15 Matrices and Linear transformations

We have been thinking of matrices in connection with solutions to linear systems of equations like $A\mathbf{x} = \mathbf{y}$. It is time to broaden our horizons a bit and start thinking of matrices as *functions*.

In general, a function $\mathbf{f}$ whose domain is $\mathbb{R}^n$ and which takes values in $\mathbb{R}^m$ is a "rule" or recipe that associates to each $\mathbf{x} \in \mathbb{R}^n$ a vector $\mathbf{y} \in \mathbf{r}^m$. We can write either

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) \text{ or, equivalently } \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m.$$

The first expression is more familiar, but the second is more useful: it tells us something about the domain and range of the function $\mathbf{f}$.

Examples:

- $f : \mathbb{R} \to \mathbb{R}$ is a real-valued function of one real variable - the sort of thing you studied in calculus. $f(x) = \sin(x) + xe^x$ is an example.
- $\mathbf{f} : \mathbb{R} \to \mathbb{R}^3$ defined by
$$\mathbf{f}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} t \\ 3t^2 + 1 \\ \sin(t) \end{pmatrix}$$
assigns to each real number t the point $\mathbf{f}(t) \in \mathbb{R}^3$; this sort of function is called a *parametric curve*. Depending on the context, it could represent the position or the velocity of a mass point.
- $f : \mathbb{R}^3 \to \mathbb{R}$ defined by
$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x^2 + 3xyz)/z^2.$$
Note that here, since the function takes values in $\mathbb{R}^1$ it is customary to write f rather than $\mathbf{f}$.

- An example of a function from $\mathbb{R}^2$ to $\mathbb{R}^3$ is

$$\mathbf{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ \cos(xy) \\ x^2y^2 \end{pmatrix}$$

In this course, we're primarily interested in functions that can be defined using matrices. In particular, if A is $m \times n$, we can use A to define a function $\mathbf{f}_A$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ which sends $\mathbf{x} \in \mathbb{R}^n$ to $A\mathbf{x} \in \mathbb{R}^m$. That is, $\mathbf{f}_A(\mathbf{x}) = A\mathbf{x}$.

Example: Let

$$A_{2\times 3} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

If

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3,$$

then we define

$$\mathbf{f}_A(\mathbf{x}) = A\mathbf{x} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y+3z \\ 4x+5y+6z \end{pmatrix}$$

sends the vector $\mathbf{x} \in \mathbb{R}^3$ to $A\mathbf{x} \in \mathbb{R}^2$. Notice that if the function goes from $\mathbb{R}^3$ to $\mathbb{R}^2$, then the matrix is 2×3 (not 3×2).

Definition: A function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **linear** if

- $\mathbf{f}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{f}(\mathbf{x}_1) + \mathbf{f}(\mathbf{x}_2)$, and
- $\mathbf{f}(c\mathbf{x}) = c\mathbf{f}(\mathbf{x})$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ and for all scalars c.

A linear function $\mathbf{f}$ is also known as a **linear transformation**.

Proposition: $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$ is linear $\iff$ for all vectors $\mathbf{x}_1, \mathbf{x}_2$ and all scalars c_1, c_2, $\mathbf{f}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1\mathbf{f}(\mathbf{x}_1) + c_2\mathbf{f}(\mathbf{x}_2)$.

Proof: Exercise

Examples:

- Define $f:\mathbb{R}^3 \to \mathbb{R}$ by
$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3x - 2y + z.$$
Then f is linear because for any
$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \quad \text{and } \mathbf{x}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix},$$
we have
$$f(\mathbf{x}_1 + \mathbf{x}_2) = f\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} = 3(x_1 + x_2) - 2(y_1 + y_2) + (z_1 + z_2).$$
And the right hand side can be rewritten as $(3x_1 - 2y_1 + z_1) + (3x_2 - 2y_2 + z_2)$, which is the same as $f(\mathbf{x}_1) + f(\mathbf{x}_2$. So the first property holds. So does the second, since $f(c\mathbf{x}) = 3cx - 2cy + cz = c(3x - 2y + z) = cf(\mathbf{x})$.

- Notice that the function f is actually f_A for the right A: if $A_{1\times 3} = (3, -2, 1)$, then $f(\mathbf{x}) = A\mathbf{x}$.

- If $A_{m\times n}$ is a matrix, then $\mathbf{f}_A:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation because $\mathbf{f}_A(\mathbf{x}_1+\mathbf{x}_2) = A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{f}_A(\mathbf{x}_1) + \mathbf{f}_A(\mathbf{x}_2)$. And $A(c\mathbf{x}) = cA\mathbf{x} \Rightarrow \mathbf{f}_A(c\mathbf{x}) = c\mathbf{f}_A(\mathbf{x})$. (These are two fundamental properties of matrix multiplication.)

- It can be shown(next section) that *any* linear transformation on a finite-dimensional space can be written as $\mathbf{f}_A$ for a suitable matrix A.

- The derivative (see Lecture 9) is a linear transformation. $D\mathbf{f}(\mathbf{a})$ is the linear approximation to $\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})$.

- There are many other examples of linear transformations; some of the most interesting ones do *not* go from $\mathbb{R}^n$ to $\mathbb{R}^m$:

  1. If f and g are differentiable functions, then
  $$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}, \text{ and } \frac{d}{dx}(cf) = c\frac{df}{dx}.$$
  Thus the function $\mathcal{D}(f) = df/dx$ is linear.

  2. If f is continuous, then we can define
  $$If(x) = \int_0^x f(s)\ ds,$$
  and I is linear, by well-known properties of the integral.

  3. The Laplace operator, Δ, defined before, is linear.

  4. Let y be twice continuously differentiable and define
  $$L(y) = y'' - 2y' - 3y.$$
  Then L is linear, as you can (and should!) verify.

  Linear transformations acting on functions, like the above, are generally known as **linear operators**. They're a bit more complicated than matrix multiplication operators, but they have the same essential property of linearity.

Exercises:

1. Give an example of a function from $\mathbb{R}^2$ to itself which is not linear.

2. Which of the functions on the first page of this chapter are linear? Answer: none. Be sure you understand why!

3. Identify all the linear transformations from $\mathbb{R}$ to $\mathbb{R}$.

4. If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is linear then the **kernel** of $\mathbf{f}$ is defined by

$$\mathrm{Ker}(\mathbf{f}) := \{\mathbf{v} \in \mathbb{R}^n \text{ such that } \mathbf{f}(\mathbf{v}) = \mathbf{0}\}.$$

   Show that $\mathrm{Ker}(\mathbf{f})$ is a subspace of $\mathbb{R}^n$.

5. If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, then the **range** of $\mathbf{f}$ is defined by

$$\mathrm{Range}(\mathbf{f}) = \{\mathbf{y} \in \mathbb{R}^m \text{ such that } \mathbf{y} = \mathbf{f}(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$$

   Show that, if $\mathbf{f}$ is linear, then $\mathrm{Range}(\mathbf{f})$ is a subspace of $\mathbb{R}^m$.

15.1 The matrix of a linear transformation

In this section, we'll show that if $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then there exists an $m \times n$ matrix A such that $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$.

Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1}, \cdots, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}$$

be the standard basis for $\mathbb{R}^n$. And suppose $\mathbf{x} \in \mathbb{R}^n$. Then

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

If $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then

$$\begin{aligned} \mathbf{f}(\mathbf{x}) &= \mathbf{f}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n) \\ &= x_1\mathbf{f}(\mathbf{e}_1) + x_2\mathbf{f}(\mathbf{e}_2) + \cdots + x_n\mathbf{f}(\mathbf{e}_n) \end{aligned}$$

This is a linear combination of $\{\mathbf{f}(\mathbf{e}_1), \ldots, \mathbf{f}(\mathbf{e}_n)\}$.

Now think of $\mathbf{f}(\mathbf{e}_1), \ldots, \mathbf{f}(\mathbf{e}_n)$ as n column vectors, and form the matrix

$$A = (\mathbf{f}(\mathbf{e}_1)|\mathbf{f}(\mathbf{e}_2)|\cdots|\mathbf{f}(\mathbf{e}_n))_{m\times n}.$$

(It's $m \times n$ because each vector $\mathbf{f}(e_i)$ is a vector in $\mathbb{R}^m$, and there are n of these vectors making up the columns.) To get a linear combination of the vectors $\mathbf{f}(e_1), \ldots, \mathbf{f}(\mathbf{e}_n)$, all we have to do is to multiply the matrix A on the right by a vector. And, in fact, it's apparent that

$$\mathbf{f}(\mathbf{x}) = x_1\mathbf{f}(\mathbf{e}_1) + x_2\mathbf{f}(\mathbf{e}_2) + \cdots + x_n\mathbf{f}(\mathbf{e}_n) = (\mathbf{f}(\mathbf{e}_1)|\cdots|\mathbf{f}(\mathbf{e}_n))\mathbf{x} = A\mathbf{x}.$$

So, given the linear transformation $\mathbf{f}$, we now know how to assemble a matrix A such that $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$. And of course the converse holds: given a matrix $A_{m\times n}$, the function $\mathbf{f}_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by $\mathbf{f}_A(\mathbf{x}) = A\mathbf{x}$ is a linear transformation.

Definition: The matrix A defined above for the function $\mathbf{f}$ is called the **matrix of f in the standard basis**.

Exercise: Show that Range($\mathbf{f}$) is the column space of the matrix A defined above, and that Ker($\mathbf{f}$) is the null space of A.

Exercise: After all the above theory, you will be happy to learn that it's almost trivial to write down the matrix A if $\mathbf{f}$ is given explicitly: If

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 2x_1 - 3x_2 + 4x_4 \\ x_2 - x_3 + 2x_5 \\ x_1 - 2x_3 + x_4 \end{pmatrix},$$

find the matrix of $\mathbf{f}$ in the standard basis. Also, find a basis for Range($\mathbf{f}$) and Ker($\mathbf{f}$).

15.2 The rank-nullity theorem - version 2

Recall that for $A_{m\times n}$, we have $n = N(A) + R(A)$. Now think of A as the linear transformation $\mathbf{f}_A : \mathbb{R}^n \to \mathbb{R}^m$. The **domain** of $\mathbf{f}_A$ is $\mathbb{R}^n$; Ker($\mathbf{f}_A$) is the null space of A, and Range($\mathbf{f}_A$) is the column space of A. We can therefore restate the rank-nullity theorem as the

Dimension theorem: Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then

$$\dim(\text{domain}(\mathbf{f}) = \dim(\text{Range}(\mathbf{f})) + \dim(\text{Ker}(\mathbf{f})).$$

Exercise: Show that the number of free variables in the system $A\mathbf{x} = \mathbf{0}$ is equal to the dimension of $\text{Ker}(\mathbf{f}_A)$. This is another way of saying that while the particular choice of free variables may depend on how you solve the system, their number is an invariant.

15.3 Choosing a useful basis for A

We now want to study *square matrices*, regarding an $n \times n$ matrix A as a linear transformation from $\mathbb{R}^n$ to itself. We'll just write $A\mathbf{v}$ for $\mathbf{f}_A(\mathbf{v})$ to simplify the notation, and to keep things really simple, we'll just talk about 2×2 matrices – all the problems that exist in higher dimensions are present in $\mathbb{R}^2$.

There are several questions that present themselves:

- Can we visualize the linear transformation $\mathbf{x} \to A\mathbf{x}$? One thing we *can't* do in general is draw a graph! Why not?
- Connected with the first question is: can we choose a better coordinate system in which to view the problem?

The answer is not an unequivocal "yes" to either of these, but we can generally do some useful things.

To pick up at the end of the last lecture, note that when we write $\mathbf{f}_A(\mathbf{x}) = \mathbf{y} = A\mathbf{x}$, we are actually using the coordinate vector of $\mathbf{x}$ in the standard basis. Suppose we change to some other basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ using the invertible matrix V. Then we can rewrite the equation in the new coordinates and basis:

We have $\mathbf{x} = V\mathbf{x}_v,$ and $\mathbf{y} = V\mathbf{y}_v$, so

$$\begin{aligned}
\mathbf{y} &= A\mathbf{x} \\
V\mathbf{y}_v &= AV\mathbf{x}_v, \text{ and} \\
\mathbf{y}_v &= V^{-1}AV\mathbf{x}_v
\end{aligned}$$

That is, the matrix equation $\mathbf{y} = A\mathbf{x}$ is given in the new basis by the equation

$$\mathbf{y}_v = V^{-1}AV\mathbf{x}_v.$$

Definition: The matrix $V^{-1}AV$ will be denoted by A_v and called the **matrix of the linear transformation f in the basis** $\{\mathbf{v}_1, \mathbf{v}_2\}$.

We can now restate the second question: Can we find a nonsingular matrix V so that $V^{-1}AV$ is particularly useful?

Definition: The matrix A is **diagonal** if the only nonzero entries lie on the main diagonal. That is, $a_{ij} = 0$ if $i \neq j$.

Example:

$$A = \begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix}$$

is diagonal. Given this diagonal matrix, we can (partially) visualize the linear transformation corresponding to multiplication by A: a vector $\mathbf{v}$ lying along the first coordinate axis is mapped to $4\mathbf{v}$, a multiple of itself. A vector $\mathbf{w}$ lying along the second coordinate axis is also mapped to a multiple of itself: $A\mathbf{w} = -3\mathbf{w}$. It's length is tripled, and its direction is reversed. An arbitrary vector $(a, b)^t$ is a linear combination of the basis vectors, and it's mapped to $(4a, -3b)^t$.

It turns out that we can find vectors like $\mathbf{v}$ and $\mathbf{w}$, which are mapped to multiples of themselves, *without* first finding the matrix V. This is the subject of the next few sections.

15.4 Eigenvalues and eigenvectors

Definitions: If a vector $\mathbf{x} \neq \mathbf{0}$ satisfies the equation $A\mathbf{x} = \lambda\mathbf{x}$, for some real number λ, then λ is said to be an **eigenvalue** of the matrix A, and $\mathbf{x}$ is said to be an **eigenvector** of A corresponding to the eigenvalue λ.

Example: If

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \text{ and } \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

then

$$A\mathbf{x} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5\mathbf{x}.$$

So $\lambda = 5$ is an eigenvalue of A, and $\mathbf{x}$ an eigenvector corresponding to this eigenvalue.

Remark: Note that the definition of eigenvector *requires* that $\mathbf{v} \neq \mathbf{0}$. The reason for this is that if $\mathbf{v} = \mathbf{0}$ were allowed, then any number λ would be an eigenvalue since the statement $A\mathbf{0} = \lambda\mathbf{0}$ holds for any λ. On the other hand, we *can* have $\lambda = 0$, and $\mathbf{v} \neq \mathbf{0}$. See the exercise below.

Those of you familiar with some basic chemistry have already encountered eigenvalues and eigenvectors in your study of the hydrogen atom. The electron in this atom can lie in any one of a countable infinity of orbits, each of which is labelled by a different value of the energy of the electron. These quantum numbers (the possible values of the energy) are in fact the eigenvalues of the Hamiltonian (a differential operator involving the Laplacian Δ). The allowed values of the energy are those numbers λ such that $H\psi = \lambda\psi$, where the eigenvector ψ is the "wave function" of the electron in this orbit. (This is the correct description of the hydrogen atom as of about 1925; things have become a bit more sophisticated since then, but it's still a good picture.)

Exercises:

1. Show that
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
is also an eigenvector of the matrix A above. What's the eigenvalue?

2. Show that
$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
is an eigenvector of the matrix
$$\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$
What is the eigenvalue?

3. Eigenvectors are not unique. Show that if $\mathbf{v}$ is an eigenvector for A, then so is $c\mathbf{v}$, for any real number $c \neq 0$.

4. Suppose λ is an eigenvalue of A.

   Definition:
$$E_\lambda = \{\mathbf{v} \in \mathbb{R}^n \text{ such that } A\mathbf{v} = \lambda\mathbf{v}\}$$
   is called the **eigenspace** of A corresponding to the eigenvalue λ.

   Show that E_λ is a subspace of $\mathbb{R}^n$. (N.b: the definition of E_λ does not require $\mathbf{v} \neq \mathbf{0}$. E_λ consists of all the eigenvectors *plus* the zero vector; otherwise, it wouldn't be a subspace.)

5. $E_0 = \mathrm{Ker}(\mathbf{f}_A)$ is just the null space of the matrix A.

Example: The matrix
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix}$$

represents a counterclockwise rotation through the angle $\pi/2$. Apart from $\mathbf{0}$, there is no vector which is mapped by A to a multiple of itself. So not every matrix has eigenvectors.
Exercise: What are the eigenvalues of this matrix?