COMS 331: Theory of Computing, Fall 2014
Homework Assignment 1

Due at the beginning class on Friday, September 5

Problem 1. Prove or disprove: If $A = \{0^n 1^n \mid n \in \mathbb{N}\}$, then $A^* = A$.

Problem 2. Prove or disprove: If $B = \{x \in \{0,1\}^* \mid \#(0,x) = \#(1,x)\}$, then $B^* = B$.

Note: The notation $\#(0,x)$ is used to denote the number of 0's in x. Likewise, $\#(1,x)$ is used to denote the number of 1's in x.

Problem 3. Prove: For every positive integer n,

$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$

Problem 4. Prove: For every language A, $A^{**} = A^*$.

Problem 5. Prove: If $S = \{0,1\}$ and $T \subseteq \{0,1\}^*$, then

$$S^* = T^* \Rightarrow S \subseteq T.$$

Problem 6. Exhibit languages $S, T \subseteq \{0,1\}^*$ such that $S^* = T^*$ and $\{0,1\} \subseteq S \subsetneq T$.

Problem 7. Define an (infinite) binary sequence $s \in \{0,1\}^\infty$ to be *prefix-repetitive* if there are infinitely many strings $w \in \{0,1\}^*$ such that $ww \sqsubseteq s$.

Prove: If the bits of a string $s \in \{0,1\}^\infty$ are chosen by independent tosses of a fair coin, then

$$Prob[s \text{ is prefix-repetitive}] = 0.$$

Note: $x \sqsubseteq y$ means that x is a prefix of y where x and y are strings.

Problem 8. Define a 2-*coloring* of $\{0,1\}^*$ to be a function $\chi : \{0,1\}^* \to \{\text{red}, \text{blue}\}$. (For example, if $\chi(1101) = \text{red}$, we say that 1101 is red in the coloring χ.)

Prove: For every 2-coloring χ of $\{0,1\}^*$ and every (infinite) binary sequence $s \in \{0,1\}^\infty$, there is a sequence

$$w_0, w_1, w_2, \cdots$$

of strings $w_n \in \{0,1\}^*$ such that
(i) $s = w_0 w_1 w_2 \cdots$, and
(ii) $w_1, w_2, w_3, \cdots$ are all the same color. (The string w_0 may or may not be this color.)