Quantum-Measurement

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ — "Bra"

$\rightarrow \begin{bmatrix}\alpha \\ \beta\end{bmatrix}$

$\langle\psi| = [\alpha^*, \beta^*]$ — "Ket" Cojugate & Transpose Matrix

Measurement = Probability

"What is the probability of the state When the system is observed = collapsed?"

→ Measure Quantum State $|\psi\rangle$ with respect to the orthonormal basis $|\phi_i\rangle = \{|\phi_0\rangle \cdots |\phi_i\rangle\}$

$P(\phi_i) = |\langle\phi_i|\psi\rangle|^2$

Note, the state should Unitery means, $\sum_{i=0}^{n} P(\phi_i) = 1$

$P(\phi_0) + P(\phi_1) + \cdots P(\phi_n) = 1$ //

Intuitively, the total of existence should be $1 = 100\%$

e.g.1

Measure Superposition $|+\rangle$ with standard basis. What is probability of $|0\rangle$ when the syste gets collapsed?

$P(0) = \langle 0|+\rangle = |\langle 0|+\rangle|^2$?

Superposition $\vec{|+\rangle} = H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$\Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 \\ 1 & -1\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix}$

Standard Basis $|0\rangle \Rightarrow \langle 0|$

$\begin{bmatrix}1 \\ 0\end{bmatrix} \Rightarrow [1\ 0]$ — like $\begin{bmatrix}\alpha \\ \beta\end{bmatrix} \Rightarrow [\alpha, \beta]$ only real # so no sign change

$|\langle 0|+\rangle|^2 = \left|[1, 0]\begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{bmatrix}\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} = 50\%$

btw, also

$|\langle 1|+\rangle|^2 = \cdots \quad \frac{1}{2} = 50\%$

So $P(0) + P(1) = \frac{1}{2} + \frac{1}{2} = 1$ Unitery //

Measuring Standard basis

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

You can write

$P(\alpha) + P(\beta) = |\alpha|^2 + |\beta|^2 = 1$

e.g.2

Let's meas some Random State $|\psi\rangle$ in standard basis, and find $P(|0\rangle_\psi)$?

$|\psi\rangle = \frac{1-\sqrt{2}i}{4}|0\rangle - \frac{3-2i}{4}|1\rangle$

So what is probability of $|0\rangle$ when measure in standard basis? $\Rightarrow \left\{\begin{matrix}|\langle 0|\psi\rangle|^2 ? \\ |\langle 1|\psi\rangle|^2 ?\end{matrix}\right\}$

shortcut from above,

$$\begin{cases} P(0) = \left|\frac{1-\sqrt{2}i}{4}\right|^2 = \frac{(1-\sqrt{2}i)(1+\sqrt{2}i)}{16} = \frac{3}{16} \text{//} \\ P(1) = \left|\frac{-3+2i}{4}\right|^2 = \frac{(-3+2i)(-3-2i)}{16} = \frac{9+4}{16} = \frac{13}{16} \text{//} \end{cases}$$

$\boxed{\text{Note: } |z|^2 = z \cdot \bar{z}}$

e.g.3

Measure state $|\psi\rangle$ in Hadamard state $|+\rangle$? (Use above $|\psi\rangle$)

$P(+) = |\langle +|\psi\rangle|^2$?

$|\psi\rangle = \frac{1-\sqrt{2}i}{4}|0\rangle - \frac{3-2i}{4}|1\rangle$

$|+\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{bmatrix} \quad \langle +| = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

$|\langle +|\psi\rangle|^2 = \left|\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]\begin{bmatrix}\frac{1-\sqrt{2}i}{4} \\ -\left(\frac{3-2i}{4}\right)\end{bmatrix}\right|^2$

$= \left|\frac{1}{\sqrt{2}} \cdot \frac{1-\sqrt{2}i}{4} + \frac{1}{\sqrt{2}} \cdot \frac{-3+2i}{4}\right|^2$

$= \left|\frac{\sqrt{2}}{2}\left(\frac{1-\sqrt{2}i}{4}\right) + \frac{\sqrt{2}}{2}\left(\frac{-3+2i}{4}\right)\right|^2$

$= \left|\frac{\sqrt{2}-2i}{8} + \frac{-3\sqrt{2}+2\sqrt{2}i}{8}\right|^2$

$= \left|\frac{-2\sqrt{2}-2(1-\sqrt{2})i}{8}\right|^2$

$= \left|\frac{-\sqrt{2}-(1-\sqrt{2})i}{4}\right|^2 \quad |z|^2 = z \cdot \bar{z}$

$= \left(\frac{-\sqrt{2}-(1-\sqrt{2})i}{4}\right)\left(\frac{-\sqrt{2}+(1-\sqrt{2})i}{4}\right)$

$= \frac{2+(1-\sqrt{2})^2}{16} = \frac{2+1-2\sqrt{2}+2}{16} = \frac{5-2\sqrt{2}}{16}$ //

On the other hand, How about $|-\rangle$

$|\langle -|\psi\rangle|^2$?

$= \cdots\cdots$

$= \left|\frac{\sqrt{2}-2i - (-3\sqrt{2}+2\sqrt{2}i)}{8}\right|^2$

$= \left|\frac{\sqrt{2}-2i+3\sqrt{2}-2\sqrt{2}i}{8}\right|^2$

$= \left|\frac{4\sqrt{2}-2(1+\sqrt{2})i}{8}\right|^2$

$= \left|\frac{2\sqrt{2}-(1+\sqrt{2})i}{4}\right|^2$

$|z|^2 = z\bar{z} = \left(\frac{2\sqrt{2}-(1+\sqrt{2})i}{4}\right)\left(\frac{2\sqrt{2}+(1+\sqrt{2})i}{4}\right)$

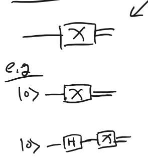
$= \frac{4 \cdot 2 + (1+\sqrt{2})^2}{16} = \frac{8+1+2\sqrt{2}+2}{16}$

$= \frac{11+2\sqrt{2}}{16}$ //

$P(+) + P(-) = \frac{5-2\sqrt{2}}{16} + \frac{11+2\sqrt{2}}{16} = \frac{16}{16} = 1$ //

Unitary

Circuit



double lines as classical bit

$0 = 1.0 = 100\%$

$\begin{cases} 0 = 50\% \\ 1 = 50\% \end{cases}$

the result is classical 0 or 1

Formula

Probability of measurement i. M - Hermitian Operator Measurement

$$\boxed{P(i) = \langle\psi|M_i^\dagger M_i|\psi\rangle}$$

$\sum_i M_i^\dagger M = I$

$\sum P(i) = \sum_i \langle\psi|M_i^\dagger M_i|\psi\rangle = 1$

$M_i = |M_i\rangle\langle M_i|$

If input is $|\psi\rangle$, then post-measure the State $|\bar{\psi}\rangle$

$|\bar{\psi}\rangle = \frac{M_i|\psi\rangle}{\sqrt{P(i)}}$

e.g

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Let's measure the Probability for $0 \rightarrow P(0)$?

Measurement Operator M_0

$M_0 = |0\rangle\langle 0| = \begin{bmatrix}1 \\ 0\end{bmatrix}[1\ 0] = \begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}$

$M_0^\dagger M_0$

$P(0) = \langle\psi||0\rangle\langle 0||\psi\rangle = [\alpha\ \beta]\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix}$

$= [\alpha\ \beta]\begin{bmatrix}\alpha \\ 0\end{bmatrix} = |\alpha|^2$

How does Measurement Modify the State?

before $|\psi\rangle$

after $|\bar{\psi}\rangle = \frac{M_0|\psi\rangle}{\sqrt{P(0)}} = \frac{|0\rangle\langle 0||\psi\rangle}{\sqrt{P(0)}}$

$= \frac{\begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\alpha \\ \beta\end{bmatrix}}{|\alpha|}$

$= \frac{\begin{bmatrix}\alpha \\ 0\end{bmatrix}}{|\alpha|} = \begin{bmatrix}1 \\ 0\end{bmatrix} = |0\rangle$

BTW. Try $\sum_i M_i^\dagger M_i = I$?

$= |0\rangle\langle 0| + |1\rangle\langle 1|$

$= \begin{bmatrix}1 \\ 0\end{bmatrix}[1\ 0] + \begin{bmatrix}0 \\ 1\end{bmatrix}[0\ 1]$

$= \begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 \\ 0 & 1\end{bmatrix}$

$= \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} = I$ //

$P(0) = |\alpha|^2$

$P(1) = |\beta|^2$

$P(0) + P(1) = |\alpha|^2 + |\beta|^2 = 1$

$\Rightarrow \sum_i P(i) = 1$

$\rightarrow |m_0|^2 + |m_1|^2 + \cdots |m_i|^2 = 1$