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Weingarten rotation surfaces in 3-dimensional de Sitter space

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Abstract. In the 3-dimensional de Sitter Space $\mathbf{S}_1^3$, a surface is said to be a spherical (resp. hyperbolic or parabolic) rotation surface, if it is a orbit of a regular curve under the action of the orthogonal transformations of the 4-dimensional Minkowski space $\mathbf{E}_1^4$ which leave a timelike (resp. spacelike or degenerate) plane pointwise fixed. In this paper, we give all spacelike and timelike Weingarten rotation surfaces in $\mathbf{S}_1^3$.

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Key words: Weingarten surface, de Sitter space, rotation surface, principal curvature.

0. Introduction

In differential geometry, for the study of the hypersurfaces theory in space forms, it is a very important and interesting problem to construct or classify the constant mean curvature hypersurfaces. Furthermore, let $\mathbf{M}$ be a connected, oriented C^∞-manifold of dimension $n \geq 2$ and $x : \mathbf{M} \to \overline{\mathbf{M}}(\bar{\kappa})$ a hypersurface immersion into a space form of constant curvature $\bar{\kappa}$. Denote by $H_1, \ldots, H_n$ the normed elementary symmetric functions of the principal curvatures $\kappa_1, \ldots, \kappa_n$ of x. It is a well known extension of the theorem egregium that the curvature functions H_r for $r \geq 2$ even and $(H_s)^2$ for $s \geq 3$ odd are "intrinsic" in the sense that they can be expressed in terms of $\bar{\kappa}$ and the geometry of the first fundamental form of x, while the mean curvature H_1 is extrinsic ([1]). Then the investigation of the hypersurfaces in space forms with some constant curvature functions is also meaningful. A hypersurface x is called the Weingarten hypersurface, if there is a principal curvature κ_i of x such that $\kappa_i = f(\kappa_1, \kappa_2, \ldots, \kappa_{i-1}, \kappa_{i+1}, \ldots, \kappa_n)$ for some function f. So, the constant mean curvature surfaces and constant curvature functions hypersurfaces are Weingarten hypersurface.

Let $\mathbf{S}_1^3$ be the 3-dimensional de Sitter Space and $\mathbf{E}_1^4$ the 4-dimensional Minkowski space. A surface in $\mathbf{S}_1^3$ is called to be the spherical rotation surface, if it is a orbit of a regular curve under the action of the orthogonal transformations on $\mathbf{E}_1^4$ which leave a timelike plane pointwise fixed; a surface in $\mathbf{S}_1^3$ is called to be the hyperbolic rotation surface, if it is a orbit of a regular curve under the action of the orthogonal transformations on $\mathbf{E}_1^4$ which leave a

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spacelike plane pointwise fixed; a surface in $\mathbf{S}_1^3$ is called to be the parabolic rotation surface, if it is a orbit of a regular curve under the action of the orthogonal transformations on $\mathbf{E}_1^4$ which leave a degenerate plane pointwise fixed. In this paper we will prove the following theorem.

THEOREM. *Let* $\mathbf{M}$ *be a Weingarten spherical, hyperbolic or parabolic rotation surface in the 3-dimensional de Sitter space* $\mathbf{S}_1^3$, *the principal curvature of* κ_1 *and* κ_2 *of* $\mathbf{M}$ *satisfy* $\kappa_2 = f(\kappa_1)$, *then it can be written as the one (or a part) of the following surfaces:*

(i) *the spherical rotation surface*

$$\begin{cases} r(u,v) = (y(u)\sin(v), y(u)\cos(v), z(u), w(u)), \quad u \in J, v \in [0, 2\pi] \\ z(u) = (y(u)^2 - 1)^{\frac{1}{2}} \sinh\varphi(u) \\ w(u) = (y(u)^2 - 1)^{\frac{1}{2}} \cosh\varphi(u) \\ \varphi(u) = \int_0^u \frac{(\varepsilon y(t)^2 + y'(t)^2 - \varepsilon)^{\frac{1}{2}}}{(y(t)^2 - 1)} dt \\ y(u) = \exp(\int_0^u \frac{dt}{\varepsilon f(t) - t}), \quad |y(u)| > 1 \end{cases} \tag{1.1}$$

or

$$\begin{cases} r(u,v) = (y(u)\sin(v), y(u)\cos(v), z(u), w(u)), \quad u \in J, v \in [0, 2\pi] \\ z(u) = (1 - y(u)^2)^{\frac{1}{2}} \cosh\varphi(u) \\ w(u) = (1 - y(u)^2)^{\frac{1}{2}} \sinh\varphi(u) \\ \varphi(u) = \int_0^u \frac{(\varepsilon y(t)^2 + y'(t)^2 - \varepsilon)^{\frac{1}{2}}}{(y(t)^2 - 1)} dt \\ y(u) = \exp(\int_0^u \frac{dt}{\varepsilon f(t) - t}), \quad |y(u)| < 1, \end{cases} \tag{1.2}$$

where $\varepsilon = \pm 1$; *when* $\varepsilon = 1$ *the surface is spacelike, when* $\varepsilon = -1$ *the surface is timelike;*

(ii) *hyperbolic rotation surface*

$$\begin{cases} r(u,v) = (x(u), y(u), w(u)\sinh(v), w(u)\cosh(v)), \quad u \in J, v \in \mathbf{R} \\ x(u) = (w(u)^2 + 1)^{\frac{1}{2}} \cos\varphi(u) \\ y(u) = (w(u)^2 + 1)^{\frac{1}{2}} \sin\varphi(u) \\ \varphi(u) = \int_0^u \frac{(\varepsilon w(t)^2 + w'(t)^2 + \varepsilon)^{\frac{1}{2}}}{(w(t)^2 + 1)} dt \\ w(u) = \exp(\int_0^u \frac{dt}{\varepsilon f(t) - t}), \end{cases} \tag{1.3}$$

when $\varepsilon = 1$ *the surface is spacelike, when* $\varepsilon = -1$ *the surface is timelike;*

or

$$\begin{cases} r(u,v) = (x(u), y(u), w(u)\cosh(v), w(u)\sinh(v)), & u \in J, v \in \mathbf{R} \\ x(u) = (1 - w(u)^2)^{\frac{1}{2}} \cos\varphi(u) \\ y(u) = (1 - w(u)^2)^{\frac{1}{2}} \sin\varphi(u) \\ \varphi(u) = \int_0^u \frac{(1-w(t)^2-w'(t)^2)^{\frac{1}{2}}}{(1-w(t)^2)} dt \\ w(u) = \exp(\int_0^u \frac{dt}{-f(t)-t}), & |w(u)| < 1, \end{cases} \tag{1.4}$$

the surface is timelike;

(iii) *parabolic rotation surface*

$$\begin{cases} r(u,v) = (x(u), vz(u), z(u), -\frac{-1+v^2z(u)^2+x(u)^2}{2z(u)}), & u \in J, v \in \mathbf{R} \\ x(u) = \pm z(u) \int \frac{\sqrt{|\varepsilon z(u)^2 + z'(u)^2|}}{z(u)^2} du \\ z(u) = \exp(\int_0^u \frac{dt}{\varepsilon f(t)-t}), \end{cases} \tag{1.5}$$

when $\varepsilon = 1$ *the surface is spacelike, when* $\varepsilon = -1$ *the surface is timelike.*

2. Preliminaries

Let $\mathbf{E}_1^{n+1}$ be the $(n+1)$-dimensional Minkowski space with the natural basis $e_1, \ldots, e_{n+1}$, its metric $\langle\ ,\ \rangle$ is given by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}, \quad x, y \in \mathbf{E}_1^{n+1},$$

where $x = (x_1, x_2, \ldots, x_{n+1})$, $y = (y_1, y_2, \ldots, y_{n+1})$. The n-dimensional de Sitter space $\mathbf{S}_1^n$ is defined by

$$\mathbf{S}_1^n = (x \in \mathbf{E}_1^{n+1} : \langle x, x \rangle = 1).$$

It is well known that $\mathbf{S}_1^n$ is the complete simply connected Pseudo-Riemannian hypersurface with constant sectional curvature 1 in $\mathbf{E}_1^{n+1}$ ([4], [5], [6]).

Let **N** be a pseudo-Riemannian manifold with the pseudo-Riemannian metric $\bar{G}$ and **M** be a submanifold of **N**. If the pseudo-Riemannian metric $\bar{G}$ of N induces a pseudo-Riemannian metric G (resp., a Riemannian metric, a degenerate quadric form) on **M**, then **M** is called a timelike (resp., spacelike, degenerate) submanifold.

Let $\mathbf{P}^k (k = 2, 3)$ denote an k-dimensional subspace of $\mathbf{E}_1^4$ passing through the origin and $\mathbf{O}(\mathbf{P}^2)$ the group of orthogonal transformations of $\mathbf{E}_1^4$ with positive determinant that leave $\mathbf{P}^2$ pointwise fixed.

DEFINITION (cf. [2]). Choose $\mathbf{P}^2$ and $\mathbf{P}^3$ such that $\mathbf{P}^2 \subset \mathbf{P}^3$ and $\mathbf{P}^3 \cap \mathbf{S}_1^3 \neq \emptyset$. Let C be a regular C^2-curve in $\mathbf{P}^3 \cap \mathbf{S}_1^3$ that does not meet $\mathbf{P}^2$. The orbit of C under the action of

$\mathbf{O}(\mathbf{P}^2)$ is called a rotation surface $\mathbf{M}$ in $\mathbf{S}_1^3$ generated by C around $\mathbf{P}^2$ if the induced metric G of $\mathbf{M}$ from $\mathbf{E}_1^4$ is nondegenerate. The surface $\mathbf{M}$ is said to be spherical (resp., hyperbolic or parabolic) if the restriction $\bar{G}/\mathbf{P}^2$ (where $\bar{G}$ is the metric of $\mathbf{E}_1^4$) is a pseudo-Riemannian metric (resp., a Riemannian metric or a degenerate quadric form).

We choose the basis $\{e_k\}$ of $\mathbf{E}_1^4$, $\mathbf{P}^2$ and $\mathbf{P}^3$, satisfying certain conditions for the spherical, hyperbolic or parabolic rotation surface (cf. [2], [4]). Let $C(u)$, $u \in J$, be an equation of the curve C in $\mathbf{P}^3 \cap \mathbf{S}_1^3$ which is parametrized by arc length and whose domain of definition J is an open interval of the set $\mathbf{R}$ of real numbers. Put $\varepsilon = \bar{G}(C'(u), C'(u)) = \pm 1$. The spherical rotation surface $\mathbf{M}_1$ is defined by

$$r(u, v) = y(u)\sin(v)e_1 + y(u)\cos(v)e_2 + z(u)e_3 + w(u)e_4, \tag{2.1}$$

where $u \in J$, $v \in [0, 2\pi]$. The induced metric G on $\mathbf{M}_1$ is given by

$$G = \varepsilon \mathrm{d}u^2 + y(u)^2 \mathrm{d}v^2, \quad \varepsilon = \pm 1. \tag{2.2}$$

When $\varepsilon = 1$, the surface is spacelike; when $\varepsilon = -1$, the surface is timelike. The functions $y(u)$, $z(u)$ and $w(u)$ satisfy

$$y(u)^2 + z(u)^2 - w(u)^2 = 1, \quad y'(u)^2 + z'(u)^2 - w'(u)^2 = \varepsilon. \tag{2.3}$$

If $\mathbf{M}_1$ is not contained in a hyperplane of $\mathbf{E}_1^4$ (i.e., $\mathbf{M}_1$ is proper), then $z'(u) \not\equiv 0$ and $w'(u) \not\equiv 0$.

The hyperbolic rotation surface $\mathbf{M}_2$ of the first kind is defined by

$$r(u, v) = x(u)e_1 + y(u)e_2 + w(u)\sinh(v)e_3 + w(u)\cosh(v)e_4, \tag{2.4}$$

where $u \in J$, $v \in \mathbf{R}$. The induced metric G on $\mathbf{M}_2$ is given by

$$G = \varepsilon \mathrm{d}u^2 + w(u)^2 \mathrm{d}v^2, \quad \varepsilon = \pm 1. \tag{2.5}$$

When $\varepsilon = 1$, the surface is spacelike; when $\varepsilon = -1$, the surface is timelike. The functions $x(u)$, $y(u)$ and $w(u)$ satisfy

$$x(u)^2 + y(u)^2 - w(u)^2 = 1, \quad x'(u)^2 + y'(u)^2 - w'(u)^2 = \varepsilon. \tag{2.6}$$

If $\mathbf{M}_2$ is not contained in a hyperplane of $\mathbf{E}_1^4$ (i.e., $\mathbf{M}_2$ is proper), then $x'(u) \not\equiv 0$ and $y'(u) \not\equiv 0$.

The hyperbolic rotation surface $\mathbf{M}_3$ of the second kind is defined by

$$r(u, v) = x(u)e_1 + y(u)e_2 + w(u)\cosh(v)e_3 + w(u)\sinh(v)e_4, \tag{2.7}$$

where $u \in J$, $v \in \mathbf{R}$. The induced metric G on $\mathbf{M}_3$ is given by

$$G = \mathrm{d}u^2 - w(u)^2 \mathrm{d}v^2. \tag{2.8}$$

The surface is timelike. The functions $x(u)$, $y(u)$ and $w(u)$ satisfy

$$x(u)^2 + y(u)^2 + w(u)^2 = 1, \;\; x'(u)^2 + y'(u)^2 + w'(u)^2 = 1. \tag{2.9}$$

If $\mathbf{M}_3$ is not contained in a hyperplane of $\mathbf{E}_1^4$ (i.e., $\mathbf{M}_3$ is proper), then $x'(u) \not\equiv 0$ and $y'(u) \not\equiv 0$.

The parabolic rotation surface $\mathbf{M}_4$ is defined by

$$r(u, v) = x(u)e_1 + vz(u)e_2 + z(u)e_3 + \left(-\frac{1}{2}v^2 z(u) + w(u)\right) e_4, \tag{2.10}$$

where $u \in J$, $v \in \mathbf{R}$. The induced metric G on $\mathbf{M}_4$ is given by

$$G = \varepsilon \mathrm{d}u^2 + z(u)^2 \mathrm{d}v^2, \;\; \varepsilon = \pm 1. \tag{2.11}$$

When $\varepsilon = 1$, the surface is spacelike; when $\varepsilon = -1$, the surface is timelike. The functions $x(u)$, $z(u)$ and $w(u)$ satisfy

$$2z(u)w(u) + x(u)^2 = 1, \;\; 2z'(u)w'(u) + x'(u)^2 = \varepsilon. \tag{2.12}$$

If $\mathbf{M}_4$ is not contained in a hyperplane of $\mathbf{E}_1^4$ (i.e., $\mathbf{M}_4$ is proper), then $z'(u) \not\equiv 0$ and $x'(u) \not\equiv 0$.

REMARK. Let vectors $x = \sum_k x_k e_k$ and $y = \sum_k y_k e_k$. For the spherical and the hyperbolic rotation surface, we have

$$\bar{G}(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4; \tag{2.13}$$

for the parabolic rotation surface, we have

$$\bar{G}(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_4 + x_4 y_3 \tag{2.14}$$

(cf. [2], [4]).

3. The proof of the Theorem

We denote by $\widetilde{\nabla}$ the covariant differentiation with respect to the indefinite Riemannian metric of $\mathbf{E}_1^4$ and by $\overline{\nabla}$ and ∇ the covariant differentiations with respect to the induced metric of $\mathbf{S}_1^3$ and $\mathbf{M}_i$ ($i = 1, 2, 3, 4$), respectively. We denote by $\eta(x)$, $x \in \mathbf{S}_1^3$, the position vector of x with respect to the origin of $\mathbf{E}_1^4$ which is a field of normal vectors of $\mathbf{S}_1^3$ in $\mathbf{E}_1^4$;

ξ_i, the normal vector field of $\mathbf{M}_i$ in $\mathbf{S}_1^3$. Then, considering that $\mathbf{M}_i$ is locally embedded in $\mathbf{S}_1^3$, we have the following Gauss's and Weingarten's formulas.

$$\begin{cases} \widetilde{\nabla}_X Y = \overline{\nabla}_X Y + \langle X, Y\rangle \eta \\ \overline{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi_i \\ \overline{\nabla}_X \xi_i = -A_{\xi_i}(X), \end{cases} \tag{3.1}$$

where X and Y are tangent vector fields on $\mathbf{M}_i$, and A_{ξ_i} is a field of type (1,1) tensor on $\mathbf{M}_i$ corresponding to ξ_i, i.e.,

$$\langle A_{\xi_i}(X), Y\rangle = h(X, Y), \quad i = 1, 2, 3, 4.$$

For the surface $\mathbf{M}_1$, the normal vector field ξ_1 is

$$\begin{aligned} \xi_1(u, v) = &((z'(u)w(u) - w'(u)z(u))\sin(v), (z'(u)w(u) - w'(u)z(u))\cos(v), \\ &w'(u)y(u) - y'(u)w(u), z'(u)y(u) - y'(u)z(u)) \end{aligned} \tag{3.2}$$

and

$$\langle \xi_1, r_u\rangle = \langle \xi_1, r_v\rangle = \langle \xi_1, r\rangle = 0, \quad \langle \xi_1, \xi_1\rangle = -\varepsilon,$$

where

$$r_u = \frac{dr(u, v)}{du}, \quad r_v = \frac{dr(u, v)}{dv}.$$

From (3.1) and (3.2), we have

$$\begin{cases} \langle A_{\xi_1}(r_u), r_u\rangle = -\langle \overline{\nabla}_{r_u}\xi_1, r_u\rangle = \langle \xi_1, \overline{\nabla}_{r_u} r_u\rangle = \langle \xi_1, \widetilde{\nabla}_{r_u} r_u\rangle \\ \qquad = y''(z'w - w'z) + z''(w'y - y'w) - w''(z'y - y'z) \\ \qquad = y(z''w' - w''z') - y'(z''w - w''z) + y''(z'w - w'z) \\ \langle A_{\xi_1}(r_v), r_v\rangle = = -\langle \overline{\nabla}_{r_v}\xi_1, r_v\rangle = \langle \xi_1, \overline{\nabla}_{r_v} r_v\rangle = \langle \xi_1, \widetilde{\nabla}_{r_v} r_v\rangle \\ \qquad = -y(z'w - w'z) \\ \langle A_{\xi_1}(r_u), r_v\rangle = -\langle \overline{\nabla}_{r_u}\xi_1, r_v\rangle = \langle \xi_1, \overline{\nabla}_{r_u} r_v\rangle = \langle \xi_1, \widetilde{\nabla}_{r_u} r_v\rangle = 0. \end{cases} \tag{3.3}$$

Since $\langle r_u, r_u\rangle = \varepsilon$, $\langle r_u, r_v\rangle = 0$, $\langle r_v, r_v\rangle = y^2$, then from (3.3), we obtain the principal curvatures κ_1 and κ_2 of $\mathbf{M}_1$:

$$\begin{cases} \kappa_1 = y(z''w' - w''z') - y'(z''w - w''z) + y''(z'w - w'z) \\ \kappa_2 = -y^{-1}(z'w - w'z). \end{cases} \tag{3.4}$$

For the surface $\mathbf{M}_2$, the normal vector field ξ_2 is

$$\begin{cases} \xi_2(u, v) = (y'(u)w(u) - w'(u)y(u), w'(u)x(u) - x'(u)w(u), \\ \qquad (y'(u)x(u) - x'(u)y(u))\sinh(v), (y'(u)x(u) - x'(u)y(u))\cosh(v)) \end{cases} \tag{3.5}$$

and

$$\langle \xi_2, r_u\rangle = \langle \xi_2, r_v\rangle = \langle \xi_2, r\rangle = 0, \quad \langle \xi_2, \xi_2\rangle = -\varepsilon,$$

where

$$r_u = \frac{dr(u,v)}{du}, \quad r_v = \frac{dr(u,v)}{dv}.$$

From (3.1) and (3.5), we have

$$\begin{cases} \langle A_{\xi_2}(r_u), r_u\rangle = -\langle \overline{\nabla}_{r_u}\xi_2, r_u\rangle = \langle \xi_2, \overline{\nabla}_{r_u} r_u\rangle = \langle \xi_2, \widetilde{\nabla}_{r_u} r_u\rangle \\ \qquad = x''(y'w - w'y) + y''(w'x - x'w) - w''(y'x - x'y) \\ \qquad = w(x''y' - y''x') + w'(y''x - x''y) - w''(y'x - x'y) \\ \langle A_{\xi_2}(r_v), r_v\rangle = -\langle \overline{\nabla}_{r_v}\xi_2, r_v\rangle = \langle \xi_2, \overline{\nabla}_{r_v} r_v\rangle = \langle \xi_2, \widetilde{\nabla}_{r_v} r_v\rangle \\ \qquad = -w(y'x - x'y) \\ \langle A_{\xi_2}(r_u), r_v\rangle = -\langle \overline{\nabla}_{r_u}\xi_2, r_v\rangle = \langle \xi_2, \overline{\nabla}_{r_u} r_v\rangle = \langle \xi_2, \widetilde{\nabla}_{r_u} r_v\rangle = 0. \end{cases} \tag{3.6}$$

Since $\langle r_u, r_u\rangle = \varepsilon$, $\langle r_u, r_v\rangle = 0$, $\langle r_v, r_v\rangle = w^2$, then from (3.6), we obtain the principal curvatures κ_1 and κ_2 of $\mathbf{M}_2$:

$$\begin{cases} \kappa_1 = w(x''y' - y''x') + w'(y''x - x''y) - w''(y'x - x'y) \\ \kappa_2 = -w^{-1}(y'x - x'y). \end{cases} \tag{3.7}$$

For the surface $\mathbf{M}_3$, the normal vector field ξ_3 is

$$\begin{aligned} \xi_3(u,v) &= (y'(u)w(u) - w'(u)y(u), w'(u)x(u) - x'(u)w(u), \\ &\quad (x'(u)y(u) - y'(u)x(u))\cosh(v), (x'(u)y(u) - y'(u)x(u))\sinh(v)) \end{aligned} \tag{3.8}$$

and

$$\langle \xi_3, r_u\rangle = \langle \xi_3, r_v\rangle = \langle \xi_3, r\rangle = 0, \quad \langle \xi_3, \xi_3\rangle = 1,$$

where

$$r_u = \frac{dr(u,v)}{du}, \quad r_v = \frac{dr(u,v)}{dv}.$$

From (3.1) and (3.8), we have

$$\begin{cases} \langle A_{\xi_3}(r_u), r_u\rangle = -\langle \overline{\nabla}_{r_u}\xi_3, r_u\rangle = \langle \xi_3, \overline{\nabla}_{r_u} r_u\rangle = \langle \xi_3, \widetilde{\nabla}_{r_u} r_u\rangle \\ \qquad = x''(y'w - w'y) + y''(w'x - x'w) + w''(x'y - y'x) \\ \qquad = w(x''y' - y''x') + w'(y''x - x''y) + w''(x'y - y'x) \\ \langle A_{\xi_3}(r_v), r_v\rangle = -\langle \overline{\nabla}_{r_v}\xi_3, r_v\rangle = \langle \xi_3, \overline{\nabla}_{r_v} r_v\rangle = \langle \xi_3, \widetilde{\nabla}_{r_v} r_v\rangle \\ \qquad = w(x'y - y'x) \\ \langle A_{\xi_3}(r_u), r_v\rangle = -\langle \overline{\nabla}_{r_u}\xi_3, r_v\rangle = \langle \xi_3, \overline{\nabla}_{r_u} r_v\rangle = \langle \xi_3, \widetilde{\nabla}_{r_u} r_v\rangle = 0. \end{cases} \tag{3.9}$$

Since $\langle r_u, r_u\rangle = 1$, $\langle r_u, r_v\rangle = 0$, $\langle r_v, r_v\rangle = -w^2$, then from (3.9), we obtain the principal curvatures κ_1 and κ_2 of $\mathbf{M}_3$:

$$\begin{cases} \kappa_1 = w(x''y' - y''x') + w'(y''x - x''y) + w''(x'y - y'x) \\ \kappa_2 = w^{-1}(x'y - y'x). \end{cases} \tag{3.10}$$

For the surface $\mathbf{M}_4$, by (2.12) it can be written as

$$r(u, v) = \left(x(u), vz(u), z(u), -\frac{-1 + v^2 z(u)^2 + x(u)^2}{2z(u)}\right).$$

Then

$$r_u = \left(x'(u), vz'(u), z'(u), -\frac{v^2 z(u)^2 z'(u) + 2z(u)x(u)x'(u) + z'(u) - x(u)^2 z'(u)}{2z(u)^2}\right)$$

$$r_v = (0, z(u), 0, -vz(u))$$

$$ds^2 = \frac{(x'z - z'x)^2 - z'^2}{z^2} du^2 + z^2 dv^2 = \varepsilon du^2 + z^2 dv^2.$$

From (2.12) we can get

$$x'(u)z(u) - z'(u)x(u) = \pm\sqrt{|\varepsilon z(u)^2 + z'(u)^2|},$$

then

$$x(u) = \pm z(u) \int \frac{\sqrt{|\varepsilon z(u)^2 + z'(u)^2|}}{z(u)^2} du.$$

The normal vector field ξ_4 is

$$\xi_4(u, v) = \left(x, vz, z, \frac{-1 - x^2 - z^2 v^2}{2z}\right) + \frac{\varepsilon z'}{z} r_u \tag{3.11}$$

and

$$\langle \xi_4, r_u \rangle = \langle \xi_4, r_v \rangle = \langle \xi_4, r \rangle = 0, \quad \langle \xi_4, \xi_4 \rangle = -\varepsilon \frac{\varepsilon z^2 + z'^2}{z^2},$$

where

$$r_u = \frac{dr(u, v)}{du}, \quad r_v = \frac{dr(u, v)}{dv}.$$

From (3.1) and (3.11), we have

$$\begin{cases} \langle A_{\xi_4}(r_u), r_u \rangle = -\langle \overline{\nabla}_{r_u} \xi_4, r_u \rangle = \langle \xi_4, \overline{\nabla}_{r_u} r_u \rangle = \langle \xi_4, \widetilde{\nabla}_{r_u} r_u \rangle \\ \qquad = -\frac{\varepsilon z^2 + z'^2}{z^2} - \varepsilon^2 (\frac{z'}{z})' \\ \qquad = -\frac{z'' + \varepsilon z}{z} \\ \langle A_{\xi_4}(r_v), r_v \rangle = -\langle \overline{\nabla}_{r_v} \xi_4, r_v \rangle = \langle \xi_4, \overline{\nabla}_{r_v} r_v \rangle = \langle \xi_4, \widetilde{\nabla}_{r_v} r_v \rangle \\ \qquad = -\varepsilon(\varepsilon z^2 + z'^2) \\ \langle A_{\xi_4}(r_u), r_v \rangle = -\langle \overline{\nabla}_{r_u} \xi_4, r_v \rangle = \langle \xi_4, \overline{\nabla}_{r_u} r_v \rangle = \langle \xi_4, \widetilde{\nabla}_{r_u} r_v \rangle = 0. \end{cases} \tag{3.12}$$

Since $\langle r_u, r_u\rangle = \varepsilon$, $\langle r_u, r_v\rangle = 0$, $\langle r_v, r_v\rangle = z^2$, then from (3.12), we obtain the principal curvatures κ_1 and κ_2 of $\mathbf{M}_4$:

$$\begin{cases} \kappa_1 = -\sqrt{\frac{z^2}{|\varepsilon z^2+z'^2|}}(\frac{z''+\varepsilon z}{z}) \\ \kappa_2 = \frac{-\varepsilon\sqrt{|\varepsilon z^2+z'^2|}}{z}. \end{cases} \tag{3.13}$$

We assume that the principal curvatures of $\mathbf{M}_i$ satisfy $\kappa_2 = t$, $\kappa_1 = f(\kappa_2) = f(t)$. For the surface $\mathbf{M}_4$, we have

$$\begin{cases} \kappa_1 = -\sqrt{\frac{z^2}{|\varepsilon z^2+z'^2|}}(\frac{z''+\varepsilon z}{z}) = f(t) \\ \kappa_2 = \frac{-\varepsilon\sqrt{|\varepsilon z^2+z'^2|}}{z} = t. \end{cases} \tag{3.14}$$

Then

$$\begin{cases} z' = \sqrt{z^2t^2 - \varepsilon z^2} \\ z'' = \frac{zz't^2+z^2tt'-\varepsilon zz'}{z'} = zt^2 - \varepsilon z + z^2t\frac{dt}{dz}. \end{cases} \tag{3.15}$$

(3.14) and (3.15) yield

$$z\frac{dt}{dz} = \varepsilon f(t) - t. \tag{3.16}$$

Therefore

$$z(u) = \exp\left(\int_0^u \frac{1}{\varepsilon f(t) - t}dt\right). \tag{3.17}$$

Then we get (1.5) of the theorem.

For the surface $\mathbf{M}_2$, from (2.6.i) we may put

$$\begin{cases} x(u) = (w(u)^2 + 1)^{\frac{1}{2}}\cos\varphi(u) \\ y(u) = (w(u)^2 + 1)^{\frac{1}{2}}\sin\varphi(u) \end{cases} \tag{3.18}$$

and then determine the function $\varphi(u)$ satisfying (2.6.ii).

Since $x'^2+y'^2-w'^2 = (ww')^2/(w^2+1)+(w^2+1)\varphi'^2-w'^2$, then from (2.6.ii) it follows that

$$\varphi'(u)^2 = \frac{\varepsilon w^2 + w'^2 + \varepsilon}{(w^2+1)^2}. \tag{3.19}$$

We assume that $\varepsilon w^2 + w'^2 + \varepsilon > 0$ on J (when $\varepsilon w^2 + w'^2 + \varepsilon = 0$, φ is constant). Therefore the function $\varphi(u)$ is of the form

$$\varphi(u) = \pm\int_0^u \frac{(\varepsilon w(t)^2 + w'(t)^2 + \varepsilon)^{\frac{1}{2}}}{(w(t)^2 + 1)}dt \tag{3.20}$$

and without loss of generality we may assume that the signature is positive.

From (3.18) and (3.20), we can show that

$$\begin{cases} y'x - x'y = (w^2+1)\varphi' = (\varepsilon w^2 + w'^2 + \varepsilon)^{\frac{1}{2}} \\ y''x - x''y = (y'x - x'y)' = (\varepsilon w w' + w'w'')/(y'x - x'y). \end{cases} \tag{3.21}$$

Differentiating (2.6.i) and (2.6.ii) we obtain

$$xx' + yy' = ww',$$
$$xx'' + yy'' = ww'' - \varepsilon,$$
$$x'x'' + y'y'' = w'w''.$$

Solving above equations for x'' and y'' we get

$$(y'x - x'y)x'' = y'(ww'' - \varepsilon) - yw'w'',$$
$$(y'x - x'y)y'' = -x'(ww'' - \varepsilon) + xw'w''.$$

So

$$x''y' - y''x' = (\varepsilon ww'' - \varepsilon w'^2 - 1)/(y'x - x'y). \tag{3.22}$$

Putting (3.21) and (3.22) into (3.7), then we get

$$\begin{cases} \kappa_1 = \frac{-w - \varepsilon w''}{(\varepsilon w^2 + w'^2 + \varepsilon)^{\frac{1}{2}}} = \frac{w + \varepsilon w''}{tw} = f(t) \\ \kappa_2 = -w^{-1}(\varepsilon w^2 + w'^2 + \varepsilon)^{\frac{1}{2}} = t. \end{cases} \tag{3.23}$$

Then $-w^{-1}(\varepsilon w^2 + w'^2 + \varepsilon)^{\frac{1}{2}} = t$ yields

$$\begin{cases} w' = \sqrt{t^2w^2 - \varepsilon w^2 - \varepsilon} \\ w'' = \frac{tw^2t' + t^2ww' - \varepsilon ww'}{\sqrt{t^2w^2 - \varepsilon w^2 - \varepsilon}} = tw^2\frac{\mathrm{d}t}{\mathrm{d}w} + t^2w - \varepsilon w. \end{cases} \tag{3.24}$$

From (3.23) and (3.24) we get

$$w\frac{\mathrm{d}t}{\mathrm{d}w} = \varepsilon f(t) - t.$$

Therefore

$$w(u) = \exp\left(\int_0^u \frac{1}{\varepsilon f(t) - t}\mathrm{d}t\right). \tag{3.25}$$

Then we get (1.3) of the theorem.

For the surface $\mathbf{M}_3$, from (2.9) we may put

$$\begin{cases} w(u)^2 - 1 < 0 \\ x(u) = (1 - w(u)^2)^{\frac{1}{2}} \cos\varphi(u) \\ y(u) = (1 - w(u)^2)^{\frac{1}{2}} \sin\varphi(u). \end{cases} \tag{3.26}$$

Then

$$\varphi'(u)^2 = \frac{1 - w^2 - w'^2}{(1 - w^2)^2}. \tag{3.27}$$

We assume that $1 - w^2 - w'^2 > 0$ on J (when $1 - w^2 - w'^2 = 0$, φ is constant). Therefore the function $\varphi(u)$ is of the form

$$\varphi(u) = \pm \int_0^u \frac{(1 - w(t)^2 - w'(t)^2)^{\frac{1}{2}}}{(1 - w(t)^2)} \mathrm{d}t \tag{3.28}$$

and without loss of generality we may assume that the signature is positive. From (2.9), (3.26) and (3.28) we can show that

$$\begin{cases} x'y - y'x = -(1 - w^2)\varphi' = -(1 - w^2 - w'^2)^{\frac{1}{2}} \\ x''y - y''x = (x'y - y'x)' = -(ww' + w'w'')/(x'y - y'x) \\ x''y' - y''x' = (1 + ww'' - w'^2)/(x'y - y'x). \end{cases} \tag{3.29}$$

Putting (3.29) into (3.10), then we get

$$\begin{cases} \kappa_1 = \frac{-w-w''}{(1-w^2-w'^2)^{\frac{1}{2}}} = \frac{w+w''}{tw} = f(t) \\ \kappa_2 = -w^{-1}(1 - w^2 - w'^2)^{\frac{1}{2}} = t. \end{cases} \tag{3.30}$$

Then $-w^{-1}(1 - w^2 - w'^2)^{\frac{1}{2}} = t$ yields

$$\begin{cases} w' = \sqrt{1 - t^2w^2 - w^2} \\ w'' = \frac{-tw^2t' - t^2ww' - ww'}{\sqrt{1-t^2w^2-w^2}} = -tw^2\frac{\mathrm{d}t}{\mathrm{d}w} - t^2w - w. \end{cases} \tag{3.31}$$

From (3.30) and (3.31) we get

$$w\frac{\mathrm{d}t}{\mathrm{d}w} = -f(t) - t.$$

Therefore

$$w(u) = \exp\left(\int_0^u \frac{1}{-f(t) - t}\mathrm{d}t\right). \tag{3.32}$$

Then we get (1.4) of the theorem.

For the surface $\mathbf{M}_1$ from (2.3) we may put

$$\begin{cases} y(u)^2 - 1 > 0 \\ z(u) = (y(u)^2 - 1)^{\frac{1}{2}} \sinh \varphi(u) \\ w(u) = (y(u)^2 - 1)^{\frac{1}{2}} \cosh \varphi(u) \end{cases} \tag{3.33.i}$$

or

$$\begin{cases} y(u)^2 - 1 < 0 \\ z(u) = (1 - y(u)^2)^{\frac{1}{2}} \cosh \varphi(u) \\ w(u) = (1 - y(u)^2)^{\frac{1}{2}} \sinh \varphi(u) \end{cases} \tag{3.33.ii}$$

and then determine the function $\varphi(u)$ satisfying $y'(u)^2 + z'(u)^2 - w'(u)^2 = \varepsilon$:

$$\varphi'(u)^2 = \frac{\varepsilon y^2 + y'^2 - \varepsilon}{(y^2-1)^2}.$$

We assume that $\varepsilon y^2 + y'^2 - \varepsilon > 0$ on J (when $\varepsilon y^2 + y'^2 - \varepsilon = 0$, φ is constant). Therefore the function $\varphi(u)$ is of the form

$$\begin{cases} \varepsilon y^2 + y'^2 - \varepsilon > 0 \\ \varphi(u) = \pm \int_0^u \frac{(\varepsilon y(t)^2 + y'(t)^2 - \varepsilon)^{\frac{1}{2}}}{|y(t)^2-1|} dt. \end{cases}$$

Without loss of generality we may assume that the signature is positive when $y(u)^2 - 1 > 0$ and negative when $y(u)^2 - 1 > 0$, then

$$\begin{cases} \varepsilon y^2 + y'^2 - \varepsilon > 0 \\ \varphi(u) = \int_0^u \frac{(\varepsilon y(t)^2 + y'(t)^2 - \varepsilon)^{\frac{1}{2}}}{y(t)^2-1} dt. \end{cases} \tag{3.34}$$

From (2.3), (3.33) and (3.34), we can show that

$$\begin{cases} z'w - w'z = (\varepsilon y^2 + y'^2 - \varepsilon)^{\frac{1}{2}} \\ z''w - w''z = (\varepsilon y y' + y'y'')/(z'w - w'z) \\ z''w' - w''z' = (-\varepsilon y y'' + \varepsilon y'^2 - 1)/(z'w - w'z). \end{cases} \tag{3.35}$$

Putting (3.35) into (3.4), then we get

$$\begin{cases} \kappa_1 = \frac{-\varepsilon y'' - y}{(\varepsilon y^2 + y'^2 - \varepsilon)^{\frac{1}{2}}} = \frac{\varepsilon y'' + y}{t y} = f(t) \\ \kappa_2 = -y^{-1}(\varepsilon y^2 + y'^2 - \varepsilon)^{\frac{1}{2}} = t. \end{cases} \tag{3.36}$$

By $t = -y^{-1}(\varepsilon y^2 + y'^2 - \varepsilon)^{\frac{1}{2}}$, we get

$$\begin{cases} y' = \sqrt{t^2y^2 - \varepsilon y^2 + \varepsilon} \\ y'' = \dfrac{t^2 y y' + y^2 t t' - \varepsilon y y'}{\sqrt{t^2y^2 - \varepsilon y^2 + \varepsilon}} = t^2 y - \varepsilon y + y^2 t \dfrac{t'}{y'}. \end{cases} \tag{3.37}$$

(3.36) and (3.37) yield

$$y \frac{dt}{dy} = \varepsilon f(t) - t. \tag{3.38}$$

Therefore

$$y(u) = \exp\left(\int_0^u \frac{1}{\varepsilon f(t) - t} dt \right). \tag{3.39}$$

Then we get (1.1) and (1.2) of the theorem.

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