(Gomory cuts Solving integer linear programs) This week's exercise will involve you learning how to solve linear programs where the desired solution must use integer-valued optimization variables. For this we need to cover a tiny bit of theory of *Gomory cuts.*

- I suggest you begin with a quick websearch (image search will help too) to see what a Gomory cut is.

Here's my crash course in Gomory cuts. We start with a final tableau to a linear program:

Final tableau:

		3	4	0	0	
c_B	Basis	y_1	y_2	s_1	s_2	bfs
3	y_1	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{16}{5}$
4	y_2	0	1	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{12}{5}$
		3	4	$\frac{2}{5}$	$\frac{9}{5}$	$\frac{96}{5}$
		0	0	$-\frac{2}{5}$	$-\frac{9}{5}$	

Optimal solution for $y \in \mathbb{R}^2$ is $y = (16/5, 12/5)$ with objective value $96/5$. However, what's the best $y^* \in \mathbb{N}^2$?

Reading off the y_1 row of the tableau we have

$$\begin{pmatrix} 1 & 0 & \frac{2}{5} & -\frac{1}{5} & \Big| & \frac{16}{5} \end{pmatrix}$$

this represents the following equality

$$y_1 + \frac{2}{5}s_1 - \frac{1}{5}s_2 = \frac{16}{5}.$$

Here we make two very clever observations: (i) since all variables are non-negative rounding down the left-hand side will make the lhs no larger, i.e.

$$y_1 + 0s_1 - 1s_2 \leq \frac{16}{5},$$

but at the best $y \in \mathbb{N}^2$ the left-hand side is now an integer so,

$$y_1 + 0s_1 - 1s_2 \leq 3,$$

by rounding down the right hand side! We now insert this new constraint with a new slack s_3 back into our problem. Notice the answer will no longer be $y = (16/5, 12/5)$.

If interested, you could think about why this procedure ensures that no feasible solutions $y \in \mathbb{N}^2$ become infeasible with the introduction of this new constraint.

- This new constraint is called a *Gomory cut*, which we can introduce to the problem.
- If the newly created problem has an optimal solution in $\mathbb{N}^2$ we're finished. Otherwise, repeat with the new problem.

Exercise 7 (Hypatia)**.** Gomory cuts – by example

(a) After a little [Internet] research write out a few bullet points to explain the steps involved in using Gomory cuts to find *integer solutions* to *linear programs.*

Now consider again the advertising example from the notes, Application 2.17. You are going to resolve an issue which arose in that example concerning integer solutions.

(b) Solve the problem as described in Application 2.17 using the lp function. Confirm it agrees with the solution found in the notes.

(c) Modify the second constraint to permit up to 13 slots to be purchased and re-solve the problem. Confirm it agrees with the solution found in the notes. This was unsatisfactory since it involves non-integer values for the numbers of adverts to purchase.

(d) Using the final tableau in the notes, **modified with the** $b_2 = 13$ **change above**, identify a Gomory cut from the x_3 row. (No need to use R)

(e) Add your Gomory cut to the constraint matrix A and re-solve the problem using lp. (Hint: since the slack variables may not appear in your original formulation, if you didn't augment in the first part then one approach would be to go back and augment to tackle this part. An alternative is to eliminate s_1 and s_2 using the original constraints.)

(f) Comment on what you have found. For example: Were you surprised in any way with the new solution? What is the best real-world (integer) solution to the problem when permitted to buy up to 13 slots? Did inserting the Gomory cut reduce or increase the objective value and why?

(g) Extension*: In general one Gomory cut is not always enough to reach the optimal integer solution. What would have happened if we had made our cut using the s_1 variable? Or the x_3 variable?