Ch 7: Stable populations

Part I:

- Introduction
- Stable age distribution using the Leslie matrix

Stable population model

- Generalization of stationary population model
- Applications:
    - To examine long-term implications of maintaining a current set of fertility and mortality rates (what if ...?)
    - Estimate the implications of changes in mortality and fertility on the population age structure, growth rates, death rates and birth rates
    - Estimation of demographic parameters in populations that can be assumed to be stable or stationary

Stationary versus stable populations

Stationary population:

1. Age-specific death rates are constant over time
2. Closed to migration (net migration rates are zero at all ages)
3. The flow of births is constant (the same # of births is added per unit of time)

Stable Population:

1. Age-specific death rates are constant over time
2. Closed to migration (net migration rates are zero at all ages)
3. Births are growing at a constant annualized growth rate: $B(t) = B(0) \cdot e^{rt}$

Stationary versus stable populations: implications

Stationary population:

1. Growth rate $= 0$
2. Constant CBR and CDR
3. Population in each age interval is the same from year to year.
4. Unchanging number of people in each age group

Stable Population:

1. Constant growth rate r
2. Constant CBR and CDR
3. Population in each age interval is growing at the same rate, the rate at which births are growing.
4. Unchanging proportion of the total population in each age group.

A simple example of a stable population

Imagine a population with the following mortality regime:

Exact Age	l_x	$p(x) = l_x/l_0$
0	1000	1.0
1	600	.60
2	400	.40
3	200	.20
4	50	.05
5	0	.00

In addition, we assume that from t_0 onwards, the annual # of births is growing at a constant rate r. For simplicity, assume all births are on Jan 1st, then # people at the start of each year $t_0, t_1, \ldots$:

Age	t_0	t_1	t_2	t_3	t_4	t_5	t_6
0	1000	$1000 \cdot e^r$	$1000 \cdot e^{2r}$	$1000 \cdot e^{3r}$	$1000 \cdot e^{4r}$	$1000 \cdot e^{5r}$	$1000 \cdot e^{6r}$
1	...	600	$600 \cdot e^r$	$600 \cdot e^{2r}$	$600 \cdot e^{3r}$	$600 \cdot e^{4r}$	$600 \cdot e^{5r}$
2	...	...	400	$400 \cdot e^r$	$400 \cdot e^{2r}$	$400 \cdot e^{3r}$	$400 \cdot e^{4r}$
3	...	...	...	200	$200 \cdot e^r$	$200 \cdot e^{2r}$	$200 \cdot e^{3r}$
4	...	...	...	...	50	$50 \cdot e^r$	$50 \cdot e^{2r}$
5	...	...	...	...	...	0	0

Simple example

Age	t_0	t_1	t_2	t_3	t_4	t_5	t_6
0	1000	$1000 \cdot e^r$	$1000 \cdot e^{2r}$	$1000 \cdot e^{3r}$	$1000 \cdot e^{4r}$	$1000 \cdot e^{5r}$	$1000 \cdot e^{6r}$
1	...	600	$600 \cdot e^r$	$600 \cdot e^{2r}$	$600 \cdot e^{3r}$	$600 \cdot e^{4r}$	$600 \cdot e^{5r}$
2	...	...	400	$400 \cdot e^r$	$400 \cdot e^{2r}$	$400 \cdot e^{3r}$	$400 \cdot e^{4r}$
3	...	...	...	200	$200 \cdot e^r$	$200 \cdot e^{2r}$	$200 \cdot e^{3r}$
4	...	...	...	...	50	$50 \cdot e^r$	$50 \cdot e^{2r}$
5	...	...	...	...	...	0	0

After t_5 the population is stable, refer to t_5 as $t = 0$, then for $t > 0$:

- Births in year t: $B(t) = B(0) \cdot e^{rt}$
- The population is growing at rate r too:

$$N(t) = \sum_a N(a,t) = \sum_a N(a, t-1) \cdot e^r = e^r \cdot N(t-1),$$

  thus $N(t) = N(0) \cdot e^{rt}$ and CBR is constant.
- The population in year t consists of people from different cohorts, with # people in age group $[a, a+1)$:

$$N(a,t) \quad = \quad B(t-a) \cdot p(a) = B(0) \cdot e^{r(t \quad a)} \cdot p(a)$$

  the # people who were born a years ago and survived until age a.

Simple example

Age	t_0	t_1	t_2	t_3	t_4	t_5	t_6
0	1000	$1000 \cdot e^{r}$	$1000 \cdot e^{2r}$	$1000 \cdot e^{3r}$	$1000 \cdot e^{4r}$	$1000 \cdot e^{5r}$	$1000 \cdot e^{6r}$
1	...	600	$600 \cdot e^{r}$	$600 \cdot e^{2r}$	$600 \cdot e^{3r}$	$600 \cdot e^{4r}$	$600 \cdot e^{5r}$
2	...	...	400	$400 \cdot e^{r}$	$400 \cdot e^{2r}$	$400 \cdot e^{3r}$	$400 \cdot e^{4r}$
3	...	...	...	200	$200 \cdot e^{r}$	$200 \cdot e^{2r}$	$200 \cdot e^{3r}$
4	...	...	...	...	50	$50 \cdot e^{r}$	$50 \cdot e^{2r}$
5	...	...	...	...	...	0	0

After t_5 the population is stable, refer to t_5 as $t = 0$, then for $t > 0$:

- Births in year t: $B(t) = B(0) \cdot e^{rt}$
- Total population $N(t) = N(0) \cdot e^{rt}$.
- # people in age group $[a, a+1)$: $N(a,t) = B(0) \cdot e^{r(t-a)} \cdot p(a)$
- The relative age distribution is constant:

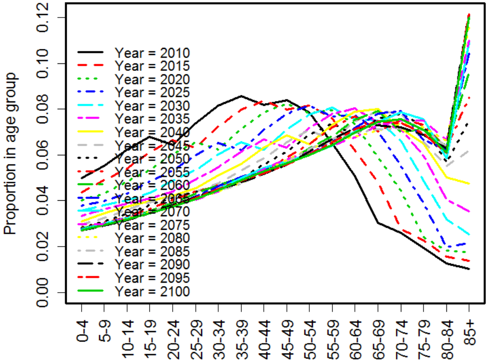
$$
\begin{aligned}
c(a,t) &= \frac{N(a,t)}{N(t)} = \frac{B(0) \cdot e^{r(t-a)} \cdot p(a)}{N(0) \cdot e^{rt}} \\
&= \frac{B(0)}{N(0)} \cdot e^{-ra} \cdot p(a) = c(a)
\end{aligned}
$$

Lotka's (1939) renewal equation

- From the example we've seen that a stable population will emerge if three conditions prevail for long enough:
  1. Growth rate in the annual # of births is constant
  2. Constant age-specific mortality rates
  3. Zero migration at all ages (or net migration needs to be constant over time at all ages)
- Alfred Lotka demonstrated that (1) can be replaced by constant ASFRs
- Implications:
  - Populations with unchanging vital rates would eventually become stable
  - Any set of vital rates has an implied stable population that would eventually be produced by those rates
- Lotka showed it with algebra, but we already knew that too!

Really?

Tutorial 6: When keeping ASFRs and survival ratios constant, we found that the age structure of women in Singapore becomes constant.



Explanation with the Leslie matrix

- Recall the CC-projection with the Leslie matrix $\mathbf{L}$ for projection period n:

$$\begin{aligned}\mathbf{W}(t+n) &= \mathbf{L}\cdot\mathbf{W}(t),\\ \mathbf{W}(t+P\cdot n) &= \mathbf{L}^{P}\cdot\mathbf{W}(t)\end{aligned}$$

- For longgg term projections, we get:

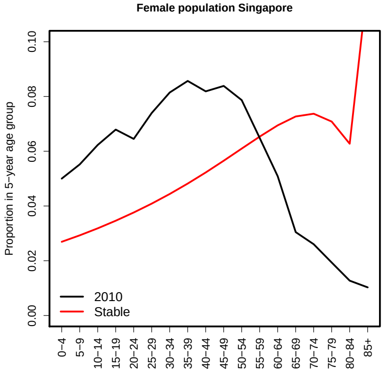
$$\lim_{P\,!\,1} \mathbf{L}^{P}\cdot\mathbf{W}(t) \;\propto\; \mathbf{V},$$

  with $\mathbf{V}$ the eigenvector corresponding to the largest eigenvalue λ of $\mathbf{L}$, such that $\mathbf{L}\cdot\mathbf{V}=\lambda\cdot\mathbf{V}$.
- This means that in long term projections, indeed the age structure has become constant, with $\mathbf{W}^{S}(t)\propto\mathbf{V}$ for all t.
- Moreover $\lambda = N^{S}(t+n)/N^{S}(t)$ (with $N^{S}(t)$ the total # females at time t for the stable population):

$$\mathbf{W}^{S}(t+n) \;=\; \mathbf{L}\cdot\mathbf{W}^{S}(t)=\lambda\cdot\mathbf{W}^{S}(t)=N^{S}(t+n)/N^{S}(t)\cdot\mathbf{W}^{S}(t).$$

Singapore females

Goal: Find age distribution and $N(t+n)/N(t)$ for stable population (if mortality and fertility rates would remain constant)

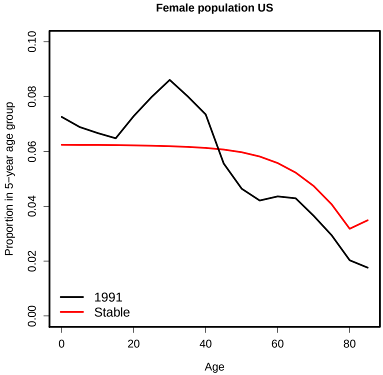


Steps:

- Construct Leslie matrix
- Find largest eigenvalue λ and eigenvector $\mathbf{V}$
- Age distribution $= \mathbf{V}/\sum_{a=1}^{A} V_i$
- $N(t+n)/N(t) = \lambda = 0.919$

Example: Population in the US 1991 (Box 7.2)

Goal: Find age distribution and $N(t+n)/N(t)$ for stable population (if mortality and fertility rates would remain constant)



Steps:

- Construct Leslie matrix
- Find largest eigenvalue λ and eigenvector $\mathbf{V}$
- Age distribution $= \mathbf{V}/\sum_{a=1}^{A} V_i$
- $N(t+n)/N(t) = \lambda = 0.999$

Note: next week we'll discuss an alternative way to calculate these outcomes