前回 ($C^\infty$級多様体上の) $C^\infty$級関数, 接ベクトル, 接空間

今回 $C^\infty$級写像/写像の微分/微分同相写像/逆関数定理

**復習** $D \subseteq \mathbb{R}^m$: 領域, $f: D \to \mathbb{R}^n$: 写像 とする

($f$ は $D$上の $n$個の関数 $y^1 = y^1(x^1,\dots,x^m), \dots, y^n = y^n(x^1,\dots,x^m)$ を用いて $f = (y^1,\dots,y^n)$ と表される) 偏微分可能

もし, $y^1,\dots,y^n$ がすべて $(x^1,\dots,x^m)$ に関する $C^\infty$級関数のとき $f$ を $C^\infty$級写像 という

## $C^\infty$級写像 ($C^\infty$-map)

$M^m, N^n$: $C^\infty$-mfds

写像 $f: M \to N$ が $C^\infty$級

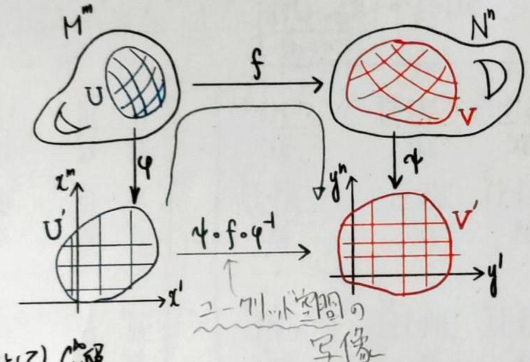
$\Leftrightarrow$ $^\forall(U,\varphi)$: $M$の座標近傍

$^\forall(V,\psi)$: $N$の 〃

に対して

$\boxed{\psi\circ f\circ\varphi^{-1}}: U' \to V'$ ($U' \subset \mathbb{R}^m$, $V' \subset \mathbb{R}^n$)

が(ユークリッド空間の間の写像として) $C^\infty$級




写像前 $V'$の座標 (写像後)

$$\psi\circ f\circ\varphi^{-1}(x^1,\dots,x^m) = (y^1,\dots,y^n)$$

$y^1(x^1,\dots,x^m)$ $\quad$ $y^n(x^1,\dots,x^m)$

$$\begin{cases} y^1 = f^1(x^1,\dots,x^m) \\ y^2 = f^2(x^1,\dots,x^m) \\ \vdots \\ y^n = f^n(x^1,\dots,x^m) \end{cases}$$

$f$の局所座標表示 という

Ex

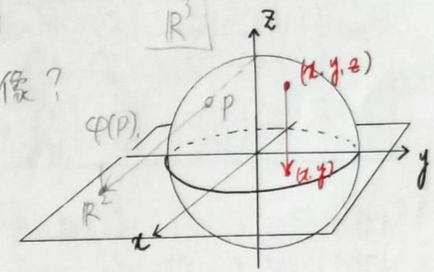
$f(u,v) = (\overset{C^\infty}{u}, \overset{C^\infty}{v}, \overset{C^\infty}{e^{u+v}})$ とすると, $f: \mathbb{R}^2 \to \mathbb{R}^3$ は $C^\infty$級写像

**Ex1** $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$

$f: S^2 \to \mathbb{R}^2$ を

$f(x,y,z) = (x,y)$ $C^\infty$級写像?

と定める (直交射影)




$\begin{cases}(U,\varphi) = (U; u,v) \\ (V,\psi) = (V; \xi,\eta)\end{cases}$ : 立体射影による座標近傍

$U = S^2 \setminus \{N\}$

$\varphi: U \to \mathbb{R}^2$; $\varphi(x,y,z) = \dfrac{1}{1-z}(x,y)$

$\varphi^{-1}: \mathbb{R}^2 \to U$; $\varphi^{-1}(u,v) = \dfrac{1}{1+u^2+v^2}(2u, 2v, u^2+v^2-1)$

$V = S^2 \setminus \{S\}$

$\psi: U \to \mathbb{R}^2$; $\psi(x,y,z) = \dfrac{1}{1+z}(x,y)$

$\psi^{-1}: \mathbb{R}^2 \to U$; $\psi^{-1}(\xi,\eta) = \dfrac{1}{1+\xi^2+\eta^2}(2\xi, 2\eta, 1-\xi^2-\eta^2)$

$C^\infty$級写像

$f: S^2 \to \mathbb{R}^2$: $C^\infty$級写像 証明すればよいこと

$\Longleftrightarrow$ $f\circ\varphi^{-1}, f\circ\psi^{-1}$: $C^\infty$級写像

$f\circ\varphi^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$ は

$f\circ\varphi^{-1}(u,v) = \dfrac{1}{1+u^2+v^2}(2u, 2v)$

Ex

$f\circ\varphi^{-1} = \left(\underset{y_1(u,v)}{\underset{\shortparallel}{\dfrac{2u}{1+u^2+v^2}}}, \underset{y_2(u,v)}{\underset{\shortparallel}{\dfrac{2v}{1+u^2+v^2}}}\right)$ であり,

$y_1$ と $y_2$ は $C^\infty$級 ~~写像~~ 関数 なので, $f\circ\varphi^{-1}$ は $\mathbb{R}^2 \to \mathbb{R}^2$ として $C^\infty$級

$f\circ\psi^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$ は

$f\circ\psi^{-1}(\xi,\eta) = \dfrac{1}{1+\xi^2+\eta^2}(2\xi, 2\eta)$

$\therefore$ $f: S^2 \to \mathbb{R}^2$: $C^\infty$級写像

証明したいこと

**Ex2** $F : \mathbb{R}^3 \to \mathbb{R}^3$ を次で定める:

$$F(x,y,z) := \frac{1}{1+z^2}\left(x^2-y^2,\ 2xy,\ 2z\right) \quad ((x,y,z)\in\mathbb{R}^3)$$

もし $(x,y,z)\in S^2 \Longrightarrow F(x,y,z)\in S^2$ がわかる

$$\because\ F(x,y,z)\cdot F(x,y,z) = \frac{1}{(1+z^2)^2}\left\{(x^2-y^2)^2+4x^2y^2+4z^2\right\}$$

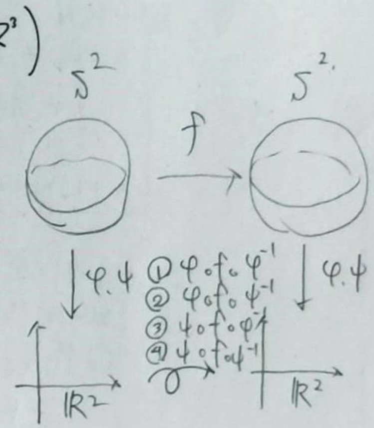
内積をとった。

$$= \frac{1}{(1+z^2)^2}\left\{(x^2+y^2)^2+4z^2\right\}$$

$x^2+y^2+z^2=1$
$\Rightarrow x^2+y^2=1-z^2$

$$= \frac{1}{(1+z^2)^2}\left\{(1-z^2)^2+4z^2\right\}$$

$$= \frac{1}{(1+z^2)^2}(1+z^2)^2 = 1$$




$\therefore\ f : S^2 \to S^2$ を $f = F|_{S^2}$ と定める。

制限写像（Fの定義域を$S^2$に限定）

$f : S^2 \to S^2$ : $C^\infty$級写像

$\Longleftrightarrow$ $\varphi\circ f\circ\varphi^{-1},\ \psi\circ f\circ\varphi^{-1}$, $\varphi\circ f\circ\psi^{-1},\ \psi\circ f\circ\psi^{-1}$ : $C^\infty$級

$$F(x,y,z) = \left(\frac{x^2-y^2}{1+z^2},\ \frac{2xy}{1+z^2},\ \frac{2z}{1+z^2}\right)$$

$\varphi^{-1}$の1変数　$\varphi^{-1}$の第2変数

$$f\circ\varphi^{-1}(u,v) = \frac{1}{1+\left(\frac{u^2+v^2-1}{1+u^2+v^2}\right)^2}\left(\left(\frac{2u}{1+u^2+v^2}\right)^2-\left(\frac{2v}{1+u^2+v^2}\right)^2,\ 2\frac{2u}{1+u^2+v^2}\frac{2v}{1+u^2+v^2},\ 2\frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

$\varphi^{-1}$の第3変数

$$= \frac{1}{(1+u^2+v^2)^2+(u^2+v^2-1)^2}\left(4u^2-4v^2,\ 8uv,\ 2(u^2+v^2-1)(u^2+v^2+1)\right)$$

$$= \frac{1}{(u^2+v^2)^2+1}\left(2u^2-2v^2,\ 4uv,\ (u^2+v^2)^2-1\right)$$

$$\varphi(x,y,z) = \left(\frac{x}{1-z},\ \frac{y}{1-z}\right)$$

$$\varphi\circ f\circ\varphi^{-1}(u,v) = \frac{1}{1-\frac{(u^2+v^2)^2-1}{(u^2+v^2)^2+1}}\left(\frac{2u^2-2v^2}{(u^2+v^2)^2+1},\ \frac{4uv}{(u^2+v^2)^2+1}\right)$$

$$= \frac{1}{2}\left(2u^2-2v^2,\ 4uv\right) = (\underbrace{u^2-v^2}_{f^1(u,v)},\ \underbrace{2uv}_{f^2(u,v)})$$

（$\because$ $u^2-v^2$, $2uv$ は、どっちも $C^\infty$級函数.）

同様に $\psi\circ f\circ\varphi^{-1}(u,v)$, $\varphi\circ f\circ\psi^{-1}(\xi,\eta)$, $\psi\circ f\circ\psi^{-1}(\xi,\eta)$ : $C^\infty$級写像 とわかる

$\therefore\ f : S^2\to S^2$ は $C^\infty$級写像（実は $f(z)=z^2$ の $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ への拡張）

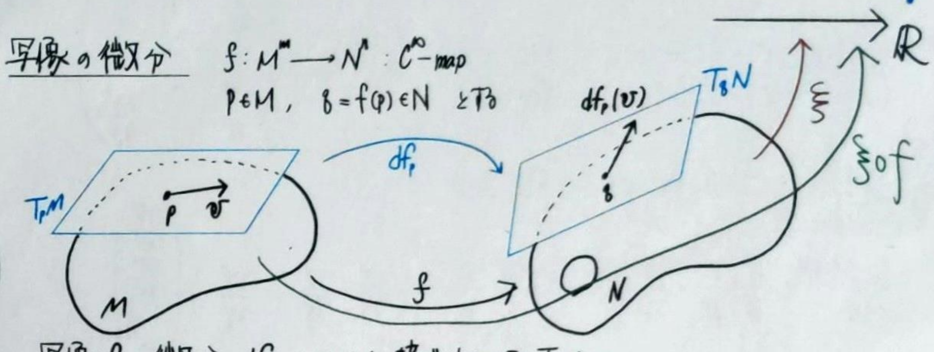
$\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$ : リーマン球面　　$\hat{f} : \hat{\mathbb{C}}\to\hat{\mathbb{C}}$ : $C^\infty$級写像

$f(z) = a_n z^n + \cdots + a_1 z + a_0$　　$\hat{f}(z) = \begin{cases} f(z) & (z\in\mathbb{C}) \\ \infty & (z=\infty)\end{cases}$



**写像の微分**　$f : M^m \to N^n$ : $C^\infty$-map

$p\in M$, $q=f(p)\in N$ とする




写像 $f$ の微分 $df_p$ とは、M の接ベクトル $v\in T_pM$ に
N の 〃 $df_p(v)\in T_qN$ を対応させるもの

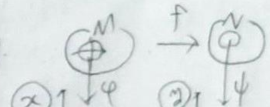
$\forall\xi\in C^\infty(N)$ に対して
$$(df_p(v))(\xi) := v(\underbrace{\xi\circ f}_{M\text{上の関数}})$$

微分のルールを対応させる

$\varphi\circ f\circ\varphi^{-1}(x^1,\cdots,x^m) = (y^1,y^2,\cdots,y^n)$

$y^j = y^j(x^1,\cdots,x^m) = f^j(x^1,\cdots,x^m)$

**局所座標表示**　$df_p : T_pM \to T_qN$




**Prop** $\begin{cases}(U,\varphi)=(U;x^1,\cdots,x^m) : M^m\text{の座標近傍},\ p\in U \\ (V,\psi)=(V;y^1,\cdots,y^n) : N^n\text{の 〃},\ f(p)=q\in V\end{cases}$

このとき
$$df_p\left(\left(\frac{\partial}{\partial x^i}\right)_p\right) = \frac{\partial f^1}{\partial x^i}(p)\left(\frac{\partial}{\partial y^1}\right)_q + \cdots + \frac{\partial f^n}{\partial x^i}(p)\left(\frac{\partial}{\partial y^n}\right)_q = \sum_{j=1}^{n}\frac{\partial f^j}{\partial x^i}(p)\left(\frac{\partial}{\partial y^j}\right)_q$$

(proof) $w\in T_qN$ とおく

$w = \sum_{j=1}^{n} w(y^j)\left(\frac{\partial}{\partial y^j}\right)_q$　$w(y^j) = \frac{\partial f^j}{\partial x^i}(p)$ を示せばok

$$w(y^j) = \left(df_p\left(\frac{\partial}{\partial x^i}\right)_p\right)(y^j) = \frac{\partial(y^j\circ f)}{\partial x^i}(p) = \frac{\partial f^j}{\partial x^i}(p)$$

//

**Ex** Ex2 の $f : S^2\to S^2$ に対して $\varphi\circ f\circ\varphi^{-1}(u,v) = (\underbrace{u^2-v^2}_{f^1},\ \underbrace{2uv}_{f^2})$

$\begin{cases}(f^1)_u = 2u,\ (f^1)_v = -2v \\ (f^2)_u = 2v,\ (f^2)_v = 2u\end{cases}$　$\therefore\ df\left(\frac{\partial}{\partial u}\right) = 2u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v}$

$f^1_u = 2u$
$f^2_v = 2$
$f^2_u = 2v$

**Prop** $\begin{cases} (U,\varphi)=(U;x^1,\dots,x^m) : M^m \text{の座標近傍}, \ p\in U \\ (V,\psi)=(V;y^1,\dots,y^n) : N^n \text{の} \ 〃, \ f(p)=q\in V \end{cases}$

このとき $df_p\left(\left(\frac{\partial}{\partial x^i}\right)_p\right) = \frac{\partial f^1}{\partial x^i}(p)\left(\frac{\partial}{\partial y^1}\right)_q + \cdots + \frac{\partial f^n}{\partial x^i}(p)\left(\frac{\partial}{\partial y^n}\right)_q = \sum_{j=1}^{n}\frac{\partial f^j}{\partial x^i}(p)\left(\frac{\partial}{\partial y^j}\right)_q$

$\begin{cases} (U,\varphi)=(U;x^1,\dots,x^m) : M^m \text{の座標近傍}, \ p\in U \\ (V,\psi)=(V;y^1,\dots,y^n) : N^n \text{の} \ 〃, \ f(p)=q\in V \end{cases}$

$$df_p\left(\left(\frac{\partial}{\partial x^1}\right)_p, \left(\frac{\partial}{\partial x^2}\right)_p, \dots, \left(\frac{\partial}{\partial x^m}\right)_p\right)$$

$$= \left(df_p\left(\frac{\partial}{\partial x^1}\right)_p, df_p\left(\frac{\partial}{\partial x^2}\right)_p, \dots, df_p\left(\frac{\partial}{\partial x^m}\right)_p\right)$$

$$= \left(\frac{\partial f^1}{\partial x^1}(p)\left(\frac{\partial}{\partial y^1}\right)_q + \cdots + \frac{\partial f^n}{\partial x^1}(p)\left(\frac{\partial}{\partial y^n}\right)_q, \ \frac{\partial f^1}{\partial x^2}(p)\left(\frac{\partial}{\partial y^1}\right)_q + \cdots + \frac{\partial f^n}{\partial x^2}(p)\left(\frac{\partial}{\partial y^n}\right)_q, \dots, \ \frac{\partial f^1}{\partial x^m}(p)\left(\frac{\partial}{\partial y^1}\right)_q + \cdots + \frac{\partial f^n}{\partial x^m}(p)\left(\frac{\partial}{\partial y^n}\right)_q\right)$$

$$= \left(\left(\frac{\partial}{\partial y^1}\right)_q, \left(\frac{\partial}{\partial y^2}\right)_q, \dots, \left(\frac{\partial}{\partial y^n}\right)_q\right) \underbrace{\begin{pmatrix} \frac{\partial f^1}{\partial x^1}(p) & \frac{\partial f^1}{\partial x^2}(p) & \cdots & \frac{\partial f^1}{\partial x^m}(p) \\ \frac{\partial f^2}{\partial x^1}(p) & \frac{\partial f^2}{\partial x^2}(p) & \cdots & \frac{\partial f^2}{\partial x^m}(p) \\ \vdots & & & \vdots \\ \frac{\partial f^n}{\partial x^1}(p) & \frac{\partial f^n}{\partial x^2}(p) & \cdots & \frac{\partial f^n}{\partial x^m}(p) \end{pmatrix}}_{\substack{n\text{行}m\text{列の行列} \\ \text{ヤコビ行列 } (Jf)_p \text{ と表す}}}$$

∴ 基底 $\left\{\left(\frac{\partial}{\partial x^i}\right)_p\right\}, \left\{\left(\frac{\partial}{\partial y^j}\right)_q\right\}$ に関する $(df_p)$ の表現行列はヤコビ行列

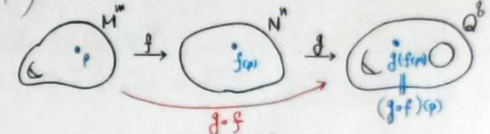


prop $p\in M$

$$d(g\circ f)_p = (dg_{f(p)})\circ(df_p)$$

**合成写像の微分** $f: M\to N : C^\infty\text{-map}$

$g: N\to Q : 〃$




**Prop** $p\in M$ に対して

$df_p : T_pM \longrightarrow T_{f(p)}N$

$dg_{f(p)} : T_{f(p)}N \longrightarrow T_{g(f(p))}Q$

$d(g\circ f)_p : T_pM \longrightarrow T_{g(f(p))}Q$

$\boxed{d(g\circ f)_p = (dg_{f(p)})\circ(df_p)}$

**Cor** $\underset{s\times m}{J(g\circ f)_p} = \underbrace{\underset{s\times n}{(Jg)_{f(p)}}\underset{n\times m}{(Jf)_p}}_{\text{行列の積}}$

$$(Jf)_p = \begin{array}{c} \quad m \\ n\begin{pmatrix} \frac{\partial f^1}{\partial x^1}(p) & \frac{\partial f^1}{\partial x^2}(p) & \cdots & \frac{\partial f^1}{\partial x^m}(p) \\ \frac{\partial f^2}{\partial x^1}(p) & \frac{\partial f^2}{\partial x^2}(p) & \cdots & \frac{\partial f^2}{\partial x^m}(p) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f^n}{\partial x^1}(p) & \frac{\partial f^n}{\partial x^2}(p) & \cdots & \frac{\partial f^n}{\partial x^m}(p) \end{pmatrix} \end{array}$$

$$\underbrace{\begin{array}{c} m \\ s\begin{pmatrix} \frac{\partial(g\circ f)^1}{\partial x^1} & \frac{\partial(g\circ f)^1}{\partial x^2} & \cdots & \frac{\partial(g\circ f)^1}{\partial x^m} \\ \frac{\partial(g\circ f)^2}{\partial x^1} & \frac{\partial(g\circ f)^2}{\partial x^2} & \cdots & \frac{\partial(g\circ f)^2}{\partial x^m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial(g\circ f)^s}{\partial x^1} & \frac{\partial(g\circ f)^s}{\partial x^2} & \cdots & \frac{\partial(g\circ f)^s}{\partial x^m} \end{pmatrix}\end{array}}_{J(g\circ f)_p} = \underbrace{\begin{array}{c} n \\ s\begin{pmatrix} \frac{\partial g^1}{\partial y^1} & \frac{\partial g^1}{\partial y^2} & \cdots & \frac{\partial g^1}{\partial y^n} \\ \frac{\partial g^2}{\partial y^1} & \frac{\partial g^2}{\partial y^2} & \cdots & \frac{\partial g^2}{\partial y^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g^s}{\partial y^1} & \frac{\partial g^s}{\partial y^2} & \cdots & \frac{\partial g^s}{\partial y^n} \end{pmatrix}\end{array}}_{(Jg)_{f(p)}} \underbrace{\begin{array}{c} m \\ n\begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^m} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \cdots & \frac{\partial f^2}{\partial x^m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f^n}{\partial x^1} & \frac{\partial f^n}{\partial x^2} & \cdots & \frac{\partial f^n}{\partial x^m} \end{pmatrix}\end{array}}_{(Jf)_p}$$

$(U;x^1,\dots,x^m) : M^m$ の座標近傍, $p\in U$

$(V;y^1,\dots,y^n) : N^n$ の 〃, $f(p)=q\in V$

$(W;z^1,\dots,z^s) : Q^s$ の 〃, $g(f(p))=r\in W$

$\begin{cases} f(x^1,\dots,x^m) = (y^1,\dots,y^n) = (f^1(x^1,\dots,x^m),\dots,f^n(x^1,\dots,x^m)) \\ g(y^1,\dots,y^n) = (z^1,\dots,z^s) = (g^1(y^1,\dots,y^n),\dots,g^s(y^1,\dots,y^n)) \end{cases}$

$y^1 = f^1(x^1,\dots,x^m)$
$y^2 = f^2(x^1,\dots,x^m)$
$\vdots$
$y^n = f^n(x^1,\dots,x^m)$

$z^1 = g^1(y^1,\dots,y^n)$
$z^2 = g^2(y^1,\dots,y^n)$
$\vdots$
$z^s = g^s(y^1,\dots,y^n)$

局所座標表示

$$\therefore (g\circ f)(x^1,\dots,x^m)$$
$$= g(f(x^1,\dots,x^m))$$
$$= g(f^1(x^1,\dots,x^m),\dots,f^n(x^1,\dots,x^m))$$
$$= (g^1(f^1(x^1,\dots,x^m),\dots,f^n(x^1,\dots,x^m)),\dots,g^s(f^1(x^1,\dots,x^m),\dots,f^n(x^1,\dots,x^m)))$$

$$
\begin{cases}
(g\circ f)^1(x^1,\cdots,x^m) = g^1(f^1(x^1,\cdots,x^m),\cdots,f^n(x^1,\cdots,x^m)) \\
(g\circ f)^2(x^1,\cdots,x^m) = g^2(f^1(x^1,\cdots,x^m),\cdots,f^n(x^1,\cdots,x^m)) \\
\quad\vdots \qquad\qquad\qquad \vdots \\
(g\circ f)^\ell(x^1,\cdots,x^m) = g^\ell(f^1(x^1,\cdots,x^m),\cdots,f^n(x^1,\cdots,x^m))
\end{cases}
\qquad
(Jg)_{f(p)} = \begin{pmatrix} \frac{\partial g^1}{\partial y^1} & \cdots & \frac{\partial g^1}{\partial y^n} \\ \vdots & & \vdots \\ \frac{\partial g^\ell}{\partial y^1} & \cdots & \frac{\partial g^\ell}{\partial y^n} \end{pmatrix}
$$

$$
\frac{\partial (g\circ f)^1}{\partial x^1} = \frac{\partial g^1}{\partial y^1}\frac{\partial y^1}{\partial x^1} + \frac{\partial g^1}{\partial y^2}\frac{\partial y^2}{\partial x^1} + \cdots + \frac{\partial g^1}{\partial y^n}\frac{\partial y^n}{\partial x^1} = \begin{pmatrix} \frac{\partial g^1}{\partial y^1} & \frac{\partial g^1}{\partial y^2} & \cdots & \frac{\partial g^1}{\partial y^n} \end{pmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial x^1} \\ \frac{\partial f^2}{\partial x^1} \\ \vdots \\ \frac{\partial f^n}{\partial x^1} \end{pmatrix}
$$

$$
\frac{\partial (g\circ f)^1}{\partial x^1} = \begin{pmatrix} \frac{\partial g^1}{\partial y^1} & \frac{\partial g^1}{\partial y^2} & \cdots & \frac{\partial g^1}{\partial y^n} \end{pmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial x^1} \\ \frac{\partial f^2}{\partial x^1} \\ \vdots \\ \frac{\partial f^n}{\partial x^1} \end{pmatrix} \quad \cdots \quad \frac{\partial (g\circ f)^1}{\partial x^m} = \begin{pmatrix} \frac{\partial g^1}{\partial y^1} & \frac{\partial g^1}{\partial y^2} & \cdots & \frac{\partial g^1}{\partial y^n} \end{pmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial x^m} \\ \frac{\partial f^2}{\partial x^m} \\ \vdots \\ \frac{\partial f^n}{\partial x^m} \end{pmatrix}
$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$
\frac{\partial (g\circ f)^\ell}{\partial x^1} = \begin{pmatrix} \frac{\partial g^\ell}{\partial y^1} & \frac{\partial g^\ell}{\partial y^2} & \cdots & \frac{\partial g^\ell}{\partial y^n} \end{pmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial x^1} \\ \frac{\partial f^2}{\partial x^1} \\ \vdots \\ \frac{\partial f^n}{\partial x^1} \end{pmatrix} \quad \cdots \quad \frac{\partial (g\circ f)^\ell}{\partial x^m} = \begin{pmatrix} \frac{\partial g^\ell}{\partial y^1} & \frac{\partial g^\ell}{\partial y^2} & \cdots & \frac{\partial g^\ell}{\partial y^n} \end{pmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial x^m} \\ \frac{\partial f^2}{\partial x^m} \\ \vdots \\ \frac{\partial f^n}{\partial x^m} \end{pmatrix}
$$

---

$$
\underbrace{\begin{pmatrix} \frac{\partial (g\circ f)^1}{\partial x^1} & \frac{\partial (g\circ f)^1}{\partial x^2} & \cdots & \frac{\partial (g\circ f)^1}{\partial x^m} \\ \frac{\partial (g\circ f)^2}{\partial x^1} & \frac{\partial (g\circ f)^2}{\partial x^2} & \cdots & \frac{\partial (g\circ f)^2}{\partial x^m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial (g\circ f)^\ell}{\partial x^1} & \frac{\partial (g\circ f)^\ell}{\partial x^2} & \cdots & \frac{\partial (g\circ f)^\ell}{\partial x^m} \end{pmatrix}}_{J(g\circ f)_p}
=
\underbrace{\begin{pmatrix} \frac{\partial g^1}{\partial y^1} & \frac{\partial g^1}{\partial y^2} & \cdots & \frac{\partial g^1}{\partial y} \\ \frac{\partial g^2}{\partial y^1} & \frac{\partial g^2}{\partial y^2} & \cdots & \frac{\partial g^2}{\partial y^n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g^\ell}{\partial y^1} & \frac{\partial g^\ell}{\partial y^2} & \cdots & \frac{\partial g^\ell}{\partial y^n} \end{pmatrix}}_{(Jg)_{f(p)}}
\underbrace{\begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^m} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^2}{\partial x^2} & \cdots & \frac{\partial f^2}{\partial x^m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f^n}{\partial x^1} & \frac{\partial f^n}{\partial x^2} & \cdots & \frac{\partial f^n}{\partial x^m} \end{pmatrix}}_{(Jf)_p}
$$

**Cor** $\underset{\ell\times m}{J(g\circ f)_p} = \underset{\ell\times n}{(Jg)_{f(p)}}\,\underset{n\times m}{(Jf)_p}$



ビ分同

同相写像 [ ビ分同相写像 ]

## 微分同相写像

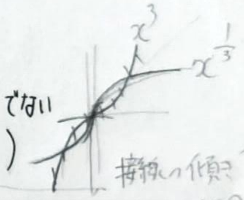
**Def** $M, N$ : $C^\infty$-mfd とする

- 写像 $f: M \longrightarrow N$ が 微分同相写像 (diffeomorphism)
  - (i) $f$ は同相写像 (つまり $f$:連続, 全単射, $f^{-1}$も連続)
  - (ii) $f, f^{-1}$ : ともに $C^\infty$級写像
- $M$と$N$が微分同相 (diffeomorphic) $\Longleftrightarrow$ $\exists f: M \longrightarrow N$ : 微分同相写像

**Ex** $M = N = \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$

(1) $f(x) = 2x$ ($\forall x \in \mathbb{R}$) は微分同相写像
($\because$ $f^{-1}(y) = \frac{1}{2}y$)

(2) $f(x) = x^3$ ($\forall x \in \mathbb{R}$) は微分同相写像でない
($\because$ $f^{-1}(y) = \sqrt[3]{y}$ : $y=0$で微分不可能)




接線の傾き $+\infty$

$\sqrt[3]{x}$ $\quad -\frac{1}{2}$ $\quad x^3$ $\quad \sqrt[3]{-\frac{1}{8}} = -\frac{1}{2}$

先程の **Prop** $f: M \to N$ $g: N \to Q$ : $C^\infty$-map
$p \in M$ に対して $\boxed{d(g\circ f)_p = (dg_{f(p)})\circ(df_p)}$
の系として 次を得る:

**Cor** (i) $d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM}$

$\mathrm{id}_X : X \to X$ は恒等写像
$\mathrm{id}_X(x) = x$ ($\forall x \in X$)

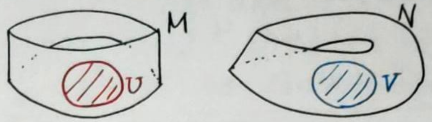
(ii) $f: M \to N$ : diffeo $\Rightarrow$ $df_p : T_pM \to T_{f(p)}N$ : 線形同型
であって, $(df^{-1})_{f(p)} = (df_p)^{-1}$

$d(g\circ f)_p = (dg_{f(p)})\circ(df_p)$

(proof)
$$
\begin{cases}
f^{-1}\circ f = \mathrm{id}_M & (df^{-1})_{f(p)}\circ(df)_p = \mathrm{id}_{T_pM} \\
f\circ f^{-1} = \mathrm{id}_N & (df)_p\circ(df^{-1})_{f(p)} = \mathrm{id}_{T_{f(p)}N}
\end{cases}
$$
(1)を使った

$$
\therefore
\begin{cases}
(df)_p : T_pM \longrightarrow T_{f(p)}N \\
(df^{-1})_{f(p)} : T_{f(p)}N \longrightarrow T_pM
\end{cases}
$$
はどちらも同型写像で互いに逆写像 //

**Def** $C^\infty$級写像 $f: M \to N$ が局所微分同相写像 (local diffeomorphism)
$\Longleftrightarrow$ ${}^\forall p \in M$ に対して
$\begin{cases} {}^\exists U : p \in M \text{ の開近傍} \\ {}^\exists V : f(p) \in N \text{ の 〃} \end{cases}$ s.t. $f: U \to V$ : 微分同相写像




$M$と$N$は同相でないが
局所微分同相写像は存在

**Cor** $f: M \to N$ : local diffeo
$\Longleftrightarrow$ $df_p : T_pM \to T_{f(p)}N$ : 線形同型 $({}^\forall p \in M)$

逆関数定理より

**Thm** (逆関数定理)
もし $df_p : T_pM \to T_{f(p)}N$ : 線形同型
$\Longrightarrow$ $\begin{cases} {}^\exists U : p \in M \text{ の開近傍} \\ {}^\exists V : f(p) \in N \text{ の 〃} \end{cases}$ s.t. $f: U \to V$ : 微分同相写像

**Ex** $f: \mathbb{C} \to \mathbb{C}$ ; $f(z) = z^2$
$df_z : T_z\mathbb{C} \to T_{f(z)}\mathbb{C}$ : 線形同型 $\Longleftrightarrow$ $z \neq 0$
$\therefore$ $f: \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ ; $f(z) = z^2$ は 局所微分同相写像
$(\mathbb{C}^\times := \mathbb{C} \setminus \{0\})$

$df_p = \sim\sim = (Jf)_p$
ヤコビ行列

$df_p$ : 線型同型写像 $\Longleftrightarrow$ ~~$(Jf)$~~ $(Jf)_p$ は可逆 $\Leftrightarrow$ $\left(\cdot\ (Jf)_p^{-1} \text{ が存在}\right)$
$\Leftrightarrow$ $\left(\cdot\ \det((Jf)_p) \neq 0\right)$

イメージ
$(x, y) \to (r, \theta)$ と同じ

$f: \mathbb{C} \longrightarrow \mathbb{C}$ , $f(z) = z^2$

$$\begin{array}{ccc} \mathbb{C} & & \mathbb{C} \\ \cup & & \cup \\ z & & w \\ = u + iv & & = \xi + i\eta \\ \leftrightarrow (u, v) & & \leftrightarrow (\xi, \eta) \end{array}$$

$$f(u+iv) = (u+iv)^2$$
$$= (u^2 - v^2) + 2iuv$$
$$= \underbrace{(u^2 - v^2)}_{\xi} + \underbrace{2uv}_{\eta} \cdot i$$

$\therefore$ $\begin{cases} \xi(u, v) = u^2 - v^2 \\ \eta(u, v) = 2uv \end{cases}$

$$(Jf)_{(u,v)} = \begin{bmatrix} \frac{\partial \xi}{\partial u} & \frac{\partial \xi}{\partial v} \\ \frac{\partial \eta}{\partial u} & \frac{\partial \eta}{\partial v} \end{bmatrix} = \begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix}$$

$\therefore$ $\det((Jf)_{(u,v)})$
$$= \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$
$$= 4(u^2 + v^2)$$
$\therefore$ $z = u + iv \neq 0 + 0i$
$\Longleftrightarrow$ $(df)_z$ は 線形同型