Ch 7 Part II

- Lotka's proof (that population becomes stable if ASFRs and mortality rates remain constant) and the intrinsic growth rate
- NRR and the intrinsic growth rate

Lotka's proof

- Population becomes stable if ASFRs and mortality rates remain constant (and there's no migration)
- Assume maternity function $m(x)$ and survival $p(x)$ constant from $t = 0$, then the # birth at time t:

$$B(t) = \int_0^t N(a,t)m(a)da + G(t),$$

with $G(t)$ the # births from women who were alive at $t = 0$, so after β years:

$$\begin{aligned} B(t) &= \int_0^t N(a,t)m(a)da \\ &= \int_0^t B(t-a) \cdot p(a) \cdot m(a)da \end{aligned}$$

- Lotka proved that solution is given by: $B(t) = B \cdot e^{rt}$ for some r, thus a stable population!
- r is called the intrinsic growth rate for that stable population

Estimating r

- For a stable population we have

$$B(t) \;=\; \int_0^t B(t-a)\cdot p(a)\cdot m(a)da$$

  and $B(t) = B \cdot e^{rt}$ such that

$$1 \;=\; \int_\alpha^\beta e^{-ra}\cdot p(a)\cdot m(a)da$$

  with (α, β) the reproductive age span.
- This can be solved iteratively using Coale's method, or faster, Newton(-Rhapson) method.
- But we already know r from our analysis with the Leslie matrix!

Estimating r using the Leslie matrix

- In Ch7, part I, we found:

$$\mathbf{W}^S(t+n) \;=\; \lambda \cdot \mathbf{W}^S(t),$$

  with $\mathbf{W}^S(t+n)$ $\#$ women in the stable population, and λ the largest eigenvalue of the Leslie matrix

- In the notation of Lotka's proof this corresponds to:

$$N(a, t+n) \;=\; \lambda \cdot N(a, t),$$

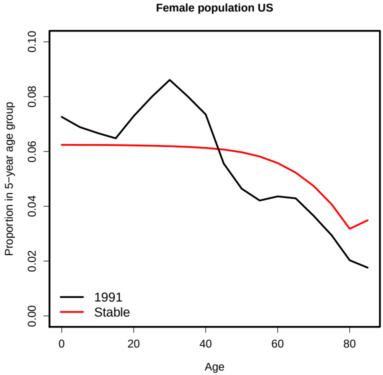
  and because the population is growing exponentially with constant growth rate r:

$$N(a, t+n) \;=\; e^{rn} \cdot N(a, t),$$

  thus $r = \ln(\lambda)/n$.

Example: Population in the US 1991 (Box 7.2)

Goal: Find age distribution, $N(t+n)/N(t)$ and r for stable population (if mortality and fertility rates would remain constant)



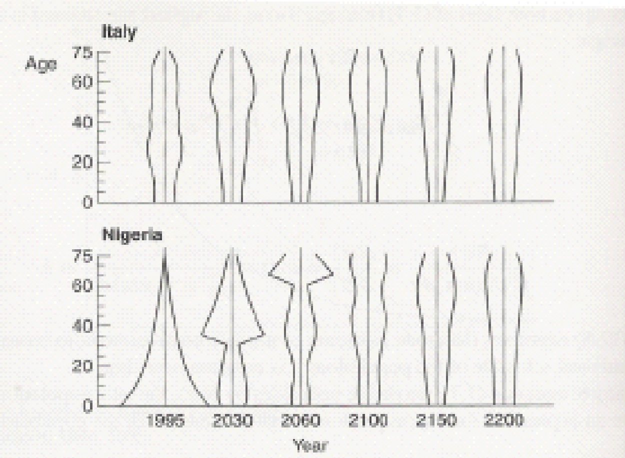
Steps:

- Construct Leslie matrix
- Find largest eigenvalue λ and eigenvector $\mathbf{V}$
- Age distribution $= \mathbf{V}/\sum_{a=1}^{A} V_i$
- $N(t+n)/N(t) = \lambda = 0.999$
- $r =$? (typo in book!)

Ergodicity

- The age distribution of the stable population, growth rate, birth rate, death rate are entirely determined by the vital rates ($m(x)$'s and $p(x)$'s), and **not** its current age distribution!
- Populations "forget their past", property known as ergodicity
- (Not true for population size, absolute numbers!)

Nigeria and Italy projected with Italy's vital rates of 1991 (fig 7.3)



How long does it take to stabilize?

NRR and intrinsic growth rate r

- NRR and r are indicators of long term growth prospects:
  - NRR = growth factor between generations

$$NRR = \int_{\alpha}^{\beta} p(a)m(a)da$$

  - r is the annual growth rate with:

$$1 = \int_{\alpha}^{\beta} e^{-ra} \cdot p(a) \cdot m(a)da$$

- What is the relation between NRR and r?
  - If $r = 0 \rightarrow$ NRR...?
  - If $r < 0 \rightarrow$ NRR...?
  - If $r > 0 \rightarrow$ NRR...?
- Interpretation?

NRR and intrinsic growth rate r

- If we define T as the time needed for a stable population to grow by factor NRR, then
$$NRR = \frac{N(T)}{N(0)} = e^{rT},$$
thus $r = \ln(NRR)/T$
  - What's T when $r = 0.01, NRR = 1.3$?
  - What's T when $r = 0.01, NRR = 1.4$?
  - What's T when $r = 0$ and NRR=1?
- T is called the mean length of a generation, and can be approximated by mean age of child bearing

r in terms of TFR, SRB and survival probabilities

- From Ch.5:

$$\begin{aligned} NRR &\approx GRR \cdot p(A_M), \\ &= TFR \cdot S \cdot p(A_M), \end{aligned}$$

  with A_M the mean age of child bearing and $S = \frac{1}{1+SRB}$
- Plug into r:

$$\begin{aligned} r &= \ln(NRR)/T \\ &= \frac{\ln(TFR) + \ln(S) + \ln(p(A_M))}{T} \end{aligned}$$

- Implications:
  - The effects of fertility and mortality on r are additive (no interaction)
  - r is a function of the log of the TFR rather than rate itself...

Changes in the TFR and r

- Suppose that we compare r before and after a decline in the TFR and keep S and $P(A_M)$ constant. Then

$$\Delta r = \frac{\ln\left(\frac{TFR(t_1)}{TFR(t_0)}\right)}{T}$$

- In other words, a reduction in TFR from 2 to 1 will have the same effect on r as a reduction from 4 to 2.
- This reflects the fact that the growth rate depends on how large one generation is compared to the previous one, and not just on the population change in absolute terms

Example

Table 7.3: *Effect of a decline of the TFR by one child on the intrinsic growth rate of various regions*

Region	*Level* 1995–2000			
	TFR	*NRR*	*r*	Δr
Africa	5.31	2.03	.026	−.008
Eastern Asia	1.78	.80	−.008	−.030
South-central Asia	3.42	1.43	.013	−.013
Southeastern Asia	2.86	1.27	.009	−.016
Western Asia	3.82	1.70	.019	−.011
Europe	1.45	.69	−.013	−.043
Latin America and Carribean	2.65	1.22	.007	−.017
Northern America	1.93	.93	−.003	−.027

Data source: United Nations, 1997.
Assumption: $T = 27.5$.

Another implication

$$r = \frac{\ln(TFR) + \ln(S) + \ln(p(A_M))}{T}$$

- r depends on T, and hence, the timing of births (tempo effects).
- Example:
  - An increase in T from 28 to 33 years would multiply r by a factor of 0.875.
  - Note that delaying childbearing make a negative r less negative, the decline is stretched over a longer period of time (unless it decreases $p(A_M)$)