THEME 4: THE CALCULUS OF PROB

- EVENTS
- SET THEORY
- AXIOMS OF PROBABILITY
- THE <u>ADDITIVE</u> RULE OF PROB

FROM DATA TO $\underbrace{\text{STATEMENTS ABOUT DATA}}_{\text{EVENTS}}$

<u>EXAMPLE</u>: THE EVENT THAT WE OBSERVE

A VALUE BIGGER OR EQUAL TO 7

IN A Poisson(1.41)

- PROBABILITY MASS FUNCTION $P_X(x)$

SPECIFIED PROBABILITIES OF WHAT

WE CALL SIMPLE EVENTS

↳ THE EVENT THAT X TAKES THE VALUE x

EVENTS ARE SUBSETS OF POSSIBLE OUTCOMES

EXAMPLES OF EVENTS

- $X = 0$ (SIMPLE)
- $X \geq 7$
- $X \neq 0$
- $X = 0$ OR $X = 1$

Basic Set Theory

- S Set of outcomes

CURLY BRACKET

$$S = \{0, 1, 2, 3, 4\}$$

$$S = \{\text{CAR}, \text{HOUSE}, \text{JOB}\}$$

- Subsets of S

e.g. $A = \{0, 2, 4\} \subseteq S$

- SET OPERATIONS PT 1

- $A \subseteq S$ A SUBSET OF S

  $B \subseteq S$

- $x \in A$ x BELONGS TO A

  E

  FOR $x \in S$ & $A \subseteq S$

  $x \notin A$ x DOES NOT BELONG TO A

- IF $A \subseteq \mathcal{J}$, $B \subseteq \mathcal{J}$ THEN

$A \subseteq B$ IF $\underbrace{x \in A \Rightarrow}_{\text{IMPLIES}} x \in B$

FOR EVERY $x \in A$

- $A = B$ IF $A \subseteq B$ & $B \subseteq A$

- ϕ EMPTY SET $\phi = \{ \}$

$\phi \subseteq A$ FOR ANY $A \subseteq \mathcal{J}$

- $A, B \subseteq J$

$A \cup B$ A UNION B IS DEFINED AS

OR

$x \in A \cup B$ IF $x \in A$ **OR** $x \in B$

- $A \cap B$ A INTERSECTED B

$x \in A \cap B$ IF $x \in A$ **AND** $x \in B$

EXAMPLE

$$\mathcal{J} = \{0, 1, 2, 3, 6, 8\}$$

$$A = \{0, 3, 6\} \subseteq \mathcal{J}$$

$$B = \{1\}$$

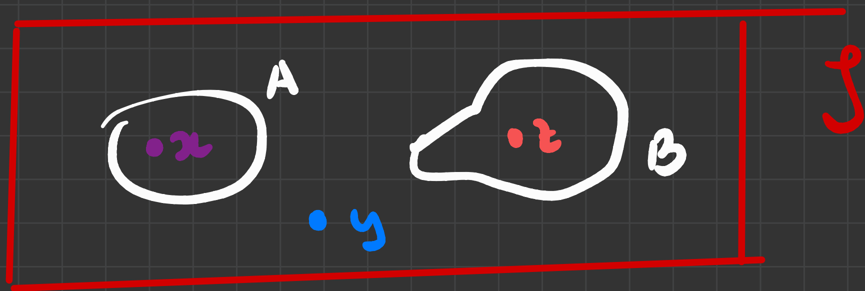
$$A \cup B = \{0, 1, 6, 3\}$$

$$A \cap B = \emptyset$$

- A^c A COMPLEMENT

if $x \in A^c$ if $x \notin A$

Venn Diagrams

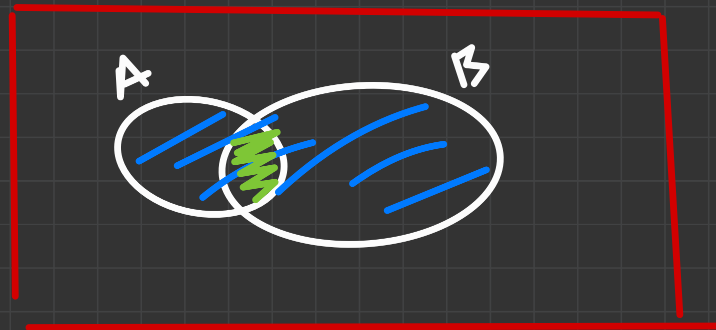


$x \in A$

$z \in B$

$y \in (A \cup B)^c$

$A \cap B = \emptyset$



$A \cup B$

$A \cap B$

$\{ X = x \}$

MATH NOTATION TO REFER TO WHAT IN NORMAL LANGUAGE WOULD BE THE EVENT THAT THE RV X TAKES VALUE x

WILL BE USED AS A SET

E.G.

$S = \{0, 1, 2, \dots\}$

SET OF ALL POSITIVE INTEGERS

$$A = \{X = 0\}$$

$$B = \{X = 1\}$$

SIMPLE EVENTS

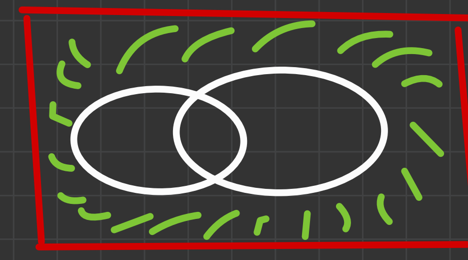
$$A \cup B = \{X = 0, \text{ OR }, X = 1\}$$

$$C = \{X \geq 7\} = D^c$$

$$D = \{X \leq 6\} = \{X = 0\} \cup \{X = 1\} \cup \dots \cup \{X = 6\}$$

SOME RESULTS IN SET THEORY

- $(A \cup B)^c = A^c \cap B^c$



PROOF :

i. $(A \cup B)^c \subseteq A^c \cap B^c$

$x \in (A \cup B)^c \Rightarrow x \notin A \cup B \Rightarrow x \notin A$ BUT ALSO $x \notin B \Rightarrow x \in A^c$ AND $x \in B^c$

$\Rightarrow x \in A^c \cap B^c$

ii. Do it yourselves

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: Do it

- NOTATION

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i$$

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^{n} A_i$$

- RESULT :

$$\left(\bigcup_{i=1}^{n} A_i \right)^c = \bigcap_{i=1}^{n} A_i^c$$

Axioms of Probability (Pt 1)

S set of possible outcomes

$A, B \subseteq S$ events

P (the probability) is a **function** from

events to $[0,1]$, s.t.

1. $P(\phi) = 0$

2. $P(S) = 1$

3. if $A \cap B = \phi$ (DISJOINT SETS)

THEN $P(A \cup B) = P(A) + P(B)$

ADDITIVE RULE OF PROB.

CONNECT PROBABILITY AS DEFINED ABOVE — IN AN ABSTRACT WAY (HARDWARE) — WITH THE PROBABILITY MODELS DISCUSSED EARLIER, DEFINED IN TERMS OF $p_X(x)$ (SOFTWARE)

$$p_X(x) = P(\{X = x\})$$

EXAMPLE

$$X \sim P_X(x)$$

$$P(\{X = x\}) = P_X(x)$$

$$P(\{X = 0 \text{ or } X = 1\}) = P(\{X=0\} \cup \{X=1\})$$

$$\{X=0\} \cap \{X=1\} = \phi$$

$$= P(\{X=0\}) + P(\{X=1\}) = P_X(0) + P_X(1)$$

Proposition $P(A^c) = 1 - P(A)$

[1] **Proof** : $A \cap A^c = \phi$

$$A \cup A^c = S$$

$$\underbrace{1 =}_{\text{Axiom 2}} P(A \cup A^c) \underbrace{=}_{\text{Axiom 3}} P(A) + P(A^c)$$

$$\Rightarrow P(A^c) = 1 - P(A) \quad \blacksquare$$

THEOREM IF $A_1, A_2, \ldots, A_n$ FOR $n \geq 2$

S.T. $A_i \cap A_j = \emptyset$ FOR ANY $i \neq j$

(MUTUALLY DISJOINT)

THEN $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

PROOF: DISCUSS LATER

EXAMPLE HOMICIDE STUDY

$$P(\{X \geq 7\}) = P(\{X \leq 6\}^c)$$

$$= 1 - P(\{X \leq 6\})$$

PREVIOUS RESULT

$$= 1 - P\left(\bigcup_{i=0}^{6} \{X = i\} \right)$$

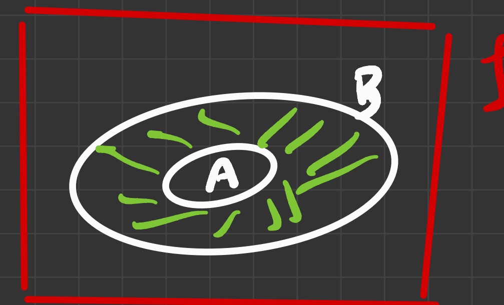
PREVIOUS THEOREM

$$= 1 - \sum_{i=0}^{6} P(\{X = i\})$$

$$= 1 - \left(p_X(0) + p_X(1) + \dots + p_X(6) \right) = 0.0017$$

Proposition if $A \subseteq B$ then $P(A) \leq P(B)$

[1] **Proof**



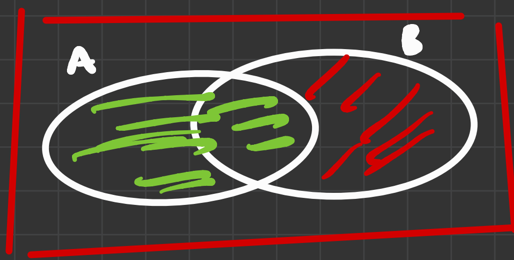
(a) $B = A \cup (A^c \cap B)$

(b) $A \cap (A^c \cap B) = \emptyset$

(c) $P(B) = P(A) + P(A^c \cap B)$

$\geq P(A)$ since $P(A^c \cap B) \geq 0$ by assumption

Proposition : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



[1] Proof :

(a) $A \cup B = A \cup (B \cap A^c)$

(b) $A \cap (A^c \cap B) = \emptyset$

(c) $P(A \cup B) \underset{\text{Axiom 3}}{=} P(A) + P(A^c \cap B)$

(d) $B = (A \cap B) \cup (B \cap A^c)$

$$P(B) = P(A \cap B) + P(A^c \cap B)$$

$$\Rightarrow \quad P(A^c \cap B) = P(B) - P(A \cap B)$$

BY PLUGGING IN PREVIOUS EQ. COMPLETED THE PROOF

$\blacksquare$

Theorem if $A_1, A_2, \ldots, A_n$ for $n \geq 2$

s.t. $A_i \cap A_j = \emptyset$ for any $i \neq j$

(mutually disjoint)

then $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

Sketch of the proof

LET'S CHECK $n=3$

(a) $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$

$$A_1 \cup A_2 \cup A_3 = \underbrace{(A_1 \cup A_2)}_{B} \cup A_3$$

$$B \cap A_3 = (A_1 \cup A_2) \cap A_3 = (A_1 \cap A_3) \cup (A_2 \cap A_3)$$

$$= \phi \cup \phi = \phi$$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1 \cup A_2) + P(A_3)$$

$$= P(A_1) + P(A_2) + P(A_3)$$

THE FULL PROOF IS BASED ON INDUCTION

WHAT ABOUT $P\left(\bigcup_{i=1}^{\infty} A_i\right) \stackrel{?}{=} \sum_{i=1}^{\infty} P(A_i)$

∞ : INFINITY WHEN $A_i \cap A_j = \phi$

FOR ALL $i \neq j$

$$\bigcup_{i=1}^{\infty} A_i = \lim_{n \to \infty} \bigcup_{i=1}^{n} A_i$$

Axioms of Prob Pt 2

Ω space of outcomes. Then, probability is

a function that maps events to $[0,1]$ s.t.

1. $P(\emptyset) = 0$
2. $P(\Omega) = 1$
3. If $A_1, A_2, \ldots$ is a seq of mutually disjoint events, i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$

   then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$