CS311: Homework #1

Due on September 5, 2014 at 05:00pm

Professor Lathrop 12:40pm

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Problem 1

Prove:

a) $12|\mathbb{N} \subseteq 3|\mathbb{N}$
For $n \in 12|\mathbb{N}, \exists k \in \mathbb{N}$ such that $n = 12k$
then $n = 3 * (4k)$ then $n \in 3|\mathbb{N}$
Therefore $12|\mathbb{N} \subseteq 3|\mathbb{N}$

b) $35|\mathbb{N} = 5|\mathbb{N} \cap 7|\mathbb{N}$

(1) we prove $35|\mathbb{N} \subseteq 5|\mathbb{N} \cap 7|\mathbb{N}$
For $n \in 12|\mathbb{N}, \exists k \in \mathbb{N}$ such that $n = 35k$
Hence, we have $n = 5 * (7k)$ and $n = 7 * (5k)$ then $n \in 5|\mathbb{N}$ and $n \in 7|\mathbb{N}$.
So $n \in 5|\mathbb{N} \cap 7|\mathbb{N}$ (done for-arrow)

(2) we prove $5|\mathbb{N} \cap 7|\mathbb{N} \subseteq 35|\mathbb{N}$
For $n \in 5|\mathbb{N} \cap 7|\mathbb{N}$, we can have $n \in 5|\mathbb{N}$ and $n \in 7|\mathbb{N}$. Since $n \in 5|\mathbb{N} \exists k \in \mathbb{N}$ such that $n = 5k$
Then $5k \in 7|\mathbb{N}$ then $5k$ is divisible by 7
Because $(5, 7) = 1$ (largest common divisor), then k is divisible by 7.
So $\exists h \in \mathbb{N}$ such that $k = 7h$. Then $n = 5 * (7h)$ or $n = 35h$. Then $n \in 35|\mathbb{N}$ (done reversed-arrow).
Therefore, $12|\mathbb{N} \subseteq 3|\mathbb{N}$.

c) $20|\mathbb{N} \not\subseteq 3|\mathbb{N}$
In order for $20|\mathbb{N} \subseteq 3|\mathbb{N}$, every single element in $20|\mathbb{N}$ must be in $3|\mathbb{N}$.
Since $20 \in 20|\mathbb{N}$, but $20 \notin 3|\mathbb{N} \Rightarrow 20|\mathbb{N} \not\subseteq 3|\mathbb{N}$

Problem 2

For arbitrary sets A,B, prove:

a) $A \cup B = B \iff A \subseteq B$

b) $A \cap B = B \iff B \subseteq A$

c) $A - (A - B) \subseteq B$
And prove there exists sets A, B such that:

d) $B \nsubseteq A - (A - B)$

Proof. (a) ($\Rightarrow$) Prove: If $A \cup B = B \Rightarrow A \subseteq B$
For $\forall x \in A$, $x \in A \cup B$. But we also have $A \cup B = B \Rightarrow x \in B$.
So $\forall x \in A$, also we have $x \in B$.
Therefore, $A \subseteq B$. (done forward arrow)
($\Leftarrow$) Prove: If $A \subseteq B \Rightarrow A \cup B = B$
By definition, $A \cup B = \{x | x \in A \parallel x \in B\}$.
We also have for every $x \in A$, $x \in B$ also (because $A \subseteq B$.)
$\Rightarrow A \cup B = \{x | x \in B\} = B$. (done reversed arrow)

(b) ($\Rightarrow$) Prove: If $A \cap B = B \Rightarrow B \subseteq A$
For $\forall x \in B$, we also have $x \in A \cap B$ because $B = A \cap B$. Therefore, also we have $x \in A$, $\Rightarrow B \subseteq A$. (done forward arrow)
($\Leftarrow$) Prove: If $B \subseteq A \Rightarrow A \cap B = B$
By definition, $A \cap B = \{x | x \in A \wedge x \in B$.
Because $B \subseteq A \Rightarrow \forall x \in B$, then apparently $x \in A$.
Therefore, $A \cap B = \{x | x \in B\} = B$. (done reversed arrow)

(c) Using venn diagrams

(d) $A = \{0, 1\}, B = \{1, 2\} \Rightarrow A - (A - B) = \{1\}$.
Therefore, $B \nsubseteq A - (A - B)$.

□

Problem 3

Give an example of a function $f : \mathbb{Z} \to \mathbb{N}$ that is both one-to-one and onto

Proof. Choose f such that:

$$f(n) = \begin{cases} 2n+1 & \text{if } n \geq 0 \\ 2|n| & \text{if } n < 0 \end{cases}$$

(a) Prove f is one-to-one
For any number $n, m \in \mathbb{Z}$, we have $f(n) = f(m) \iff n, m \geq 0 \parallel n, m < 0$
If $n, m \geq 0$, then $f(n) = f(m) \iff 2n + 1 = 2m + 1 \iff n = m$.
If $n, m < 0$, then $f(n) = f(m) \iff 2|n| = 2|m \iff n = m$.
Therefore $f(n) = f(m) \iff n = m \Rightarrow f$ is one-to-one.

(b) Prove f is onto
For any number $v \in \mathbb{N}$,
If v is odd, we can choose $n = (v - 1)/2$ such that $f(n) = v$.
If v is even, we can choose $n = -v/2$ such that $f(n) = v$.
Therefore f is onto.

$\square$

Problem 4

Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined as $f(n) = 3x + 7$.Prove:

(a) f is one to one

(b) f is NOT onto

Proof. (a) For every $x, y \in \mathbb{Z}$, we have: $f(x) = f(y) \Leftrightarrow 3x + 7 = 3y + 7 \Leftrightarrow x = y \Rightarrow f$ is one to one.

(b) Let $v = 8$. Assume that there is a value $x \in Z$ such that $f(x) = v$ or $f(x) = 8$
$\Rightarrow 3x + 7 = 8 \Rightarrow x = 1/7 \Rightarrow x \notin \mathbb{Z} \Rightarrow$ contradiction.
Therefore f is NOT onto.

$\square$

Problem 5

Let r be a relation over real numbers such that for $a, b \in \mathbb{R}, a \ r \ b$ if and only if $a - b \in \mathbb{Z}$. Prove that r is an equivalence relation.

Proof. (1) Prove r is reflective
For $\forall x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z} \Rightarrow r$ is reflective.

(2) Prove r is symmetric
For $x, y \in \mathbb{R}$, assume that $x - y \in \mathbb{Z} \Rightarrow x - y = z \in \mathbb{Z} \Rightarrow y - x = -z \in \mathbb{Z} \Rightarrow r$ is symmetric.

(3) Prove r is transitive
For $x, y, z \in \mathbb{R}$, assume that $x - y = a \in \mathbb{Z}$ and $y - z = b \in \mathbb{Z}$
$\Rightarrow x - z = (x - y) + (y - z) = a + b \in \mathbb{Z} \Rightarrow r$ is transitive.

So, r is an equivalence relation. $\square$

Problem 6

Use induction to prove the following:

a) For all $n \in \mathbb{Z}^+$,

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2$$

Base case: $n = 1 \Rightarrow 1 + 3 + 5 + \cdots + 2n - 1 = 1 = n^2 \Rightarrow$ statement is true for $n = 1$.
Induction: Assume the statement is true for n, that is $1 + 3 + 5 + \cdots + 2n - 1 = n^2$, we'll prove that the statement also holds for $n + 1$, that is $1 + 3 + 5 + \cdots + 2(n + 1) - 1 = (n + 1)^2$.
$1+3+5+\cdots+2(n+1)-1 = 1+3+5+\cdots+2n-1+2(n+1)-1 = n^2+2(n+1)-1 = n^2+2n+1 = (n+1)^2$.
(done)

b) For all $n \in \mathbb{Z}^+$,

$$3^n > 2^n$$

Base case: $n = 1 \Rightarrow 3^n = 3$ and $2^n = 2 \Rightarrow 3^n > 2^n \Rightarrow$ statement is true for $n = 1$.
Induction: Assume the statement is true for n, that is $3^n > 2^n$, we'll prove that the statement is also holds for $n + 1$, that is $3^{n+1} > 2^{n+1}$.
We have $3^{n+1} = 3^n3$ and $2^{n+1} = 2^n2$.
Since $3^n > 2^n$ and $3 > 2$, then $3^n3 > 2^n2 \Rightarrow 3^{n+1} > 2^{n+1}$. (done)

c) For all $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Base case: $n = 1$, $\sum\limits_{i=1}^{n} i = 1 = \frac{n(n+1)}{2} \Rightarrow$ statement is true for $n = 1$.
Induction: Assume the statement is true for n, that is $\sum\limits_{i=1}^{n} i = \frac{n(n+1)}{2}$, we'll prove that the statement is also holds for $n + 1$, that is $\sum\limits_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n^2 + n + 2n + 2}{2} = \frac{(n+1)(n+2)}{2}\end{aligned}$$

(done)

d) For all $n \in \mathbb{Z}^+$,

$$n^3 + 2n \text{ is divisible by } 3$$

Base case: $n = 1$,$n^3 + 2n = 3$ is divisible by 3. So, statement is true for $n = 1$.
Induction: Assume the statement is true for n, that is $n^3 + 2n = 3k$, where $k \in \mathbb{Z}^+$, we'll prove that the statement is also holds for $n + 1$, that is $(n + 1)^3 + 2(n + 1)$ is also divisible by 3.
$(n+1)^3+2(n+1) = n^3+3n^2+3n+1+2n+2 = (n^3+2n)+(3n^2+3n+3) = 3k+3(n^2+n+1) = 3(k+n^2+n+1)$ is divisible by 3. (done)