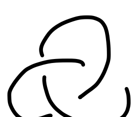
A diagrammatic approach to the Rasmussen invariant via tangles & cobordisms.

Taketo Sano (RIKEN)

2025-04-18 / 横国大幾何トポロジーセミナー

(Reference : arxiv:2503.05414 / slides are available online)

Overview

$K = $ 

a knot / link in $\mathbb{R}^3$

$$K \xmapsto{\ J\ } J(K)$$

(Jones, 1984)

$$J(K) = q^1 + q^3 + q^5 - q^9$$

— the <u>Jones polynomial</u> of K.

Kh

Categorify

(Khovanov, 2000)

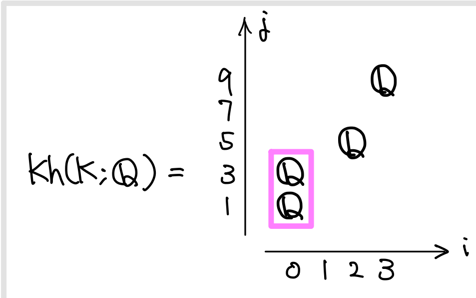
$Kh(K;\mathbb{Q}) =$

$j \backslash i$	0	1	2	3
9				$\mathbb{Q}$
7				
5			$\mathbb{Q}$	
3	$\mathbb{Q}$			
1	$\mathbb{Q}$			

— the <u>Khovanov homology</u> of K

$$\hat{\chi} = \sum_{i,j} (-1)^i q^j \dim_{\mathbb{Q}}(-)$$

Overview



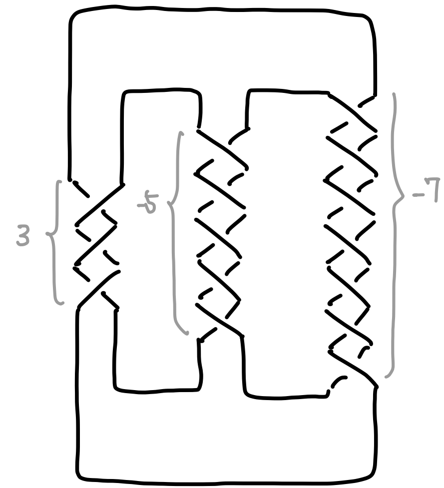
$\rightsquigarrow$ $\boxed{s(K) = 2}$

— the Rasmussen invariant of K.

s is algorithmically computable,
but its geometric interpretation remains a mystery.

We give a "diagrammatic description" of s based on
Bar-Natan's reformulation of Kh. homology via tangles & cobordisms.

Overview. (Application)



$P = P(3,-5,-7)$

computable by hand!

- $\boxed{s(P) \neq 0.}$

  $\Longrightarrow$ P is NOT smoothly slice.

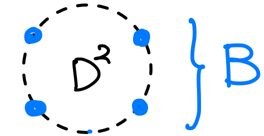
- $\Delta_P = 1$ (Alexander poly).

  $\Longrightarrow$ P is topologically slice

$\rightsquigarrow$ Such knot gives rise to an exotic $\mathbb{R}^4$.

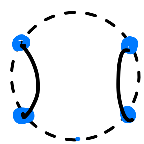
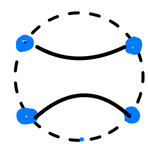
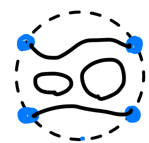
Contents.

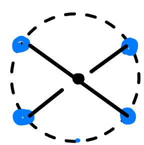
NEW!

§1. Fix $B \subset \partial D^2$, a set of even num. of pts. 

$\rightsquigarrow$ $\boxed{\mathrm{Cob}_\bullet(B)}$ — the (preadditive) category of (1+1)-dotted cobordisms

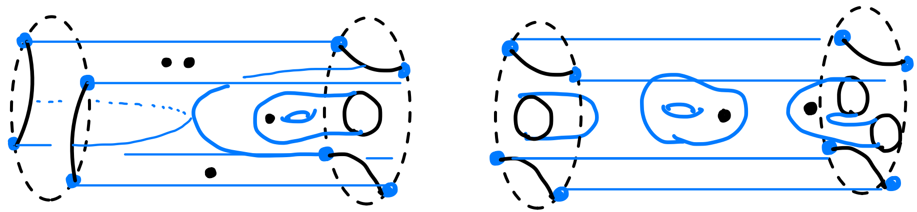
objects ... $\{\text{compact } \overbrace{\text{1-mfd}}^{\text{unoriented}}\ X \subset D^2,\ \ \partial X = B\}$

 ,  ,  , ...

( $\notin$ $\mathrm{Cob}_\bullet(B)$)

morphisms ... (next page)

morphisms $\cdots$ $\mathbb{Z}\left\{ \begin{array}{l} S : X \longrightarrow Y \;; \\ \text{dotted surface } S \subset D^2 \times I, \\ \partial S = X \times \{0\} \cup B \times I \cup Y \times \{1\} \end{array} \right\} \Big/ \begin{array}{l}\text{isotopy}\\ \text{rel } \partial.\end{array}$

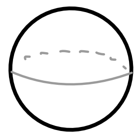
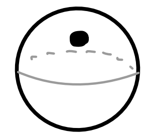


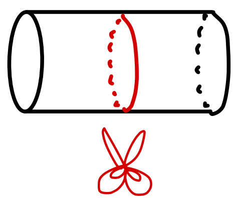
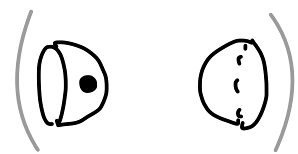
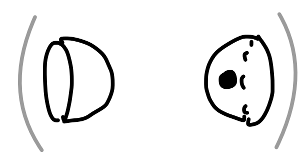
$X \xrightarrow{S} Y \qquad Y \xrightarrow{S'} Z$

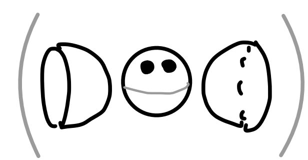
$S' \circ S$ (composition)

<u>Remark</u> Objects of $Cob_\bullet(B)$ are NOT sets!

Local relations in $\mathrm{Hom}_{\mathrm{Cob}_\bullet(B)}$

(S)  $= 0$, (S.)  $= 1$,

(NC)  $=$  $+$ 

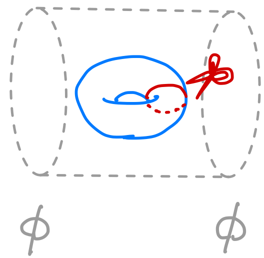
$-$ 

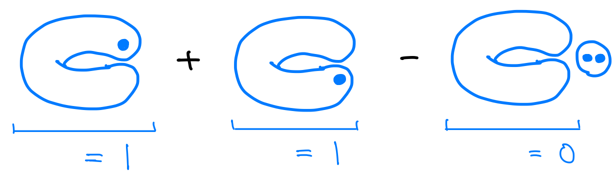
$\boxed{\mathrm{Cob}_{\bullet/\ell}(B)} := \mathrm{Cob}_\bullet(B)$ modulo local relations.

Consider when $B = \emptyset$; $X, Y = \emptyset$.

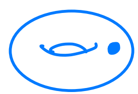
$$\mathrm{Hom}_{Cob_{\bullet/\ell}(\emptyset)}(\emptyset, \emptyset) = \mathbb{Z}\{\underline{\text{closed}}\ \text{dotted surface}\} \Big/ \substack{\text{isotopy} \\ \text{\& loc. rel}}$$

e.g.

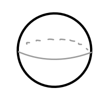
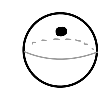
1) 

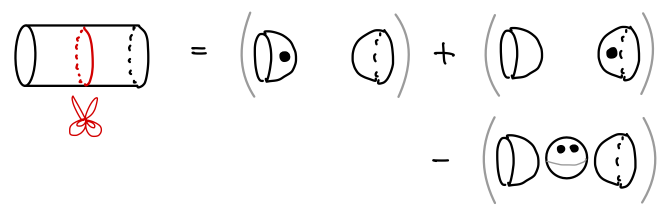
$\overset{(NC)}{=}$ 

$= 2 \cdot \emptyset$

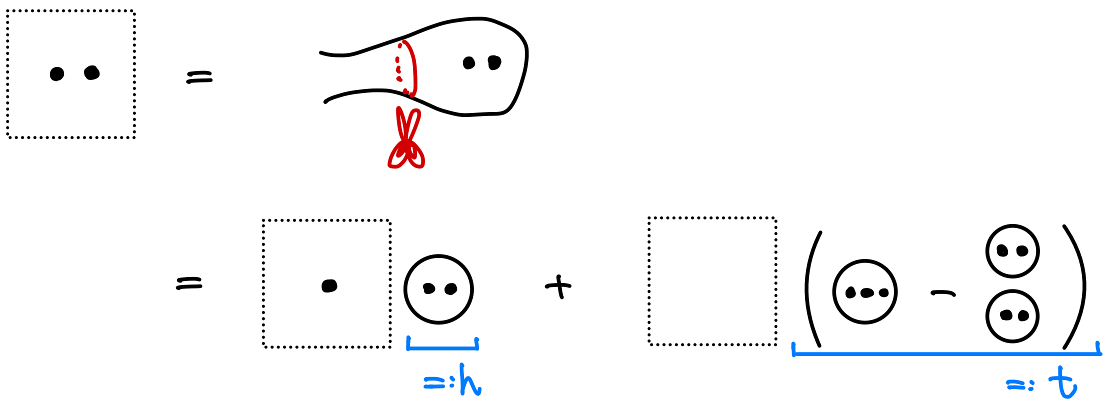
2)  $= 2$ 

3)  $=$ 

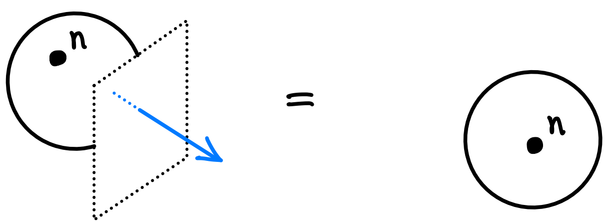
(S)  $= 0$, (S.)  $= 1$,

(NC) 

Lem

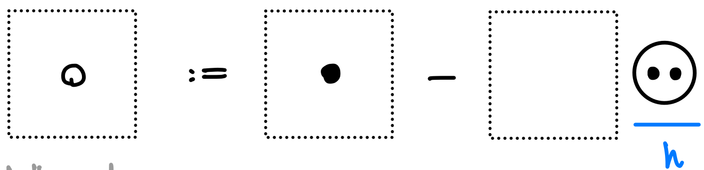


Lem

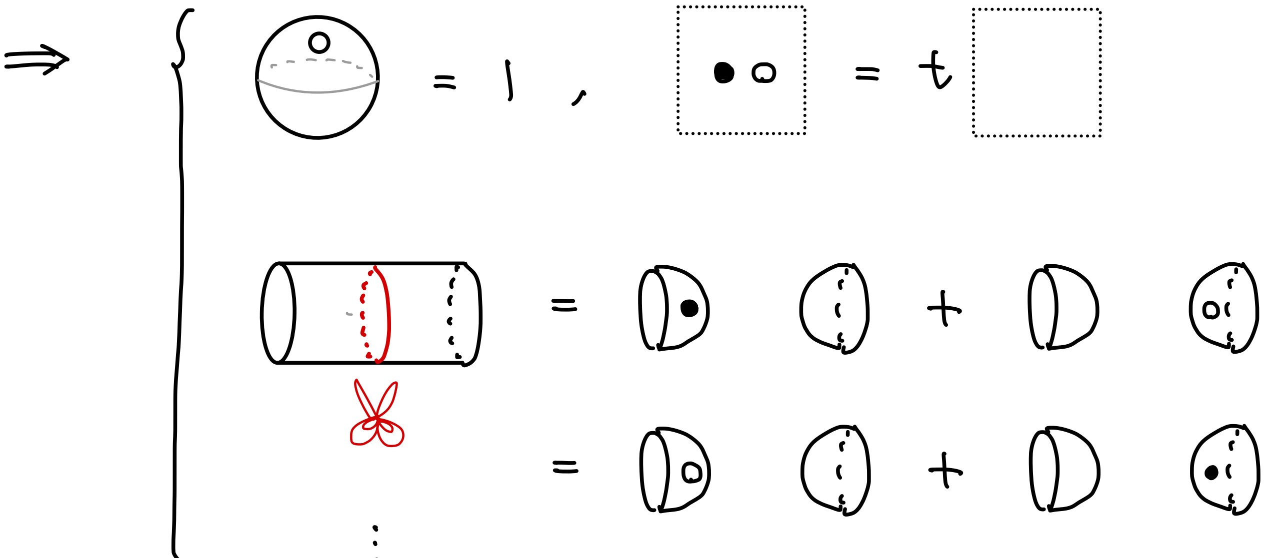


(dotted spheres can pass through walls.)

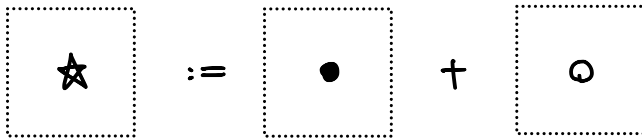
Notation



hollow dot

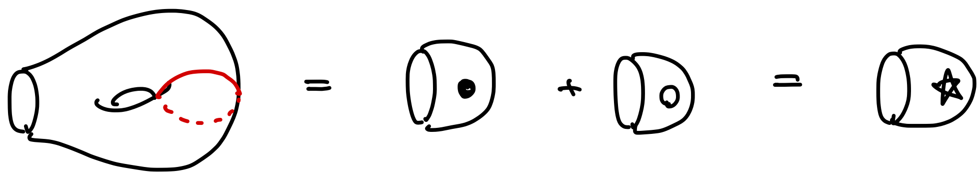


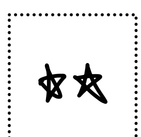
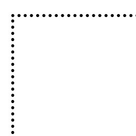
Notation

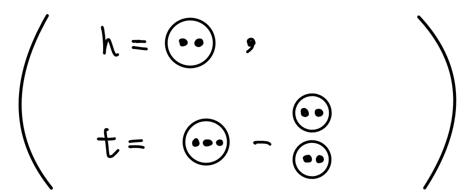


star

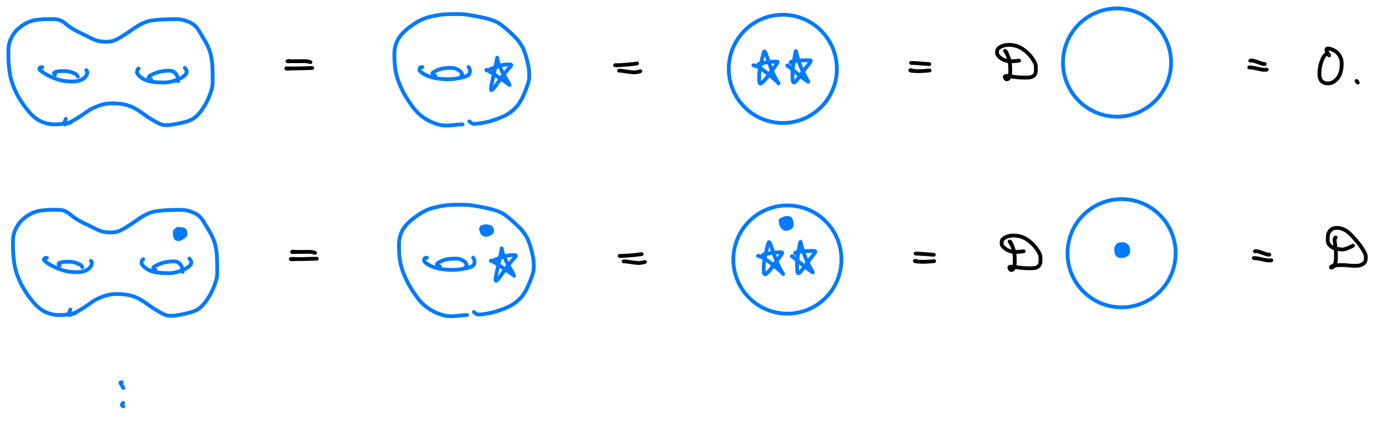
$\Longrightarrow$



 =  $\underbrace{(h^2 + 4t)}_{=:\, D}$



Now, we can easily compute ...



Prop

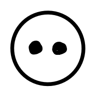
$$\left.\begin{array}{ll} \Sigma_{2g} = 0, & \Sigma_{2g+1} = 2\mathcal{D}^{g} \\ \Sigma^{\bullet}_{2g} = \mathcal{D}^{g}, & \Sigma^{\bullet}_{2g+1} = h\,\mathcal{D}^{g} \end{array}\right\} \in \mathbb{Z}[h,t]$$

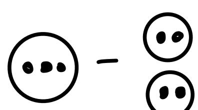
Prop. The evaluation of closed surfaces give

$$\mathrm{Hom}_{\mathrm{Cob}_{\bullet/\ell}(\emptyset)}(\emptyset, \emptyset) \cong \mathbb{Z}[h,t] \quad (=: R)$$

as graded rings. $\left(\deg S = \chi(S) - 2 \cdot \left(\begin{smallmatrix}\text{number of}\\ \text{dots}\end{smallmatrix} \right) \right)$

$\emptyset \longmapsto 1$ (deg 0)

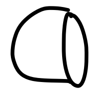
 $\longmapsto h$ (deg -2)

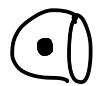
 $\longmapsto t$ (deg -4)

Prop.

grading shift

$$\mathrm{Hom}_{Cob_{\bullet/\ell}(\phi)}(\phi, \bigcirc) \cong R\{1\} \oplus R\{-1\} \quad (=: A)$$

 (1) $\longmapsto (1, 0)$

 (-1) $\longmapsto (0, 1)$

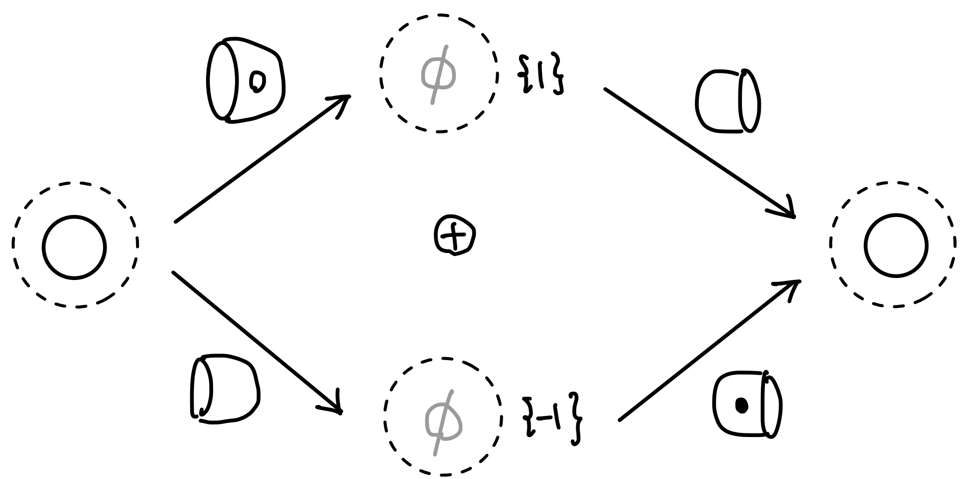
$$\mathrm{Hom}_{Cob_{\bullet/\ell}(\phi)}(\phi, \underbrace{\bigcirc \cdots \bigcirc}_{r}) \cong \underbrace{A \otimes \cdots \otimes A}_{r}$$

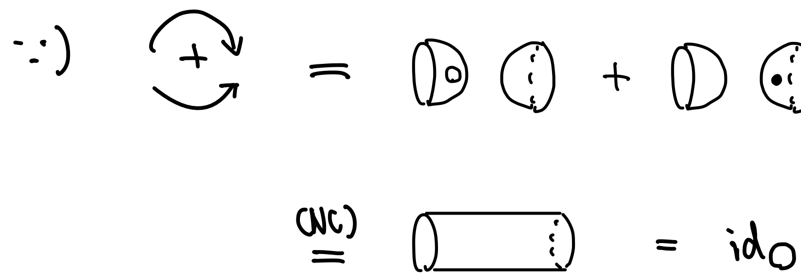
To prove this ...

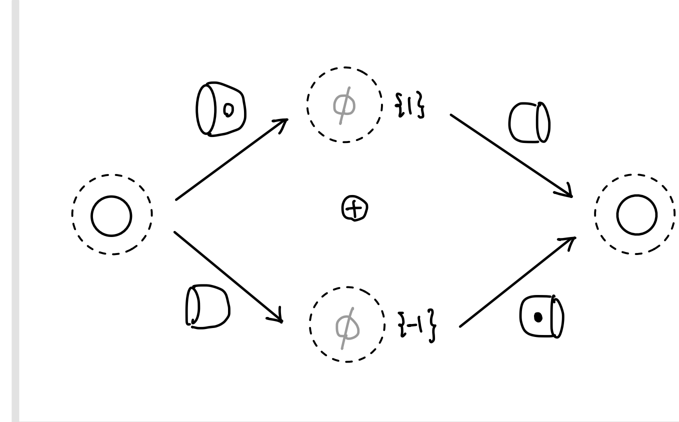
(<u>Note</u> Here, $\mathrm{Cob}_{\bullet/\ell}$ is turned into an additive cat. by taking its add. closure.)

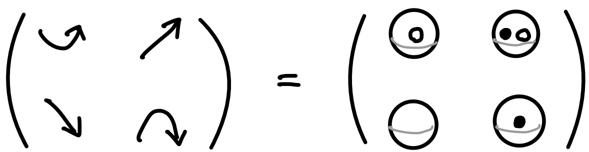
<u>Key Prop 1.</u> (Delooping)

In $\mathrm{Cob}_{\bullet/\ell}(B)$, the following isomorphisms hold.





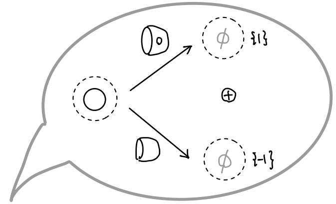




$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

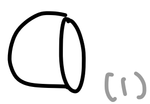
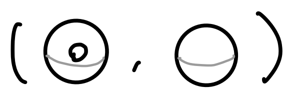
$$= \mathrm{id}_{\emptyset \oplus \emptyset}$$

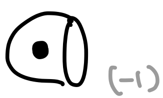
We have,



$$\mathrm{Hom}_{\mathrm{Cob}_{\bullet/\ell}(\emptyset)}(\emptyset, \bigcirc) \cong \mathrm{Hom}_{\mathrm{Cob}_{\bullet/\ell}(\emptyset)}(\emptyset, \emptyset)\{1\} \oplus \mathrm{Hom}_{\mathrm{Cob}_{\bullet/\ell}(\emptyset)}(\emptyset, \emptyset)\{-1\}$$

$$\cong R\{1\} \oplus R\{-1\}$$

 $\longmapsto$  $= (1, 0) =: 1$

 $\longmapsto$  $= (0, 1) =: X$

$A = R\langle 1, X\rangle$ can be endowed a (commutative) Frobenius algebra structure as follows:

1) R-alg. str. $\left(A = R[X]/(X^2 - hX - t) \right)$

$$m : A \otimes A \longrightarrow A \qquad , \qquad \iota : R \longrightarrow A$$

$$\begin{aligned} 1 \otimes 1 &\longmapsto 1 & \qquad 1 &\longmapsto 1 \\ 1 \otimes X,\ X \otimes 1 &\longmapsto X \\ X \otimes X &\longmapsto hX + t1 \end{aligned}$$

2) R-coalg. str.

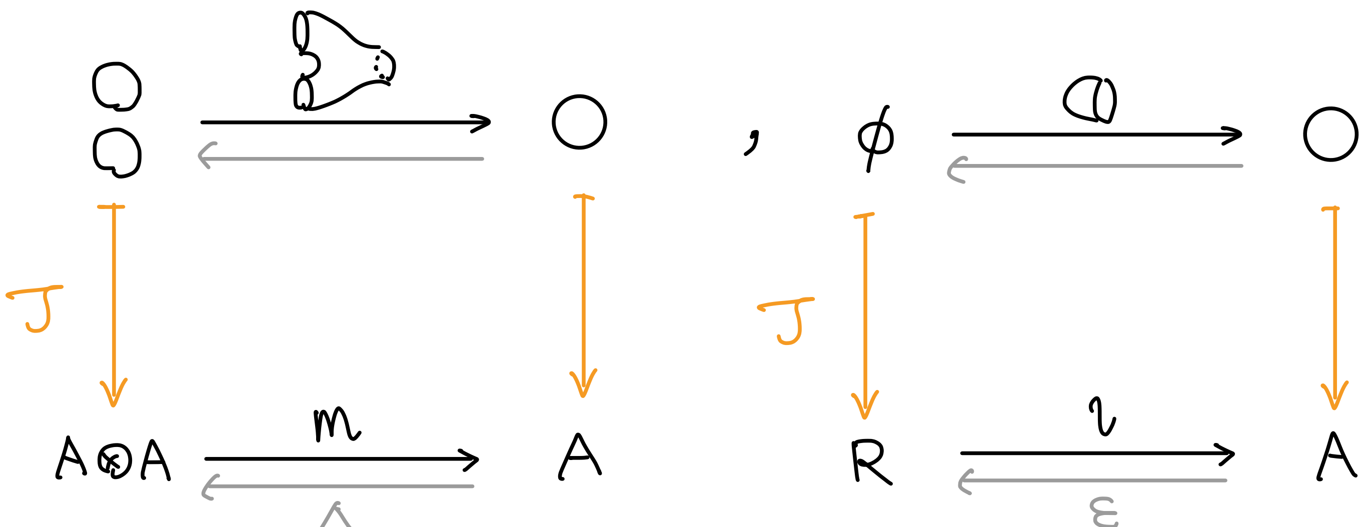
$$\Delta : A \longrightarrow A \otimes A \qquad\qquad \varepsilon : A \longrightarrow R$$

$$\begin{aligned} 1 &\longmapsto 1 \otimes X + X \otimes 1 - h(1 \otimes 1) & \qquad 1 &\longmapsto 0 \\ X &\longmapsto X \otimes X + t(1 \otimes 1) & X &\longmapsto 1. \end{aligned}$$

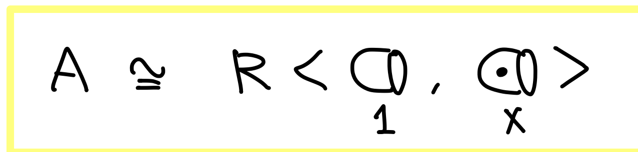
Prop The "tautological functor" (representable functor)

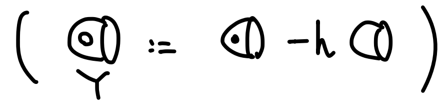
$$\mathcal{J} = \mathrm{Hom}_{\mathrm{Cob}_{/\!\ell}(\phi)}(\phi, -) : \mathrm{Cob}_{/\!\ell}(\phi) \longrightarrow \mathrm{Mod}_R$$

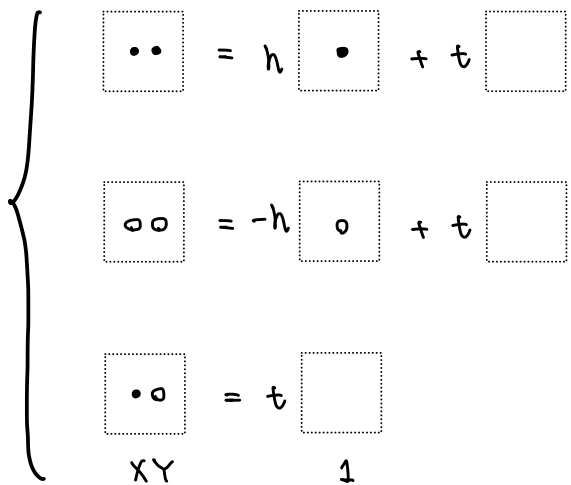
coincides with the $(1+1)$-TQFT $\mathcal{F}_A$ obtained from the Frobenius algebra A.



Conclusion The algebraic ingredients used to construct Khovanov homology is recovered from the loc. relations !







$(X^2 = hX + t)$

$(Y^2 = -hY + t)$

$(XY = t)$

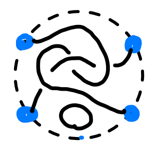
Contents.

§2. Diag(B) ... the cat. of tangle diagrams with $\partial = B$.

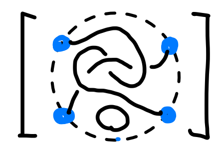
The formal Khovanov bracket

The cat. of chain complexes in $\mathrm{Cob}^3_{/l}(B)$.

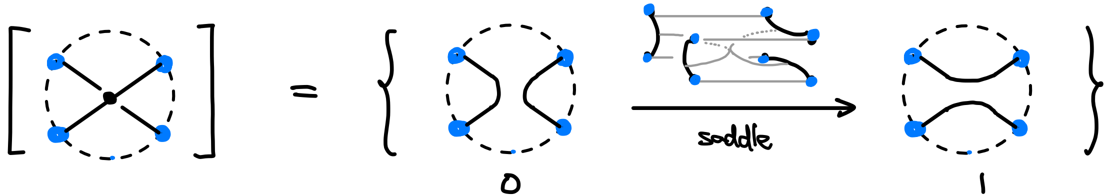
$$[-] : \mathrm{Diag}(B) \longrightarrow \mathrm{Kom}(\mathrm{Cob}^3_{/l}(B))$$

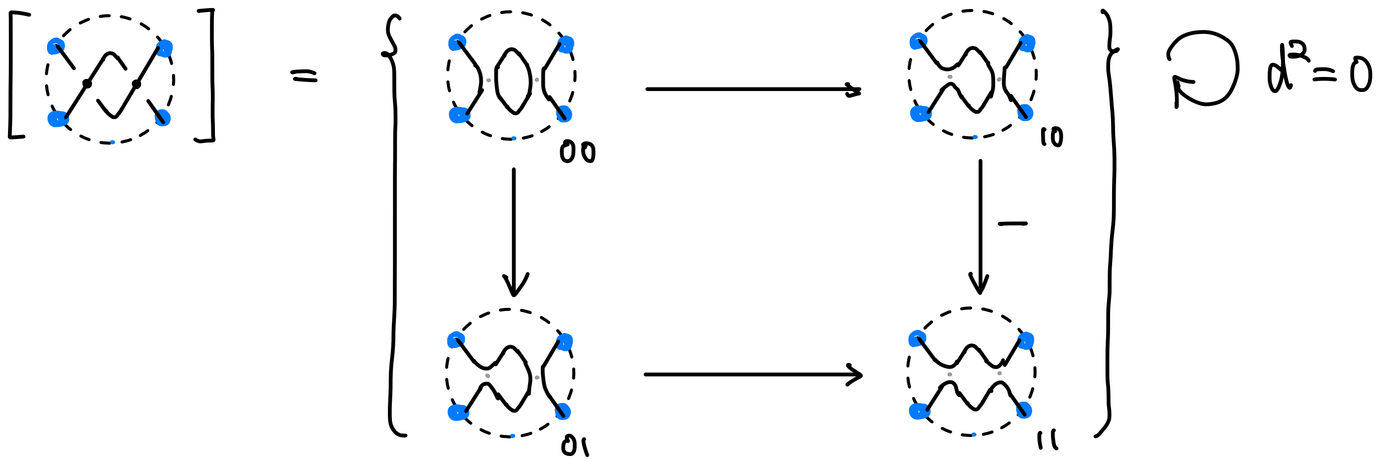


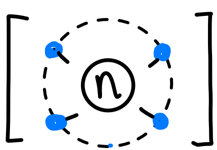
$\longmapsto$



is defined as follows ...





 $=$ { n-dim. anti-commutative cube in $Cob_{/\ell}(B)$ } $d^2 = 0$

<u>Remark</u>. The "chain complex" $[T]$ is NOT a set!

i.e. $\boxed{x \in [T]}$ does NOT make sense.

However, $\boxed{d^2 = 0}$ makes sense in $\mathrm{Cob}^{\circ}_{/e}(B)$.

Furthermore,

$$\left\{ \begin{array}{lll} - & \text{chain maps} & (f \circ d = d \circ f) \\ - & \text{chain homotopies} & (f - g = d \circ h + h \circ d) \\ - & \text{chain homotopy equivalences} & (g \circ f \simeq id,\ f \circ g \simeq id) \end{array} \right.$$

do make sense!

Thm [Bar-Natan '05]

The chain homotopy type of $[T]$ is invariant under the Reidemeister moves.

In particular, when $B = \emptyset$, the composition

$$\begin{array}{ccccccc}
\mathrm{Diag}(\emptyset) & \xrightarrow{[-]} & \mathrm{Kom}(\mathrm{Cob}_{/\ell}(\emptyset)) & \xrightarrow[\text{(TQFT)}]{\mathcal{F}_A} & \mathrm{Kom}(\mathrm{Mod}_R) & \xrightarrow[\text{(homology)}]{H} & \mathrm{Mod}_R \\
\in & & \in & & \in & & \in \\
K & \longmapsto & [K] & \longmapsto & CKh(K) & \longmapsto & Kh(K)
\end{array}$$

coincides with the Khovanov homology functor.

To prove this ...

Key Prop 2 (Gaussian elimination)

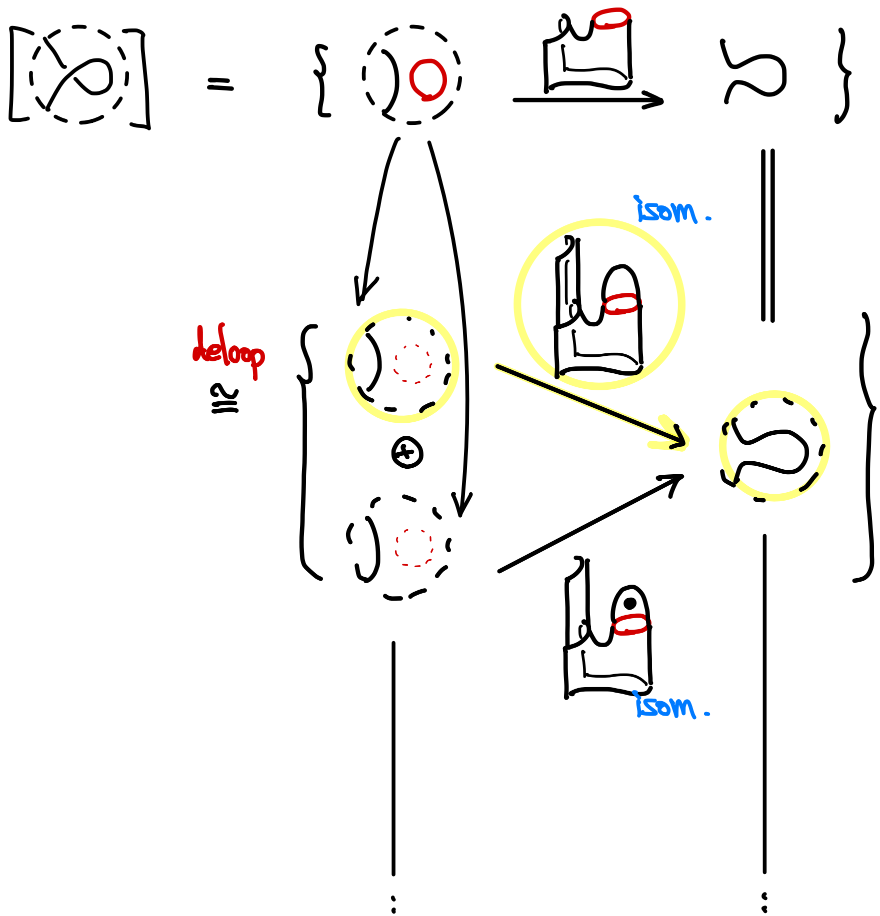
In $\mathrm{Kom}(\mathrm{Cob}_{\bullet/l}(B))$, we have

$$\begin{tikzcd}
\Omega \arrow[d, "\cong"', "\text{isom.}"] & = & \{\cdots \arrow[r] & (X) \arrow[r, "\binom{x}{y}"] \arrow[d, equal] & \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \arrow[r, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & a^{-1}b \\ & 1 \end{pmatrix}"'] & \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix} \arrow[r, "(z\ w)"] \arrow[d, "\begin{pmatrix} 1 & \\ -ca^{-1} & 1 \end{pmatrix}"] & W \arrow[r] \arrow[d, equal] & \cdots\} \\
\Omega' \arrow[d, "\simeq"', "\text{htpy eq.}"] & = & \{\cdots \arrow[r] & (X) \arrow[r, "\binom{0}{y}"] \arrow[d, equal] & \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \arrow[r, "\begin{pmatrix} a & 0 \\ 0 & s \end{pmatrix}"] \arrow[d, "\simeq"'] & \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix} \arrow[r, "(0\ w)"] \arrow[d, "\simeq"'] & W \arrow[r] \arrow[d, equal] & \cdots\} \\
\Omega'' & = & \{\cdots \arrow[r] & (X) \arrow[r, "y"] & (Y_1) \arrow[r, "s"] & (Z_1) \arrow[r, "w"] & W \arrow[r] & \cdots\}
\end{tikzcd}$$

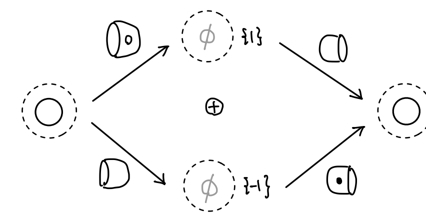
invertible

$s = d - ca^{-1}b$

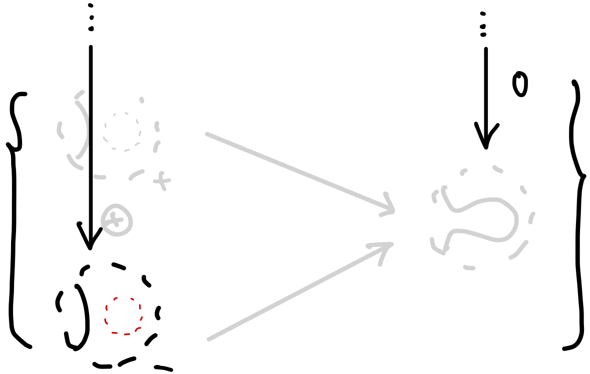
Proof of Thm (only for R1)



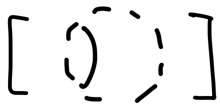
Delooping



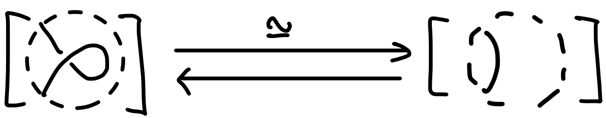
elim.
$\simeq$



$\simeq$



$\therefore$ We get explicit ch. htpy equivalences

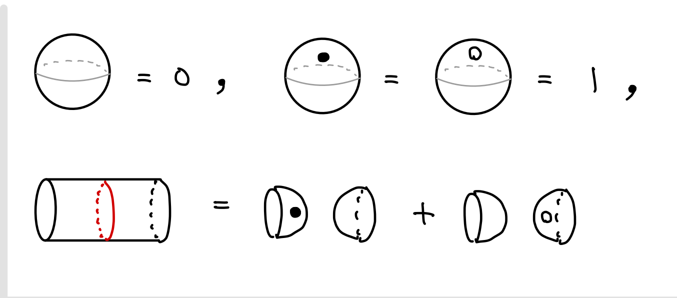


$\blacksquare$

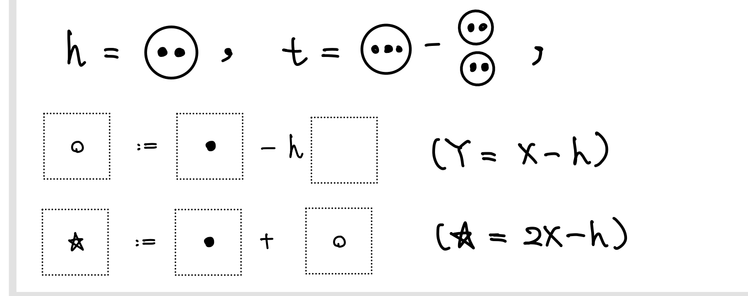
Contents.

Cheet Sheet

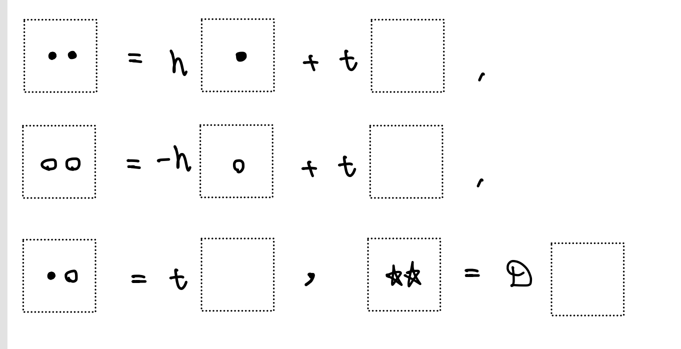
Local relations



Notations



Reductions



Delooping