10/23 (水), (da Silva?) arXiv: 2403.10716

- Ruled Surface
- Curves and Surfaces of Constant angle.

↳ 曲面の特ちょうはどれだけある?

$\boxed{M^3 \text{の性質による}}$

Ex) $M = \mathbb{E}^3 \Rightarrow$ ~~sphere~~ 球体は totally umbilical

$\mathbb{S}^3$

$\mathbb{H}^3$

Main Theorem 1979, 81, 2020.

全ての球面 $\subseteq M^3$ $(r \ll 1)$ が totally umbilical である

$\Rightarrow$ M は定曲率をもつ

Ex.2 In $\mathbb{E}^3$, $\begin{pmatrix} \mathbb{S}^3 \\ \mathbb{H}^3 \end{pmatrix}$ $\overset{\text{カッパ}}{K} = 0 \Rightarrow$ ruled (surface)

(外在的)

Definition

Theorem in M^3

(I) $\exists$ external flat surface tangent to any given $\Pi \subset T_p(M)$

(II) $\overset{\text{カッパ}}{K}_{ext} = 0 \Rightarrow$ ruled (~~then~~, M^3 has 定曲率)

$\Longleftrightarrow$

proof $M^3(c)$ を考える.

形作用素 $\begin{bmatrix} K_1 & 0 \\ 0 & 0 \end{bmatrix}$

$\cdot K_1 = 0 \Rightarrow \Sigma^2$ 全測地的

$\therefore \Sigma^2$ is ruled.

$\cdot K_1 \neq 0 \Rightarrow$ 曲率テンソルが以下を満たす.

$$R_{uvvu} = -h_{uu}\Gamma^u_{vv}$$

☆何？

以上より, $\nabla_{\partial_v}\partial_v = \Gamma^u_{vv} + \cancel{\Gamma^v_{vv}}\partial_v + \cancel{h_{vv}}\nu$

out it　　　0.

vが弧長.

$\therefore$ v曲線は測地線である $\Longleftrightarrow R_{uvvu} = 0$

(定曲率上では, three distinct directions は $R_{ijkl} = 0$ を表す),

系: "$K_{ext} = 0 \Rightarrow$ ruled"

$$\Longleftrightarrow R_{uvvu}\Big|_{\Sigma^2} = 0$$

$(\Rightarrow)$. $R_p(\partial_v, \partial_u)\partial_v = \nabla_{\partial_u}(\nabla_{\partial_v}\partial_v) - \nabla_{\partial_v}\nabla_{\partial_u}\partial_v + \nabla_{\underbrace{[\partial_u, \partial_v]}_{=0}}\partial_v$

なに?

☆ $\nabla_{\partial_u}\partial_v = A\partial_u + B\nu$

$$B = \langle \nu, \nabla_{\partial_u}\partial_v \rangle = \langle \nu, \nabla_{\partial_v}\partial_u \rangle$$

$$= -\langle \nabla_{\partial_v}\nu, \partial_u \rangle = 0.$$

∂_r

$$\therefore R_p(\partial_r, \partial_u)\partial_r = -\nabla_{\partial_r}(A\partial_u)$$

$$= -\left(\frac{\partial A}{\partial r} + A^2\right)\partial_1$$

∴ Sectional curvature

$$K(\partial_u, \partial_r) = -\left(\frac{\partial A}{\partial r} + A^2\right)\Bigg|_p$$

∴ M^3 は isotropic(同型)、つまり

$$K(\Pi_p) = K(p)$$

$\Longrightarrow$ M^3 は space form ▨.

§ Const Angle (キリング場) $S^2 \times \mathbb{R}$

M が unisotropic のとき、固定した方向($M^2 \times \mathbb{R}$)を定ぎできる

(M が特別な場(killing field)の場合).

・V が N における全ての曲線の平行移動となる場合,

$\Rightarrow$ V は $N \subset M$ に対して fixed direction である

Theorem

一般化されたhelix $\mathbb{E}^3$: $\langle T, V \rangle = \cos\theta$ (const),

$\Longleftrightarrow$ $\tau / \kappa = \cot\theta$ (const)

$$V = \cos\theta\, T + \sin\theta\, B$$

任意の

1931年、 M^n に 拡張

cte ? const ?

constant

Proof $\langle T, V\rangle = cte$

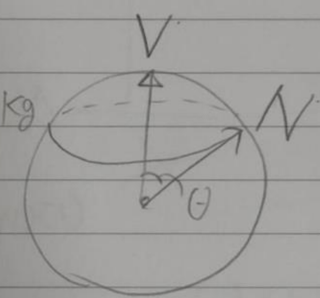
$$\langle D_T T, V\rangle + \underbrace{\langle T, D_T V\rangle}_{=0} = 0$$

$$\kappa\langle N, V\rangle = 0$$

$$\therefore \langle N, V\rangle = 0$$

Theorem

Slant helix : $\langle N, V\rangle = \cos\theta$



$$\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'$$

$= cte$ (const ?)

$$V = \cos\theta\, N + \sin\theta\, D,$$

$$D = \frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}}$$ (true on any M^3)

$\Sigma^2 \subset M^3$ is constant angle $\Longleftarrow$ ノルムν が const angle.

Theorem

Σ^2 が一定 angle $\Longrightarrow K_{ext} = 0$ & ruled

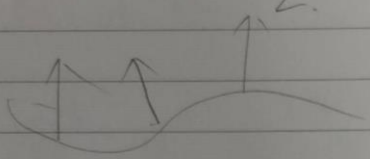
$(R_{term} = 0)$

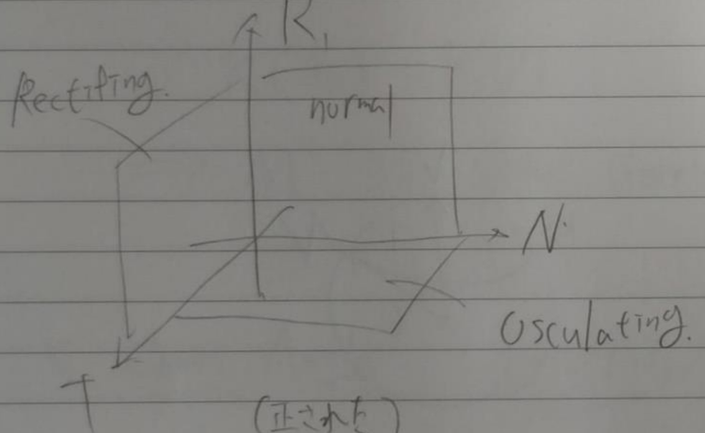
slant helices

Theorem

$\mathbb{E}^3$, Σ^2, constant angle を持つ

$\Longleftrightarrow$ 測地線が slant helices である.

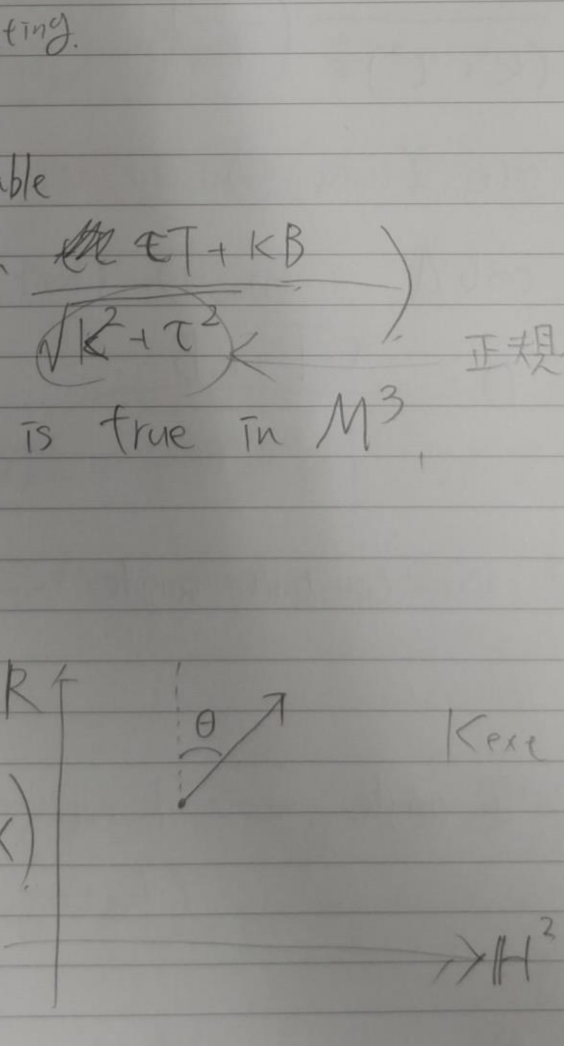




(正された)
Rectifying developable

$$\exp^{M^3}_{\alpha(s)}\left(v\,\frac{\tau T+\kappa B}{\sqrt{\kappa^2+\tau^2}}\right)$$ 正規化

Theorem If $\angle XO-Y$ is true in M^3, M^3 is flat.



$(\nabla_X \partial_t = 0,\ {}^\forall X)$

$\kappa_{ext} = 0$

$$(X-P)\cdot N = 0$$

$$T\cdot N + \underbrace{(X-P)\cdot(-\kappa T + \tau B)}_{\frac{\tau T + \kappa B}{\sqrt{\tau^2+\kappa^2}}} = 0$$

<u>Lemma</u> Σ^2が性質(i)(ii)を持ってるときΣ^2は rectifying developable である.

(i) $\gamma \subseteq \Sigma^2$が測地線である,

(ii) $K_{ext}|_{\gamma} = 0$