R403: Probabilistic and Statistical Computations with R

Topic 6: Discrete and Continuous Markov Chains

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Lecture Contents

Introduction

Introduction

- Just a brief introduction to the topic(s) will be given
- Reason: insufficient time, no further usage in subsequent courses
- The latter does not mean irrelevance or inapplicability
- For intermediate and advanced treatments, references will be provided

Stochastic Processes

Stochastic Processes

Definition 1

A stochastic process is a collection of random variables $\{X(t),\ t \in T\}$ where t is a (time) index. $X(t)$ is called the *state* of the process at time t, and T is the *index set* of the process.

Definition 2

If T is countable, the stochastic process is called a *discrete-time process*. If T is not countable, the process is a *continuous-time* one.

- Notation:

$$\begin{array}{lcl} X(t) & - & \text{continuous-time} \\ X_t & - & \text{discrete-time} \end{array}$$

Definition 3

The *state space* of a stochastic process is the set of all possible values of $X(t)$.

Discrete-time Markov Chains

Discrete-time Markov Chains

- Let X_t be a discrete-time stochastic process that generates a value for each time $t \in T$
- Assume that the random variables that constitute it $X_0, X_1, \ldots$ are *not* independent
- Instead, values at time t are determined by values in previous periods

Definition 4 (Markov Property)

A stochastic process has the Markov property[a] if the conditional distribution of future states depends only on the present state, and not on the events that precede it.

[a] Also known as the *memoryless property*.

Discrete-time Markov Chains (2)

- Let the set of of possible values of the stochastic process $\{X_t, t = 0, 1, 2, \ldots\}$ be a finite or a countable number
- Usually that set of possible values is taken to be the set of non-negative integers, i.e. $\mathcal{S} = \{0, 1, 2, \ldots\}$
- The process is said to be in state i at time t if $X_t = i$

Definition 5

A *Markov chain* is a stochastic process for which the conditional distribution of any future state X_{t+1} depends only on the present state X_t, and not on the past states $X_0, X_1, \ldots, X_{t-1}$, i.e.

$$P(X_{t+1} = j | X_t = i, X_{t-1} = i_{t-1}, \ldots, X_1 = i_1, X_0 = i_0) =$$

$$= P(X_{t+1} = j | X_t = i) = P_{ij}$$

Discrete-time Markov Chains (3)

- P_{ij} is fixed and is interpreted as the probability of transitioning from state i at time t to state j at time $t+1$
- Note that it is possible that $i = j$, i.e. the process can remain in the same state in the next period
- P_{ij} has the following properties:
  - $P_{ij} \geq 0, \quad i, j \geq 0$
  - $\sum_{j=0}^{\infty} P_{ij} = 1, \quad i = 0, 1, 2, \ldots$

Discrete-time Markov Chains (4)

- All such probabilities of transition can be arranged in a matrix called the *transition matrix*

$$\mathbf{P} = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \dots \\ P_{10} & P_{11} & P_{12} & \dots \\ \vdots & \vdots & \vdots & \\ P_{i0} & P_{i1} & P_{i2} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

- Take a numerical example borrowed from Chiang and Wainwright (2005), Exercise 4.7. (in the next slide)

Discrete-time Markov Chains (5)

Chiang and Wainwright (2005), Exercise 4.7.

Consider a situation of mass layoff (i.e. a factory shuts down) where 1200 people become unemployed and now begin a job search. In this case there are two states: employed (E) and unemployed (U) with an initial vector

$$x_0' = \begin{pmatrix} \text{E} & \text{U} \end{pmatrix} = \begin{pmatrix} 0 & 1200 \end{pmatrix}$$

Suppose that in any given period an unemployed person will find a job with probability 0.7 and will therefore remain unemployed with a probability of 0.3. Additionally, persons who find themselves employed in any given period may lose their job with a probability of 0.1 (and will have a 0.9 probability of remaining employed).

- The Markov transition matrix for this problem is (solving part (a) of the example)

$$\mathbf{P} = \begin{pmatrix} 0.9 & 0.1 \\ 0.7 & 0.3 \end{pmatrix}$$

Chapman–Kolmogorov Equations

- The probabilities P_{ij} defined so far are probabilities of *one-step* transitions
- How about probabilities of *multi-step* ones?
- The k-step transition probabilities P_{ij}^k are the probabilities that in k periods a process will be in state j is presently it is in state i

$$P_{ij}^k = P(X_{t+k} = j | X_t = i), \quad k \geq 0, \, j \geq 0 \quad (*)$$

- To compute those k-step transition probabilities, the Chapman–Kolmogorov equations are used:

$$P_{ij}^{k+r} = \sum_{m=0}^{\infty} P_{im}^k P_{mj}^r, \quad \forall k, r \geq 0, \, \forall i, j$$

- Here, m denotes the intermediate states that the process will go into after it starts in state i and before it ends up in state j
- To arrive at the latter, in other words it takes $k + r$ transitions

Chapman–Kolmogorov Equations (2)

- In more detail, the latter can be more thoroughly written as

$$\begin{aligned}
P_{ij}^{k+r} &= P(X_{k+r} = j | X_0 = i) = \\
&= \sum_{m=0}^{\infty} P(X_{k+r} = j, X_k = m | X_0 = i) = \\
&= \sum_{m=0}^{\infty} P(X_{k+r} = j | X_k = m, X_0 = i) P(X_k = m | X_0 = i) = \\
&= \sum_{m=0}^{\infty} P_{im}^{k} P_{mj}^{r}
\end{aligned}$$

Chapman–Kolmogorov Equations (3)

- Denote by $\mathbf{P}^{(k)}$ the matrix of k-step transition probabilities P_{ij}^k
- Then, using $(*)$,

$$\mathbf{P}^{(k+r)} = \mathbf{P}^{(k)}\mathbf{P}^{(r)}$$

- It is easy to show that $\mathbf{P}^{(k)} = \mathbf{P}^k$
- Return now to the example
- Part (b) requires to find what will be the number of unemployed people after (i) two periods; (ii) 3 periods; (iii) 5 periods; (iv) 10 periods
- To do that, simply use the powers of $\mathbf{P}$
- You will find out that the levels of employment and unemployment reach a steady-state (this is part (c) of the example)

Continuous-time Markov Chains

Continuous-time Markov Chains

- Let $\{X(t),\ t \geq 0\}$ be a continuous-time process taking on non-negative integers as values
- Such a process is a *continuous-time Markov chain* if $\forall s, t \geq 0$ and $\forall i, j, x(u) \geq 0,\ u \in [0, s)$,

$$P(X(t+s) = j | X(s) = i, X(u) = x(u),\ u \in [0, s)) =$$

$$= P(X(t+s) = j | X(s) = i)$$

- Here, $X(s)$ is the present state, while $X(u)$ relates to the past

Continuous-time Markov Chains (2)

- If the probability
$$P(X(t+s) = j|X(s) = i)$$
is independent of s, then the Markov chain is said to have *stationary/homogeneous* transition probabilities
- Applications of continuous-time Markov chains: birth and death processes, population processes, migration processes, etc.
- The **markovchain** package: handles discrete-time *and* continuous-time Markov chains (the vignette contains lots of examples)

References

- Privault, N. (2018): *Understanding Markov Chains*, Springer, 2nd ed.
- Ross, S. (2019): *Introduction to Probability Models*, Academic Press, 12th ed.
- https://en.wikipedia.org/wiki/Markov_property