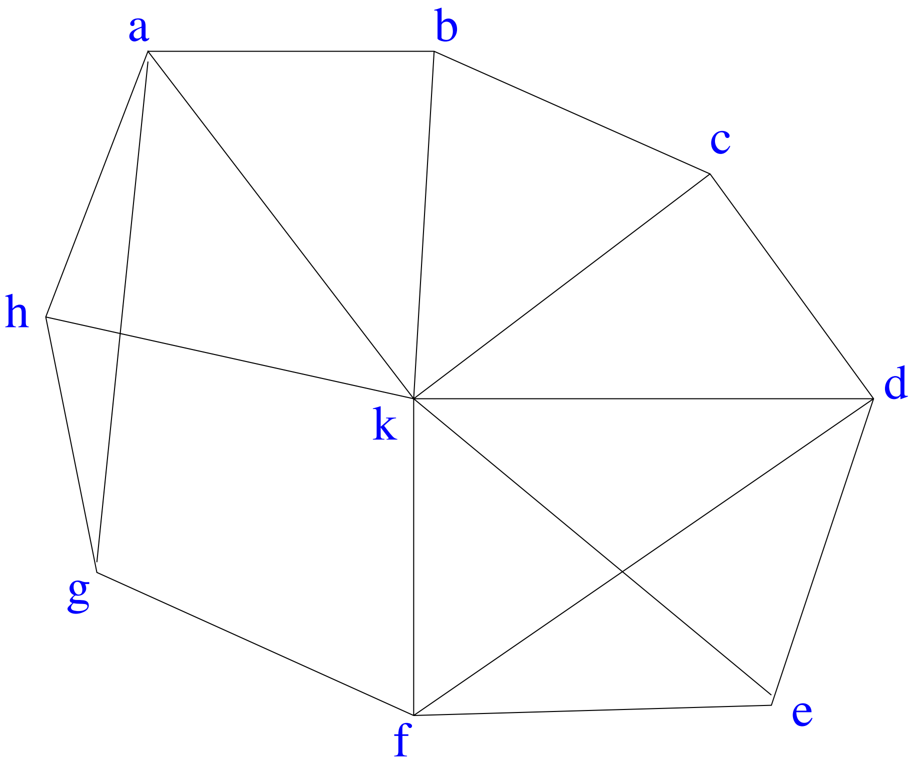
Graph Theory

Simple Graph $G = (V, E)$.
V={vertices}, E={edges}.

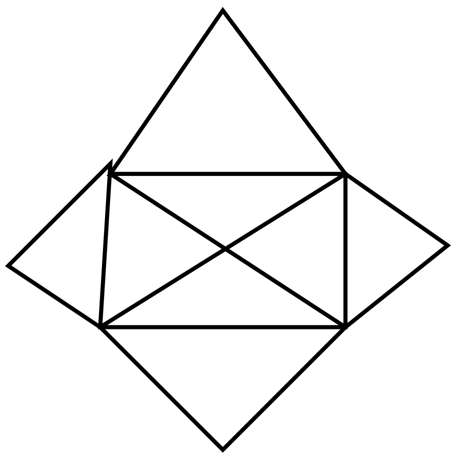


V={a,b,c,d,e,f,g,h,k}

E={(a,b),(a,g),(a,h),(a,k),(b,c),(b,k),...,(h,k)} |E|=16.

Eulerian Graphs

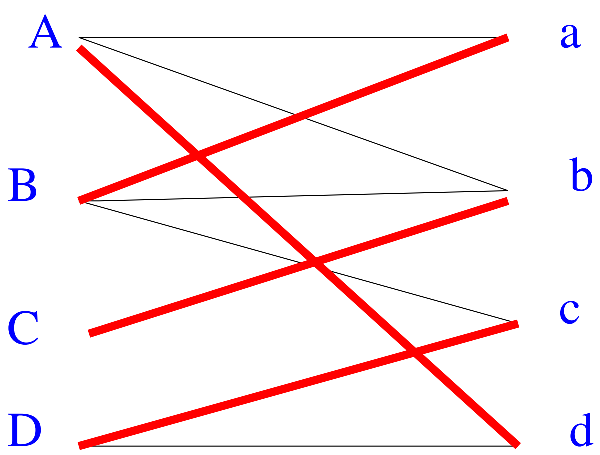
Can you draw the diagram below without taking your pen off the paper or going over the same line twice?



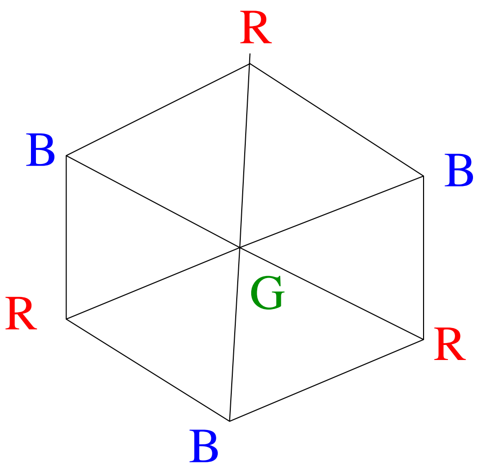
Bipartite Graphs

G is bipartite if $V = X \cup Y$ where X and Y are disjoint and every edge is of the form (x, y) where $x \in X$ and $y \in Y$.

In the diagram below, A,B,C,D are women and a,b,c,d are men. There is an edge joining x and y iff x and y like each other. The thick edges form a “perfect matching” enabling everybody to be paired with someone they like. Not all graphs will have perfect matching!



Vertex Colouring



Colours {R,B,G}

Let $C = \{colours\}$. A vertex colouring of G is a map $f : V \rightarrow C$. We say that $v \in V$ gets coloured with $f(v)$.

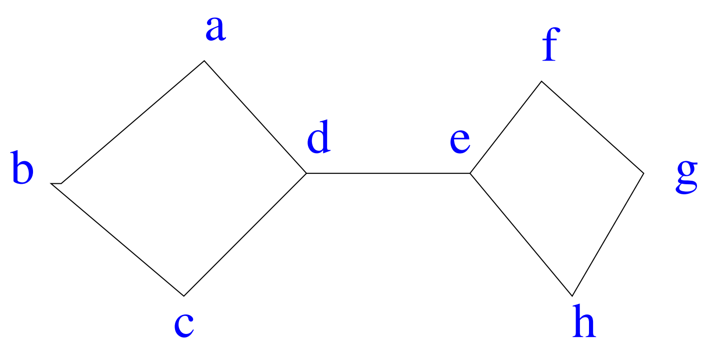
The colouring is *proper* iff $(a, b) \in E \Rightarrow f(a) \neq f(b)$.

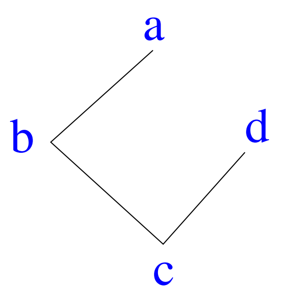
The *Chromatic Number* $\chi(G)$ is the minimum number of colours in a proper colouring.

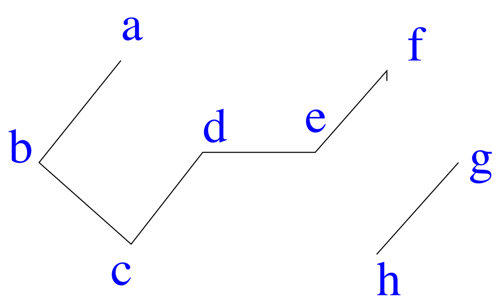
Application: V={exams}. (a, b) is an edge iff there is some student who needs to take both exams. $\chi(G)$ is the minimum number of periods required in order that no student is scheduled to take two exams at once.

Subgraphs

$G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.
G' is a *spanning* subgraph if $V' = V$.







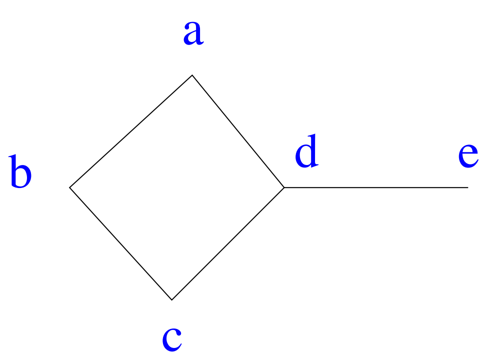
NOT SPANNING

SPANNING

If $V' \subseteq V$ then

$$G[V'] = (V', \{(u,v) \in E : \ u,v \in V'\})$$

is the subgraph of G *induced* by V'.



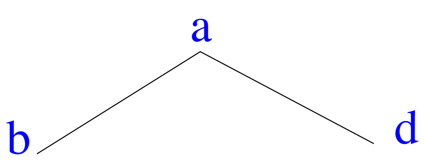
G[{a,b,c,d,e}]

Similarly, if $E_1 \subseteq E$ then $G[E_1] = (V_1, E_1)$ where

$$V_1 = \{v \in V \ : \ \exists e \in E_1 \text{ such that } v \in e\}$$

is also *induced* (by E_1).

$E_1 = \{(a,b), (a,d)\}$

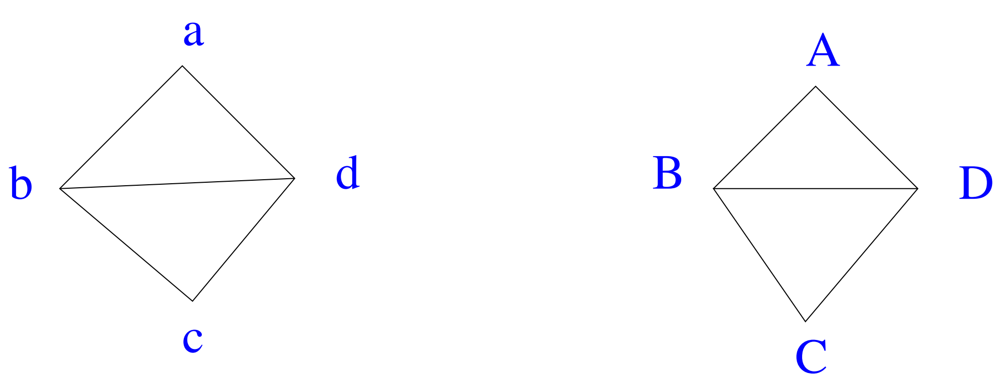


G[E$_1$]

Isomorphism

$G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection $f : V_1 \to V_2$ such that

$$(v, w) \in E_1 \leftrightarrow (f(v), f(w)) \in E_2.$$



f(a)=A etc.

Complete Graphs

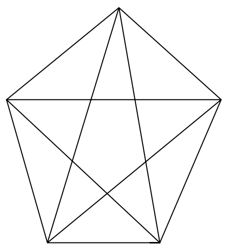
$$K_n = ([n], \{(i,j) : 1 \leq i < j \leq n\})$$

is the complete graph on n vertices.

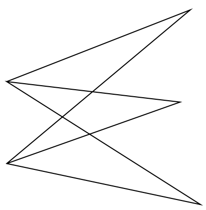
$$K_{m,n} = ([m] \cup [n], \{(i,j) : i \in [m], j \in [n]\})$$

is the complete bipartite graph on $m + n$ vertices.

(The notation is a little imprecise but hopefully clear.)



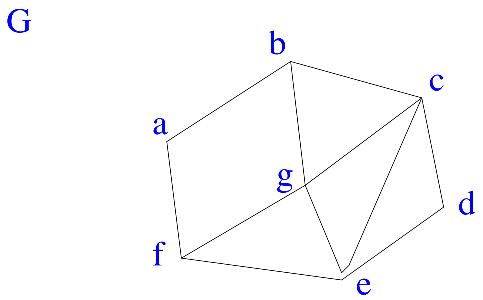
K_5



$K_{2,3}$

Vertex Degrees

$$
\begin{aligned}
d_G(v) &= \text{degree of vertex } v \text{ in } G \\
&= \text{number of edges incident with } v \\
\delta(G) &= \min_v d_G(v) \\
\Delta(G) &= \max_v d_G(v)
\end{aligned}
$$



$d_G(a)=2$, $d_G(g)=4$ etc.

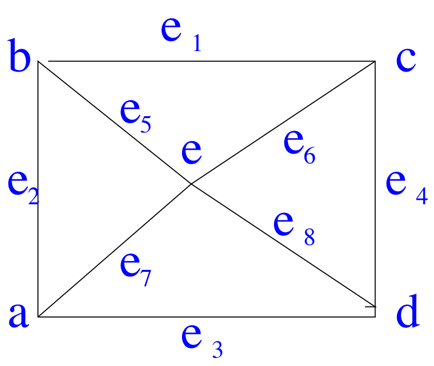
$\delta(G)=2$, $\Delta(G)=4$.

Matrices and Graphs

Incidence matrix M: $V \times E$ matrix.

$$M(v,e) = \begin{cases} 1 & v \in e \\ 0 & v \notin e \end{cases}$$

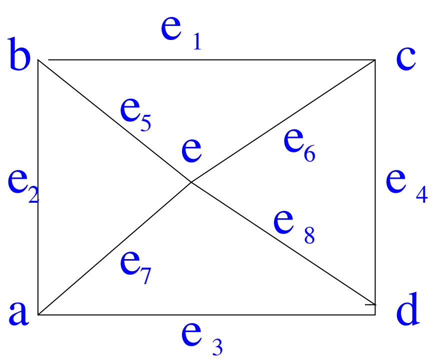
	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
a		1	1				1	
b	1	1			1			
c	1			1		1		
d			1	1				1
e					1	1	1	1



Adjacency matrix A: $V \times V$ matrix.

$$A(v,w) = \begin{cases} 1 & v, w \text{ adjacent} \\ 0 & \text{otherwise} \end{cases}$$

	a	b	c	d	e
a		1		1	1
b	1		1		1
c		1		1	1
d	1		1		1
e	1	1	1	1	



Theorem 1

$$\sum_{v \in V} d_G(v) = 2|E|$$

Proof Consider the incidence matrix M. Row v has $d_G(v)$ 1's. So

$$\#\text{ 1's in matrix } M \text{ is } \sum_{v \in V} d_G(v).$$

Column e has 2 1's. So

$$\#\text{ 1's in matrix } M \text{ is } 2|E|.$$

$\square$

Corollary 1 *In any graph, the number of vertices of odd degree, is even.*

Proof Let $ODD = \{\text{odd degree vertices}\}$ and $EVEN = V \setminus ODD$.

$$\sum_{v \in ODD} d(v) = 2|E| - \sum_{v \in EVEN} d(v)$$

is even.

So $|ODD|$ is even. $\square$