Entanglement / Bell State

Create multi - qubits state which
can't be factored out in tensor product terms

2 - qubits system

$|\psi\rangle \otimes |\phi\rangle = |\psi\phi\rangle$

$|\psi\rangle = a|0\rangle + b|1\rangle$
$|\phi\rangle = c|0\rangle + d|1\rangle$

$|\psi\rangle \otimes |\phi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$

$= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$

in tensor product

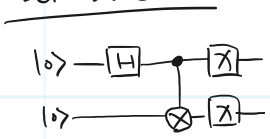
$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$, $|\phi\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$

$|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a\begin{bmatrix} c \\ d \end{bmatrix} \\ b\begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix}$

$= \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}$

Bell - State Circuit

"Put 1st q-bit in superposition and relate the 2nd q-bit using CNOT"



$|Bell\rangle = CNOT\,(H \otimes I)\,|00\rangle$

Remember
H is H
I

To keep 1x4 matrix

Commutative Rule:
$(A \otimes B)(C \otimes D)$
$= (AB) \otimes (CD)$

$= CNOT \cdot (H \otimes I)\,(|0\rangle \otimes |0\rangle)$

$= CNOT\,(H|0\rangle) \otimes (I|0\rangle)$

$= CNOT\,(H|0\rangle) \otimes |0\rangle$

$= CNOT\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle$

$= \frac{1}{\sqrt{2}}\,CNOT\,(|0\rangle + |1\rangle) \otimes |0\rangle$

$= \frac{1}{\sqrt{2}}\,CNOT\,(|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle)$

$= \frac{1}{\sqrt{2}}\,CNOT\,(|00\rangle + |10\rangle)$

$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

CNOT "If 1st q-bit is 1, flip the 2nd q-bit"
$|00\rangle \to |00\rangle$
$|01\rangle \to |01\rangle$
$|10\rangle \to |11\rangle$
$|11\rangle \to |10\rangle$

Question, is factoring possible?
⇒ Can you express in tensor prod.?
like $|\psi\rangle \otimes |\phi\rangle$
$[?] \otimes [!]$ ..... Nope.

Think all possible 4 terms

$\frac{1}{\sqrt{2}}(ab|00\rangle + ac|01\rangle + bc|10\rangle + bd|11\rangle)$

What we got here is

$\frac{1}{\sqrt{2}}((1)|00\rangle + (0)|01\rangle + (0)|10\rangle + (1)|11\rangle)$

that is;

$ab = \frac{1}{\sqrt{2}}$
$ac = 0$
$bc = 0$
$cd = \frac{1}{\sqrt{2}}$

$\Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$|00\rangle = 50\%$
$|01\rangle = 0\%$
$|10\rangle = 0\%$
$|11\rangle = 50\%$

(or simply ignore $\frac{1}{\sqrt{2}}$
no solution for $ab = cd = 1$
$ac = bc = 0$)

Try in tensor Prod way

$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\quad H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$= \frac{1}{\sqrt{2}}\left(\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \otimes \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right)$

$= \frac{1}{\sqrt{2}}\left(\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \otimes \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right)$

$= \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

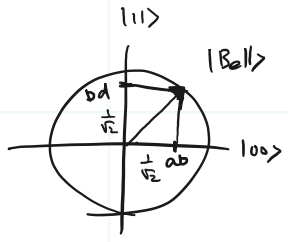
$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Unit Circle



$\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

⇒ Can't find solution $a, b, c,$ and d
→ no factorization
→ ENTANGLED.