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NOTES, COMMENTS, AND LETTERS TO THE EDITOR

On Renegotiation-Proof Collusion under Imperfect Public Information*

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We construct a weakly renegotiation-proof equilibrium in Green and Porter's (1984, *Econometrica* **52**, 87–100) imperfect public information model, which Pareto-dominates the Cournot-Nash equilibrium. *Journal of Economic Literature* Classification Numbers: C72, D43, D82.

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1. INTRODUCTION

In a recent paper, Chen [1] shows an example of a weakly renegotiation-proof equilibrium (WRPE) in Green-Porter [4]'s imperfect public information model, where firms are engaged in a Cournot competition and the observable market price is determined by the unobservable aggregate output chosen by the firms and a random variable. That example was to falsify van Damme [2]'s conjecture that the only WRPE in that model is the repetition of the Cournot-Nash equilibrium (CNE). In doing so, he first proves a proposition [1, Proposition 1] which, among other things, implies that firms cannot sustain a collusion in any stage of a WRPE. That is, in no stage of a WRPE, can firms simultaneously entertain higher profits than the CNE profits. He also concludes, after constructing an asymmetric equilibrium, that the only symmetric WRPE is the repetition of the CNE.

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In this note, contrary to his proposition and conclusion, we construct a symmetric WRPE such that firms can sustain a collusion in some stages of the equilibrium. Moreover, this equilibrium is shown to Pareto-dominate the CNE, thus partially answering the open question of how high total profits can be in a WRPE.

The best way to understand how we are able to construct the asserted equilibrium is to look at Chen's Proposition 1. Chen's proof relies on the fact that players' continuation payoffs depend only on the previous period's price realization. But players' actions, and thus continuation payoffs, can depend on other things, e.g., different phases. Chen himself does use two phases when he shows an example of WRPE, and the continuation payoffs are different across phases with the same price realization p.

In constructing the equilibrium, we introduce a simple bi-valued public randomizing device which is used when determining who should be punished in case of an unfavorable public outcome, i.e., low price.[1] Maintaining a collusion implies that there is at least one stage of the equilibrium where firms' payoff vector Pareto-dominates the CNE payoff vector. Consequently, there are periods in which firms' outputs are less than the CNE outputs. In those periods, each firm has an incentive to increase its output. To support the collusion, there must be a punishment scheme when the price is low enough. Since we are interested in finding a renegotiation-proof equilibrium, this punishment should not give every player a lower continuation payoff. That means one firm has to entertain a higher continuation payoff after a low price. By making the punishment random and the expected value of the continuation payoffs from the punishment lower than that from collusion, we can deter deviations.[2]

2. THE MODEL

The model presented here is the same as that in Chen [1]. There are two firms (firm 1 and firm 2) producing a homogeneous good with zero cost, and they are engaged in a Cournot competition. When the output produced by firm i is q_i, the price realized in the market is $\tilde{P} = P(Q) + \tilde{\eta}$, where $Q = q_1 + q_2$ and $\tilde{\eta}$ is a random variable with density g. We assume g is differentiable, single-peaked at, and symmetric around 0. Let G be the cumulative distribution function.

Each firm observes the price realized in the market and its own output, but not the quantity of output produced by the other firm.

[1] We can approximate this device by using the price realization of the past period judiciously. See Section 3 for a discussion of the randomizing device.

[2] Please refer to Section 3 for a detailed discussion.

2.1. *Stage Game Γ*

Both firms simultaneously choose their respective outputs q_1 and q_2 ($q_i \in \mathbb{R}_+$). The payoff (profit) for firm i is

$$\tilde{\pi}_i(q_i, \tilde{P}) = q_i \tilde{P},$$

and the expected payoff is

$$\pi_i(q_i; q_j) = q_i P(Q).$$

The stage game is denoted by $\Gamma = (\mathbb{R}^2_+; \pi_1, \pi_2; G)$.

2.2. *Repeated Game $\Gamma^\infty(\delta)$*

$\Gamma^\infty(\delta)$ is the infinite repetition of the stage game Γ with the common discount factor $\delta \in (0, 1)$. Period t runs from 1 to ∞. For $t \geqslant 2$, a public history h^t is $(p(1), p(2), ..., p(t-1))$, that is, past price realizations up to, but not including, t. A private history for firm i, denoted h_i^t, is $(q(1), q(2), ..., q(t-1))$. Let H^t and H_i^t be the sets of all public and private histories in period t respectively. A strategy for firm i is $\sigma_i = (\sigma_i^t)_{t=1}^\infty$ where $\sigma_i^1 \in \mathbb{R}_+$ and $\sigma_i^t: (H^t, H_i^t) \to \mathbb{R}_+$ for $t = 2, 3, ...$ A strategy pair $\sigma = (\sigma_1, \sigma_2)$ will be called a strategy profile.

Given a strategy profile σ, we can determine the probability distribution over all the possible histories in each period, and consequently the probability distribution over stage-game payoffs. Firm i's objective in $\Gamma^\infty(\delta)$ is to maximize the expected value of the discounted average payoff

$$(1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_i^t,$$

where π_i^t is the period t stage-game payoff. We will deal only with public strategy profiles, that is, the strategy profiles that depend only on public histories. In this case, each firm's belief regarding exactly at which node it is, given a public history, does not matter to the subsequent play. Besides, we can unambiguously refer to the continuation payoffs, since firms have identical probability distributions over stage-game payoffs after any public history.

We will use the perfect Bayesian equilibrium (PBE) concept in this note. Especially, we use the weakest notion of PBE: each player's strategy after any history (i.e., at any information set) maximizes his expected payoff given others' strategies and his belief about nodes, and beliefs after any history to which the strategy profile assigns positive probability are obtained using Bayes' rule. A PBE is called a WRPE if none of its continuation payoff vector is strictly Pareto-dominated by another.[3]

3. AN EXAMPLE

Let $P(Q)=a-bQ$ if $Q \leqslant (a/b)$, and 0 otherwise. Then the best response function of player i is

$$q_i = R(q_j) = \frac{a-bq_j}{2b} \qquad i, j=1, 2, \quad i \neq j.$$

Note that, given the other players' output q_j, player i's expected stage game payoff as a function of his own output, $\pi_i(q_i; q_j)$, is single-peaked.

We find a WRPE which Pareto-dominates the Cournot-Nash equilibrium. Let $a=6$, $b=1$, that is, $P(Q)=6-Q$ as in Chen [1]. Then the CNE is $(q_1, q_2)=(2, 2)$ and $(\pi_1, \pi_2)=(4, 4)$. The strategy profile σ below is a WRPE. In the profile, we use parameters κ, τ, p^1, and p^2 whose values will be determined shortly.

Phase 1: Play $(q_1, q_2)=(2-\kappa, 2-\kappa)$. The expected price is $E(p)=2+2\kappa$ and the expected payoff (π_1, π_2) is $(4+2\kappa-2\kappa^2, 4+2\kappa-2\kappa^2)$.

Keep playing it until p goes below p^1. If $p<p^1$, then go to phase 2-1 with probability 1/2, and to phase 2-2 with probability 1/2.[4]

Phase 2-1: Play $(q_1, q_2)=(1+\tau, 3)$. Then $E(p)=2-\tau$ and (π_1, π_2) is $(2+\tau-\tau^2, 6-3\tau)$.

If $p>p^2$, then go to phase 2-2; otherwise go to phase 1.

Phase 2-2: Play $(q_1, q_2)=(3, 1+\tau)$. Then $E(p)=2-\tau$ and (π_1, π_2) is $(6-3\tau, 2+\tau-\tau^2)$.

If $p>p^2$, then go to phase 2-1; otherwise go to phase 1.

We assume that $0<\kappa<1$ and $0<\tau<1/2$. Let V_i^h be firm i's continuation payoff at phase h ($i=1, 2$, $h=1, 21, 22$).

Before proving the claim, let us explain the intuition. In phase 1, firm 1 has a short-term incentive to increase its output. But doing so will increase the probability of going out of phase 1 and entering into phase 2-1 or phase 2-2 with equal probability. Since $V_1^1 > \frac{1}{2}(V_1^{21}+V_1^{22})$ (Eq. (1) below) by construction, firm 1's short-term incentive is checked by a long-term consideration. Phase 2-1 is a phase in which firm 1 is being punished. In

[3] See Farrel and Maskin [3] for this concept.

[4] See below how this can be done.

this phase, firm 1 has a short-term incentive to increase its output. But doing so will decrease the probability of entering phase 2-2 instead of phase 1. Since $V_1^{22} > V_1^1$ (Eq. (2) below) by construction, this short-term incentive is checked. In phase 2-2, firm 1 has a short-term incentive to decrease its output. Since doing so will increase the probability of entering phase 2-1 instead of phase 1 and $V_1^1 > V_1^{21}$ by Eq. (3) below, this incentive is checked. A symmetric argument applies for firm 2.

Let us briefly discuss the randomizing device. When price p is below the threshold level p^1 in phase 1, a randomizing device determines which phase among phase 2-1 or 2-2 to enter. The probability of entering either phase is 1/2. We assume it is independent across periods. It would not be difficult to find such a device in the real world. Alternatively, we can partition the interval $[0, p^1]$ into $2N$ equal-sized subintervals, with enumerating them from 1 to $2N$. Taking N to be sufficiently large and arranging to punish player 1 if the price falls in the odd-numbered intervals and player 2 if it falls in the even-numbered intervals, we can make the punishment approximately random.

Observe that the implementation of the randomizing device should not be under firms' control in each period. Otherwise, firms would like to back off from implementing it and thus entering phase 2-1 or phase 2-2 when they have to, which upsets the proposed σ. We can think of many schemes which do not give the firms the right to decide whether to implement the randomizing device–sunspots, simultaneous determination of a coin toss and the price realization, etc. The interval-partitioning scheme described above also supports the weak renegotiation-proofness of the equilibrium.

3.1. *Renegotiation-proofness*

Note that

$$\mathrm{Prob}(\tilde{P} \leqslant p^1 \mid E(p) = 2 + 2\kappa) = G(p^1 - 2 - 2\kappa).$$

Similarly,

$$\mathrm{Prob}(\tilde{P} \leqslant p^2 \mid E(p) = 2 - \tau) = G(p^2 - 2 + \tau).$$

Then we have:

[1] Phase 1.

$$\begin{aligned} V_1^1 &= (1-\delta)(4 + 2\kappa - 2\kappa^2) + \delta[(1 - G(p^1 - 2 - 2\kappa))\, V_1^1 \\ &\quad + \tfrac{1}{2} G(p^1 - 2 - 2\kappa)(V_1^{21} + V_1^{22})] \\ V_2^1 &= (1-\delta)(4 + 2\kappa - 2\kappa^2) + \delta[(1 - G(p^1 - 2 - 2\kappa))\, V_2^1 \\ &\quad + \tfrac{1}{2} G(p^1 - 2 - 2\kappa)(V_2^{21} + V_2^{22})] \end{aligned}$$

[2] Phase 2-1.

$$V_1^{21} = (1-\delta)(2+\tau-\tau^2) + \delta[G(p^2-2+\tau)\ V_1^1 + (1-G(p^2-2+\tau))\ V_1^{22}]$$

$$V_2^{21} = (1-\delta)(6-3\tau) + \delta[G(p^2-2+\tau)\ V_2^1 + (1-G(p^2-2+\tau))\ V_2^{22}]$$

[3] Phase 2-2.

$$V_1^{22} = (1-\delta)(6-3\tau) + \delta[G(p^2-2+\tau)\ V_1^1 + (1-G(p^2-2+\tau))\ V_1^{21}]$$

$$V_2^{22} = (1-\delta)(2+\tau-\tau^2) + \delta[G(p^2-2+\tau)\ V_2^1 + (1-G(p^2-2+\tau))\ V_2^{21}]$$

It is clear that $V_1^1 = V_2^1$, $V_1^{21} = V_2^{22}$, and $V_1^{22} = V_2^{21}$. From now on, we only deal with firm 1's continuation payoffs explicitly. Let $\alpha = G(p^1 - 2 - 2\kappa)$ and $\beta = G(p^2 - 2 + \tau)$. Since the value functions V_1^1, V_1^{21}, and V_1^{22} form a system of linear equations, we can solve them to get

$$V_1^1 - \frac{1}{2}(V_1^{21} + V_1^{22}) = \frac{(1-\delta)(4\kappa - 4\kappa^2 + 2\tau + \tau^2)}{2(1-\delta+\alpha\delta+\beta\delta)} \tag{1}$$

$$V_1^{22} - V_1^1 = \frac{(1-\delta)\ \Delta_1}{2(1+\delta-\beta\delta)(1-\delta+\alpha\delta+\beta\delta)} \tag{2}$$

$$V_1^1 - V_1^{21} = \frac{(1-\delta)\ \Delta_2}{2(1+\delta-\beta\delta)(1-\delta+\alpha\delta+\beta\delta)} \tag{3}$$

where

$$\Delta_1 = 4(1-\kappa+\kappa^2) - 6\tau + \delta(1-\beta)[2\tau(1-\tau) - 4(1+\kappa-\kappa^2)] + \delta\alpha(2-\tau)^2 \tag{4}$$

$$\Delta_2 = 4(1+\kappa-\kappa^2) - 2\tau(1-\tau) + \delta(1-\beta)[6\tau - 4(1-\kappa+\kappa^2)] + \delta\alpha(2-\tau)^2 \tag{5}$$

Since $0<\kappa<1$ and $0<\tau<1/2$ by assumption, $V_1^1 - (V_1^{21} + V_1^{22})/2 > 0$. We will choose α and β such that $0<\alpha<1/2$ and $1/2<\beta<1$. In that case, Δ_1 is positive for all $0<\delta<1$ and (κ, τ)'s satisfying $2 - 6\kappa + 6\kappa^2 > 5\tau + \tau^2$. We will choose κ and τ such that

$$2 - 6\kappa + 6\kappa^2 > 5\tau + \tau^2 \tag{†}$$

in the following. Therefore, $V_1^{22} - V_1^1 > 0$. Since $\Delta_2 - \Delta_1 = 2(1+\delta-\beta\delta)(4\kappa - 4\kappa^2 + 2\tau + \tau^2) > 0$, we have $V_1^1 - V_1^{21} > V_1^{22} - V_1^1$. In particular, $V_1^1 - V_1^{21} > 0$.

In summary, we have $V_1^{22} > V_1^1 > V_1^{21}$ and $V_2^{21} > V_2^1 > V_2^{22}$. Since no pair of continuation payoff vectors are Pareto ranked, the proposed strategy profile σ is a WRPE if we can show that no firm has an incentive to deviate from the proposed strategy profile σ.

3.2. *No Incentive to Deviate*

We will show that there is no profitable one-shot deviation at any phase. Then, by the unimprovability principle, σ is a PBE. Let $V_i^h(q_i)$ be the continuation payoff of firm i when it produces q_i in this period at phase h and then follows the proposed strategy thereafter, assuming that the other firm is following the proposed strategy.

[1] Phase 1.

$$V_1^1(q_1) = (1-\delta)\, q_1(4+\kappa-q_1) + \delta[(1-G(p^1-4-\kappa+q_1))\, V_1^1 + \tfrac{1}{2}G(p^1-4-\kappa+q_1)(V_1^{21}+V_1^{22})]$$

$$V_1^{1\prime}(q_1) = (1-\delta)(4+\kappa-2q_1) + \delta g(p^1-4-\kappa+q_1)[\tfrac{1}{2}(V_1^{21}+V_1^{22}) - V_1^1]$$

$$V_1^{1\prime\prime}(q_1) = -2(1-\delta) + \delta g'(p^1-4-\kappa+q_1)[\tfrac{1}{2}(V_1^{21}+V_1^{22}) - V_1^1]$$

If we show $V_1^{1\prime}(2-\kappa)=0$ and $V_1^{1\prime\prime}<0$ for all $q_1 \in \mathbb{R}_+$, then firm 1 has no incentive to deviate at phase 1 and then conform afterwards. Since

$$V_1^{1\prime}(2-\kappa) = 3\kappa(1-\delta) + \delta g(p^1-2-2\kappa)[\tfrac{1}{2}(V_1^{21}+V_1^{22}) - V_1^1] > 0$$

as $p^1 \to \infty$ or $p^1 \to -\infty$, we need

$$3\kappa(1-\delta) + \delta g(0)[\tfrac{1}{2}(V_1^{21}+V_1^{22}) - V_1^1] < 0$$

to have a p^1 which solves $V_1^{1\prime}(2-\kappa)=0$. That is, by Eq. (1) above,

$$g(0) > \frac{6\kappa(1-\delta+\alpha\delta+\beta\delta)}{\delta(4\kappa-4\kappa^2+2\tau+\tau^2)}. \tag{C1}$$

Also we need

$$|g'(\eta)| < \frac{4(1-\delta+\alpha\delta+\beta\delta)}{\delta(4\kappa-4\kappa^2+2\tau+\tau^2)}, \qquad \forall \eta \in (-\infty, \infty). \tag{C2}$$

A similar argument for firm 2 will lead to the same conditions (C1) and (C2).

[2] Phase 2-1.

$$V_1^{21}(q_1)=(1-\delta)\,q_1(3-q_1)+\delta[G(p^2-3+q_1)\,V_1^1 + (1-G(p^2-3+q_1))\,V_1^{22}]$$

$$V_1^{21\prime}(q_1)=(1-\delta)(3-2q_1)+\delta g(p^2-3+q_1)[V_1^1-V_1^{22}]$$

$$V_1^{21\prime\prime}(q_1)=-2(1-\delta)+\delta g'(p^2-3+q_1)[V_1^1-V_1^{22}]$$

Since $0<\tau<1/2$ by assumption,

$$V_1^{21\prime}(1+\tau)=(1-\delta)(1-2\tau)+\delta g(p^2-2+\tau)[V_1^1-V_1^{22}]>0$$

as $p^2\to\infty$ or $p^2\to-\infty$. Thus we need, by Eqs. (2) and (4),

$$g(0)>\frac{2(1-2\tau)(1+\delta-\beta\delta)(1-\delta+\alpha\delta+\beta\delta)}{\delta\Delta_1}. \tag{C3}$$

Also we need

$$|g'(\eta)|<\frac{4(1+\delta-\beta\delta)(1-\delta+\alpha\delta+\beta\delta)}{\delta\Delta_1}, \qquad \forall\eta\in(-\infty,\infty). \tag{C4}$$

[3] Phase 2-2.

$$V_1^{22}(q_1)=(1-\delta)\,q_1(5-\tau-q_1)+\delta[G(p^2-5+\tau+q_1)\,V_1^1 + (1-G(p^2-5+\tau+q_1))\,V_1^{21}]$$

$$V_1^{22\prime}(q_1)=(1-\delta)(5-\tau-2q_1)+\delta g(p^2-5+\tau+q_1)[V_1^1-V_1^{21}]$$

$$V_1^{22\prime\prime}(q_1)=-2(1-\delta)+\delta g'(p^2-5+\tau+q_1)[V_1^1-V_1^{21}]$$

Since

$$V_1^{22\prime}(3)=-(1-\delta)(1+\tau)+\delta g(p^2-2+\tau)[V_1^1-V_1^{21}]<0$$

as $p^2\to\infty$ or $p^2\to-\infty$, we need, by Eqs. (3) and (5),

$$g(0)>\frac{2(1+\tau)(1+\delta-\beta\delta)(1-\delta+\alpha\delta+\beta\delta)}{\delta\Delta_2}. \tag{C5}$$

Also we need

$$|g'(\eta)|<\frac{4(1+\delta-\beta\delta)(1-\delta+\alpha\delta+\beta\delta)}{\delta\Delta_2}, \qquad \forall\eta\in(-\infty,\infty). \tag{C6}$$

A similar argument for firm 2 will lead to the same conditions (C3) $\sim$ (C6). For firm 2, conditions (C5) and (C6) will apply to phase 2-1, and conditions (C3) and (C4) will apply to phase 2-2. In the following, we will denote the right-hand side of condition (Ci) ($i = 1, ..., 6$) by c_i.

It is easy to see that (C6) implies (C2) and (C4). Summing up, our problem is to find p^1, p^2, κ, and τ such that equations[5]

$$6\kappa(1-\delta+\alpha\delta+\beta\delta) = \delta g(p^1-2-2\kappa)(4\kappa-4\kappa^2+2\tau+\tau^2),$$
$$2(1-2\tau)(1+\delta-\beta\delta)(1-\delta+\alpha\delta+\beta\delta) = \delta g(p^2-2+\tau)\,\Delta_1,$$

and

$$2(1+\tau)(1+\delta-\beta\delta)(1-\delta+\alpha\delta+\beta\delta) = \delta g(p^2-2+\tau)\,\Delta_2$$

hold, and both (†) and (C6) are satisfied. (If we find such p^1, p^2, κ, and τ, then conditions (C1), (C3), and (C5) will be automatically satisfied.) For a concrete example, consider a logistic distribution with density

$$g(\eta) = \frac{10}{3}\frac{\exp(-(10/3)\,\eta)}{(1+\exp(-(10/3)\,\eta))^2}$$

and $\delta = 0.99$. Note that $G(\eta) = (1+\exp(-\frac{10}{3}\eta))^{-1}$.

A solution for the given specification is $(p^1, p^2, \kappa, \tau) = (1.99759, 1.98801, 0.1, 0.202521)$. The maximum of $|g'(\eta)|$ is achieved at $\eta = \frac{3}{10}\ln(2 \pm \sqrt{3})$, which is 1.06917, and c_6 is 1.2551. Thus, (C6) is satisfied. It is also easy to see that (†) is satisfied. We have $V_1^1 = 4.04416$, $V_1^{21} = 4.02807$, and $V_1^{22} = 4.05213$, implying that the proposed equilibrium σ Pareto-dominates the Cournot Nash equilibrium.

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[5] Recall that $\alpha = G(p^1 - 2.2)$ and $\beta = G(p^2 - 2 + \tau)$.