5/30(金) キ何セミナー

・§1. イントロ

設定

$f \in L^1(\mathbb{R}^n)$: 非負, $\|f\|=1$,

$|x|^2 f(x) \in L^1(\mathbb{R}^n)$

⊛

$$\Rightarrow -\int_{\mathbb{R}^n} f \ln f \, dx \le \frac{n}{2} \ln\left(\frac{2\pi e}{n}\int_{\mathbb{R}^n} |x|^2 f(x)\, dx\right)$$

Ⓔ Ⓥ

(等号成立: $f(x) = G(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$)

⊛ の解釈

(左辺) = −(エントロピー)

(つまり, $\mathrm{Ent}_{\mathcal{L}^n}(f\mathcal{L}^n) := \int f \ln f \, dx$)

(右辺) = μ の分散

(つまり, $\mathrm{Var}(\mu) = \min_{y\in\mathbb{R}^n} \int |x-y|^2 \, d\mu(x)$)

Theorem 1 (K)

(M^n, g) : ~~完備~~ cplt (完備: complete), Rieman mfd, $\mathrm{Ric}_g \ge 0$

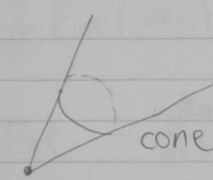
$\forall \nu \in P_2(M)$ のとき,

└ 2次モーメント有界

$$-\mathrm{Ent}_{\mathrm{vol}_g}(\nu) \le \frac{n}{2} \ln\left(\frac{2\pi e}{n} \mathrm{Var}(\nu)\right),$$

Theorem 2

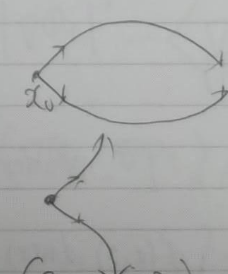
等号成立条件: $M = \mathbb{R}^n$



§2. RCD(0, N) condition

(M^n, g) : 完備, リーマン mfd

sec ≥ 0



sec < 0

$$\mathrm{Ric}_g(v) = \sum_{i=1}^{n-1} \sec(\{v, e_i\}) \ : \ 平均$$

ONB

"測地線のあつまり" のコントロールはできる

例

$$\mu_i := \frac{\mathbb{I}_{B_r(x_i)}}{\mathrm{Vol}(B_r(x_i))} \mathrm{Vol}_g \quad i=0,1.$$

$$\mathrm{Ent}_{\mathrm{vol}}(\mu_i) = \log\left(\frac{1}{\mathrm{Vol}(B_r(x_i))}\right)$$

μ_t ; 確率測度　support$(\mu_t) = A_t$

$(\mu_t = \rho_t \,\mathrm{vol})$

$$(1-t)\,\mathrm{Ent}(\mu_0) + t\,\mathrm{Ent}(\mu_1)$$

$$\geq \mathrm{Ent}(\mu_t) = \int_{A_t} \rho_t \ln \rho_t \, d\mathrm{vol}$$

$$= \mathrm{Vol}(A_t) \int_{A_t} \rho_t \ln \rho_t \frac{d\mathrm{vol}}{\mathrm{Vol}(A_t)}$$

$$\geq \mathrm{vol}(A_t) \left(\int_{A_t} \rho_t \frac{d\mathrm{vol}}{\mathrm{vol}(A_t)}\right) \ln\left(\int_{A_t} \rho_t \frac{d\mathrm{vol}}{\mathrm{vol}(A_t)}\right)$$

$\mu_0, \mu_1 \in P_2(M)$

$\rightsquigarrow W_2(\mu_0, \mu_1)$: L^2-Wasserstein 距離

$$W_2^2(\mu, \delta_x) = \int d^2(x,y)\, d\mu(x)$$

$\mu_0, \mu_1 \in \mathcal{D}(\mathrm{Ent})$

μ_t: L^2-Wasserstein geodesic, $\mu_0 \to \mu_1$

∂

$$|\dot{\mu}_t| = \lim_{h \downarrow 0} \frac{W_2(\mu_{t+h}, \mu_t)}{h}$$

$$\begin{cases} |\dot{\mu}_t|^2 = \int |\nabla {}^{\exists}\varphi_t|^2 \, d\mu_t \\ \partial_t \varphi_t = -\frac{1}{2} |\nabla \varphi_t|^2 \end{cases}$$

$$\Rightarrow \frac{d}{dt} \mathrm{Ent}(\mu_t) = -\int \Delta \varphi_t \, d\mu_t$$

$$\mathrm{Hess}\,\mathrm{Ent} = \frac{d^2}{dt^2} \mathrm{Ent}(\mu_t)$$

$$= \frac{d}{dt}\left(-\int \Delta \varphi_t \, d\mu_t\right)$$

$$= -\int -\frac{1}{2} \Delta |\nabla \varphi_t|^2 + \langle \nabla \Delta \varphi_t, \nabla \varphi_t \rangle \, d\mu_t$$

$$= \int \frac{1}{2} \Delta |\nabla \varphi_t|^2 - \langle \nabla \Delta \varphi_t, \nabla \varphi_t \rangle \, d\mu_t$$

$$= \int \mathrm{Ric}(\nabla \varphi_t, \nabla \varphi_t) + \|\mathrm{Hess}\,\varphi_t\|^2 \, d\mu_t$$

$$\geq \int K |\nabla \varphi_t|^2 \, d\mu_t + \int \frac{1}{n} (\Delta \varphi_t)^2 \, d\mu_t$$

$$\geq K \int |\nabla \varphi_t|^2 \, d\mu_t + \frac{1}{n} \left(\int \Delta \varphi_t \, d\mu_t\right)^2$$

$\therefore \ \mathrm{Hess}\,\mathrm{Ent} - \frac{1}{n} d\mathrm{Ent} \otimes d\mathrm{Ent} \geq K,$

$U_n := \exp\left(-\frac{1}{n}\mathrm{Ent}\right)$

$\mathrm{Hess}\, U_n \leq -\frac{K}{n} U_n$

$K=0$ のとき: $\mathrm{Hess}\, U_n \leq 0.$

$U_n(\mu_t) \geq (1-t) U_n(\mu_0) + t U_n(\mu_1)$

Def (Erbar-Kuwada- 15)

(X, d, m): metric measure sp

$\Longleftrightarrow$ (X, d): 完備 sep met. sp

m: locally finite Borel meas on X,

(X, d, m): $CD^e(0, N)$

$\Longleftrightarrow$ $\forall \mu_0, \mu_1 \in \mathcal{P}_2(X) \cap \mathcal{D}(\mathrm{Ent}_m)$

$\exists \mu_t$: 測地線

$U_N(\mu_t) \geq (1-t) U_N(\mu_0) + t U_N(\mu_1)$

$\forall t \in [0,1]$

Theorem

(M^n, g): cplt. リーマン mfd

(M, d_g, vol_g): $CD^e(0, N)$

$\Longleftrightarrow$ $\mathrm{Ric}_g \geq 0$ かつ $n \leq N$

$S: M \longrightarrow \mathbb{R}$: C^∞: $(0, N)$

$U_N(x) := \exp\left(-\frac{1}{N} S(x)\right)$

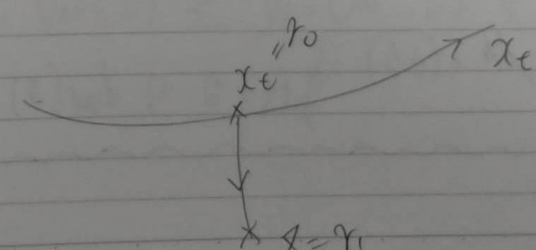
$U_N(\gamma_t) \geq (1-t) U_N(\gamma_0) + t U_N(\gamma_1)$

$$\frac{U_N(\gamma_t) - U_N(\gamma_0)}{t} \geq U_N(\gamma_1) - U_N(\gamma_0)$$

$$\left.\frac{d}{dt} U_N(\gamma_t)\right|_{t=0} \geq U_N(\gamma_1) - U_N(\gamma_0)$$

x_t: S の gradient flow

つまり $\dot{x}_t = -\nabla S(x_t)$



$$U_N(z) - U_N(x_t)$$

$$\leq \frac{d}{ds} U_N(\gamma_s)\Big|_{s=0} = \frac{-1}{N} U_N(\gamma_0) \langle \nabla S, \dot{\gamma}_0 \rangle$$

$$= \frac{1}{N} U_N(\gamma_0) \langle \dot{x}_t, \dot{\gamma}_t \rangle = -\frac{1}{2N} U_N(\underset{x_t}{\gamma_0}) \frac{d}{dt} d^2(x_t, z)$$

$$\frac{d}{dt} d^2(x_t, z) \leq 2N\left(1 - \frac{U_N(z)}{U_N(x_t)}\right) \quad \forall z$$

(x_t) : $EVI_{(0,N)}$ - flow

known

(M^n, g) : $Ric_g \geq 0$

$\Leftarrow$ Ent vol の $EVI_{(0,n)}$ - flow

$(\mu_t)_{t>0}$, $\lim_{t \downarrow 0} \mu_t = \mu_0$.

$P(x, y, t)$: 熱核

$\left(\mu_t(dx) = \int P(x, y, t)\, \mu_0(dy)\right)$

$\mu_0 = \delta_x \Rightarrow \mu_t(dy) = \int P(x, y, t)\, d\mathrm{vol}(y)$

$\left(P_{x,t} = P(x, y, t)\, dy\right)$

§3 シャノン不等式

- $\mathbb{R}^n$ のとき

$(\mathbb{R}^n, \underbrace{d_E}_{ユークリッド距離}, \mathcal{L}^n)$: $RCD(0, n)$ 空間

$$P(x, y, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{(x-y)^2}{4t}}$$

$$\mathrm{Var}(P_{x,t}) = \int |x-y|^2 P(x, y, t)\, dy$$

$f \geq 0$, L^1, $\|f\| = 1$, $|x|^2 f \in L^1$

$\nu = f\mathcal{L}^n \in P_2(\mathbb{R}^n)$

$\omega \in b(\nu) \iff \mathrm{Var}(\nu) = \int |x-\omega|^2 d\nu(x)$

$\mu_t^\omega = P(\omega, y, t)\, dy$: $EVI_{(0,n)}$ - flow

$$\frac{d}{dt} W_2^2(\mu_t^\omega, \nu) \leq 2n\left(1 - \frac{U_n(\nu)}{U_n(\mu_t^\omega)}\right)$$

$$-\mathrm{Var}(\nu) \leq \underbrace{W_2^2(\mu_t^\omega, \nu)}_{\geq 0} \leq 2n\left(t - U_n(\nu)\underbrace{\int_0^t \frac{ds}{U_n(\mu_s^\omega)}}_{=: F(t)}\right)$$

NO.

DATE

∂_x

$$U_n(\nu) \le \frac{1}{F(t)}\left(t + \frac{Var(\nu)}{2n}\right)$$

$$\sqrt{\frac{t}{\pi e}} \quad \because t = \frac{Var(\nu)}{2n}$$

$$= \sqrt{\frac{2n\pi e}{Var(\nu)} \times \frac{Var(\nu)}{n}}$$

$$= \sqrt{\frac{2\pi e}{n} Var(\nu)}$$

(λ_n つけて)

$\exp\left(-\frac{1}{n}E_n t\right)$

3.2 リーマン mfd

(M^n, g): 完備 リーマン mfd, $Ric \ge 0$

$m = vol_g$

Lemma 3.1

$x_0 \in M$, $\mu_t^{x_0} := P(x_0, y, t)\, m(dy)$

$$\Longrightarrow V_t^2 := W_2^2(\mu_t^{x_0}, \delta_{x_0}) \le 2nt$$

$\because$

$$V_t^2 = \int_0^t \frac{d}{ds} V_s^2\, ds = \int_0^t \frac{d}{ds} W_2^2(\mu_s^{x_0}, \delta_{x_0})\, ds$$

$\frac{d}{ds} W_2^2(\mu_s^{x_0}, \delta_{x_0})$

$$= \frac{d}{ds}\int d^2(x_0, y)\, P(x_0, y, s)\, dm(y)$$

$$= \int d^2(x_0, y)\, \underbrace{\partial_s P(x_0, y, s)}_{= \Delta_y P(x, y, s)}\, dm(y)$$

$$\text{"="} \int \Delta d^2_{x_0}\, P(x_0, y, s)\, dm(y)$$

$$\le 2n \int P(x_0, y, s)\, dm(y)$$

$= 2n$ //

Lemma 3.2

$x_0 \in M$, $\mu_t^{x_0} = P(x_0, y, t)\, dm(y)$

$$F(t) = \int_0^t \frac{ds}{U_n(\mu_s^{x_0})} \Longrightarrow \lim_{t \downarrow 0} \frac{F(t)}{\sqrt{t}} = (\pi e)^{-\frac{1}{2}}$$

$$\left(\frac{F(t)}{\sqrt{t}}\right)' \ge 0, \quad \therefore F(t) \ge \sqrt{\frac{t}{\pi e}}$$

Theorem I の証明

$$U_n(\nu) \leq \frac{1}{F(t)}\left(t + \frac{Var(\nu)}{2n}\right)$$

$$\leq \sqrt{\frac{2n\pi e}{Var(\nu)} \times \frac{Var(\nu)}{n}}$$

$$= \sqrt{\frac{2\pi e}{n} Var(\nu)}$$

//

§4 Rigid

$N \geq 2$

Def 4.1

(Y, d_Y, m_Y) : $RCD(N-2, N-1)$

$(N=2,\ diam Y \leq \pi)$

$C(Y) := [0, \infty) \times Y / \{0\} \times Y$

$d_C((t_1, y_1)(t_2, y_2)) := \sqrt{t_1^2 + t_2^2 - 2t_1t_2 \cos(d(y_1, y_2))}$

$dm_C(t, y) := t^{N-1} dt \otimes dm_Y(y)$

2)

Assumption

$\exists x_0 \in X,\ \exists D_0 > 0$ s.t.

$$(*)\quad U_N(\mu_t^{x_0}) = \sqrt{\frac{2D_0^2}{N} W_2^2(\mu_t^{x_0}, \delta_{x_0})}$$

$\forall t > 0.$

Theorem 2

$(*) \iff X : N$-m.m. Cone

$(C(Y), d_C, m_C)$: N-m.m. cone over Y

: $RCD(0, N)$ sp

Cor

$(*),\ x_0 \in b(\mu_t^{x_0}) \Rightarrow X = \mathbb{R}^N$

Cor

$\exists \nu \in P_2(X),\ \exists x_0 \in b(\nu)$

$\mu_t^{x_0} = P(x_0, y, t)\, dm(y)$

$\lim_{t \downarrow 0} \frac{F(t)}{\sqrt{t}} = \exists D_0^{-1} > 0$

$X = \mathbb{R}^N$ かつ $\nu = \mu_t^{x_0}$, $\exists t > 0$

$\Uparrow$

$U_N(\nu) = \sqrt{\frac{2D_0^2}{N} Var(\nu)}$