Implementability with contingent contracts

Kiho Yoon

Department of Economics, Korea University

145 Anam-ro, Seongbuk-gu, Seoul, Korea 02841

kiho@korea.ac.kr

http://econ.korea.ac.kr/~kiho

Abstract

We establish the equivalence of implementability by a contingent contract, implementability by a linear contract, acyclicity, and scaled cycle monotonicity when the type space is a compact metric space. We employ graph-theoretic arguments.

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1. Introduction

One of the fundamental results in mechanism design is the characterization of implementability via monotonicity. Myerson (1981) has shown for one-dimensional type spaces that a necessary and sufficient condition for implementability is the monotonicity of allocation rule (together with an integrability requirement). Rochet (1987) has established for general type spaces that a necessary and sufficient condition for implementability is cycle monotonicity. Several papers, including Saks and Yu (2005), Bikhchandani *et al.* (2006), Ashlagi *et al.* (2010), have examined the type spaces in which weak monotonicity, which is closely connected to the monotonicity of the allocation rule, is both necessary and sufficient for implementability. In particular, Archer and Kleinberg (2014) have shown that local weak monotonicity and vortex-freeness together imply and are implied by cycle monotonicity, and hence are necessary and sufficient for implementability, when the type space is convex.

While these papers have considered implementability when only upfront lump-sum payments are possible, Deb and Mishra (2014) recently studied implementability with contingent contracts, i.e., when the payments can also depend on realized outcomes. This work is significant since contingent contracts are widely used in many practical settings. In oil and gas lease auctions, for instance, buyers pay a fixed percentage of revenues in royalties in addition to upfront cash payments. Other examples include intercorporate asset sales, licensing agreements for intellectual property, and build-operate-transfer highway construction contracts in procurements.[1] They showed that (i) a necessary and sufficient condition for implementability by a contingent contract is acyclicity, and that (ii) implementability by a contingent contract is equivalent to implementability by a linear contract. They also qualitatively described implementable allocation rules. They proved the results, however, under the finite type space assumption.

[1] Please refer to Skrzypacz (2013), Deb and Mishra (2014) and the references therein for detailed discussion on contingent contracts.

In this paper, we extend their results to more general type spaces. In particular, we establish the equivalence of implementability by a contingent contract, implementability by a linear contract, acyclicity, and scaled cycle monotonicity when the type space is a compact metric space. We employ a graph-theoretic approach to implementability, which not only makes this extension possible but also gives a new and simple proof for the finite type space case. In particular, we utilize the topological sort of a directed acyclic graph.

The extension beyond finite type spaces is not merely an exercise in mathematical completeness. In most practical situations, including the ones listed above, it is simply implausible that the agents have only finite possible realizations of their private information: If agents think they may have 'one more' value distinct from the others currently deemed possible, then the finite type space assumption is not appropriate.[2]

2. Characterization of implementability

We consider a single-agent setting without loss of generality.[3] Let A denote the set of alternatives. A type of the agent is given by a map $v : A \to I\!R$. Let V denote the set of types. An allocation rule is a map $f : V \to A$.

The fundamental difference separating mechanism design with contingent contracts from the standard mechanism design setting with only upfront lump-sum payments is that ex post payoff of the agent is contractible. The concept of dominant strategy incentive compatibility in this setting is defined below.

Definition 1. An allocation rule f is implementable by a linear contract if there exists a linear contract (r, t), where $r : V \to (0, \infty)$ and $t : V \to I\!R$, such that

$$r(v)v(f(v)) - t(v) \geq r(v')v(f(v')) - t(v')$$

for all $v, v' \in V$. Then we say that the linear mechanism (f, r, t) is incentive compatible.

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[2] See McAfee and Reny (1992) for a related discussion.

[3] It is straightforward to extend the results to the multi-agent setting. We focus on the single-agent setting for notational convenience.

Note that, when $r(v) = 1$ for all $v \in V$, this reduces to implementability by a lump-sum transfer scheme, and the corresponding mechanism (f, t) is the standard dominant strategy incentive compatible mechanism.

Definition 2. An allocation rule f is implementable (by a contingent contract) if there exists a contingent contract $s : I\!R \times V \to I\!R$ that is strictly increasing in the first argument such that

$$s(v(f(v)), v) \geq s(v(f(v')), v')$$

for all $v, v' \in V$. Then we say that the contingent mechanism (f, s) is incentive compatible.

Observe that implementability by a lump-sum transfer scheme implies implementability by a linear contract, which in turn implies implementability (by a contingent contract). We now employ the graph-theoretic approach to implementability. Representative works of this approach include Müller *et al.* (2007) and Heydenreich *et al.* (2009). The type graph G_f given an allocation rule f has the node set V and contains a directed edge from any node v to any other v' of length[4]

$$l(v, v') = v(f(v)) - v'(f(v)).$$

Definition 3. An allocation rule f satisfies *cycle monotonicity* if, for any finite cycle $v^1, \ldots, v^k, v^{k+1} = v^1$ of types in V, we have

$$\sum_{i=1}^{k} l(v^i, v^{i+1}) \geq 0.$$

Definition 4. An allocation rule f satisfies *scaled cycle monotonicity* if there exists a function $\lambda : V \to (0, \infty)$ such that, for any finite cycle $v^1, \ldots, v^k, v^{k+1} = v^1$ of types in V, we have

$$\sum_{i=1}^{k} \lambda(v^i) l(v^i, v^{i+1}) \geq 0.$$

[4] Note that the weight, rather than the length, is a more standard term in the graph theory. We use the length to respect the convention in this literature.

Definition 5. An allocation rule f satisfies *acyclicity* if, for any finite cycle $v^1, \ldots, v^k$, $v^{k+1} = v^1$ of types in V, we have

$$l(v^1, v^2) \leq 0, \ldots, l(v^{k-1}, v^k) \leq 0 \Rightarrow l(v^k, v^1) \geq 0.$$

Observe that cycle monotonicity (CM henceforth) implies scaled cycle monotonicity (SCM henceforth), which in turn implies acyclicity (AC henceforth): CM implies SCM since we can set $\lambda(v) = 1$ for all $v \in V$. For AC, assume $l(v^1, v^2) \leq 0, \ldots, l(v^{k-1}, v^k) \leq 0$. Since $\sum_{i=1}^{k} \lambda(v^i) l(v^i, v^{i+1}) \geq 0$ by SCM and $\lambda(v^i) > 0$, we must have $l(v^k, v^1) \geq 0$.

Rochet (1987) has established that a necessary and sufficient condition for the implementability of an allocation rule f by a lump-sum transfer is CM. Similarly, we establish below that a necessary and sufficient condition for the implementability of an allocation rule f (by a contingent contract) is AC, and moreover, every implementable allocation rule can be implemented by a linear contract. We first state the following proposition without proof, since it is a reproduction of the results in Deb and Mishra (2014).

Proposition 1. *(a) (Lemma 1 of Deb and Mishra) If an allocation rule f is implementable (by a contingent contract), then it is acyclic.*
(b) (Proposition 2 of Deb and Mishra) If an allocation rule f satisfies SCM, then it is implementable by a linear contract.

This proposition shows that (i) SCM implies implementability by a linear contract, (ii) which obviously implies implementability (by a contingent contract), and (iii) which in turn implies AC. Thus, if we can show that AC implies SCM then all these properties are equivalent. We have the following proposition for infinite type spaces. We need to assume that A is a topological space so that the continuity of $f : V \to A$ and $v : A \to I\!R$ can be defined.

Proposition 2. *Assume that the set V of types is a compact metric space. Assume also that the allocation rule f and each $v \in V$ is continuous. If f satisfies AC, then it satisfies SCM.*

Proof. For a fixed $\delta > 0$, consider an open ball $B(v, \delta)$ of radius δ about $v \in V$. The set $\bigcup_{v \in V} B(v, \delta)$ covers V. Since V is compact, there is a finite subcover $\{B(v_1, \delta), \ldots, B(v_n, \delta)\}$ of V. Let $\widehat{B}_1(\delta) = B(v_1, \delta)$ and $\widehat{B}_j(\delta) = B(v_j, \delta) \setminus \big(B(v_1, \delta) \cup \cdots \cup B(v_{j-1}, \delta)\big)$ for $j = 2, \ldots, n$. Then, $\{\widehat{B}_1(\delta), \ldots, \widehat{B}_n(\delta)\}$ also covers V, and $\widehat{B}_j(\delta)$'s are mutually disjoint. We can choose δ sufficiently small such that, for any $\epsilon > 0$, we have

(i) $|v(a) - v'(a)| < \epsilon$ for all $a \in A$ and for all $v, v' \in \widehat{B}_j(\delta)$ and

(ii) $|\tilde{v}(f(v)) - \tilde{v}(f(v'))| < \epsilon$ for all $\tilde{v} \in V$ and $v, v' \in \widehat{B}_j(\delta)$.

Note in particular that the second inequality follows from the Heine-Cantor theorem that every continuous function on a compact set is uniformly continuous.

Pick an arbitrary w^j from each $\widehat{B}_j(\delta)$ for $j = 1, \ldots, n$. Let $\widehat{V} = \{w^1, \ldots, w^n\}$. Then, the finite graph $\widehat{G}_f$ whose node set is $\widehat{V}$ and whose directed edges are just those in the original graph, i.e., $l(w^j, w^{j'}) = w^j(f(w^j)) - w^{j'}(f(w^j))$ for all $j, j' \in \{1, \ldots, n\}$, obviously satisfies AC. Hence, it satisfies SCM by Proposition A in the appendix. That is, there is a function $\lambda : \widehat{V} \to (0, \infty)$ such that, for any finite cycle $w^{\pi(1)}, \ldots, w^{\pi(k)}, w^{\pi(k+1)} = w^{\pi(1)}$ in $\widehat{G}_f$ where $(\pi(1), \ldots, \pi(k))$ is an ordered collection of distinct members of the set $\{1, \ldots, n\}$, we have $\sum_{i=1}^{k} \lambda(w^{\pi(i)}) l(w^{\pi(i)}, w^{\pi(i+1)}) \geq 0$. Observe that we can choose $\lambda : \widehat{V} \to (0, \infty)$ such that $\lambda(w^j) \leq \epsilon$ holds for all $w^j \in \widehat{V}$ since $\widehat{V}$ is finite and only relative magnitude matters.[5] We now extend the domain of the function λ to V: for each $v \in V$, define $\lambda(v) = \lambda(w^j)$ where $w^j \in \widehat{V}$ is the node picked from the set $\widehat{B}_j$ to which v belongs. Since each $v \in V$ belongs to exactly one $\widehat{B}_j$ by construction, the function $\lambda : V \to (0, \infty)$ is well-defined.

Consider any finite cycle $v^1, \ldots, v^k$, $v^{k+1} = v^1$ in the original graph G_f. Let $w^{j(i)}$ be the node picked from the set $\widehat{B}_{j(i)}$ to which i belongs. We have $w^{j(i)}(f(w^{j(i)})) - v^i(f(v^i)) = w^{j(i)}(f(w^{j(i)})) - v^i(f(w^{j(i)})) + v^i(f(w^{j(i)})) - v^i(f(v^i)) < 2\epsilon$ as well as $v^{i+1}(f(v^i)) - w^{j(i+1)}(f(w^{j(i)})) = v^{i+1}(f(v^i)) - v^{i+1}(f(w^{j(i)})) + v^{i+1}(f(w^{j(i)})) - w^{j(i+1)}(f(w^{j(i)})) < 2\epsilon$

5 To be specific, let $\overline{\lambda} = \max_{w^j \in \widehat{V}} \lambda(w^j)$ and reset $\lambda(w^j)$ as $\epsilon \lambda(w^j)/\overline{\lambda}$ for all $w^j \in \widehat{V}$.

by the inequalities (i) and (ii) above. So, we have $l(w^{j(i)}, w^{j(i+1)}) < l(v^i, v^{i+1}) + 4\epsilon$. Hence,

$$\sum_{i=1}^{k} \lambda(v^i) l(v^i, v^{i+1}) > \sum_{i=1}^{k} \lambda(w^{j(i)}) \big(l(w^{j(i)}, w^{j(i+1)}) - 4\epsilon \big)$$
$$\geq \sum_{i=1}^{k} \lambda(w^{j(i)}) l(w^{j(i)}, w^{j(i+1)}) - 4k\epsilon^2 \geq -4k\epsilon^2,$$

where the first inequality follows from $\lambda(v^i) = \lambda(w^{j(i)})$ and the inequality just obtained, the second inequality follows from $\lambda(w^j) \leq \epsilon$ for all $w^j \in \widehat{V}$, and the last inequality follows from the fact that $\widehat{G}_f$ satisfies SCM.[6]

Since ϵ is arbitrary, we can conclude that f satisfies SCM. To see this, suppose to the effect of contradiction that f satisfies AC but not SCM. By the definition of SCM, this contradicts the argument above. *Q.E.D.*

Note that the proof essentially collapses the problem into a finite space problem and uses the results obtained for the latter. Note also that a discrete space is compact if and only if it is finite, and that every function is continuous on a discrete domain. Hence, this proposition applies also to finite type spaces. We cannot extend this characterization to the case when (i) V is not compact, (ii) f is not continuous, or (iii) v is not continuous. We give counterexamples for each of these cases.

Example 1. (AC may not imply SCM when V is not compact.) Let $V = \{v^\theta : A \to I\!R \mid \theta \in [0,1]\}$, where

$$v^\theta(a) = \begin{cases} a/\theta & \text{for } \theta \in (0,1]; \\ 2 & \text{for } \theta = 0. \end{cases}$$

Let $f(v^\theta) = \theta$ for $\theta \in [0,1]$. We have

$$l(v^\theta, v^{\theta'}) = \begin{cases} 1 - \theta/\theta' & \text{when } \theta, \theta' \in (0,1]; \\ 2 & \text{when } \theta = 0 \text{ and } \theta' \in (0,1]; \\ -1 & \text{when } \theta \in (0,1] \text{ and } \theta' = 0. \end{cases}$$

Thus, $l(v^\theta, v^{\theta'}) > 0$ if and only if $\theta < \theta'$ and so AC is satisfied. We now show that f cannot be implemented by a linear contract. Suppose to the effect of contradiction that it can

[6] Observe that it is possible to have $w^{j(i)} = w^{j(i+1)}$ for some $i = 1, \ldots, k$ but this does not affect the argument since only a cycle involving less nodes results.

be implemented by a linear contract (r, t). Then, adding the two incentive compatibility conditions for v^θ and $v^{\theta'}$, we have

$$r(v^\theta)l(v^\theta, v^{\theta'}) + r(v^{\theta'})l(v^{\theta'}, v^\theta) \geq 0.$$

Observe first that $r(v^0) \geq r(v^\theta)/2$ for $\theta \in (0, 1]$ since $r(v^0)l(v^0, v^\theta) + r(v^\theta)l(v^\theta, v^0) = 2r(v^0) - r(v^\theta) \geq 0$ for $\theta \in (0, 1]$. Observe next that $r(v^\theta) \geq r(v^1)/\theta$ for $\theta \in (0, 1)$ since $r(v^\theta)/r(v^{\theta'}) \geq -l(v^{\theta'}, v^\theta)/l(v^\theta, v^{\theta'}) = \theta'/\theta$ for $0 < \theta < \theta' \leq 1$. Hence, $r(v^0) \geq r(v^\theta)/2 \geq r(v^1)/(2\theta)$ for $\theta \in (0, 1)$. The right-hand side goes to infinity as θ goes to zero. This implies $r(v^0) = \infty$, which is a contradiction. By Proposition 1(b), f does not satisfy SCM.

Example 2. (AC may not imply SCM when f is not continuous.) Let $V = \{v^\theta : A \to \mathbb{R} \mid \theta \in [-1, 1]\}$, where $v^\theta(a) = a\theta$ for $\theta \in [-1, 1]$. Let $A = \mathbb{R} \cup \{\infty\}$ and

$$f(v^\theta) = \begin{cases} 1/\theta & \text{for } \theta \in [-1, 1] \setminus \{0\}; \\ \infty & \text{for } \theta = 0. \end{cases}$$

We have

$$l(v^\theta, v^{\theta'}) = \begin{cases} (\theta - \theta')/\theta & \text{when } \theta \in [-1, 1] \setminus \{0\} \text{ and } \theta' \in [-1, 1]; \\ -\infty & \text{when } \theta = 0 \text{ and } \theta' \in (0, 1]; \\ \infty & \text{when } \theta = 0 \text{ and } \theta' \in [-1, 0). \end{cases}$$

Observe that $l(v^\theta, v^{\theta'}) > 0$ if (i) $\theta > 0$ and $\theta > \theta'$ or (ii) $\theta < 0$ and $\theta < \theta'$. It is straightforward to check that AC is satisfied. Now consider the cycle v^0, v^1, v^0. For any $\lambda : V \to (0, \infty)$, we have $\lambda(v^0)l(v^0, v^1) + \lambda(v^1)l(v^1, v^0) = -\infty + \lambda(v^1) = -\infty < 0$. Hence, f does not satisfy SCM.

Example 3. (AC may not imply SCM when v is not continuous.) Let $V = \{v^\theta : A \to \mathbb{R} \mid \theta \in [-1, 1]\}$, where

$$v^\theta(a) = \begin{cases} \theta/a & \text{if } a \in [-1, 1] \setminus \{0\}; \\ \infty & \text{if } a = 0 \end{cases}$$

for $\theta \in [-1, 1] \setminus \{0\}$, and $v^0(a) = 0$ for all $a \in [-1, 1]$. Let $f(v^\theta) = \theta$ for $\theta \in [-1, 1]$. We have

$$l(v^\theta, v^{\theta'}) = \begin{cases} (\theta - \theta')/\theta & \text{when } \theta \in [-1, 1] \setminus \{0\} \text{ and } \theta' \in [-1, 1]; \\ -\infty & \text{when } \theta = 0 \text{ and } \theta' \in [-1, 1] \setminus \{0\}. \end{cases}$$

Observe that $l(v^\theta, v^{\theta'}) > 0$ if (i) $\theta > 0$ and $\theta > \theta'$ or (ii) $\theta < 0$ and $\theta < \theta'$. It is straightforward to check that AC is satisfied. Now consider the cycle v^0, v^1, v^0. For any $\lambda : V \to (0, \infty)$, we have $\lambda(v^0)l(v^0, v^1) + \lambda(v^1)l(v^1, v^0) = -\infty + \lambda(v^1) = -\infty < 0$. Hence, f does not satisfy SCM.

Summarizing the previous arguments, we have:

Theorem. *Assume that the set V of types is a compact metric space. Assume also that the allocation rule f and each $v \in V$ is continuous. The following statements are equivalent:*

(i) *f is implementable (by a contingent contract).*

(ii) *f is implementable by a linear contract.*

(iii) *f satisfies acyclicity.*

(iv) *f satisfies scaled cycle monotonicity.*

3. Conclusion

We have characterized implementability when contingent payments are possible. Taking a graph-theoretic approach, we have established the equivalence of implementability by a contingent contract, implementability by a linear contract, acyclicity, and scaled cycle monotonicity under the assumption that the type space is compact metric space and the allocation rule and the valuations are continuous.[7] We have shown by counterexamples that this characterization is tight.

A drawback of the current setup is that the reports of the types can be detected ex-post. As Deb and Mishra (2014) have argued, however, this deterministic model is merely for expositional purposes. We can easily extend the analysis to a stochastic model in which the ex-post payoff is a random variable and so does not reveal the types.

[7] We note that Theorem B in the supplement to Deb and Mishra (2014) contains an equivalence result with the additional technical conditions that the set A of alternatives is a metric space and the allocation rule f is implemented by a contingent contract $s : \mathbb{R} \times V \to \mathbb{R}$ that is twice continuously (partially) differentiable in the first argument.

Appendix

In the appendix, we establish the following proposition. We want to note that the proof utilizes the topological sort of a directed acyclic graph.

Proposition A. *Assume that the set V of types is finite. If an allocation rule f satisfies AC, then it satisfies SCM.*

We start with a lemma, which is instrumental to the proof of the main proposition. This modular approach presents the main idea of the proof in a clear fashion. We introduce a slightly stronger notion of acyclicity.

Definition A. An allocation rule f satisfies *strong acyclicity* if, for any finite cycle $v^1, \dots, v^k, v^{k+1} = v^1$ of types in V, we have

$$l(v^1, v^2) \leq 0, \dots, l(v^{k-1}, v^k) \leq 0 \Rightarrow l(v^k, v^1) > 0.$$

We note that this stronger assumption is used only for the lemma. The main proposition does not impose this assumption, but the assumption of acyclicity (AC).

Lemma 1. *Assume that the set V of types is finite. If an allocation rule f satisfies strong acyclicity, then it satisfies SCM.*

Proof. Fix the allocation rule f, and define a new graph H which is derived from the type graph G_f as follows. The node set is V, and for any $v, v' \in V$, there is an unweighted directed edge from v to v' if and only if $l(v, v') \leq 0$ in G_f. By strong acyclicity, H does not contain a cycle, that is, it is a DAG (directed acyclic graph).

Then, as is well-known in the field of computer science, there is a *topological sort* of H: there is a linear ordering of all nodes $v \in V$ such that if there is a directed edge from v to v' then the node v appears before the node v' in this ordering.[8] Rename, if needed, the nodes in V (for both G_f and H) in an increasing order according to this topological sort.

[8] See, for example, chapter 22 of Cormen *et al.* (2009).

An important fact about a DAG is that it contains no *back edge.* Thus, any edge from v^i to v^j with $i > j$ in our original graph G_f has a positive length $l(v^i, v^j) > 0$.

Let n be the cardinality of V, and let

$$m = \min_{v,v' \in V} \{l(v, v') | l(v, v') > 0\} \quad \text{and} \quad M = \max_{v,v' \in V} |l(v, v')|,$$

where $|l(v, v')|$ is the absolute value of $l(v, v')$. Note that both m and M belong to the interval $(0, \infty)$ due to strong acyclicity and the finiteness of V. Let $\lambda(v^1) = 1$, and recursively define for $i = 2, \ldots, n$ that

$$\lambda(v^i) = \frac{M}{m} \sum_{j=1}^{i-1} \lambda(v^j).$$

Now, a cycle in G_f can be represented by $(v^{\pi(1)}, \ldots, v^{\pi(k)})$, where $(\pi(1), \ldots, \pi(k))$ is an ordered collection of distinct members of the set $\{1, \ldots, n\}$. This is a cycle with directed edges from $v^{\pi(i)}$ to $v^{\pi(i+1)}$ for $i = 1, \ldots, k$ with the convention that $v^{\pi(k+1)} = v^{\pi(1)}$. When $\pi(i) < \pi(i + 1)$, we have $l(v^{\pi(i)}, v^{\pi(i+1)}) \leq 0$ if there is an edge in H from $v^{\pi(i)}$ to $v^{\pi(i+1)}$; otherwise, we have $l(v^{\pi(i)}, v^{\pi(i+1)}) > 0$. On the other hand, when $\pi(i) > \pi(i + 1)$, we always have $l(v^{\pi(i)}, v^{\pi(i+1)}) > 0$ since there is no back edge in H.

Consider any cycle $(v^{\pi(1)}, \ldots, v^{\pi(k)})$ in G_f, and let $\pi(i^*)$ be the largest number among $\{\pi(1), \ldots, \pi(k)\}$, i.e., $\pi(i^*) > \pi(i)$ for all $i \in \{1, \ldots, k\} \setminus \{i^*\}$. Thus, we have $l(v^{\pi(i^*)}, v^{\pi(i^*+1)}) > 0$. By our construction of $\lambda(v^i)$, we have

$$\begin{aligned}
&\lambda(v^{\pi(i^*)}) l(v^{\pi(i^*)}, v^{\pi(i^*+1)}) \geq m\lambda(v^{\pi(i^*)}) = M \sum_{i=1}^{\pi(i^*)-1} \lambda(v^j) \\
\geq &\sum_{i \in \{1,\ldots,k\} \setminus \{i^*\}} -\lambda(v^{\pi(i)}) l(v^{\pi(i)}, v^{\pi(i+1)}).
\end{aligned}$$

Hence, $\sum_{i=1}^{k} \lambda(v^{\pi(i)}) l(v^{\pi(i)}, v^{\pi(i+1)}) \geq 0$ and SCM is satisfied. *Q.E.D.*

In the proof, (i) we first define a DAG (directed acyclic graph) H, (ii) topologically sort the nodes, and then (iii) assign progressively larger $\lambda(v^i)$ to the sorted nodes such that the scaled length of a later node would dominate the sum of the scaled lengths of the previous ones. This works since there is no back edge in the topological sort when strong acyclicity holds. In comparison, we need one more step of limiting argument when the allocation rule f satisfies only acyclicity, not strong acyclicity, since there may exist a back edge of non-positive length in G_f. We first state the following straightforward fact.

Lemma 2. *Assume that an allocation rule f satisfies AC. If $l(v^i, v^{i+1}) \leq 0$ for a finite cycle $v^1, \ldots, v^k, v^{k+1} = v^1$, then $l(v^i, v^{i+1}) = 0$ for all $i = 1, \ldots, k$.*

Proof. Suppose to the effect of contradiction that there is some i for which $l(v^i, v^{i+1}) < 0$. Then, the cycle $v^{i+1}, v^{i+2}, \ldots, v^k, v^1, \ldots, v^i, v^{i+1}$ does not satisfy AC. *Q.E.D.*

We now establish our assertion for finite spaces.

Proof of Proposition A. For any $\epsilon > 0$, define a new type graph G_f^ϵ as the one obtained by changing the length of all edges in G_f with $l(v, v') = 0$ to $l^\epsilon(v, v') = \epsilon$. Then, G_f^ϵ satisfies strong acyclicity by Lemma 2 since any possible negative cycle with $l(v^i, v^{i+1}) \leq 0$ for all $i = 1, \ldots, k$ is eliminated by the changes, and so satisfies SCM by Lemma 1.

Since G_f is the limit of G_f^ϵ as $\epsilon \to 0$, it also satisfies SCM. To see this, suppose to the effect of contradiction that G_f does not satisfy SCM. Then, for any given $\lambda : V \to (0, \infty)$, there exists a finite cycle $v^1, \ldots, v^k, v^{k+1} = v^1$ with $\sum_{i=1}^k \lambda(v^i) l(v^i, v^{i+1}) < 0$. Note that there are only a finite number of cycles since V is finite. Thus, ranging over λ only gives a finite number of such cycles. We observe that we can restrict the range of λ as $(0, 1)$ instead of $(0, \infty)$ since what really matters here is only relative magnitude. Hence, for sufficiently small $\epsilon > 0$, there is a graph G_f^ϵ such that for all $\lambda : V \to (0, 1)$ we have $\sum_{i=1}^k \lambda(v^i) l^\epsilon(v^i, v^{i+1}) < 0$. This contradicts the fact that G_f^ϵ for all $\epsilon > 0$ satisfies SCM. *Q.E.D.*

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References

Archer, A., and Kleinberg, R., 2014. Truthful germs are contagious: a local to global characterization of truthfulness. Games and Economic Behavior 86, 340-366.

Ashlagi, I., Braverman, M., Hassidim, A., and Monderer, D., 2010. Monotonicity and implementability. Econometrica 78, 1749-1772.

Bikhchandani, S., Chatterji, S., Lavi, R., Mu'alem, A., Nisan, N., and Sen, A., 2006. Weak monotonicity characterizes deterministic dominant-strategy implementation. Econometrica 74, 1109-1132.

Cormen, T., Leiserson, C., Rivest, R., and Stein, C., 2009. Introduction to Algorithms, MIT Press.

Deb, R., and Mishra, D., 2014. Implementation with contingent contracts. Econometrica 82, 2371-2393.

Heydenreich, B., Müller, R., Uetz, M., and Vohra, R., 2009. Characterization of revenue equivalence. Econometrica 77, 307-316.

McAfee, P., and Reny, P., 1992. Correlated information and mechanism design. Econometrica 60, 395-421.

Müller, R., Perea, A., and Wolf, S., 2007. Weak monotonicity and Bayes-Nash incentive compatibility. Games and Economic Behavior 61, 344-358.

Myerson, R., 1981. Optimal auction design. Mathematics of Operations Research 6, 58-73.

Rochet, J.-C., 1987. A necessary and sufficient condition for rationalizability in a quasi-linear context. Journal of Mathematical Economics 16, 191-200.

Saks, M.E., and Yu, L., 2005. Weak monotonicity suffices for truthfulness on convex domains. in: Proceedings of the 7th ACM Conferences on Electronic Commerce, ACM Press, 286-293.

Skrzypacz, A., 2013. Auctions with contingent payments - An overview. International Journal of Industrial Organization 31, 666-675.