CS311: Homework #1

Due on September 5, 2014 at 09:00am

Professor Lutz 09:00am

Kien Nguyen

Problem 1

Prove or disapprove: If $A = \{0^n1^n | n \in \mathbb{N}\}$, then $A^* = A$

Proof. We'll disapprove it.
Assume that $A = \{0^n1^n | n \in \mathbb{N}\}$, we have $0101 \in A^*$, but $0101 \notin A$. Therefore, $A^* \neq A$ $\square$

Problem 2

Prove or disapprove: If $B = \{x \in \{0,1\}^* | \#(0,x) = \#(1,x)\}$, then $B* = B$.
Note: The notation $\#(0,x)$ is used to denote the number of $0's$ in x. Likewise, $\#(1,x)$ is used to denote the number of $1's$ in x.

Proof. (1) Prove $B* \in B$.

For every $y \in B^*$, we have y is a combination of some elements in B. Therefore we can write:

$$y = x_1 x_2 \ldots x_n,\, where\, n \in \mathbb{N},\, and\, x_i \in B, i = 1 \ldots n$$

then,

$$\#(0,y) = \#(0,x_1) + \#(0,x_2) + \cdots + \#(0,x_n)$$

and

$$\#(1,y) = \#(1,x_1) + \#(0,x_2) + \cdots + \#(1,x_n)$$

Since we have $x_i \in B$ then $\#(0,x_i) = \#(1,x_i) \forall i = 1 \ldots n$, then $\#(0,y) = \#(1,y) \Rightarrow y \in B$. Therefore $B^* \subseteq B$.

(2) Prove $B \in B^*$.

For any alphabet A, each element in A can itself represent a string in A^*, therefore $A \in A^*$ for every A. So, we have $B \in B^*$.

From (1) and (2), we have $B = B^*$. $\square$

Problem 3

Prove: For every positive integer n,

$$\sum_{i=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$$

Proof. We'll prove this problem by induction on n.

(1) Base case: $n = 1$, we need

$$\frac{1}{1^1} \leq 2 - \frac{1}{1}$$

and that is true.

(2) We assume that the statement is true for n, that is $\sum\limits_{i=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}$, we will prove that the statement also holds for n+1, that is $\sum\limits_{i=1}^{n+1} \frac{1}{k^2} \leq 2 - \frac{1}{n+1}$.

We have

$$\begin{aligned} \sum_{i=1}^{n+1} \frac{1}{k^2} &= \sum_{i=1}^{n} \frac{1}{k^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \\ &= 2 - \frac{n^2+n+1}{n(n+1)^2} = 2 - \frac{n(n+1)+1}{n(n+1)^2} \\ &\leq 2 - \frac{n(n+1)}{n(n+1)^2} = 2 - \frac{1}{n+1} \end{aligned}$$

So,

$$\sum_{i=1}^{n+1} \frac{1}{k^2} \leq 2 - \frac{1}{n+1}$$

□

Problem 4

Prove: For every language $A, A^{**} = A^*$.

Proof. (1) Since each element in A^* can itself represent a string in A^{**}, then $A^* \subseteq A^{**}$.

(2) We'll prove that $A^{**} \subseteq A^*$.
Each element $y \in A^{**}$ can be represented as $y = x_1 x_2 \ldots x_n$ where $n \in \mathbb{N}$ and $x_i \in A^*$ for $i = 1 \ldots n$.
Because $x_i \in A^*$, x_i is a concatenation from elements from $A \Rightarrow x_1 x_2 \ldots x_n$ is also a concatenation from elements from $A \Rightarrow y$ is also a concatenation from elements from $A \Rightarrow y \in A^*$.
Therefore, $A^{**} \subseteq A^*$.

From (1) and (2), we have $A^{**} = A^*$. $\square$

Problem 5

Prove: If $S = \{0, 1\}$ and $T \subseteq \{0, 1\}^*$, then

$$S^* = T^* \Rightarrow S \subseteq T$$

Proof. Since $S = \{0, 1\}$, $0 \in S^*$ and $1 \in S^*$. Followed by the fact that $S^* = T^* \Rightarrow 0 \in T^*$ and $1 \in T^*$.
$T \subseteq \{0, 1\}^* \Rightarrow T$ is a list of strings where each string is a concatenation of $0's$ and $1's$. Since T^* contains 0 and 1, and 0 and 1 are single characters, then there is no way that elements in T can produce a single 0 and a single 1 in T^* without T actually contains 0 and 1 itself $\Rightarrow 0 \in T$ and $1 \in T$.
So, all the elements in S are in $T \Rightarrow S \subseteq T$. $\square$

Problem 6

Exhibit languages $S, T \subseteq \{0,1\}^*$ such that $S^* = T^*$ and $\{0,1\} \subseteq S \subset T$.

Proof. We pick $S = \{0, 1, 01\}$ and $T = \{0, 1, 01, 10\}$. Both S and T contain 0 and 1 $\Rightarrow \{0,1\} \subseteq S$ and $\{0,1\} \subseteq T$.
We can also see that $S \subset T$.
We'll prove that $S^* = T^*$.

(1) Prove $S^* \subseteq T^*$
For $y \in S^*$, y is sure a concatenation of 0 and 1. And we have $0 \in T$ and $1 \in T$.
Therefore $y \in T^* \Rightarrow S^* \subseteq T^*$.

(2) Similarly, we also have $T^* \subseteq S^*$.

So, $S^* = T^*$. □

Problem 7

Define an (infinite) binary sequence $s \in \{0,1\}^\infty$ to be $prefix - repetitive$ if there are infinitely many strings $w \in \{0,1\}^*$ such that $ww \sqsubseteq s$.
Prove: If the bits of a strings $s \in \{0,1\}^\infty$ are chosen by independent tosses of a fair coin, then

$$Prob[s\ is\ prefix - repetitive] = 0.$$

Note: $x \sqsubseteq y$ means that x is a prefix of y where x and y are strings.

Proof. Because if s is prefix-repetitive, there are infinitely many $w \in \{0,1\}^*$ such that $ww \sqsubseteq s$, then we can have the length of w can grow to ∞.
Assume that after n tosses, we have a string w, the probabilty such that after another n tosses, we have the whole string as ww is $(\frac{1}{2})^n$ (since it is a fair coin $\Rightarrow$ the probability to have head or tail is 1/2, and since we want the second n tosses to be exactly the same as the first n tosses $\Rightarrow$ each toss of the second n tosses must be the same as the correspond toss from the first n tosses $\Rightarrow$ for each one, we have only 1/2 chance it happens).
Because n can grow to ∞, $(\frac{1}{2})^n$ can grow to 0.
$\Rightarrow$ Prob[having ww in the string] is 0 $\Rightarrow$ Prob[s is prefix-repetitive] is 0. $\square$

Problem 8

Define 2-*coloring* of $\{0,1\}^*$ to be a function $\mathcal{X} : \{0,1\}^* \to \{red, blue\}$. (For examle, if $\mathcal{X}(1101)$ = red, we say that 1101 is red in coloring $\mathcal{X}$.)
Prove: For every 2-coloring $\mathcal{X}$ and every (infinite) binary sequence $s \in \{0,1\}^\infty$, there is a sequence

$$w_0, w_1, w_2, ...$$

of strings $w_n \in \{0,1\}^*$ such that

(i) $s = w_0 w_1 w_2 \ldots$, and

(ii) $w_1, w_2, w_3, \ldots$ are all the same color. (The string w_0 may not be this color.)

Proof. I don't know how to solve this problem $\square$