7

$g_{ij} = g(\partial_i, \partial_j) = \langle \partial_i, \partial_j \rangle$ $\to g_{ij} \in C^\infty(U)$

1

前回 $g = \sum_{i,j=1}^{m} g_{ij}\, dx^i dx^j$ $(g_{ij} = g(\partial_i, \partial_j))$ $\boxed{g(X,Y) = \langle X, Y \rangle}$ と書く $(X, Y \in \mathfrak{X}(M))$

└ 対称積 $dx^i \odot dx^j$ が省略されてる。

ユークリッド空間 $\mathbb{E}^m = (\mathbb{R}^m, g = (dx^1)^2 + \cdots + (dx^m)^2)$

↑ $g_{11} = g_{22} = \cdots = g_{mm} = 1$

$g_{ij} = 0 \quad (i \neq j)$

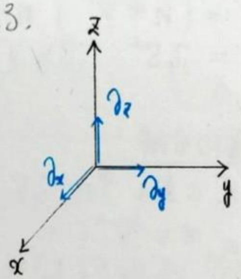
$g_{ij} = g(\partial_i, \partial_j)$

$= \delta_{ij}$: ユークリッド計量

$m=3$ $\mathbb{E}^3 = (\mathbb{R}^3, dx^2 + dy^2 + dz^2)$

↑ $\underset{g(\partial_x,\partial_x)}{g_{11}} = \underset{g(\partial_y,\partial_y)}{g_{22}} = \underset{g(\partial_z,\partial_z)}{g_{33}} = 1$

$\begin{cases} x^1 = x \\ x^2 = y \\ x^3 = z \end{cases}$



$\partial_x \leftrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix},\ \partial_y \leftrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix},\ \partial_z \leftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$ と同一視すると

$a\partial_x + b\partial_y + c\partial_z \longleftrightarrow a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix}$

$\langle a\partial_x + b\partial_y + c\partial_z, A\partial_x + B\partial_y + C\partial_z \rangle$

$= a\langle \partial_x, A\partial_x + B\partial_y + C\partial_z \rangle + b\langle \partial_y, A\partial_x + B\partial_y + C\partial_z \rangle + c\langle \partial_z, A\partial_x + B\partial_y + C\partial_z \rangle$

1 0 0 　 0 1 0 　 0 0 1

$= aA + bB + cC$

同一視

$\left\langle \begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}A\\B\\C\end{pmatrix} \right\rangle = aA + bB + cC$ (通常の) ユークリッド内積

e_1 　 e_2 　 e_3

$\partial_x \leftrightarrow \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\partial_y \leftrightarrow \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\partial_z \leftrightarrow \begin{bmatrix}0\\0\\1\end{bmatrix}$ と同一視。

$m=3$. $\mathbb{E}^3 = (\mathbb{R}^3, dx^2 + dy^2 + dz^2)$

$\hookrightarrow g(\partial_x, \partial_x) = g(\partial_y, \partial_y) = g(\partial_z, \partial_z) = 1.$

$\langle \underbrace{a\partial_x + b\partial_y + c\partial_z}_{= \begin{bmatrix}a\\b\\c\end{bmatrix}}, \underbrace{A\partial_x + B\partial_y + C\partial_z}_{= \begin{bmatrix}A\\B\\C\end{bmatrix}} \rangle$

$= aA + bB + cC$ (通常のユークリッド内積)

2

部分多様体と第1基本形式

Def $m \leq n$ とする

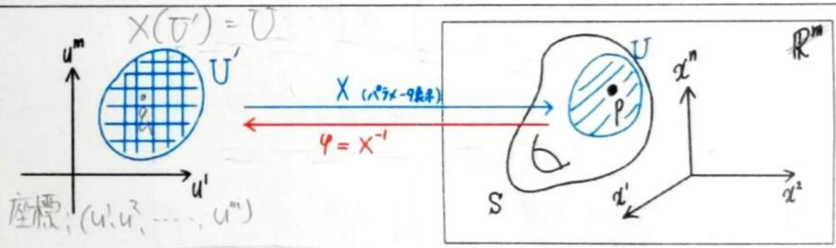
$\mathbb{R}^n$ の部分集合 $S\ (\subset \mathbb{R}^n)$ が以下の条件をみたすとき

S を m次元部分多様体 という (submanifold)

$\forall p \in S$ に対して $\begin{cases} \exists U \subset S : p \text{ の } (S \text{ における}) \text{ 開近傍} \\ \exists U' \subset \mathbb{R}^m : \text{開集合} \\ \exists X : U' \longrightarrow \mathbb{R}^n : C^\infty \text{級写像} \end{cases}$

s.t. $\begin{cases} \text{(i) } X : U' \longrightarrow U : \text{同相写像} \\ \text{(ii) } \forall q \in U' \text{ に対して } \{X_{u^1}(q), X_{u^2}(q), \cdots, X_{u^m}(q)\} \text{ は1次独立} \end{cases}$

全単射, 連続, 逆写像も連続

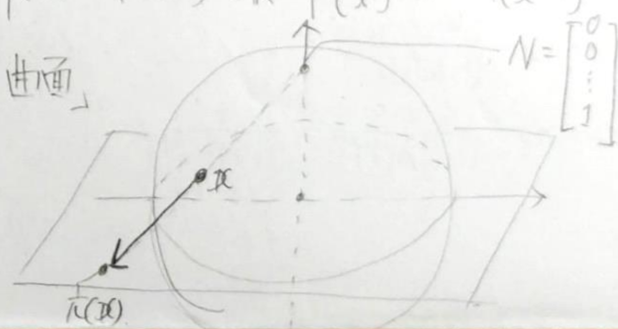


- このとき S は C^∞級多様体
- $\varphi := X^{-1} : U \longrightarrow U'$ と定めると (U, φ) は S の m次元座標近傍
- $n = m+1$ のとき $S = S^m$ を $\mathbb{R}^{m+1}$ の 超曲面 (hypersurface) という (とくに $m=2$ のときは曲面という)

Ex m次元球面

$S^m := \{(x^1, \cdots, x^{m+1}) \in \mathbb{R}^{m+1} \mid (x^1)^2 + \cdots + (x^{m+1})^2 = 1\}$

S^m は「超曲面」



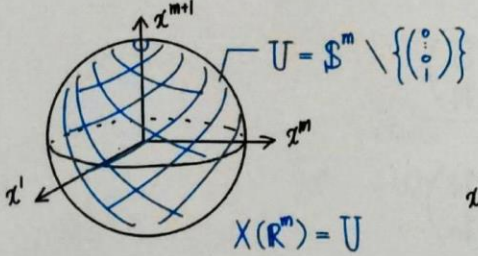
$\pi : U \to \mathbb{R}^{m+1}$, $S = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix}$

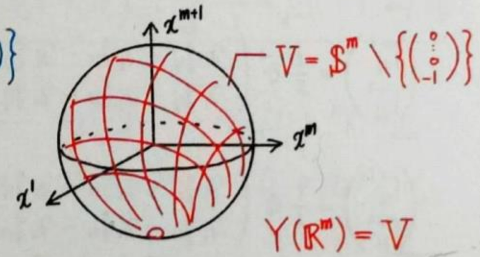
例 $\mathbb{S}^m := \{(x^1, x^2, \cdots, x^{m+1}) \in \mathbb{R}^{m+1} \mid (x^1)^2 + (x^2)^2 + \cdots + (x^{m+1})^2 = 1\}$

m次元球面 は 超曲面

$$\begin{cases} X(u^1, \cdots, u^m) = \dfrac{1}{1+(u^1)^2+\cdots+(u^m)^2} \begin{pmatrix} 2u^1 \\ \vdots \\ 2u^m \\ -1+(u^1)^2+\cdots+(u^m)^2 \end{pmatrix} \\ Y(u^1, \cdots, u^m) = \dfrac{1}{1+(u^1)^2+\cdots+(u^m)^2} \begin{pmatrix} 2u^1 \\ \vdots \\ 2u^m \\ 1-(u^1)^2-\cdots-(u^m)^2 \end{pmatrix} \end{cases}$$

$X, Y : \mathbb{R}^m \longrightarrow \mathbb{R}^{m+1}$ は $\mathbb{S}^m$ のパラメータ表示





Ex $m = 2$

$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$G^{-1} = \dfrac{1}{ac-b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}$

$\therefore g^{11} = \dfrac{c}{ac-b^2}$

$g^{12} = \dfrac{-b}{ac-b^2} = g^{21}$

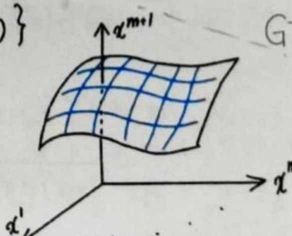
$g^{22} = \dfrac{a}{ac-b^2}$

例 $f : \mathbb{R}^m \longrightarrow \mathbb{R}$: C^∞級関数

$S := \{(x^1, x^2, \cdots, x^m, x^{m+1}) \in \mathbb{R}^{m+1} \mid x^{m+1} = f(x^1, \cdots, x^m)\}$

(fのグラフという)

Sは超曲面



$X : \mathbb{R}^m \longrightarrow \mathbb{R}^{m+1}$; $X = \begin{pmatrix} x^1 \\ \vdots \\ x^m \\ f(x^1, \cdots, x^m) \end{pmatrix}$ m+1コ

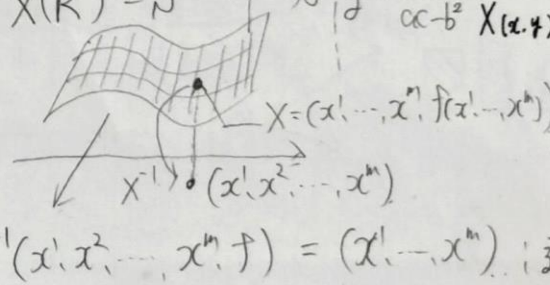
$\dfrac{\partial X}{\partial x^1}$ $\qquad$ $\dfrac{\partial X}{\partial x^m}$

$X_{x^1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ f_{x^1} \end{pmatrix}$ m+1, $\cdots$, $X_{x^m} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ f_{x^m} \end{pmatrix}$ m+1

$a^1 X_{x^1} + \cdots + a^m X_{x^m} = 0 \Longrightarrow a^1 = \cdots = a^m = 0$

$\therefore X_{x^1}, \cdots, X_{x^m}$ は 1次独立

$\therefore \begin{bmatrix} a^1 \\ a^2 \\ \vdots \\ a^m \\ a^1 f_{x^1} + \cdots + a^m f_{x^m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$



$X^{-1}(x^1, x^2, \cdots, x^m, f) = (x^1, \cdots, x^m)$: 連続

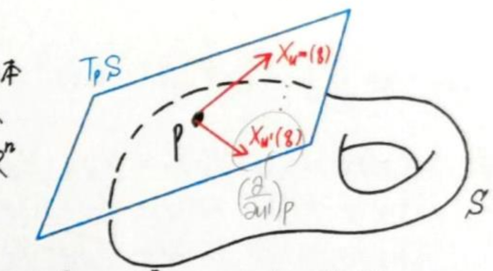
S : $\mathbb{R}^n$ の m次元部分多様体

$p \in S$ において パラメータ表示

$X : U' \longrightarrow U \subset S \subset \mathbb{R}^n$

$U' \subset \mathbb{R}^m$, $p \in U$

$P = X(q)$



$X(q) = p$ となる点 $q \in U'$

$T_pS = \mathrm{Span}\{X_{u^1}(q), \cdots, X_{u^m}(q)\}$ は $\mathbb{R}^n$ の m次元部分空間

← S の接空間 $\quad$ ↳ m次元部分空間の定義 $\quad$ ※ここでは $\left(\dfrac{\partial X}{\partial u^i}\right)_q \longleftrightarrow \left(\dfrac{\partial}{\partial u^i}\right)_p$ 同一視

$\mathbb{R}^n$ の内積を T_pS に制限

$\forall v, w \in T_pS$ に対し、 $g_p(v, w) := v \cdot w$ ← $\mathbb{R}^n$ のベクトルとして内積

このとき g は S上のリーマン計量

第一基本形式, 誘導計量 という

(first fundamental form) (induced metric)

u^m, u^1, X, U

$X_{u^i} = \dfrac{\partial X}{\partial u^i}$

座標表示

$$\boxed{U\text{上で} \quad g = \sum_{i,j=1}^m g_{ij}\, du^i du^j, \quad g_{ij} = X_{u^i} \cdot X_{u^j}}$$

$g(v, w) = v \cdot w$, X_{u^i} X_{u^j} $\therefore g_{ij} = X_{u^i} \cdot X_{u^j}$

$$G := \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1m} \\ g_{21} & g_{22} & \cdots & g_{2m} \\ & & \vdots & \\ g_{m1} & g_{m2} & \cdots & g_{mm} \end{pmatrix} = (g_{ij})_{i,j=1,\cdots,m}, \quad G^{-1} = \begin{pmatrix} g^{11} & g^{12} & \cdots & g^{1m} \\ g^{21} & g^{22} & \cdots & g^{2m} \\ & & \vdots & \\ g^{m1} & g^{m2} & \cdots & g^{mm} \end{pmatrix} = (g^{ab})_{a,b=1,\cdots,m}$$

$g_{ij} = X_{u^i} \cdot X_{u^j} = X_{u^j} \cdot X_{u^i}$

例 $\mathbb{R}^2$で定義された関数 $f(x, y)$ のグラフ

$S = \{(x, y, z) \mid z = f(x, y)\}$

$X(x, y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$, $X_x = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix}$, $X_y = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}$ $\therefore \begin{cases} g_{11} = X_x \cdot X_x = 1 + f_x^2 \\ g_{12} = X_x \cdot X_y = f_x f_y \\ g_{22} = X_y \cdot X_y = 1 + f_y^2 \end{cases}$

$$\begin{aligned} g &= g_{11}\, dx^2 + \underline{g_{12}\, dx\, dy} + \underline{g_{21}\, dy\, dx} + g_{22}\, dy^2 \\ &= (1 + f_x^2)\, dx^2 + \underline{2 f_x f_y\, dx\, dy} + (1 + f_y^2)\, dy^2 \end{aligned}$$

$G = \begin{bmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{bmatrix}$ が正定値 $\Longleftrightarrow$ $\det(G) > 0$, $\mathrm{tr}(G) > 0$

例 $\mathbb{S}^2 = \{ (x^1, x^2, x^3) \in \mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \}$

2次元球面 $\mathbb{S}^2$ のパラメータ表示 $X : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$X(u^1, u^2) = \frac{1}{1+(u^1)^2+(u^2)^2}\begin{pmatrix} 2u^1 \\ 2u^2 \\ -1+(u^1)^2+(u^2)^2 \end{pmatrix}$$

$$X_{u^1} = \left(\frac{1}{1+(u^1)^2+(u^2)^2}\right)_{u_1}\begin{pmatrix} 2u^1 \\ 2u^2 \\ -1+(u^1)^2+(u^2)^2 \end{pmatrix} + \frac{1}{1+(u^1)^2+(u^2)^2}\begin{pmatrix} 2u^1 \\ 2u^2 \\ -1+(u^1)^2+(u^2)^2 \end{pmatrix}_{u_1}$$

$$= \frac{-2u^1}{(1+(u^1)^2+(u^2)^2)^2}\begin{pmatrix} 2u^1 \\ 2u^2 \\ -1+(u^1)^2+(u^2)^2 \end{pmatrix} + \frac{1}{1+(u^1)^2+(u^2)^2}\begin{pmatrix} 2 \\ 0 \\ 2u^1 \end{pmatrix}$$

$$= \frac{2}{(1+(u^1)^2+(u^2)^2)^2}\underbrace{\begin{pmatrix} 1-(u^1)^2+(u^2)^2 \\ -2u^1u^2 \\ 2u^1 \end{pmatrix}}_{=\varphi_1 \text{とおく}}$$

$$X_{u^2} = \frac{2}{(1+(u^1)^2+(u^2)^2)^2}\underbrace{\begin{pmatrix} -2u^1u^2 \\ 1+(u^1)^2-(u^2)^2 \\ 2u^2 \end{pmatrix}}_{=\varphi_2 \text{とおく}}$$

$$\varphi_1 \cdot \varphi_1 = (1-(u^1)^2+(u^2)^2)^2 + (-2u^1u^2)^2 + (2u^1)^2 = (1+(u^1)^2+(u^2)^2)^2$$

$$\varphi_1 \cdot \varphi_2 = (1-(u^1)^2+(u^2)^2)(-2u^1u^2) - (2u^1u^2)(1+(u^1)^2-(u^2)^2) + 2u^1 \cdot 2u^2 = 0$$

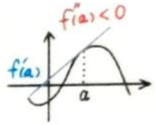
$$\varphi_2 \cdot \varphi_2 = (-2u^1u^2)^2 + (1+(u^1)^2-(u^2)^2)^2 + (2u^2)^2 = (1+(u^1)^2+(u^2)^2)^2$$

$$\therefore \begin{cases} g_{11} = X_{u^1} \cdot X_{u^1} = \dfrac{4\,\varphi_1\cdot\varphi_1}{(1+(u^1)^2+(u^2)^2)^4} = \dfrac{4}{(1+(u^1)^2+(u^2)^2)^2} \\ g_{12} = X_{u^1} \cdot X_{u^2} = \dfrac{4\,\varphi_1\cdot\varphi_2}{(1+(u^1)^2+(u^2)^2)^4} = 0 \\ g_{22} = X_{u^2} \cdot X_{u^2} = \dfrac{4\,\varphi_2\cdot\varphi_2}{(1+(u^1)^2+(u^2)^2)^4} = \dfrac{4}{(1+(u^1)^2+(u^2)^2)^2} \end{cases}$$

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \frac{4}{(1+(u^1)^2+(u^2)^2)^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

atled

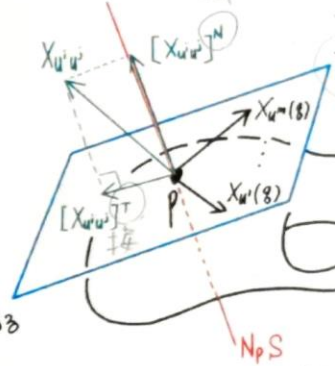
2階微分 … 曲がり具合



$N_pS := \{ n \in \mathbb{R}^n \mid n \cdot v = 0,\ {}^\forall v \in T_pS \}$

$= T_pS^\perp$: S の p における法空間

normal space

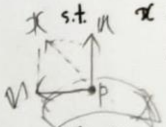


(n−m)次元

曲面の場合 m=2 n=3

☆ 直交直和分解

${}^\forall x \in \mathbb{R}^n$ に対し ${}^\exists v \in T_pS$, ${}^\exists n \in N_pS$



s.t. $x = v + n = [x]^T + [x]^N$ $\left\{ \begin{array}{l} v = [x]^T \\ n = [x]^N \end{array} \right.$ と定める

2階微分 $X_{u^iu^j} = [X_{u^iu^j}]^T + [X_{u^iu^j}]^N$

Def $\nabla_{X_{u^j}} X_{u^i} := [X_{u^iu^j}]^T$: X_{u^i} の X_{u^j} による共変微分という

ナブラ nabla

第1基本形式 g に対する レビチビタ接続

Rem $X_{u^iu^j} = X_{u^ju^i}$ より $\nabla_{X_{u^j}} X_{u^i} = \nabla_{X_{u^i}} X_{u^j}$

一般に $\nabla_V W \neq \nabla_W V$

$\nabla_{X_{u^j}} X_{u^i}(p) \in T_pS = \mathrm{Span}\{X_{u^1}(p), \cdots, X_{u^m}(p)\}$ より

$\nabla_{X_{u^j}} X_{u^i} = \Gamma_{ij}^1 X_{u^1} + \cdots + \Gamma_{ij}^m X_{u^m}$ —— (ij) と表すことができる

スカラー

$g = \sum g_{ij}\, du^i du^j$

$\nabla_V W - \nabla_W V = [V, W]$ Lie 括弧

$\partial_i = \dfrac{\partial}{\partial u^i}$

$$\boxed{\text{命題1}\quad \Gamma_{ij}^k = \frac{1}{2}\sum_{a=1}^m g^{ak}(\partial_i g_{aj} + \partial_j g_{ia} - \partial_a g_{ij})}$$

proof $X_{u^1}, \cdots, X_{u^m}$ と (ij) の両辺に内積

$X_{u^iu^j} \cdot X_{u^k} = (\nabla_{X_{u^j}} X_{u^i}) \cdot X_{u^k} + [X_{u^iu^j}]^N \cdot X_{u^k}$

$\therefore X_{u^iu^j} \cdot X_{u^k} = (\nabla_{X_{u^j}} X_{u^i}) \cdot X_{u^k}$ —— ⊛

(ij) $\cdot X_{u^k}$ より $(\nabla_{X_{u^j}} X_{u^i}) \cdot X_{u^k} = \Gamma_{ij}^1 \underbrace{X_{u^1} \cdot X_{u^k}}_{g_{1k}} + \cdots + \Gamma_{ij}^m \underbrace{X_{u^m} \cdot X_{u^k}}_{g_{mk}}$

⊛↳ $X_{u^iu^j} \cdot X_{u^k} = \Gamma_{ij}^1 g_{1k} + \cdots + \Gamma_{ij}^m g_{mk}$ —— ☆

$$\boxed{\text{Lem}\quad X_{u^iu^j} \cdot X_{u^k} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})}$$

Lem $X_{u^iu^j}\cdot X_{u^k} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)$

proof

$$
\begin{aligned}
(g_{ik})_{u^j} &= (X_{u^i}\cdot X_{u^k})_{u^j} = X_{u^iu^j}\cdot X_{u^k} + X_{u^i}\cdot X_{u^ju^k}\\
+(g_{jk})_{u^i} &= (X_{u^j}\cdot X_{u^k})_{u^i} = X_{u^iu^j}\cdot X_{u^k} + X_{u^j}\cdot X_{u^iu^k}\\
-(g_{ij})_{u^k} &= (X_{u^i}\cdot X_{u^j})_{u^k} = X_{u^iu^k}\cdot X_{u^j} + X_{u^i}\cdot X_{u^ju^k}\\
\hline
(g_{ik})_{u^j} + (g_{jk})_{u^i} - (g_{ij})_{u^k} &= 2X_{u^iu^j}\cdot X_{u^k} \quad /\!/
\end{aligned}
$$

☆ と Lem より

$$\boxed{g_{1k}\Gamma^1_{ij} + g_{2k}\Gamma^2_{ij} + \cdots + g_{mk}\Gamma^m_{ij} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)} \quad ijk$$

$(i, j, k = 1, \cdots, m)$

m

$$
\begin{cases}
k=1: & g_{11}\Gamma^1_{ij} + g_{21}\Gamma^2_{ij} + \cdots + g_{m1}\Gamma^m_{ij} = \frac{1}{2}\left(\partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij}\right) \quad \text{—— } ij1\\
k=2: & g_{12}\Gamma^1_{ij} + g_{22}\Gamma^2_{ij} + \cdots + g_{m2}\Gamma^m_{ij} = \frac{1}{2}\left(\partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij}\right) \quad \text{—— } ij2\\
\vdots & \vdots\\
k=m: & g_{1m}\Gamma^1_{ij} + g_{2m}\Gamma^2_{ij} + \cdots + g_{mm}\Gamma^m_{ij} = \frac{1}{2}\left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right) \quad \text{—— } ijm
\end{cases}
$$

$$
\underbrace{\begin{pmatrix} g_{11} & g_{21} & & g_{m1}\\ g_{12} & g_{22} & & g_{m2}\\ \vdots & \vdots & \cdots & \vdots\\ g_{1m} & g_{2m} & & g_{mm} \end{pmatrix}}_{\substack{/\!/\\ G}}
\begin{pmatrix} \Gamma^1_{ij}\\ \Gamma^2_{ij}\\ \vdots\\ \Gamma^m_{ij} \end{pmatrix}
= \frac{1}{2}
\begin{pmatrix} \partial_i g_{1j} + \partial_j g_{i1} - \partial_1 g_{ij}\\ \partial_i g_{2j} + \partial_j g_{i2} - \partial_2 g_{ij}\\ \vdots\\ \partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij} \end{pmatrix}
$$

リーマン計量なので逆行列G^{-1}が存在

$$
\begin{pmatrix} \Gamma^1_{ij}\\ \Gamma^2_{ij}\\ \vdots\\ \Gamma^m_{ij} \end{pmatrix}
= \frac{1}{2}G^{-1}
\begin{pmatrix} \partial_i g_{1j} + \partial_j g_{i1} - \partial_1 g_{ij}\\ \partial_i g_{2j} + \partial_j g_{i2} - \partial_2 g_{ij}\\ \vdots\\ \partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij} \end{pmatrix}
$$

$G^{-1} = (g^{ab})_{a,b=1,\cdots,m}$ より $\boxed{\Gamma^k_{ij} = \frac{1}{2}\sum_{a=1}^{m} g^{ak}\left(\partial_i g_{aj} + \partial_j g_{ia} - \partial_a g_{ij}\right)}$ ▨

Γ^k_{ij} を Christoffel（クリストッフェル）記号 (Christoffel's symbol) という

例 $S^2 = \{(x^1, x^2, x^3)\in\mathbb{R}^3 \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$

2次元球面 S^2 のパラメータ表示 $X : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$X(u^1, u^2) = \frac{1}{1+(u^1)^2+(u^2)^2}\begin{pmatrix} 2u^1\\ 2u^2\\ -1+(u^1)^2+(u^2)^2 \end{pmatrix}$$

$$G = \begin{pmatrix} g_{11} & g_{12}\\ g_{12} & g_{22} \end{pmatrix} = \frac{4}{\left(1+(u^1)^2+(u^2)^2\right)^2}\begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix}$$

$$G^{-1} = \frac{\left(1+(u^1)^2+(u^2)^2\right)^2}{4}\begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g^{11} & g^{12}\\ g^{12} & g^{22} \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^1_{ij}\\ \Gamma^2_{ij} \end{pmatrix} = \frac{1}{2}G^{-1}\begin{pmatrix} \partial_i g_{1j} + \partial_j g_{i1} - \partial_1 g_{ij}\\ \partial_i g_{2j} + \partial_j g_{i2} - \partial_2 g_{ij} \end{pmatrix}$$

$$
\begin{aligned}
\begin{pmatrix} \Gamma^1_{11}\\ \Gamma^2_{11} \end{pmatrix} &= \frac{1}{2}G^{-1}\begin{pmatrix} \partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\\ \partial_1 g_{21} + \partial_1 g_{12} - \partial_2 g_{11} \end{pmatrix}\\
&= \frac{1}{2}\frac{\left(1+(u^1)^2+(u^2)^2\right)^2}{4}\begin{pmatrix} \partial_1 g_{11}\\ -\partial_2 g_{11} \end{pmatrix}\\
&= \frac{2}{1+(u^1)^2+(u^2)^2}\begin{pmatrix} -u^1\\ u^2 \end{pmatrix}
\end{aligned}
$$

$$\partial_1 g_{11} = \frac{\partial}{\partial u^1}\frac{4}{\left(1+(u^1)^2+(u^2)^2\right)^2} = \frac{-16u^1}{\left(1+(u^1)^2+(u^2)^2\right)^3}$$

$$\partial\, g_{11} = \frac{\partial}{\partial u^2}\frac{4}{\left(1+(u^1)^2+(u^2)^2\right)^2} = \frac{-16u^2}{\left(1+(u^1)^2+(u^2)^2\right)^3}$$

$$
\begin{aligned}
\therefore\ \nabla_{X_{u^1}} X_{u^1} &= \Gamma^1_{11}X_{u^1} + \Gamma^2_{11}X_{u^2}\\
&= \frac{2}{1+(u^1)^2+(u^2)^2}\left(-u^1X_{u^1} + u^2X_{u^2}\right)
\end{aligned}
$$

$\boxed{\nabla_{X_{u^j}} X_{u^i} = \Gamma^1_{ij}X_{u^1} + \cdots + \Gamma^m_{ij}X_{u^m} \text{ と表される}}$