quad V = \begin{bmatrix} -0.5 & 0.53 & 0.16 & -0.66 \\ -0.5 & -0.59 & 0.13 & -0.11 \\ -0.46 & 0.49 & -0.25 & 0.65 \\ -0.39 & -0.33 & -0.68 & -0.09 \\ -0.37 & -0.17 & 0.65 & 0.35 \end{bmatrix}$$

Part c

The best $rank = 1$ approximation to X will be $X^{*1} = U_1 diag(d_1) V_1'$, where U_1 is a matrix with the first column of U, and V_1 is a matrix with the first column of V:

$$X^{*1} = U_1 diag(d_1) V_1' = \begin{bmatrix} -0.5 \\ -0.5 \\ -0.46 \\ -0.39 \\ -0.37 \end{bmatrix} \times \begin{bmatrix} 16.58 \end{bmatrix} \times \begin{bmatrix} -0.4 & -0.44 & -0.71 & -0.38 \end{bmatrix}$$

$$= \begin{bmatrix} 3.35 & 3.61 & 5.89 & 3.11 \\ 3.38 & 3.64 & 5.94 & 3.14 \\ 3.08 & 3.31 & 5.4 & 2.85 \\ 2.63 & 2.83 & 4.62 & 2.43 \\ 2.45 & 2.64 & 4.31 & 2.27 \end{bmatrix}$$

The best $rank = 2$ approximation to X will be $X^{*2} = U_2 diag(d_1, d_2) V_2'$, where U_2 is a matrix with the first two columns of U, and V_2 is a matrix with the first two columns of V:

$$X^{*2} = U_2 diag(d_1, d_2) V_2' = \begin{bmatrix} -0.5 & 0.53 \\ -0.5 & -0.59 \\ -0.46 & 0.49 \\ -0.39 & -0.33 \\ -0.37 & -0.17 \end{bmatrix} \times \begin{bmatrix} 16.58 & 0 \\ 0 & 3.78 \end{bmatrix} \times \begin{bmatrix} -0.4 & -0.44 & -0.71 & -0.38 \\ -0.43 & 0.3 & 0.45 & -0.72 \end{bmatrix}$$

$$= \begin{bmatrix} 2.49 & 4.2 & 6.78 & 1.66 \\ 4.35 & 2.97 & 4.95 & 4.75 \\ 2.28 & 3.86 & 6.23 & 1.51 \\ 3.16 & 2.46 & 4.07 & 3.33 \\ 2.73 & 2.44 & 4.02 & 2.75 \end{bmatrix}$$

Part d

Note that

$$\tilde{X} = \begin{bmatrix} -1 & 0.8 & 1.8 & -0.8 \\ 1 & -0.2 & -0.2 & 2.2 \\ 0 & 0.8 & 0.8 & -1.8 \\ 2 & -1.2 & -1.2 & -0.8 \\ -2 & -0.2 & -1.2 & 1.2 \end{bmatrix}$$

The Singular Value Decomposition of $\tilde{X}$ is:

$$\tilde{U} = \begin{bmatrix} -0.57 & 0.17 & 0.29 & -0.6 \\ 0.51 & 0.11 & 0.7 & 0.21 \\ -0.5 & -0.25 & -0.09 & 0.69 \\ 0.34 & -0.69 & -0.32 & -0.33 \\ 0.22 & 0.65 & -0.57 & 0.03 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} 3.83 & 0 & 0 & 0 \\ 0 & 3.38 & 0 & 0 \\ 0 & 0 & 2.01 & 0 \\ 0 & 0 & 0 & 0.48 \end{bmatrix}$$

$$\tilde{V} = \begin{bmatrix} 0.34 & -0.81 & 0.45 & 0.18 \\ -0.37 & 0.18 & 0.26 & 0.88 \\ -0.58 & 0.03 & 0.68 & -0.45 \\ 0.64 & 0.56 & 0.52 & 0 \end{bmatrix}$$

Therefore, the principal component directions will be the columns of $\tilde{V}$, the principal components will be the inner products $z_j = \langle x_i, v_j \rangle$, the columns of:

$$z = \begin{bmatrix} -2.19 & 0.56 & 0.57 & -0.29 \\ 1.95 & 0.39 & 1.4 & 0.1 \\ -1.91 & -0.84 & -0.18 & 0.33 \\ 1.3 & -2.32 & -0.65 & -0.16 \\ 0.85 & 2.21 & -1.14 & 0.02 \end{bmatrix}$$

The “loadings” of the first principal component are the first column of the matrix $\tilde{V}$,

$$\tilde{V}_1 = \begin{bmatrix} 0.34 \\ -0.37 \\ -0.58 \\ 0.64 \end{bmatrix}$$

Part e

The best $rank = 1$ approximation to $\tilde{X}$ will be $\tilde{X}^{*1} = \tilde{U}_1 diag(\tilde{d}_1), \tilde{V}_1$, where $\tilde{U}_1$ is a matrix with first column of $\tilde{U}$ and $\tilde{V}_1$ is a matrix with the first column of $\tilde{V}$:

$$\tilde{X}^{*1} = \tilde{U}_1 diag(\tilde{d}_1)\tilde{V}_1' = \begin{bmatrix} -0.57 \\ 0.51 \\ -0.5 \\ 0.34 \\ 0.22 \end{bmatrix} \times \begin{bmatrix} 3.83 \end{bmatrix} \times \begin{bmatrix} 0.34 & -0.37 & -0.58 & 0.64 \end{bmatrix}$$

$$= \begin{bmatrix} -0.75 & 0.81 & 1.26 & -1.41 \\ 0.67 & -0.72 & -1.12 & 1.26 \\ -0.66 & 0.71 & 1.1 & -1.23 \\ 0.45 & -0.48 & -0.75 & 0.84 \\ 0.29 & -0.31 & -0.49 & 0.55 \end{bmatrix}$$

The best $rank = 2$ approximation to $\tilde{X}$ will be $\tilde{X}^{*2} = \tilde{U}_2 diag(\tilde{d}_1, \tilde{d}_2)\tilde{V}_2'$, where $\tilde{U}_2$ is a matrix with the first two columns of $\tilde{U}$, and $\tilde{V}_2$ is a matrix with the first two columns of $\tilde{V}$:

$$\tilde{X}^{*2} = \tilde{U}_2 diag(\tilde{d}_1, \tilde{d}_2)\tilde{V}_2' = \begin{bmatrix} -0.57 & 0.17 \\ 0.51 & 0.11 \\ -0.5 & -0.25 \\ 0.34 & -0.69 \\ 0.22 & 0.65 \end{bmatrix} \times \begin{bmatrix} 3.83 & 0 \\ 0 & 3.38 \end{bmatrix} \times \begin{bmatrix} 0.34 & -0.37 & -0.58 & 0.64 \\ -0.81 & 0.18 & 0.03 & 0.56 \end{bmatrix}$$

$$= \begin{bmatrix} -1.2 & 0.91 & 1.28 & -1.1 \\ 0.36 & -0.65 & -1.11 & 1.47 \\ 0.02 & 0.56 & 1.07 & -1.71 \\ 2.32 & -0.89 & -0.83 & -0.46 \\ -1.49 & 0.08 & -0.41 & 1.79 \end{bmatrix}$$

Part f

Note that since here we are dealing with a sample covariance matrix, we have a symmetric non-negative definite matrix, so an eigen analysis will also yield all the information we need for a singular value decomposition, and thus we can easily find the best 1 and 2 component approximations.

We begin with the matrix:

$$\frac{1}{5}\tilde{X}'\tilde{X} = \begin{bmatrix} 2 & -0.6 & -0.4 & -0.2 \\ -0.6 & 0.56 & 0.76 & -0.36 \\ -0.4 & 0.76 & 1.36 & -0.76 \\ -0.2 & -0.36 & -0.76 & 2.16 \end{bmatrix}$$

An eigen decomposition yields:

$$\text{Eigenvalues} = D = \begin{bmatrix} 2.94 \\ 2.29 \\ 0.81 \\ 0.05 \end{bmatrix}, \quad \text{Eigenvectors} = U = V = \begin{bmatrix} 0.34 & 0.81 & -0.45 & -0.18 \\ -0.37 & -0.18 & -0.26 & -0.88 \\ -0.58 & -0.03 & -0.68 & 0.45 \\ 0.64 & -0.56 & -0.52 & 0 \end{bmatrix}$$

Thus, a best 1 component approximation will be of the form:

$$\frac{1}{5}\tilde{X}'\tilde{X}^{*1} = \tilde{U}_1 diag(\tilde{d}_1)\tilde{V}_1' = \begin{bmatrix} 0.34 \\ -0.37 \\ -0.58 \\ 0.64 \end{bmatrix} \times \begin{bmatrix} 2.94 \end{bmatrix} \times \begin{bmatrix} 0.34 & -0.37 & -0.58 & 0.64 \end{bmatrix}$$

$$= \begin{bmatrix} 0.35 & -0.37 & -0.58 & 0.65 \\ -0.37 & 0.4 & 0.62 & -0.7 \\ -0.58 & 0.62 & 0.97 & -1.09 \\ 0.65 & -0.7 & -1.09 & 1.22 \end{bmatrix}$$

A best 2 component approximation will be of the form:

$$\frac{1}{5}\tilde{X}'\tilde{X}^{*2} = \tilde{U}_2 diag(\tilde{d}_1, \tilde{d}_2)\tilde{V}_2' = \begin{bmatrix} 0.34 & 0.81 \\ -0.37 & -0.18 \\ -0.58 & -0.03 \\ 0.64 & -0.56 \end{bmatrix} \times \begin{bmatrix} 2.94 & 0 \\ 0 & 2.29 \end{bmatrix} \times \begin{bmatrix} 0.34 & -0.37 & -0.58 & 0.64 \\ 0.81 & -0.18 & -0.03 & -0.56 \end{bmatrix}$$

$$= \begin{bmatrix} 1.84 & -0.7 & -0.64 & -0.39 \\ -0.7 & 0.47 & 0.64 & -0.47 \\ -0.64 & 0.64 & 0.97 & -1.05 \\ -0.39 & -0.47 & -1.05 & 1.94 \end{bmatrix}$$

Part d - standardized

Note that

$$\tilde{\tilde{X}} = \begin{bmatrix} -0.63 & 0.96 & 1.38 & -0.49 \\ 0.63 & -0.24 & -0.15 & 1.34 \\ 0 & 0.96 & 0.61 & -1.1 \\ 1.26 & -1.43 & -0.92 & -0.49 \\ -1.26 & -0.24 & -0.92 & 0.73 \end{bmatrix}$$

The Singular Value Decomposition of $\tilde{\tilde{X}}$ is:

$$\tilde{\tilde{U}} = \begin{bmatrix} -0.6 & 0 & 0.14 & -0.65 \\ 0.3 & 0.22 & 0.79 & 0.2 \\ -0.44 & -0.34 & -0.16 & 0.68 \\ 0.58 & -0.58 & -0.24 & -0.27 \\ 0.16 & 0.7 & -0.53 & 0.04 \end{bmatrix}, \quad \tilde{\tilde{D}} = \begin{bmatrix} 3.04 & 0 & 0 & 0 \\ 0 & 2.14 & 0 & 0 \\ 0 & 0 & 1.39 & 0 \\ 0 & 0 & 0 & 0.49 \end{bmatrix}$$

$$\tilde{\tilde{V}} = \begin{bmatrix} 0.36 & -0.69 & 0.56 & 0.29 \\ -0.63 & 0.13 & 0.19 & 0.74 \\ -0.6 & -0.17 & 0.49 & -0.61 \\ 0.33 & 0.69 & 0.64 & 0 \end{bmatrix}$$

Therefore, the principal component directions will be the columns of $\tilde{\tilde{V}}$, the principal components will be the inner products $z_j = \langle x_i, v_j \rangle$, the columns of:

$$z = \begin{bmatrix} -1.82 & -0.01 & 0.19 & -0.32 \\ 0.92 & 0.48 & 1.09 & 0.1 \\ -1.34 & -0.73 & -0.22 & 0.33 \\ 1.75 & -1.24 & -0.33 & -0.13 \\ 0.49 & 1.5 & -0.73 & 0.02 \end{bmatrix}$$

The "loadings" of the first principal component are the first column of the matrix $\tilde{\tilde{V}}$,

$$\tilde{V}_1 = \begin{bmatrix} 0.36 \\ -0.63 \\ -0.6 \\ 0.33 \end{bmatrix}$$

Part e - standardized

The best $rank = 1$ approximation to $\tilde{\tilde{X}}$ will be $\tilde{\tilde{X}}^{*1} = \tilde{\tilde{U}}_1 diag(\tilde{\tilde{d}}_1), \tilde{\tilde{V}}_1$, where $\tilde{\tilde{U}}_1$ is a matrix with first column of $\tilde{\tilde{U}}$ and $\tilde{\tilde{V}}_1$ is a matrix with the first column of $\tilde{\tilde{V}}$:

$$\tilde{\tilde{X}}^{*1} = \tilde{\tilde{U}}_1 diag(\tilde{\tilde{d}}_1)\tilde{\tilde{V}}_1' = \begin{bmatrix} -0.6 \\ 0.3 \\ -0.44 \\ 0.58 \\ 0.16 \end{bmatrix} \times \begin{bmatrix} 3.04 \end{bmatrix} \times \begin{bmatrix} 0.36 & -0.63 & -0.6 & 0.33 \end{bmatrix}$$

$$= \begin{bmatrix} -0.65 & 1.15 & 1.09 & -0.61 \\ 0.33 & -0.58 & -0.55 & 0.31 \\ -0.48 & 0.85 & 0.8 & -0.45 \\ 0.63 & -1.11 & -1.05 & 0.58 \\ 0.18 & -0.31 & -0.29 & 0.16 \end{bmatrix}$$

The best $rank = 2$ approximation to $\tilde{\tilde{X}}$ will be $\tilde{\tilde{X}}^{*2} = \tilde{\tilde{U}}_2 diag(\tilde{\tilde{d}}_1, \tilde{\tilde{d}}_2)\tilde{\tilde{V}}_2'$, where $\tilde{\tilde{U}}_2$ is a matrix with the first two columns of $\tilde{\tilde{U}}$, and $\tilde{\tilde{V}}_2$ is a matrix with the first two columns of $\tilde{\tilde{V}}$:

$$\tilde{\tilde{X}}^{*2} = \tilde{\tilde{U}}_2 diag(\tilde{\tilde{d}}_1, \tilde{\tilde{d}}_2)\tilde{\tilde{V}}_2' = \begin{bmatrix} -0.6 & 0 \\ 0.3 & 0.22 \\ -0.44 & -0.34 \\ 0.58 & -0.58 \\ 0.16 & 0.7 \end{bmatrix} \times \begin{bmatrix} 3.04 & 0 \\ 0 & 2.14 \end{bmatrix} \times \begin{bmatrix} 0.36 & -0.63 & -0.6 & 0.33 \\ -0.69 & 0.13 & -0.17 & 0.69 \end{bmatrix}$$

$$= \begin{bmatrix} -0.65 & 1.15 & 1.09 & -0.61 \\ 0 & -0.52 & -0.63 & 0.64 \\ 0.03 & 0.75 & 0.93 & -0.95 \\ 1.49 & -1.27 & -0.84 & -0.27 \\ -0.86 & -0.12 & -0.55 & 1.2 \end{bmatrix}$$

Part f - standardized

Note that since here we are dealing with a sample covariance matrix, we have a symmetric non-negative definite matrix, so an eigen analysis will also yield all the information we need for a singular value decomposition, and thus we can easily find the best 1 and 2 component approximations.

We begin with the matrix:

$$\frac{1}{5}\tilde{\tilde{X}}'\tilde{\tilde{X}} = \begin{bmatrix} 0.8 & -0.45 & -0.19 & -0.08 \\ -0.45 & 0.8 & 0.7 & -0.26 \\ -0.19 & 0.7 & 0.8 & -0.35 \\ -0.08 & -0.26 & -0.35 & 0.8 \end{bmatrix}$$

An eigen decomposition yields:

$$\text{Eigenvalues} = D = \begin{bmatrix} 1.85 \\ 0.91 \\ 0.39 \\ 0.05 \end{bmatrix}, \quad \text{Eigenvectors} = U = V = \begin{bmatrix} 0.36 & -0.69 & -0.56 & 0.29 \\ -0.63 & 0.13 & -0.19 & 0.74 \\ -0.6 & -0.17 & -0.49 & -0.61 \\ 0.33 & 0.69 & -0.64 & 0 \end{bmatrix}$$

Thus, a best 1 component approximation will be of the form:

$$\frac{1}{5}\tilde{\tilde{X}}'\tilde{\tilde{X}}^{*1} = \tilde{\tilde{U}}_1 diag(\tilde{\tilde{d}}_1)\tilde{\tilde{V}}_1' = \begin{bmatrix} 0.36 \\ -0.63 \\ -0.6 \\ 0.33 \end{bmatrix} \times \begin{bmatrix} 1.85 \end{bmatrix} \times \begin{bmatrix} 0.36 & -0.63 & -0.6 & 0.33 \end{bmatrix}$$

$$= \begin{bmatrix} 0.24 & -0.42 & -0.4 & 0.22 \\ -0.42 & 0.74 & 0.7 & -0.39 \\ -0.4 & 0.7 & 0.66 & -0.37 \\ 0.22 & -0.39 & -0.37 & 0.21 \end{bmatrix}$$

A best 2 component approximation will be of the form:

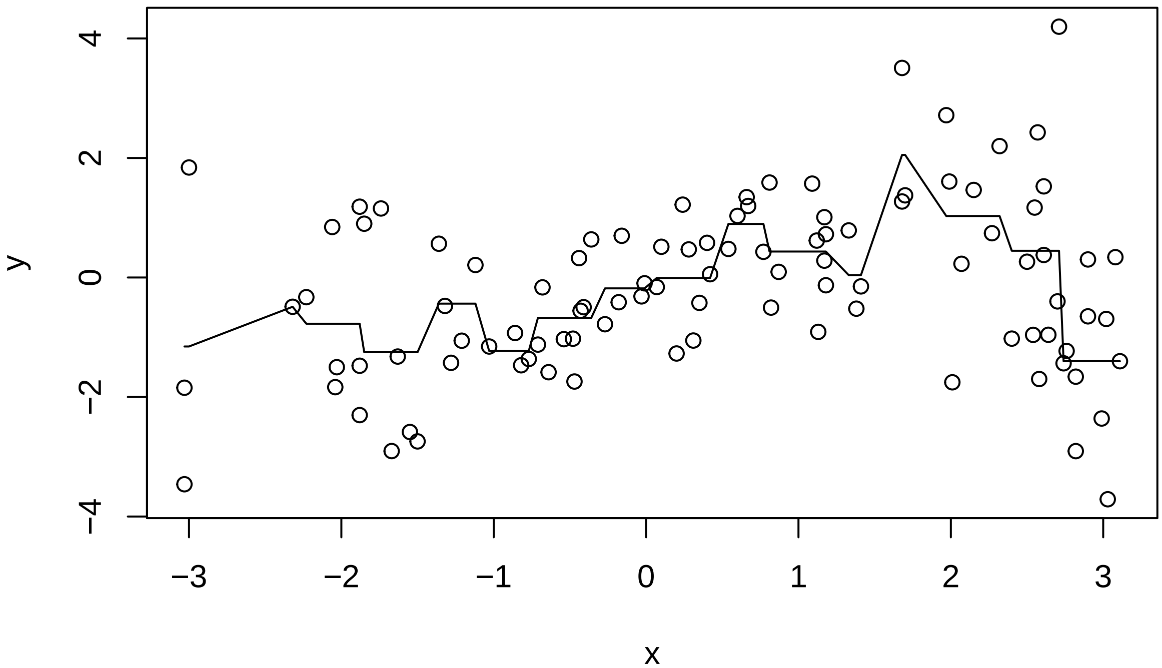
$$\frac{1}{5}\tilde{\tilde{X}}'\tilde{\tilde{X}}^{*2} = \tilde{\tilde{U}}_2 diag(\tilde{\tilde{d}}_1, \tilde{\tilde{d}}_2)\tilde{\tilde{V}}_2' = \begin{bmatrix} 0.36 & -0.69 \\ -0.63 & 0.13 \\ -0.6 & -0.17 \\ 0.33 & 0.69 \end{bmatrix} \times \begin{bmatrix} 1.85 & 0 \\ 0 & 0.91 \end{bmatrix} \times \begin{bmatrix} 0.36 & -0.63 & -0.6 & 0.33 \\ -0.69 & 0.13 & -0.17 & 0.69 \end{bmatrix}$$

$$= \begin{bmatrix} 0.68 & -0.5 & -0.29 & -0.21 \\ -0.5 & 0.76 & 0.68 & -0.31 \\ -0.29 & 0.68 & 0.69 & -0.48 \\ -0.21 & -0.31 & -0.48 & 0.64 \end{bmatrix}$$

Problem 2

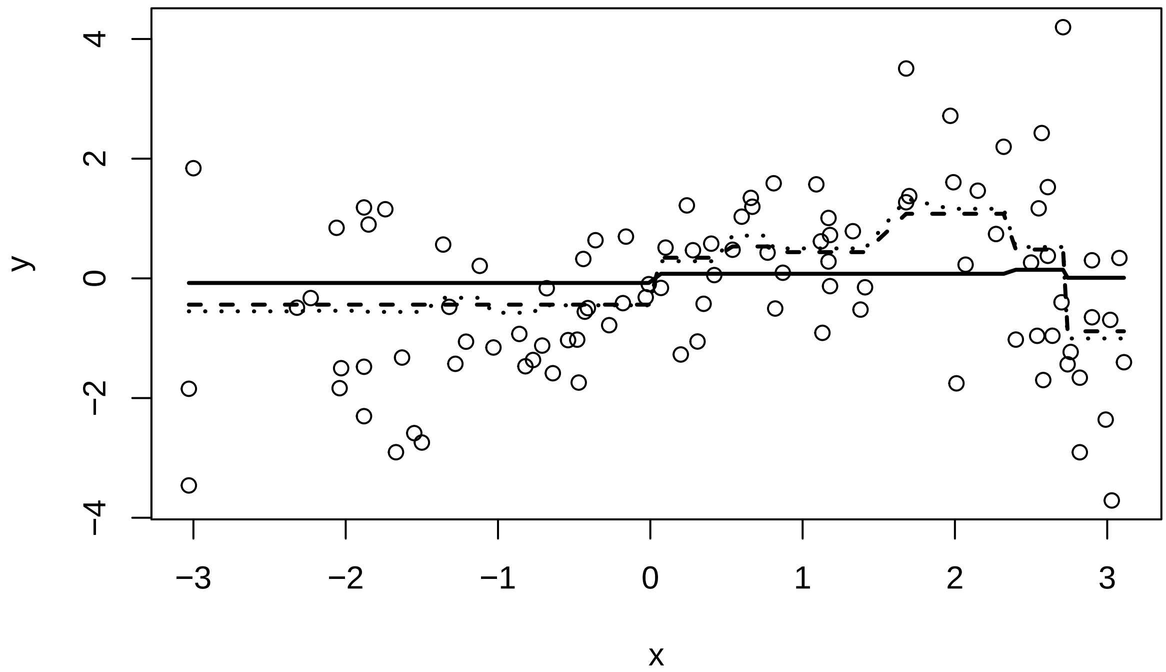
Part a

A scatterplot of the data and the fitted values from the OLS model are shown below.



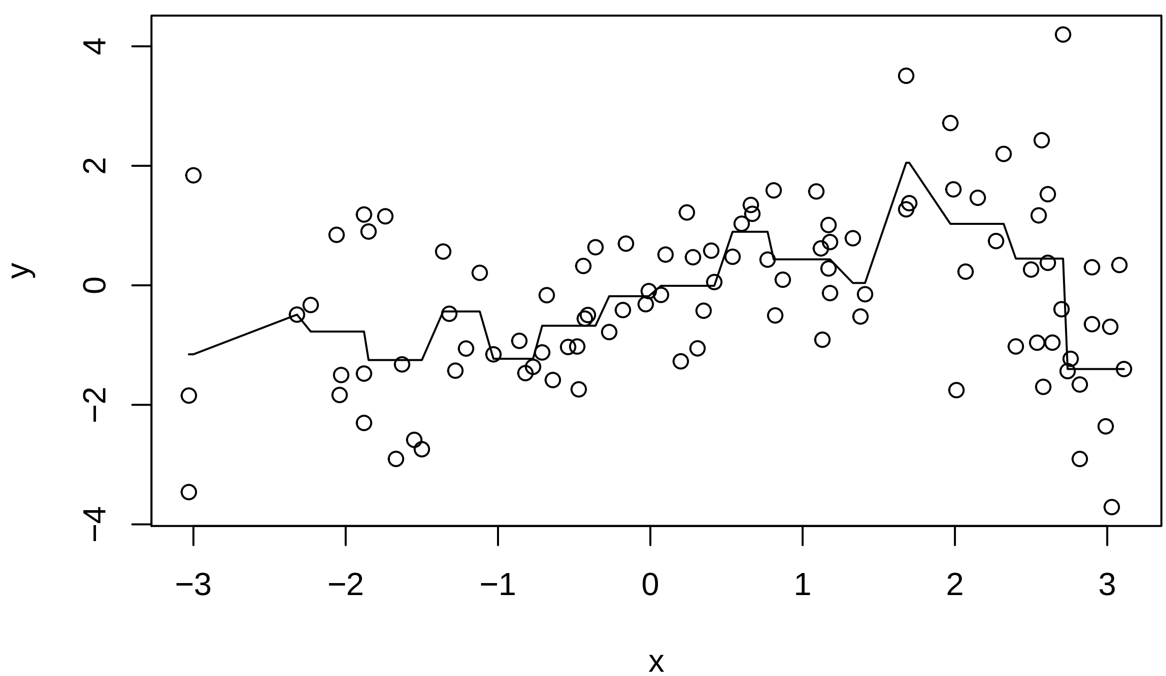
Part b

The solid line in the plot below gives (approximately) the fitted values when there are 2 nonzero coefficients, the dashed line are fitted values when there are 4 nonzero coefficients, and the dotted line are fitted values for 8 nonzero coefficients.



Problem 3

Note: This is Stat 602 HW3 from 2015 (problem 21)



Problem 4

Note: This is Stat 602 HW3 from 2015 (problem 22)

Note: There is some crossover with 502X HW2 Q15(a) here.

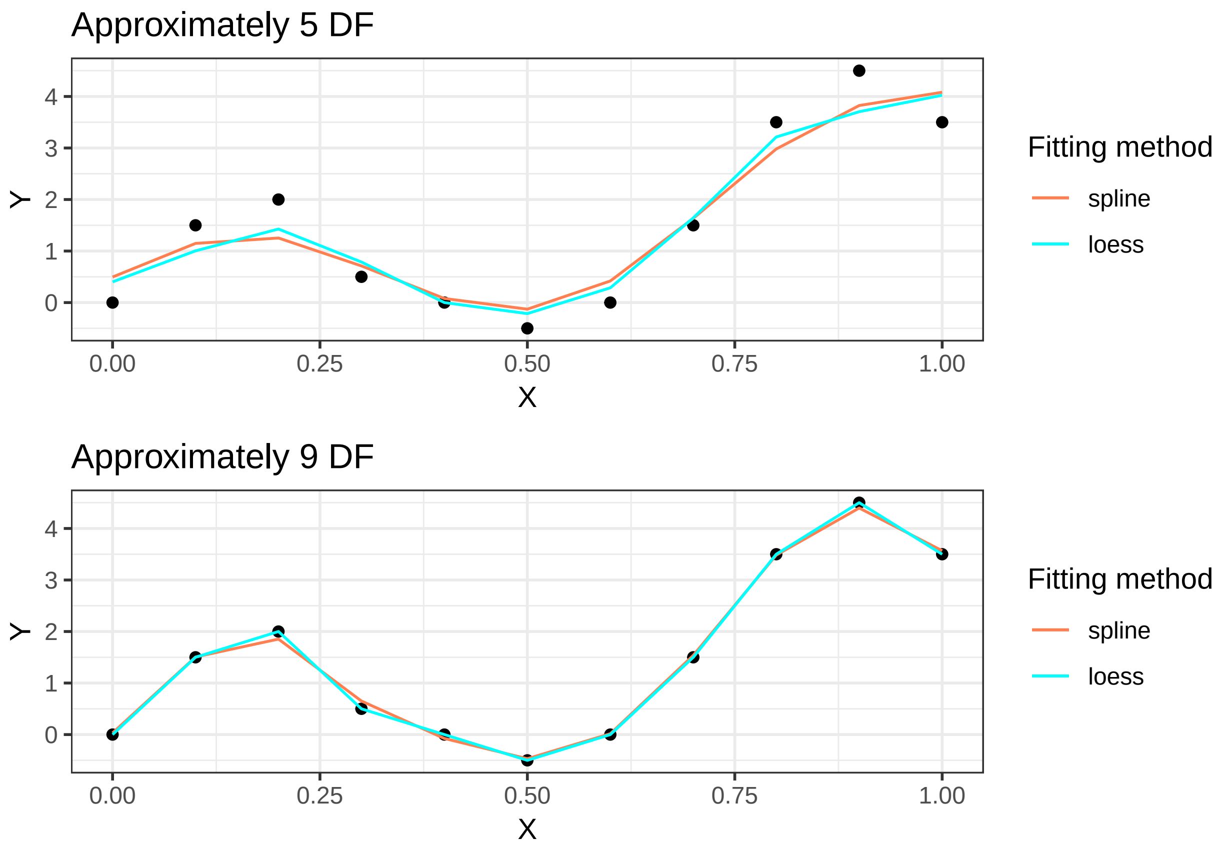


Figure 1: Plots for effective degrees of freedom 5 (top) and 9 (bottom) for Problem 6(a).

Part a

Part b

Part c

Problem 5

Note: This is Stat 602 HW3 from 2015 (problem 23)

Part a

Part b

Problem 6

Part a

The plots for this problem can be seen in Figure 1.

Problem 7

kNN

elastic net

PCR

PLS

MARS (in earth)

Problem 8

Part a

Part b

Part c

Part d