19. $\boxed{10}$ $\mathbb{R}^2_+ := \left\{ (x, y) \in \mathbb{R}^2 \,\middle|\, y > 0 \right\}$ $\quad g_H := \frac{1}{y^2} dx^2 + \frac{1}{y^2} dy^2$

$$\begin{cases} g_H = \frac{1}{y^2} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \\ g_H^{-1} = y^2 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \end{cases} \qquad \Gamma_{ij} = \Gamma_{ji}$$

→ Γ^1、Γ^2 を先に計算しておく.

$$\begin{bmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{bmatrix} = \frac{1}{2} g_H^{-1} \begin{bmatrix} \partial_1 g_{11} + \cancel{\partial_1 g_{11}} - \cancel{\partial_1 g_{11}} \\ \cancel{\partial_1 g_{21}} + \cancel{\partial_1 g_{12}} - \partial_2 g_{11} \end{bmatrix}$$

$$= \frac{1}{2} y^2 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \partial_1 g_{11} \\ -\partial_2 g_{11} \end{bmatrix}$$

$$= \frac{1}{2} y^2 \begin{bmatrix} 0 \\ 2y^{-3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \cancel{y}\,\frac{1}{y} \end{bmatrix} \qquad \therefore \Gamma^1_{11} = 0, \quad \Gamma^2_{11} = \frac{1}{y} \quad \text{ok}$$

y^{-2}

$-y^{-2}$

$2y^{-3}$

$\therefore \Gamma^1_{22} = 0$

$\Gamma^2_{22} = -\frac{1}{y}$ ok

$g_{11} = y^{-2}$

$g_{22} = y^{-2}$

$$\begin{bmatrix} \Gamma^1_{12} \\ \Gamma^2_{12} \end{bmatrix} = \frac{1}{2} g_H^{-1} \begin{bmatrix} \partial_1 \cancel{g_{12}} + \partial_2 g_{11} - \partial_1 \cancel{g_{12}} \\ \partial_1 g_{22} + \partial_2 \cancel{g_{12}} - \partial_2 \cancel{g_{12}} \end{bmatrix}$$

$i = 1 \quad j = 2$

$$= \frac{1}{2} y^2 \begin{bmatrix} \partial_2 g_{11} \\ \partial_1 g_{22} \end{bmatrix}$$

$$= \frac{1}{2} y^2 \begin{bmatrix} -2y^{-3} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -y^{-1} \\ 0 \end{bmatrix} \qquad \therefore \Gamma^1_{12} = -\frac{1}{y}, \quad \Gamma^2_{12} = 0 \quad \text{OK}$$

$$\begin{bmatrix} \Gamma^1_{22} \\ \Gamma^2_{22} \end{bmatrix} = \frac{1}{2} g_H^{-1} \begin{bmatrix} \partial_2 \cancel{g_{12}} + \cancel{\partial_2 g_{21}} - \partial_1 g_{22} \\ \partial_2 g_{22} + \cancel{\partial_2 g_{22}} - \cancel{\partial_2 g_{22}} \end{bmatrix}$$

$i = 2, \quad j = 2$

$$= \frac{1}{2} y^2 \begin{bmatrix} \overset{0}{-\partial_1 g_{22}} \\ \partial_2 g_{22} \end{bmatrix}$$

$$= \frac{1}{2} y^2 \begin{bmatrix} 0 \\ -2y^{-3} \end{bmatrix} = \begin{bmatrix} 0 \\ -y^{-1} \end{bmatrix}$$

$\boxed{10-1}$

y^{-2}

$R^1_{221} = ?$ $(i=1,\ j=2,\ k=2,\ l=1)$

$$R^1_{221} = \frac{\partial}{\partial x^1}\Gamma^1_{22} - \frac{\partial}{\partial x^2}\Gamma^1_{12} + \sum_{a=1}^{2}\left(\Gamma^1_{1a}\Gamma^a_{22} - \Gamma^1_{2a}\Gamma^a_{12}\right)$$

$$= \cancel{\frac{\partial}{\partial x^1}(0)} - \frac{\partial}{\partial x^2}(-y^{-1}) + \left(\underbrace{\cancel{\Gamma^1_{11}}}_{0}\cancel{\Gamma^1_{22}} - \underbrace{\Gamma^1_{21}\Gamma^1_{12}}_{\frac{1}{y^2}} + \underbrace{\Gamma^1_{12}\Gamma^2_{22}}_{\frac{1}{y^2}} - \underbrace{\cancel{\Gamma^1_{22}}}_{0}\cancel{\Gamma^2_{12}}\right)$$

$$= 0 - y^{-2} + \cancel{\left(-\frac{1}{y^2} + \frac{1}{y^2}\right)}$$

$$= -\frac{1}{y^2}$$ ok

(2) $R^1_{222} = ?$ $(i=1,\ j=2,\ k=2,\ l=2)$

$$R^1_{222} = \frac{\partial}{\partial x^2}\Gamma^1_{22} - \frac{\partial}{\partial x^2}\Gamma^1_{22} + \sum_{a=1}^{m}\left(\Gamma^1_{2a}\Gamma^a_{22} - \Gamma^1_{2a}\Gamma^a_{22}\right)$$

$$= \frac{\partial}{\partial x^2}(0) - \frac{\partial}{\partial x^2}(0) + \left(\cancel{\Gamma^1_{21}}\underbrace{\cancel{\Gamma^1_{22}}}_{0} - \Gamma^1_{21}\underbrace{\Gamma^1_{22}}_{0} + \underbrace{\Gamma^1_{22}}_{0}\Gamma^2_{22} - \underbrace{\Gamma^1_{22}}_{0}\Gamma^2_{22}\right)$$

$$= 0 - 0 + 0$$

$$= 0$$

(3) $\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) \overset{?}{=} \nabla_{\frac{\partial}{\partial x^j}}\left(\frac{\partial}{\partial x^i}\right)$

(∇: レビチビタ接続)

(かっこ積 $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$)

∂x

(L4) $\nabla_{\partial^i}\partial^j - \nabla_{\partial^j}\partial^i = [\partial^i, \partial^j]$

$= 0.$

$\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial^i$

↑

局所座標系だから成立

$R \neq 0$

(M, g): 平坦でない リーマンmfd

R: リーマン曲率テンソル

$\underbrace{R_{\partial^i\partial^j}}_{\neq}\partial^k \overset{?}{=} \underbrace{R_{\partial^j\partial^i}}_{\neq}\partial^k$

曲率作用素の反対称性: $R_{XY} = -R_{YX}$

$R_{\partial^i\partial^j}\partial^k := \nabla_{[\partial^i, \partial^j]}\partial^k - \left(\nabla_{\partial^i}\nabla_{\partial^j}\partial^k - \nabla_{\partial^j}\nabla_{\partial^i}\partial^k\right)$

↑

一般のベクトル場 X, Y, Z でも成立