

Lecture 6: Game theory/ Strategic reasoning

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(Version 1.4)

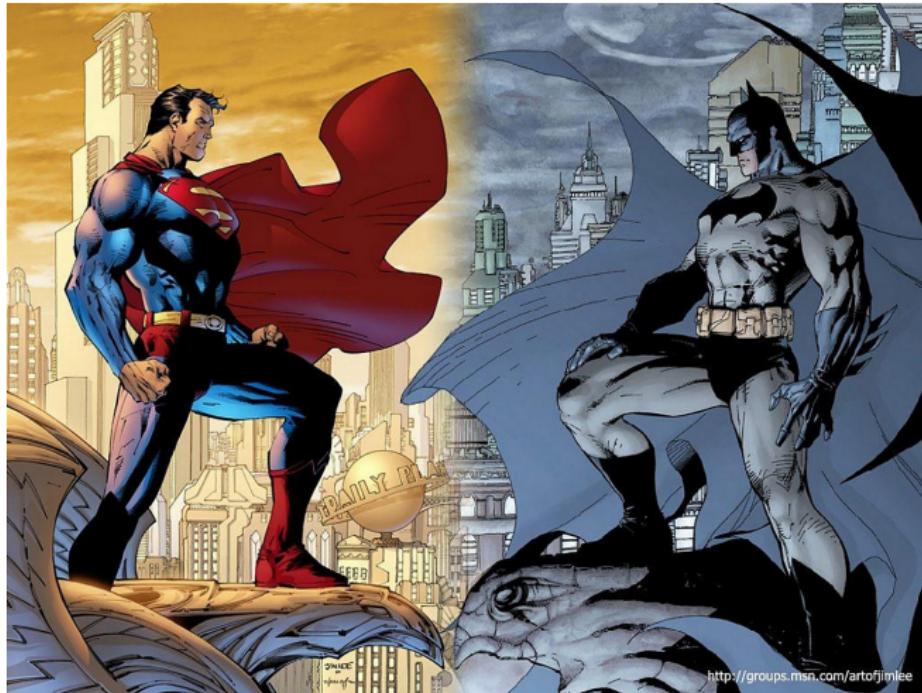


Today

- Introduction
- Probabilistic Reasoning I
- Probabilistic Reasoning II
- Sequential Decision Making
- Temporal Probabilistic Reasoning
- Game Theory
- Argumentation I
- Argumentation II
- (A peek at) Machine Learning
- AI & Ethics



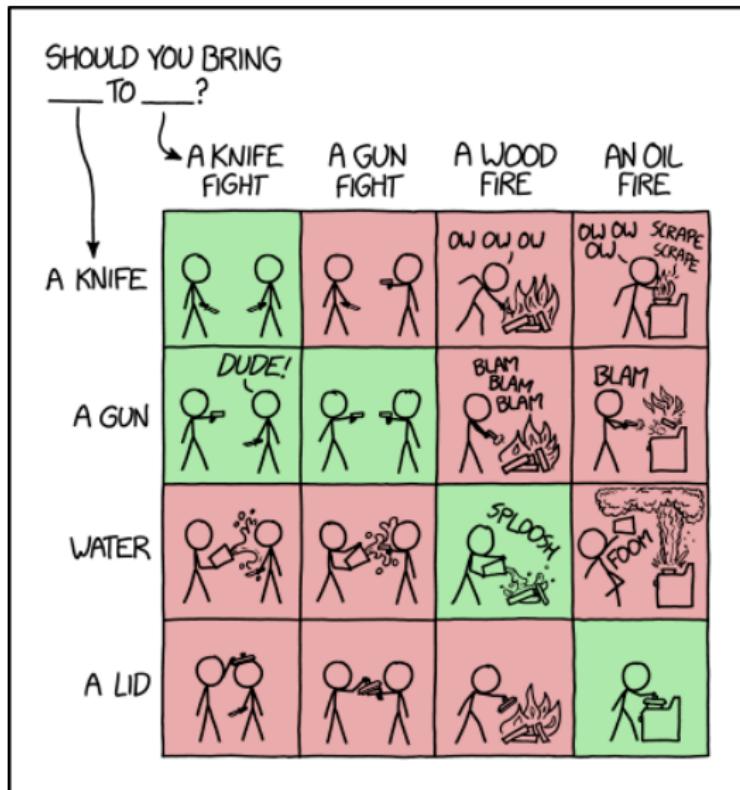
Competition between agents



(groups.msn.com/artofjimlee)

<http://groups.msn.com/artofjimlee>

Multi-party decision making



(xkcd.com/1890/)

In the real world



(U.S. Army Military History Institute)

- Strategic analysis.

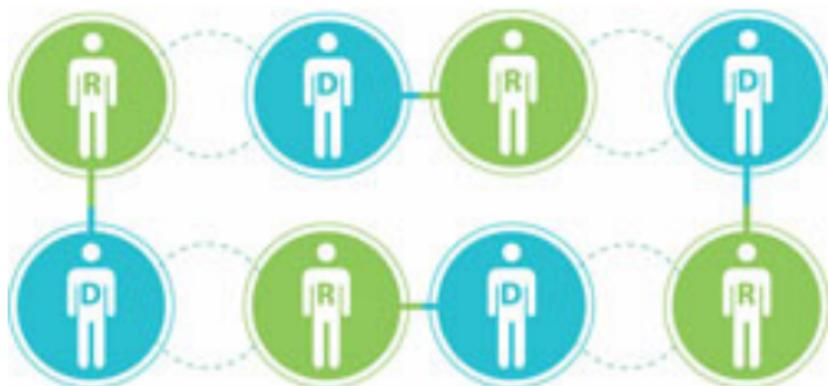
In the real world



(Adi Talwar/citylimits.org)

- School admissions

In the real world



(*froedtert.com*)

- Kidney exchange
- Longest chain involved 28 donors and recipients.

The grade game

Report for year beginning	day of	191	and Ending	day of	191	Page Reached						
	Sept.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.	May	June	Yearly Avg.	
Reading		a	a								144	
Spelling		B+	A+	B+							94	
Writing		B+	B+	B+							82	
Drawing											2	
Arithmetic		B	A+	A							2	
Grammar		D	B+	B							64	
Geography		C	A	B							2	
Physiology												
Agriculture												
Texas History												
U. S. History												
Civics												
Composition												
Physical Geography												
Literature												
General History												
Algebra												
Geometry												
Physics												
Application		C	A	R							84	
Department												
Days Absent												
Times Tardy												
Sept.												
Oct.	<i>Lyndon Johnson</i>											
Nov.	<i>Robert S. Jones</i>											
Dec.	<i>John C. Nichols</i>											
Jan.												
Feb.												
March												
April												
May												
June												

Saturday grades:

G = 92-100 C = 70-79 D = 60-69 F = 0-59

TO THE TEACHER

This report contains full information regarding the pupil's work. Fill the columns carefully so the class record book will need no revision with regard to this pupil. Return with promptness. Encourage him to make his marks with pencil only. Encourage him to make his marks in which your records are made will be a good habit. If you have any questions will prove to be a valuable medium of communication between teacher and parents. Please keep a frequent written record of work passed over by all pupils. Report them to the work. Let your students tell the honest teacher. Frequently bring or send my office a copy of the marks to this office for inspection.

SCALE OF GRADING

EXCELLENT 100-90
VERY GOOD 89-80
GOOD 79-70
SATISFACTORY 69-60
POOR 59-50
DEFICIENT 49-40
CONDUCT 39-30
UNSATISFACTORY 29-20
The parent and guardian will please sign below

(Lyndon Johnson's Report Card, US National Archives)

be in close sympathy with the community must

The grade game

- Take a piece of paper
- You are randomly paired with a partner (you do not know who!)
- You have to write X or Y on the piece of paper
- You will get a grade based on the following rules:
- If both you and your partner write X, then you both get a B
- If you write X and your partner writes Y then you get D and your partner gets A
- If you write Y and your partner writes X then you get A and your partner gets D
- If both you and your partner write Y then you both get C

(The Grade Game, Ben Polak)



The grade game

- What would you do?



The grade game

- What would you do?
- What you get depends also on the choice of your partner.



The grade game

- What would you do?
- What you get depends also on the choice of your partner.
- This is the blueprint of strategic interaction.



The grade game

<https://pollev.com/frederikm106>



Choose the side

- Which side of the road to drive on?



(haulagetoday.com)

Choose the side

- Which side of the road to drive on?



(businessinsider.com)

Choose the side

- Which side of the road to drive on?



Any fule kno that.

(Geoffrey Williams/Ronald Searle)

Choose the side

- Which side of the road to drive on?



- Same side as everyone else
(berkshireeagle.com)

Choose the side

- How do you choose when you don't know what "everyone else" is doing.



Game theory

- Game theory is a framework for analysing interactions between a set of agents.
- Abstract specification of interactions.
- Describes each agent's preferences in terms of their **utility**.
 - Assume agents want to maximise utility.
- Give us a range of **solution strategies** with which we can make some predictions about how agents will/should interact.



Payoff Matrices

- We can characterise the “choose side” scenario in a **payoff matrix**

		j	
		left	right
i	left	1	0
	right	0	1
		0	1

- we have n agents, each player picking a **(pure) strategy**
- Agent i is the **row player**
gets the lower reward in a cell.
- Agent j is the **column player**
gets the upper reward in a cell.



Payoff Matrices

- We can characterise the grade game scenario in a payoff matrix

		j	
		Y	X
		Y	2
i	Y	2	1
	X	4	3
		X	1
			3

- Payoffs are the US grade points that correspond to the problem statement.
- From the game earlier: Grade A is 4, Grade B is 3 etc.



Outcomes

- An **outcome** is what we get when we combine the actions of all the players.
- An outcome corresponds to an element of the **payoff matrix**

		j	
		left	right
		1	0
i	up	1	0
	down	0	1
		0	1

- We identify outcomes by the moves the players make:

(what i plays, what j plays)

- Thus *(up, right)* identifies the outcome in which i plays *up* and j plays *right*

Payoff Matrices

- Actually there are two matrices here, one (call it A) that specifies the payoff to i and another B that specifies the payoff to j .
- Sometimes we'll write the payoff matrix as (A, B) in recognition of this.
- $A = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ is the payoff matrix for i from the following table

		j	
		left	right
		up	2
i	up	1	0
	down	0	1
		3	1

- Note that $a_{i',j'}$ is the payoff if i picks action i' and j picks action j'



Solution Concepts

- How will a rational agent will behave in any given scenario?
- Play...
 - Dominant strategy;
 - Nash equilibrium strategy;
 - Pareto optimal strategies;
 - Strategies that maximise social welfare.



Dominant Strategies

- Given any particular strategy s (either C or D) agent i , there will be a number of possible outcomes.
- We say s_1 dominates s_2 if every outcome possible by i playing s_1 is preferred over every outcome possible by i playing s_2 .
- Thus in this game:

		j	
		D	C
i	D	1	4
	C	1	4

C dominates D for both players.

Dominant Strategies

- Two senses of “preferred”
- s_1 strongly dominates s_2 if the utility of every outcome possible by i playing s_1 is strictly greater than every outcome possible by i playing s_2 .
- In other words, $u(s_1) > u(s_2)$, for all outcomes.
- s_1 weakly dominates s_2 if the utility of every outcome possible by i playing s_1 is no less than every outcome possible by i playing s_2 .
- In other words, $u(s_1) \geq u(s_2)$, for all outcomes.



Dominant Strategies

- A rational agent will never play a dominated strategy.
- So in deciding what to do, we can **delete dominated strategies**.
- Unfortunately, there isn't always a unique undominated strategy.



Dominant Strategies

- Game with dominated strategies

	L	C	R
U	1	1	0
M	3	0	0
D	1	1	5
	1	4	0

- Can eliminate the dominated strategies and simplify the game
- Remove R (dominated by L).



Dominant Strategies

- Game with dominated strategies

	L	C
U	1	1
	3	0
M	1	1
	1	1
D	1	1
	0	4

- Can eliminate the dominated strategies and simplify the game
- Remove R (dominated by L).



Dominant Strategies

- If we are lucky, we can eliminate enough strategies so that the choice of action is obvious.
- In general we aren't that lucky.



Dominant Strategies

- Consider this scenario:

		j	
		C	D
		1	4
i	A	2	3
	B	2	3
		3	2

- Are there any dominated strategies?

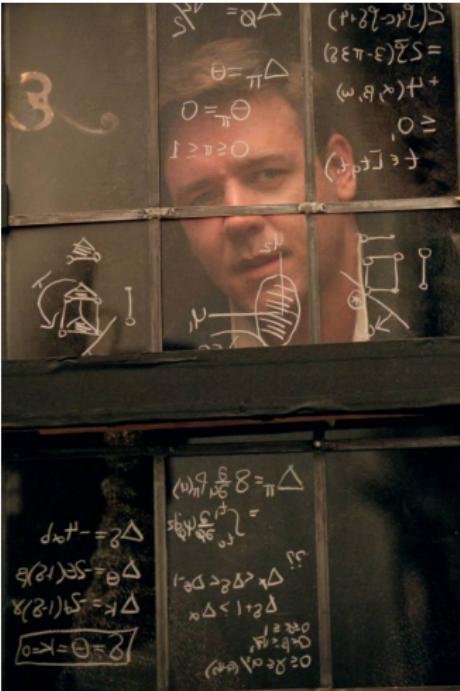
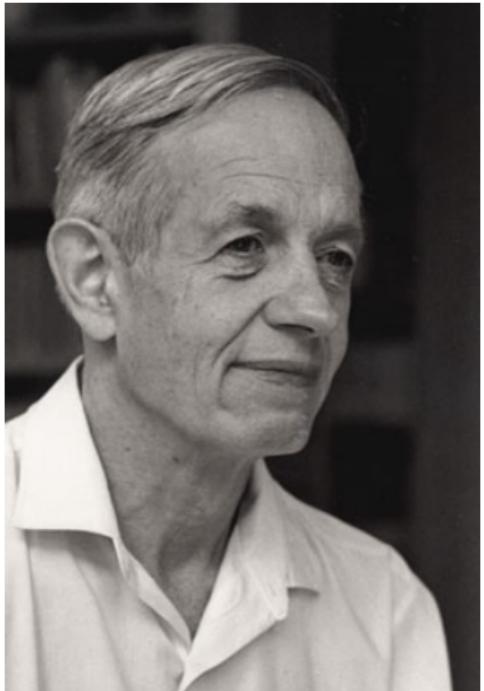
Are there any dominated strategies?

<https://pollev.com/frederikm106>

Last year's success rate: 0.28



Nash Equilibrium



John Forbes Nash.

(princeton.edu, Universal Pictures/DreamWorks)

Nash Equilibrium

- In general, we will say that two strategies s_1 and s_2 are in Nash equilibrium (NE) if:
 - ① under the assumption that agent i plays s_1 , agent j can do no better than play s_2 ; and
 - ② under the assumption that agent j plays s_2 , agent i can do no better than play s_1 .
- Neither agent has any incentive to deviate from a NE.
- Eh?



Nash Equilibrium

- Let's consider the payoff matrix for the grade game:

	j		
i	Y	2	1
	X	4	3
	1	3	

- Here the Nash equilibrium is (Y, Y) .
- If i assumes that j is playing Y , then i 's **best response** is to play Y .
- Similarly for j .

Nash Equilibrium

- If two strategies are best responses to each other, then they are in Nash equilibrium.



Nash Equilibrium

- In a game like this you can find the NE by cycling through the outcomes, asking if either agent can improve its payoff by switching its strategy.

		j	
		Y	X
		Y	1
i	Y	2	4
	X	4	3
		1	3

- Thus, for example, (X, Y) is not an NE because i can switch its payoff from 1 to 2 by switching from X to Y.

Nash Equilibrium

- More formally:

A pair of strategies (i^*, j^*) is a **Nash equilibrium solution** to the game (A, B) if:

$$\forall i, a_{i^*, j^*} \geq a_{i, j^*}$$

$$\forall j, b_{i^*, j^*} \geq b_{i^*, j}$$

- That is, (i^*, j^*) is a **Nash equilibrium** if:
 - If j plays j^* , then i^* gives the best outcome for i .
 - If i plays i^* , then j^* gives the best outcome for j .



Nash Equilibrium

- Unfortunately:
 - ① Not every interaction scenario has a pure strategy NE.
 - ② Some interaction scenarios have more than one NE.



Nash Equilibrium

- This game has two pure strategy NEs, (C, C) and (D, D) :

		j	
		D	C
		D	5
i	D	3	2
	C	0	3
		2	3

- In both cases, a single agent can't unilaterally improve its payoff.

Nash Equilibrium

- This game has no pure strategy NE:

		j	
		D	C
		D	2
i	D	2	1
	C	1	2
		C	0
		2	1

- For every outcome, one of the agents will improve its utility by switching its strategy.
- We can find a form of NE in such games, but we need to go beyond pure strategies.



Nash equilibria?

- Consider this scenario (again):

		<i>j</i>	
		C	D
		1	4
<i>i</i>	A	2	3
	B	2	3
		3	2

- Are there any Nash equilibria?
- <https://pollev.com/frederikm106>
Last year's success rate 0.29

Pareto Optimality

- An outcome is said to be **Pareto optimal** (or **Pareto efficient**) if there is no other outcome that makes one agent **better off** without making another agent **worse off**.
- If an outcome is Pareto optimal, then at least one agent will be reluctant to move away from it (because this agent will be worse off).
- If an outcome ω is **not** Pareto optimal, then there is another outcome ω' that makes **everyone** as happy, if not happier, than ω .
- “Reasonable” agents would agree to move to ω' from ω if ω is **not** Pareto optimal and ω' is.



Pareto Optimality

- This game has one Pareto efficient outcome, (D, D) .

		j	
		D	C
		D	5
i	D	3	2
	C	0	0
		2	1

- There is no solution in which either agent does better.

Pareto Optimality

- This next game has two Pareto efficient outcomes, (C, D) and (D, C) .

		<i>j</i>	
	D	D	C
<i>i</i>	D	1	4
	C	3	1
	C	1	1
		4	1

- Note that Pareto efficiency doesn't necessarily mean *fair*.
- Just that you can't move away and make one agent better off without making the other worse off.
- Different way of thinking about this: Ignore players and imagine you are buying socks online. All options cost the same
- i is the number of stars (rating) and j is the number of pairs you get.
- Trade-off: all reasonable options are Pareto efficient

Pareto optimal?

- Consider this scenario (again):

		<i>j</i>	
		C	D
		1	4
<i>i</i>	A	2	3
	B	2	3
		3	2

- Are there any Pareto optimal outcomes?
- <https://pollev.com/frederikm106>
Last year's success rate 0.51

Pareto Optimality

- Pareto optimality is a rather weak concept.



(coolfunpedia.blogspot.co.uk)

- What is the Pareto optimal way to divide a pile of money between player i and player j ?

Social Welfare

- The social welfare of an outcome ω is the sum of the utilities that each agent gets from ω :

$$\sum_{i \in Ag} u_i(\omega)$$

- Think of it as the “total amount of money in the system”.
- As a solution concept, may be appropriate when the whole system (all agents) has a single owner (then overall benefit of the system is important, not individuals).



Social Welfare

- As a solution concept it doesn't consider the benefits to individuals.



(telegraph.co.uk)

- A very skewed outcome can maximise social welfare.

Social Welfare

- In both these games, (C, C) maximises social welfare.

		j	
		D	C
		D	2
i	D	2	1
	C	3	4
		3	4

		j	
		D	C
		D	2
i	D	2	1
	C	3	9
		3	0

Normal form games



(Pendleton Ward/Cartoon Network)

Normal form games

- An n-person, finite, **normal form** game is a tuple (N, A, u) , where
 - N is a finite set of players.
 - $A = A_1 \times \dots \times A_n$ where A_i is a finite set of actions available to i .
Each $a = (a_1, \dots, a_n) \in A$ is an **action profile**.
 - $u = (u_1, \dots, u_n)$ where $u_i : A \mapsto \mathbb{R}$ is a real-valued **utility** function for i .
- Naturally represented by an n-dimensional matrix



Strategies

- We analyze games in terms of **strategies**, that is what agents decide to do.
 - Combined with what the other agent(s) do(es) this jointly determines the payoff.
- An agent's **strategy set** is its set of available choices.
- Can just be the set of actions — **pure** strategies.
- We need more than just pure strategies in many cases.
 - Will discuss this later



Payoff matrix

- Here is the payoff matrix from the “choose which side” (of the road) game:

		j	
		left	right
		left	0
i	left	1	0
	right	0	1

- We can classify games by the form of the payoff matrix.

Common payoff games

- “Choose which side” game

	left	right
left	1	0
right	0	1

Also called the *coordination game*

- Any game with $u_i(a) = u_j(a)$ for all $a \in A_i \times A_j$ is a **common payoff** game.



Common payoff games

- The misanthropes' (un)coordination game:

		left	right
		left	0 1
left	0	1	
	right	1	0
right	1	0	
	right	1	0

Here we try to avoid each other.

Misanthrope



<http://www.thebadchemicals.com>



Constant sum games

- Matching pennies

	heads	tails
heads	-1	1
	1	-1
tails	1	-1
	-1	1

- Any game with $u_i(a) + u_j(a) = c$ for all $a \in A_i \times A_j$ is a constant sum game.



Zero-sum games

- A particular category of constant sum games are zero-sum games.
- Where utilities sum to zero:

$$u_1(a_i) + u_j(\omega) = 0 \quad \text{for all } a \in A_i \times A_j$$



Zero-sum games

- Where preferences of agents are diametrically opposed we have **strictly competitive** scenarios.



(Library of Congress)

- Zero sum implies strictly competitive.

Zero-sum games

- Zero-sum encounters in real life are very rare . . . but people tend to act in many scenarios as if they were zero-sum.
- Most encounters have some room in the set of outcomes for agents to find (somewhat) mutually beneficial outcomes.



Zero-sum games

- Rock, paper, scissors:



(Google/Droga5)

is another constant/zero sum game.

- Game in two senses.

The rules

- Rules for “rock, paper, scissors”.



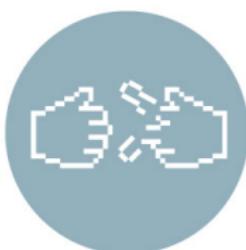
Rock



Paper



Scissors



Rock breaks scissors



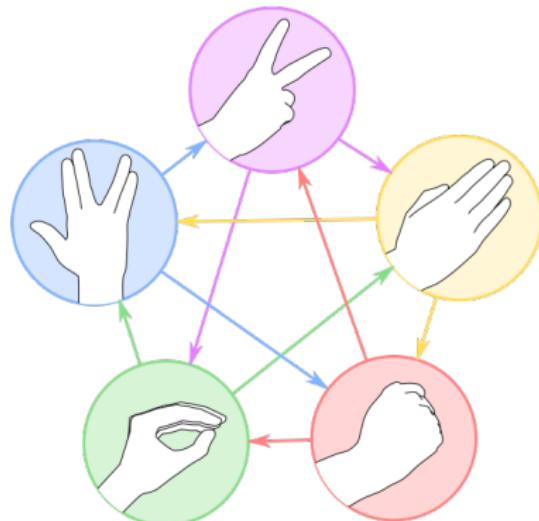
Paper covers rock



Scissors cut paper

(eyemotive.com)

Rock, paper scissors, lizard, Spock



(DMacks/Nojhan/Wikipedia)

<http://www.youtube.com/watch?v=x5Q6-wMx-K8>

- Scissors cuts paper, paper covers rock, rock crushes lizard, lizard poisons Spock, Spock smashes scissors, scissors decapitates lizard, lizard eats paper, paper disproves Spock, Spock vaporizes rock, and as it always has, rock crushes scissors.

Rock, paper, scissors

		j		
		rock	paper	scissors
i	rock	0	1	-1
	paper	0	-1	1
i	paper	-1	0	1
	scissors	1	-1	0
		-1	1	0

Mixed strategy

- Chances are you would play a **mixed** strategy.
- You would:
 - sometimes play rock,
 - sometimes play paper; and
 - sometimes play scissors.
- A fixed/pure strategy is easy for an adaptive player to beat.



Mixed strategy

- A mixed strategy is just a probability distribution across a set of pure strategies.
- So, for a game where agent i has two actions a_1 and a_2 , a mixed strategy for i is a probability distribution:

$$MS_i = \{P(a_1), P(a_2)\}$$

- Given this mixed strategy, when i comes to play, they pick action a_1 with probability $P(a_1)$ and a_2 with probability $P(a_2)$.



Mixed strategy

- To determine the mixed strategy, i needs then to compute the best values of $P(a_1)$ and $P(a_2)$.
- These will be the values which give i the highest expected payoff given the options that j can choose and the joint payoffs that result.
- We could write down the expected payoffs of different mixed strategies and pick the one that optimises expected payoff.
- There is also a simple graphical method which works for very simple cases.
- Will look at this method next.



Mixed strategy

- Let's consider the payoff matrix:

		j	
		a_3	a_4
		a_1	-3
i	a_1	3	-1
	a_2	0	-1
		0	1

- We want to compute mixed strategies to be used by the players.
- That means decide $P(a_1)$ and $P(a_2)$ etc.

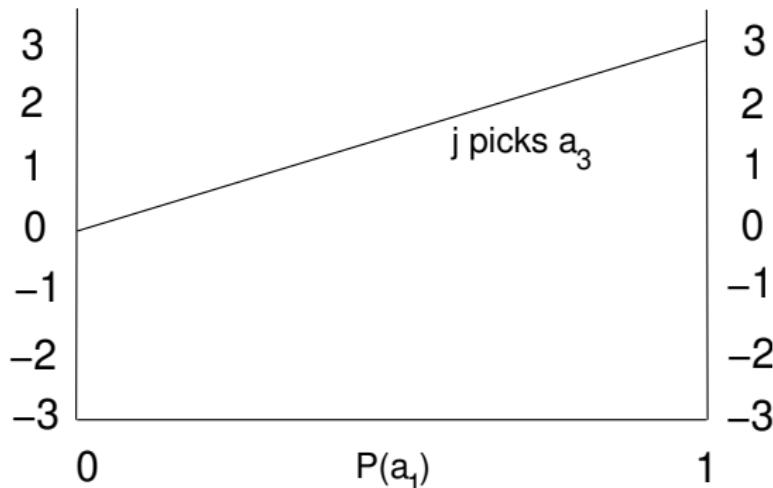


Mixed strategy

- *i*'s analysis of this game would be something like this.

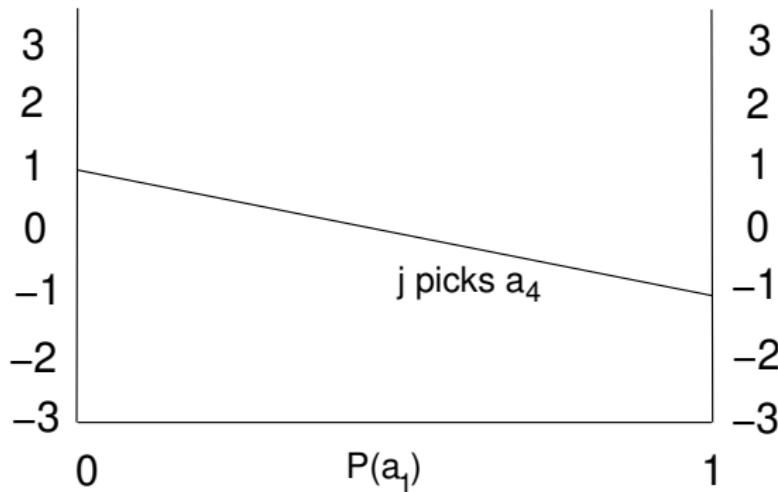


Mixed strategy



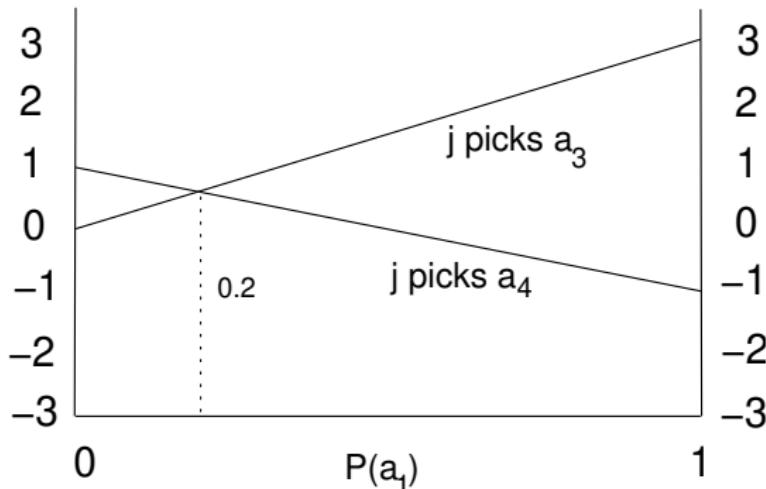
- If j picks a_3 , i 's payoff will be 3 or 0 depending on whether i picks a_1 or a_2 .
- The expected payoff therefore varies along the line, as $P(a_1)$ varies from 0 to 1.

Mixed strategy



- If j picks a_4 , i 's payoff will be -1 or 1 depending on whether i picks a_1 or a_2 .
- The expected payoff therefore varies along the line, as $P(a_1)$ varies from 0 to 1 .

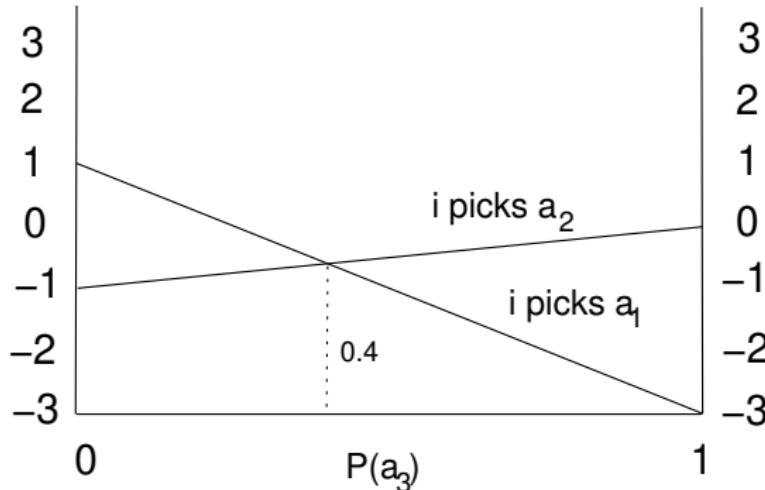
Mixed strategy



- Where the lines intersect, *i* has the same expected payoff whatever *j* does.
- This is a rational choice of mixed strategy.

Mixed strategy

- j can do the same kind of analysis:



Mixed strategy

- This analysis will help i and j choose a mixed strategy in zero-sum games.



(Archives of the Institute of Advanced Study, Princeton)

- This approach is due to von Neumann.

General sum games

- Battle of the Outmoded Gender Stereotypes
 - aka Battle of the Sexes

	this	that
this	1	0
that	2	0
	0	1



(Time-Life/Getty)

- Game contains elements of cooperation and competition.
- The interplay between these is what makes general sum games interesting.

Negotiation

- Interplay between cooperation and competition leads to **negotiation**
- See, for example, the work of Sarit Kraus.



(*law-train.eu*)

Nash equilibrium

- Earlier we introduced the notion of **Nash equilibrium** as a solution concept for general sum games.
- (We didn't describe it in exactly those terms.)
- Looked at pure strategy Nash equilibrium.
- Issue was that not every game has a pure strategy Nash equilibrium.



Nash equilibrium

- For example:

		j	
		D	C
		D	2
i	D	1	2
	C	0	1
		2	1

- Has no pure strategy NE.

Nash equilibrium

- The notion of Nash equilibrium extends to mixed strategies.
- And **every** game has at least one mixed strategy Nash equilibrium.



Nash equilibrium

- For a game with payoff matrices A (to i) and B (to j), a mixed strategy (x^*, y^*) is a Nash equilibrium solution if:

$$\begin{aligned}\forall x, x^* A y^{*T} &\geq x A y^{*T} \\ \forall y, x^* B y^{*T} &\geq x^* B y^T\end{aligned}$$

- In other words, x^* gives a higher **expected** value to i than any other strategy when j plays y^* .
- Similarly, y^* gives a higher **expected** value to j than any other strategy when i plays x^* .



Nash equilibrium

- Unfortunately, this doesn't solve the problem of **which** Nash equilibrium you should play.



The Prisoner's Dilemma



(Wolf Films/NBC)

The Prisoner's Dilemma

Two suspects are collectively charged with a crime and held in separate cells, with no way of meeting or communicating.

They are told that:

- if one confesses and the other does not, the confessor will be freed, and the other will be jailed for three years;*
- if both confess, then each will be jailed for two years.*

Both prisoners know that if neither confesses, then they will each be jailed for one year.



The Prisoner's Dilemma

- Payoff matrix for prisoner's dilemma:

		j	
		confess	lie
i	confess	3	1
	lie	4	2
		1	2

Numbers are payoffs to players, not years in jail.

- What should each agent do?

What Should You Do?

- You can minimise the jail time by picking confess (at most 3 years)
 - But you trust your partner, then you can get away with 2 years
- ...



Solution Concepts

- Payoff matrix for prisoner's dilemma:

		j	
		confess	lie
i	confess	3	1
	lie	4	2
		1	2

- There is no Nash equilibrium.
- All outcomes **except** (*confess, confess*) are Pareto optimal.
- (*lie, lie*) maximises social welfare.

Oh!



(Pendleton Ward/Cartoon Network)



Prisoner's dilemma

- Real world examples:
 - nuclear arms reduction/proliferation
 - free rider systems — public transport, file sharing;
 - in the UK — television licenses.
 - climate change — to reduce or not reduce emissions
 - doping in sport
 - ...
- The prisoner's dilemma is **ubiquitous**.
- Can we encourage cooperation?



The Shadow of the Future

- *Play the game more than once.*
If you know you will be meeting your opponent again, then the incentive to confess appears to evaporate.
 - If you confess, you can be punished (compared to the co-operation reward.)
 - If you get tricked, then what you lose can be amortised over the rest of the iterations, making it a small loss.
- Cooperation is (provably) the rational choice in the **infinitely** repeated prisoner's dilemma.
- Experiments with real prisoners in Germany show that this is not what happens though ...



The Shadow of the Future

- But what if there are a **finite** number of repetitions?

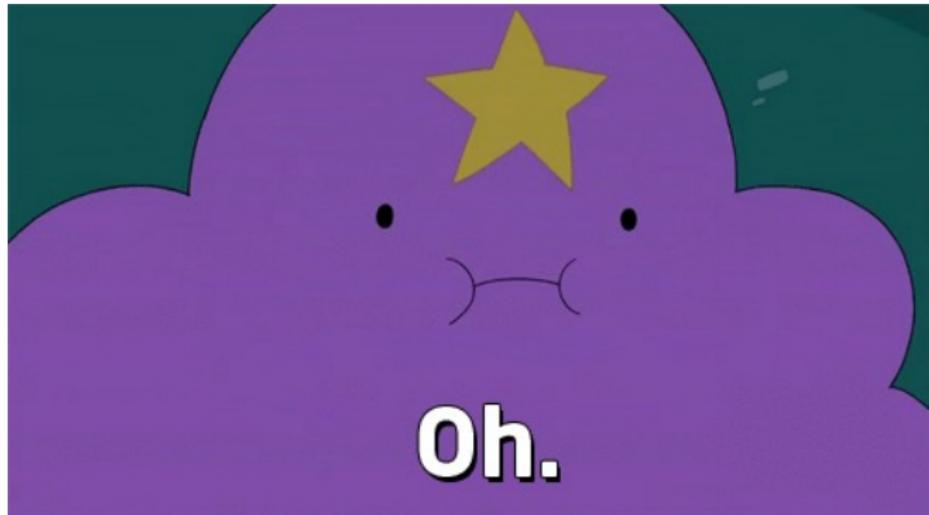


Backwards Induction

- Suppose you both know that you will play the game exactly n times.
On round n , you have an incentive to defect, to gain that extra bit of payoff.
But this makes round $n - 1$ the last “real” round, and so you have an incentive to defect there, too.
This is the **backwards induction** problem.
- Playing the prisoner’s dilemma with a fixed, finite, pre-determined, commonly known number of rounds, confessing is the best strategy.
- That seems to suggest that you should **never** lie (cooperate with partner).



Oh!



(Pendleton Ward/Cartoon Network)



But

- So how does cooperation arise? Why does it make sense?
- After all, there does seem to be such a thing as society, and even in a big city like London, people don't behave so badly.
- Or, maybe more accurately, they don't behave badly all the time.



But

- Turns out that...
- As long as you have some probability of repeating the interaction, co-operation can have a better expected payoff.
- As long as there are enough co-operative folk out there, you can come out ahead by co-operating.



Mathematical!



Summary

- Have looked at **strategic reasoning** in the presence of other agents.
- Covered some ideas from game theory and discussed what it can do for us.
- Lots more we haven't covered...
- Game theory helps us to get a handle on some of the aspects of cooperation between self-interested agents.
- Rarely any definitive answers.
- Given human interactions, that should not surprise us.

