both symmetric and skew symmetric. A *differential s-form* is a skew-symmetric covariant tensor field of type $(0, s)$. For the calculus of differential forms, see, for example, [BG].

10. Lemma. Let μ be an n-form. If $V_i = \sum_{j=1}^{n} A_{ij} W_j$ for $1 \leq i \leq n$, then

$$\mu(V_1, \ldots, V_n) = (\det A)\mu(W_1, \ldots, W_n).$$

The proof is a standard combinatorial argument.

TENSOR DERIVATIONS

Previous sections have dealt with tensor algebra; we now consider some tensor calculus.

11. Definition. A *tensor derivation* $\mathscr{D}$ on a smooth manifold M is a set of $\boldsymbol{R}$-linear functions

$$\mathscr{D} = \mathscr{D}_s^r \colon \mathfrak{T}_s^r(M) \to \mathfrak{T}_s^r(M) \qquad (r \geq 0, s \geq 0)$$

such that for any tensors A and B:

(1) $\mathscr{D}(A \otimes B) = \mathscr{D}A \otimes B + A \otimes \mathscr{D}B$,
(2) $\mathscr{D}(CA) = C(\mathscr{D}A)$ for any contraction C.

Thus $\mathscr{D}$ is $\boldsymbol{R}$-linear, preserves tensor type, obeys the usual Leibnizian product rule, and commutes with all contractions. For a function $f \in \mathfrak{F}(M)$ recall that $fA = f \otimes A$; hence $\mathscr{D}(fA) = (\mathscr{D}f)A + f\mathscr{D}A$.

In the special case $t = s = 0$, $\mathscr{D}_0^0$ is a derivation on $\mathfrak{T}_0^0(M) = \mathfrak{F}(M)$ so, as discussed in Chapter 1, there is a unique vector field $V \in \mathfrak{X}(M)$ such that

$$\mathscr{D}f = Vf \qquad \text{for all} \quad f \in \mathfrak{F}(M).$$

Since tensor derivations are generally not $\mathfrak{F}(M)$-linear the value of $\mathscr{D}A$ at a point $p \in M$ cannot usually be found from A_p alone. However it can be found from the values of A on any arbitrarily small neighborhood of p. This local character of tensor derivations can be expressed as follows.

12. Proposition. If $\mathscr{D}$ is a tensor derivation on M and $\mathscr{U}$ is an open set of M, then there is a unique tensor derivation $\mathscr{D}_{\mathscr{U}}$ on $\mathscr{U}$ such that

$$\mathscr{D}_{\mathscr{U}}(A \mid \mathscr{U}) = (\mathscr{D}A) \mid \mathscr{U} \qquad \text{for all tensors} \quad A \quad \text{on} \quad M.$$

($\mathscr{D}_{\mathscr{U}}$ is called the *restriction* of $\mathscr{D}$ to $\mathscr{U}$, and henceforth we omit the subscript $\mathscr{U}$.)

Scheme of Proof. Let $B \in \mathfrak{T}^r_s(\mathscr{U})$. If $p \in \mathscr{U}$ let f be a bump function at p with support in $\mathscr{U}$. Thus $fB \in \mathfrak{T}^r_s(M)$. Define

$$(\mathscr{D}_{\mathscr{U}} B)_p = \mathscr{D}(fB)_p.$$

Then show: (1) This definition is independent of the choice of bump function. (2) $\mathscr{D}_{\mathscr{U}} B$ is a smooth tensor field on $\mathscr{U}$. (3) $\mathscr{D}_{\mathscr{U}}$ is a tensor derivation on $\mathscr{U}$. (4) $\mathscr{D}_{\mathscr{U}}$ has the stated restriction property. (5) $\mathscr{D}_{\mathscr{U}}$ is unique. ∎

The Leibnizian formula $\mathscr{D}(A \otimes B) = \mathscr{D}A \otimes B + A \otimes \mathscr{D}B$ can be recast as follows.

13. Proposition (The Product Rule). Let $\mathscr{D}$ be a tensor derivation on M. If $A \in \mathfrak{T}^r_s(M)$ then

$$\begin{aligned}\mathscr{D}[A(\theta^1,\ldots,\theta^r, X_1,\ldots,X_s)] &= (\mathscr{D}A)(\theta^1,\ldots,\theta^r, X_1,\ldots,X_s)\\ &\quad + \sum_{i=1}^{r} A(\theta^1,\ldots,\mathscr{D}\theta^i,\ldots,\theta^r, X_1,\ldots,X_s)\\ &\quad + \sum_{j=1}^{s} A(\theta^1,\ldots,\theta^r, X_1,\ldots,\mathscr{D}X_j,\ldots,X_s).\end{aligned}$$

(The placement of parentheses is crucial here: on the left-hand side $\mathscr{D}$ is applied to a function, on the right-hand side to the tensor A, to one-forms, and to vector fields.)

Proof. For simplicity let $r = s = 1$. We assert that

$$A(\theta, X) = \bar{C}(A \otimes \theta \otimes X),$$

where $\bar{C}$ is a composition of two contractions. In fact, relative to a coordinate system $A \otimes \theta \otimes X$ has components $A^i_j \theta_k X^l$, while $A(\theta, X) = \sum A^i_j \theta_i X^j$. Thus

$$\begin{aligned}\mathscr{D}(A(\theta, X)) &= \mathscr{D}\,\bar{C}(A \otimes \theta \otimes X) = \bar{C}\,\mathscr{D}(A \otimes \theta \otimes X)\\ &= \bar{C}(\mathscr{D}A \otimes \theta \otimes X) + \bar{C}(A \otimes \mathscr{D}\theta \otimes X) + \bar{C}(A \otimes \theta \otimes \mathscr{D}X)\\ &= (\mathscr{D}A)(\theta, X) + A(\mathscr{D}\theta, X) + A(\theta, \mathscr{D}X).\end{aligned}$$ ∎

For a $(1, s)$ tensor expressed as an $\mathfrak{F}(M)$-multilinear function $A: \mathfrak{X}(M)^s \to \mathfrak{X}(M)$ the tensor derivation obeys the same formal product rule, namely,

$$\mathscr{D}(A(X_1,\ldots,X_s)) = (\mathscr{D}A)(X_1,\ldots,X_s) + \sum_{i=1}^{s} A(X_1,\ldots,\mathscr{D}X_i,\ldots,X_s).$$

Both these versions of the product rule will frequently be solved for the term involving $\mathscr{D}A$. This gives a formula for $\mathscr{D}$ of an arbitrary tensor in terms of $\mathscr{D}$ applied solely to functions, vector fields, and one-forms. But for a one-form

$$(\mathscr{D}\theta)(X) = \mathscr{D}(\theta X) - \theta(\mathscr{D}X).$$

Thus functions and vector fields suffice.

14. Corollary. If tensor derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ agree on functions $\mathfrak{F}(M)$ and vector fields $\mathfrak{X}(M)$, then $\mathscr{D}_1 = \mathscr{D}_2$.

Furthermore, from suitable data on $\mathfrak{F}(M)$ and $\mathfrak{X}(M)$ we can construct a tensor derivation.

15. Theorem. Given a vector field $V \in \mathfrak{X}(M)$ and an $\boldsymbol{R}$-linear function $\delta\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

$$\delta(fX) = Vf\, X + f\,\delta(X) \qquad \text{for all} \quad f \in \mathfrak{F}(M), \quad X \in \mathfrak{X}(M),$$

there exists a unique tensor derivation $\mathscr{D}$ on M such that $\mathscr{D}_0^0 = V\colon \mathfrak{F}(M) \to \mathfrak{F}(M)$ and $\mathscr{D}_0^1 = \delta$.

Proof. $\mathscr{D}_0^0$ and $\mathscr{D}_0^1$ are given. The formula preceding Corollary 14 shows that $\mathscr{D}$ on a one-form θ must be defined by

$$(\mathscr{D}\theta)(X) = V(\theta X) - \theta(\delta X) \qquad \text{for all} \quad X \in \mathfrak{X}(M).$$

Using the formula given for δ it is easy to check that $\mathscr{D}\theta$ is $\mathfrak{F}(M)$-linear, hence is a one-form, and that $\mathscr{D} = \mathscr{D}_1^0\colon \mathfrak{X}^*(M) \to \mathfrak{X}^*(M)$ is $\boldsymbol{R}$-linear.

By the product rule (13), $\mathscr{D}$ on an (r, s) tensor A with $r + s \geq 2$ must be defined by

$$\begin{aligned}(\mathscr{D}A)(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s) = {} & V(A(\theta^1, \ldots, \theta^r, X_1, \ldots, X_s)) \\ & - \sum_{i=1}^{r} A(\theta^1, \ldots, \mathscr{D}\theta^i, \ldots, \theta^r, X_1, \ldots, X_s) \\ & - \sum_{j=1}^{s} A(\theta^1, \ldots, \theta^r, X_1, \ldots, \delta X_j, \ldots, X_s).\end{aligned}$$

(On the right-hand side, $\mathscr{D}$ of a one-form is defined as above.)

Again it is easy to verify that $\mathscr{D}A$ is $\mathfrak{F}(M)$-multilinear hence is an (r, s) tensor, and that $\mathscr{D}\colon \mathfrak{T}_s^r(M) \to \mathfrak{T}_s^r(M)$ is $\boldsymbol{R}$-linear. Furthermore, a direct computation shows that $\mathscr{D}(A \otimes B) = \mathscr{D}A \otimes B + A \otimes \mathscr{D}B$. (Take A and B of type (1, 1) to see how this works.)

To prove that $\mathscr{D}$ commutes with contraction, consider first the case $C\colon \mathfrak{T}_1^1(M) \to \mathfrak{F}(M)$. That $\mathscr{D}C = C\mathscr{D}$ on tensor products $\theta \otimes X$ is immediate from the definition of $\mathscr{D}$ on one-forms. Hence $\mathscr{D}C = C\mathscr{D}$ on sums of terms of the form $\theta \otimes X$. Since $\mathscr{D}$ is local and C pointwise it suffices to prove $\mathscr{D}C = C\mathscr{D}$ on coordinate neighborhoods. But there Lemma 5 shows that every (1, 1) tensor can be written as such a sum.

The extension to arbitrary contractions is an exercise in parentheses. Taking $A \in \mathfrak{T}^1_2(M)$ for example,

$$\begin{aligned}(\mathscr{D}C^1_2 A)(X) &= \mathscr{D}((C^1_2 A)(X)) - (C^1_2 A)(\mathscr{D}X) \\ &= \mathscr{D}(C\{A(\cdot, X, \cdot)\}) - C\{A(\cdot, \mathscr{D}X, \cdot)\} \\ &= C\{\mathscr{D}(A(\cdot, X, \cdot)) - A(\cdot, \mathscr{D}X, \cdot)\} \\ &= C\{(\mathscr{D}A)(\cdot, X, \cdot)\} = (C^1_2 \mathscr{D}A)(X).\end{aligned}$$

Hence $\mathscr{D}C^1_2 A = C^1_2 \mathscr{D}A$. ∎

Here is an application of the theorem.

16. Definition. If $V \in \mathfrak{X}(M)$ the tensor derivation L_V such that

$$L_V(f) = Vf \qquad \text{for all} \quad f \in \mathfrak{F}(M),$$
$$L_V(X) = [V, X] \qquad \text{for all} \quad X \in \mathfrak{X}(M)$$

is called the *Lie derivative* relative to V.

The definition is valid since L_V on vector fields satisfies the hypothesis on δ in the theorem:

$$L_V(fX) = [V, fX] = Vf\,X + f[V, X] = Vf\,X + fL_V X.$$

SYMMETRIC BILINEAR FORMS

Semi-Riemannian geometry involves a particular kind of (0, 2) tensor on tangent spaces. To study these in general, let V be a real vector space (finite-dimensional where the context so indicates). A *bilinear form* on V is an **R**-bilinear function $b: V \times V \to \mathbf{R}$, and we consider only the symmetric case: $b(v, w) = b(w, v)$ for all v, w.

17. Definition. A symmetric bilinear form b on V is

(1) *positive* [*negative*] *definite* provided $v \neq 0$ implies $b(v, v) > 0$ [<0],
(2) *positive* [*negative*] *semidefinite* provided $b(v, v) \geq 0$ [≤ 0] for all $v \in V$,
(3) *nondegenerate* provided $b(v, w) = 0$ for all $w \in V$ implies $v = 0$.

Also b is *definite* [*semidefinite*] provided either alternative in (1) [(2)] holds. If b is definite then it is obviously both semidefinite and nondegenerate; the converse follows from Exercise 12.