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ISOMETRIC IMMERSIONS INTO $\mathbb{S}^n \times \mathbb{R}$ AND $\mathbb{H}^n \times \mathbb{R}$ AND APPLICATIONS TO MINIMAL SURFACES

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ABSTRACT. We give a necessary and sufficient condition for an n-dimensional Riemannian manifold to be isometrically immersed in $\mathbb{S}^n \times \mathbb{R}$ or $\mathbb{H}^n \times \mathbb{R}$ in terms of its first and second fundamental forms and of the projection of the vertical vector field on its tangent plane. We deduce the existence of a one-parameter family of isometric minimal deformations of a given minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$, obtained by rotating the shape operator.

1. INTRODUCTION

It is well known that the first and second fundamental forms of a hypersurface of a Riemannian manifold satisfy two compatibility equations called the Gauss and Codazzi equations. More precisely, let $\bar{\mathcal{V}}$ be an orientable Riemannian manifold of dimension $n+1$ and $\mathcal{V}$ a submanifold of $\bar{\mathcal{V}}$ of dimension n. Let ∇ (respectively, $\bar{\nabla}$) be the Riemannian connection of $\mathcal{V}$ (respectively, $\bar{\mathcal{V}}$), R (respectively, $\bar{\mathrm{R}}$) be the Riemann curvature tensor of $\mathcal{V}$ (respectively, $\bar{\mathcal{V}}$), i.e.,

$$\mathrm{R}(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z,$$

and S be the shape operator of $\mathcal{V}$ associated to its unit normal N, i.e., $\mathrm{S}X = -\bar{\nabla}_X N$. Then the following equations hold for all vector fields X, Y, Z, W on $\mathcal{V}$:

$$\langle \mathrm{R}(X,Y)Z, W\rangle - \langle \bar{\mathrm{R}}(X,Y)Z, W\rangle = \langle \mathrm{S}X, Z\rangle\langle \mathrm{S}Y, W\rangle - \langle \mathrm{S}Y, Z\rangle\langle \mathrm{S}X, W\rangle,$$

$$\nabla_X \mathrm{S}Y - \nabla_Y \mathrm{S}X - \mathrm{S}[X,Y] = \bar{\mathrm{R}}(X,Y)N.$$

These are respectively the Gauss and Codazzi equations.

In the case where $\bar{\mathcal{V}}$ is a space form, i.e., the sphere $\mathbb{S}^{n+1}$, the Euclidean space $\mathbb{R}^{n+1}$ or the hyperbolic space $\mathbb{H}^{n+1}$, these equations become the following:

$$\begin{aligned} \langle \mathrm{R}(X,Y)Z, W\rangle - \kappa(\langle X, Z\rangle\langle Y, W\rangle - \langle Y, Z\rangle\langle X, W\rangle) \\ = \langle \mathrm{S}X, Z\rangle\langle \mathrm{S}Y, W\rangle - \langle \mathrm{S}X, W\rangle\langle \mathrm{S}Y, Z\rangle, \end{aligned} \tag{1}$$

$$\nabla_X \mathrm{S}Y - \nabla_Y \mathrm{S}X - \mathrm{S}[X,Y] = 0, \tag{2}$$

where κ is the sectional curvature of $\bar{\mathcal{V}}$, i.e., $\kappa = 1, 0, -1$ for $\mathbb{S}^{n+1}$, $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$ respectively. Thus the Gauss and Codazzi equations only involve the first and second fundamental forms of $\mathcal{V}$; they are defined *intrinsically* on $\mathcal{V}$ (as soon as we know S). This comes from the fact that these ambient spaces are isotropic. Moreover, in this case the Gauss and Codazzi equations are also sufficient conditions

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for an n-dimensional simply connected manifold to be immersed into $\bar{\mathcal{V}}$ with given first and second fundamental forms: if $\mathcal{V}$ is a Riemannian manifold endowed with a field S of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\mathcal{V} \to \mathrm{T}_y\mathcal{V}$ such that (1) and (2) hold (where R denotes the Riemann curvature tensor of $\mathcal{V}$), then there exists an isometric immersion from $\mathcal{V}$ into $\bar{\mathcal{V}}$ with S as the shape operator. The reader can refer to [Car92] and also to [Ten71] for a proof in the case of $\mathbb{R}^{n+1}$.

In the case of a general manifold $\bar{\mathcal{V}}$, the Gauss and Codazzi equations are not defined intrinsically on $\mathcal{V}$, since the Riemann curvature tensor of the ambient space $\bar{\mathcal{V}}$ is involved. Yet, in the case where $\bar{\mathcal{V}} = \mathbb{S}^n \times \mathbb{R}$ or $\bar{\mathcal{V}} = \mathbb{H}^n \times \mathbb{R}$, these equations are well defined as soon as we know:

(1) the projection T of the vertical vector $\frac{\partial}{\partial t}$ (corresponding to the factor $\mathbb{R}$) onto the tangent space of $\mathcal{V}$,
(2) the normal component ν of $\frac{\partial}{\partial t}$, i.e., $\nu = \langle N, \frac{\partial}{\partial t}\rangle$.

Indeed, the Gauss and Codazzi equations become the following:

$$\begin{aligned} \mathrm{R}(X,Y)Z &= \langle \mathrm{S}X, Z\rangle \mathrm{S}Y - \langle \mathrm{S}Y, Z\rangle \mathrm{S}X \\ &\quad +\kappa(\langle X,Z\rangle Y - \langle Y,Z\rangle X - \langle Y,T\rangle\langle X,Z\rangle T \\ &\quad -\langle X,T\rangle\langle Z,T\rangle Y + \langle X,T\rangle\langle Y,Z\rangle T + \langle Y,T\rangle\langle Z,T\rangle X), \\ \nabla_X \mathrm{S}Y - \nabla_Y \mathrm{S}X - \mathrm{S}[X,Y] &= \kappa\nu(\langle Y,T\rangle X - \langle X,T\rangle Y), \end{aligned}$$

where $\kappa = 1$ and $\kappa = -1$ for $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ respectively.

The Gauss equation can be formulated in the following equivalent way: the sectional curvature $K(P)$ (for the metric of $\mathcal{V}$) of every plane $P \subset \mathrm{T}\mathcal{V}$ satisfies

$$K(P) = \det \mathrm{S}_P + \kappa(1 - ||T_P||^2),$$

where S_P is the restriction of S on P and T_P is the orthogonal projection of T on P.

The first aim of this paper is to give a necessary and sufficient condition in order for a Riemannian manifold with a symmetric operator S to be isometrically immersed into $\mathbb{S}^n \times \mathbb{R}$ or $\mathbb{H}^n \times \mathbb{R}$ with S as shape operator. More precisely, we prove the following Theorem.

Theorem (Theorem 3.3). *Let $\mathcal{V}$ be a simply connected Riemannian manifold of dimension n, $\mathrm{d}s^2$ its metric (which we also denote by $\langle\cdot,\cdot\rangle$) and ∇ its Riemannian connection. Let S be a field of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\mathcal{V} \to \mathrm{T}_y\mathcal{V}$, T a vector field on $\mathcal{V}$ and ν a smooth function on $\mathcal{V}$ such that $||T||^2 + \nu^2 = 1$.*

Let $\mathbb{M}^n = \mathbb{S}^n$ or $\mathbb{M}^n = \mathbb{H}^n$. Assume that $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ satisfies the Gauss and Codazzi equations for $\mathbb{M}^n \times \mathbb{R}$ and the following equations:

$$\nabla_X T = \nu \mathrm{S}X, \quad \mathrm{d}\nu(X) = -\langle \mathrm{S}X, T\rangle.$$

Then there exists an isometric immersion $f : \mathcal{V} \to \mathbb{M}^n \times \mathbb{R}$ such that the shape operator with respect to the normal N associated to f is

$$\mathrm{d}f \circ \mathrm{S} \circ \mathrm{d}f^{-1}$$

and such that

$$\frac{\partial}{\partial t} = \mathrm{d}f(T) + \nu N.$$

Moreover the immersion is unique up to a global isometry of $\mathbb{M}^n \times \mathbb{R}$ preserving the orientations of both $\mathbb{M}^n$ and $\mathbb{R}$.

The two additional conditions come from the fact that the vertical vector field $\frac{\partial}{\partial t}$ is parallel.

The method to prove this theorem is similar to that of Tenenblat ([Ten71]): it is based on differential forms, moving frames and integrable distributions.

This work was motivated by the study of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. There were many recent developments in the theory of these surfaces. Rosenberg ([Ros02b]) studied the geometry of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$, and more generally in $M \times \mathbb{R}$ where M is a surface of non-negative curvature. Nelli and Rosenberg ([NR02]) studied minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and proved a Jenkins-Serrin theorem. Hauswirth ([Hau06]) constructed many examples in $\mathbb{H}^2 \times \mathbb{R}$. Meeks and Rosenberg ([MR05]) initiated the theory of minimal surfaces in $M \times \mathbb{R}$ where M is a compact surface. Recently, Abresch and Rosenberg ([AR04]) extended the notion of a holomorphic Hopf differential to constant mean curvature surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$; using this holomorphic differential, they proved that all immersed constant mean curvature spheres are embedded and rotational.

In this paper, we use our Theorem 3.3 to prove the existence of a one-parameter family of isometric minimal deformations of a given minimal surface in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$. This family is obtained by rotating the shape operator; hence it is the analog of the associate family of a minimal surface in $\mathbb{R}^3$. This is the following theorem.

Theorem (Theorem 4.2)**.** *Let Σ be a simply connected Riemann surface and $x : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ a conformal minimal immersion. Let N be the induced normal. Let S be the symmetric operator on Σ induced by the shape operator of $x(\Sigma)$. Let T be the vector field on Σ such that $\mathrm{d}x(T)$ is the projection of $\frac{\partial}{\partial t}$ onto $\mathrm{T}(x(\Sigma))$. Let $\nu = \langle N, \frac{\partial}{\partial t} \rangle$.*

Let $z_0 \in \Sigma$. Then there exists a unique family $(x_\theta)_{\theta \in \mathbb{R}}$ of conformal minimal immersions $x_\theta : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ such that:

1. *$x_\theta(z_0) = x(z_0)$ and $(\mathrm{d}x_\theta)_{z_0} = (\mathrm{d}x)_{z_0}$,*
2. *the metrics induced on Σ by x and x_θ are the same,*
3. *the symmetric operator on Σ induced by the shape operator of $x_\theta(\Sigma)$ is $e^{\theta \mathrm{J}}\mathrm{S}$,*
4. *$\frac{\partial}{\partial t} = \mathrm{d}x_\theta(e^{\theta \mathrm{J}}T) + \nu N_\theta$, where N_θ is the unit normal to x_θ.*

Moreover we have $x_0 = x$, and the family (x_θ) is continuous with respect to θ.

In particular taking $\theta = \frac{\pi}{2}$ defines a conjugate surface; the geometric properties of conjugate surfaces in $\mathbb{M}^2 \times \mathbb{R}$ and in $\mathbb{R}^3$ are similar. Finally, we give examples of conjugate surfaces. In $\mathbb{S}^2 \times \mathbb{R}$, we show that helicoids and unduloids are conjugate. In $\mathbb{H}^2 \times \mathbb{R}$, we show that helicoids are conjugated to catenoids or to minimal surfaces foliated by horizontal curves of constant curvature belonging to the Hauswirth family (see [Hau06]).

2. Preliminaries

Notation. In this paper we will use the following index conventions: Latin letters i, j, etc., denote integers between 1 and n, and Greek letters α, β, etc., denote integers between 0 and $n+1$. For example, the notation $A^i_j = B^i_j$ means that this relation holds for all integers i, j between 1 and n, and the notation $\sum_\alpha C_\alpha$ means $C_0 + C_1 + \cdots + C_{n+1}$.

The set of vector fields on a Riemannian manifold $\mathcal{V}$ will be denoted by $\mathfrak{X}(\mathcal{V})$.

We denote by $\frac{\partial}{\partial t}$ the unit vector giving the orientation of $\mathbb{R}$ in $\mathbb{M}^n \times \mathbb{R}$; we call it the vertical vector.

2.1. **The compatibility equations in $\mathbb{M}^n \times \mathbb{R}$.** Let $\mathbb{M}^n = \mathbb{S}^n$ or $\mathbb{M}^n = \mathbb{H}^n$; in the first case we set $\kappa = 1$ and in the second case we set $\kappa = -1$. Let $\bar{\mathrm{R}}$ be the Riemann curvature tensor of $\mathbb{M}^n \times \mathbb{R}$. Let $\mathcal{V}$ be an oriented hypersurface of $\mathbb{M}^n \times \mathbb{R}$ and N the unit normal to $\mathcal{V}$.

Proposition 2.1. *For $X, Y, Z, W \in \mathfrak{X}(\mathcal{V})$ we have*

$$\begin{aligned}\langle \bar{\mathrm{R}}(X,Y)Z,W\rangle = \kappa(&\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle \\ &- \langle Y,T\rangle\langle W,T\rangle\langle X,Z\rangle - \langle X,T\rangle\langle Z,T\rangle\langle Y,W\rangle \\ &+ \langle X,T\rangle\langle W,T\rangle\langle Y,Z\rangle + \langle Y,T\rangle\langle Z,T\rangle\langle X,W\rangle), \\ \langle \bar{\mathrm{R}}(X,Y)N,Z\rangle = \kappa\nu(&\langle X,Z\rangle\langle Y,T\rangle - \langle Y,Z\rangle\langle X,T\rangle),\end{aligned}$$

where

$$\nu = \left\langle N, \frac{\partial}{\partial t}\right\rangle$$

and T is the projection of $\frac{\partial}{\partial t}$ on $\mathrm{T}\mathcal{V}$, i.e.,

$$T = \frac{\partial}{\partial t} - \nu N.$$

Proof. Any vector field on $\mathbb{M}^n \times \mathbb{R}$ can be written $X(m,t) = (X^t_{\mathbb{M}^n}(m), X^m_{\mathbb{R}}(t))$, where, for each $t \in \mathbb{R}$, $X^t_{\mathbb{M}^n}$ is a vector field on $\mathbb{M}^n$ and, for each $m \in \mathbb{M}^n$, $X^m_{\mathbb{R}}$ is a vector field on $\mathbb{R}$. Then for $X, Y, Z, W \in \mathfrak{X}(\mathbb{M}^n \times \mathbb{R})$ we have

$$\begin{aligned}\langle \bar{\mathrm{R}}(X,Y)Z,W\rangle &= \langle \bar{\mathrm{R}}_{\mathbb{M}^n}(X_{\mathbb{M}^n}, Y_{\mathbb{M}^n})Z_{\mathbb{M}^n}, W_{\mathbb{M}^n}\rangle \\ &= \kappa(\langle X_{\mathbb{M}^n}, Z_{\mathbb{M}^n}\rangle\langle Y_{\mathbb{M}^n}, W_{\mathbb{M}^n}\rangle - \langle Y_{\mathbb{M}^n}, Z_{\mathbb{M}^n}\rangle\langle X_{\mathbb{M}^n}, W_{\mathbb{M}^n}\rangle).\end{aligned}$$

We have $X_{\mathbb{M}^n} = X - \left\langle X, \frac{\partial}{\partial t}\right\rangle \frac{\partial}{\partial t}$. Thus, if $X \in \mathrm{T}\mathcal{V}$, we have $X_{\mathbb{M}^n} = X - \langle X,T\rangle \frac{\partial}{\partial t}$, and similar expressions for $Y, Z, W \in \mathrm{T}\mathcal{V}$. A computation gives the expected formula for $\langle \bar{\mathrm{R}}(X,Y)Z,W\rangle$.

Finally we have $N_{\mathbb{M}^n} = N - \nu\frac{\partial}{\partial t}$, so a computation gives the expected formula for $\langle \bar{\mathrm{R}}(X,Y)N,Z\rangle$. □

Using the fact that the vector field $\frac{\partial}{\partial t}$ is parallel, we obtain the following equations.

Proposition 2.2. *For $X \in \mathfrak{X}(\mathcal{V})$ we have*

$$\nabla_X T = \nu \mathrm{S}X, \quad \mathrm{d}\nu(X) = -\langle \mathrm{S}X, T\rangle.$$

Proof. We have $\frac{\partial}{\partial t} = T + \nu N$ and $\bar{\nabla}_X \frac{\partial}{\partial t} = 0$. Thus we get

$$0 = \bar{\nabla}_X T + (\mathrm{d}\nu(X))N + \nu\bar{\nabla}_X N = \nabla_X T + \langle \mathrm{S}X, T\rangle N + (\mathrm{d}\nu(X))N - \nu\mathrm{S}X.$$

Taking the tangential and the normal components in this equality, we obtain the expected formulas. □

Remark 2.3. In the case of an orthonormal pair (X,Y) we get

$$\langle \bar{\mathrm{R}}(X,Y)X,Y\rangle = \kappa(1 - \langle Y,T\rangle^2 - \langle X,T\rangle^2).$$

The reader can also refer to section 3.2 in [AR04].

2.2. **Moving frames.** In this section we introduce some material about the technique of moving frames. The reader can also refer to [Ros02a].

Let $\mathcal{V}$ be a Riemannian manifold of dimension n, ∇ its Levi-Civita connection, and R the Riemannian curvature tensor. Let S be a field of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\mathcal{V} \to \mathrm{T}_y\mathcal{V}$. Let $(e_1, \dots, e_n)$ be a local orthonormal frame on $\mathcal{V}$ and $(\omega^1, \dots, \omega^n)$ the dual basis of $(e_1, \dots, e_n)$, i.e.,

$$\omega^i(e_k) = \delta^i_k.$$

We also set

$$\omega^{n+1} = 0.$$

We define the forms ω^i_j, ω^{n+1}_j, ω^i_{n+1} and ω^{n+1}_{n+1} on $\mathcal{V}$ by

$$\omega^i_j(e_k) = \langle \nabla_{e_k} e_j, e_i \rangle, \quad \omega^{n+1}_j(e_k) = \langle \mathrm{S}e_k, e_j \rangle,$$
$$\omega^j_{n+1} = -\omega^{n+1}_j, \quad \omega^{n+1}_{n+1} = 0.$$

Then we have

$$\nabla_{e_k} e_j = \sum_i \omega^i_j(e_k) e_i, \quad \mathrm{S}e_k = \sum_j \omega^{n+1}_j(e_k) e_j.$$

Finally we set $R^i_{klj} = \langle \mathrm{R}(e_k, e_l)e_j, e_i \rangle$.

Proposition 2.4. *We have the following formulas:*

$$\mathrm{d}\omega^i + \sum_p \omega^i_p \wedge \omega^p = 0, \tag{3}$$

$$\sum_p \omega^{n+1}_p \wedge \omega^p = 0, \tag{4}$$

$$\mathrm{d}\omega^i_j + \sum_p \omega^i_p \wedge \omega^p_j = -\frac{1}{2} \sum_k \sum_l R^i_{klj} \omega^k \wedge \omega^l, \tag{5}$$

$$\mathrm{d}\omega^{n+1}_j + \sum_p \omega^{n+1}_p \wedge \omega^p_j = \frac{1}{2} \sum_k \sum_l \langle \nabla_{e_k} \mathrm{S}e_l - \nabla_{e_l} \mathrm{S}e_k - \mathrm{S}[e_k, e_l], e_j \rangle \omega^k \wedge \omega^l. \tag{6}$$

Proof. These are well known formulas. However, since our conventions slightly differ from those of [Ten71] and [Ros02a], we give a proof for sake of clarity.

We have $\mathrm{d}\omega^i(e_p, e_q) = -\omega^i([e_p, e_q]) = -\omega^i(\nabla_{e_p} e_q - \nabla_{e_q} e_p) = -\omega^i_q(e_p) + \omega^i_p(e_q)$ and $\sum_k \omega^i_k \wedge \omega^k(e_p, e_q) = \omega^i_q(e_p) - \omega^i_p(e_q)$, so (3) is proved. Also, we have $\sum_k (\omega^{n+1}_k \wedge \omega^k)(e_p, e_q) = \omega^{n+1}_q(e_p) - \omega^{n+1}_p(e_q) = \langle \mathrm{S}e_p, e_q \rangle - \langle \mathrm{S}e_q, e_p \rangle = 0$, so (4) is proved.

We have $\omega^i_j = \sum_k \langle e_i, \nabla_{e_k} e_j \rangle \omega^k$, so

$$\begin{aligned} \mathrm{d}\omega^i_j &= \sum_k \sum_l e_l \langle e_i, \nabla_{e_k} e_j \rangle \omega^l \wedge \omega^k + \sum_k \langle e_i, \nabla_{e_k} e_j \rangle \mathrm{d}\omega^k \\ &= \sum_k \sum_l (\langle \nabla_{e_l} e_i, \nabla_{e_k} e_j \rangle + \langle e_i, \nabla_{e_l} \nabla_{e_k} e_j \rangle) \omega^l \wedge \omega^k \\ &\quad - \sum_k \sum_l \langle e_i, \nabla_{e_k} e_j \rangle \omega^k_l \wedge \omega^l. \end{aligned}$$

Moreover we have

$$\begin{aligned}\sum_k\sum_l\langle e_i,\nabla_{e_k}e_j\rangle\omega_l^k\wedge\omega^l &= \sum_k\sum_l\sum_q\langle e_i,\nabla_{e_k}e_j\rangle\langle e_k,\nabla_{e_q}e_l\rangle\omega^q\wedge\omega^l\\ &= \sum_l\sum_q\langle e_i,\nabla_{\nabla_{e_q}e_l}e_j\rangle\omega^q\wedge\omega^l.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\sum_p\omega_p^i\wedge\omega_j^p &= \sum_k\sum_l\sum_p\langle e_i,\nabla_{e_l}e_p\rangle\langle e_p,\nabla_{e_k}e_j\rangle\omega^l\wedge\omega^k\\ &= -\sum_k\sum_l\sum_p\langle\nabla_{e_l}e_i,e_p\rangle\langle e_p,\nabla_{e_k}e_j\rangle\omega^l\wedge\omega^k\\ &= -\sum_k\sum_l\langle\nabla_{e_l}e_i,\nabla_{e_k}e_j\rangle\omega^l\wedge\omega^k.\end{aligned}$$

Thus we conclude that

$$\mathrm{d}\omega_j^i+\sum_p\omega_p^i\wedge\omega_j^p=\sum_k\sum_l\langle e_i,\nabla_{e_l}\nabla_{e_k}e_j-\nabla_{\nabla_{e_l}e_k}e_j\rangle\omega^l\wedge\omega^k.$$

Adding this equality with itself after exchanging k and l and using the fact that $\omega^k\wedge\omega^l=-\omega^l\wedge\omega^k$, we get

$$2\left(\mathrm{d}\omega_j^i+\sum_p\omega_p^i\wedge\omega_j^p\right) = \sum_k\sum_l\langle e_i,\mathrm{R}(e_k,e_l)e_j\rangle\omega^l\wedge\omega^k,$$

and finally we get (5).

We have $\omega_j^{n+1}=\sum_k\langle \mathrm{S}e_k,e_j\rangle\omega^k$, so

$$\begin{aligned}\mathrm{d}\omega_j^{n+1} &= \sum_k\sum_l e_l\langle \mathrm{S}e_k,e_j\rangle\omega^l\wedge\omega^k+\sum_k\langle \mathrm{S}e_k,e_j\rangle\mathrm{d}\omega^k\\ &= \sum_k\sum_l(\langle\nabla_{e_l}\mathrm{S}e_k,e_j\rangle+\langle \mathrm{S}e_k,\nabla_{e_l}e_j\rangle)\omega^l\wedge\omega^k-\sum_k\sum_l\langle \mathrm{S}e_k,e_j\rangle\omega_l^k\wedge\omega^l.\end{aligned}$$

Moreover we have

$$\begin{aligned}\sum_k\sum_l\langle \mathrm{S}e_k,e_j\rangle\omega_l^k\wedge\omega^l &= \sum_k\sum_l\sum_q\langle \mathrm{S}e_k,e_j\rangle\langle e_k,\nabla_{e_q}e_l\rangle\omega^q\wedge\omega^l\\ &= \sum_l\sum_q\langle \mathrm{S}e_j,\nabla_{e_q}e_l\rangle\omega^q\wedge\omega^l.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\sum_p\omega_p^{n+1}\wedge\omega_j^p &= \sum_k\sum_p\langle \mathrm{S}e_k,e_p\rangle\omega^k\wedge\omega_j^p\\ &= \sum_k\sum_p\sum_l\langle \mathrm{S}e_k,e_p\rangle\langle e_p,\nabla_{e_l}e_j\rangle\omega^k\wedge\omega^l\\ &= \sum_k\sum_l\langle \mathrm{S}e_k,\nabla_{e_l}e_j\rangle\omega^k\wedge\omega^l.\end{aligned}$$

Thus we conclude that

$$\begin{aligned} \mathrm{d}\omega_j^{n+1} + \sum_p \omega_p^{n+1} \wedge \omega_j^p &= \sum_k \sum_l (\langle \nabla_{e_l} \mathrm{S} e_k, e_j\rangle - \langle \mathrm{S} e_j, \nabla_{e_l} e_k\rangle)\omega^l \wedge \omega^k \\ &= \sum_k \sum_l \langle e_j, \nabla_{e_l} \mathrm{S} e_k - \mathrm{S}\nabla_{e_l} e_k\rangle \omega^l \wedge \omega^k. \end{aligned}$$

Adding this equality with itself after exchanging k and l and using the fact that $\omega^k \wedge \omega^l = -\omega^l \wedge \omega^k$, we get

$$2\left(\mathrm{d}\omega_j^{n+1} + \sum_p \omega_p^{n+1} \wedge \omega_j^p\right) = \sum_k \sum_l \langle e_j, \nabla_{e_l} \mathrm{S} e_k - \nabla_{e_k} \mathrm{S} e_l - \mathrm{S}[e_l, e_k]\rangle \omega^l \wedge \omega^k,$$

and finally we get (6). □

2.3. **Some facts about hypersurfaces of $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.** In this section we consider an orientable hypersurface $\mathcal{V}$ of $\mathbb{M}^n \times \mathbb{R}$ with $\mathbb{M}^n = \mathbb{S}^n$ or $\mathbb{M}^n = \mathbb{H}^n$.

We denote by $\mathbb{L}^p$ the p-dimensional Lorentz space, i.e., $\mathbb{R}^p$ endowed with the quadradic form

$$-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^{p-1})^2.$$

We will use the following inclusions: we have

$$\mathbb{S}^n = \{(x^0, \ldots, x^n) \in \mathbb{R}^{n+1}; (x^0)^2 + \sum_i (x^i)^2 = 1\},$$

and so

$$\mathbb{S}^n \times \mathbb{R} \subset \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2},$$

and we have

$$\mathbb{H}^n = \{(x^0, \ldots, x^n) \in \mathbb{L}^{n+1}; -(x^0)^2 + \sum_i (x^i)^2 = -1, x^0 > 0\},$$

and so

$$\mathbb{H}^n \times \mathbb{R} \subset \mathbb{L}^{n+1} \times \mathbb{R} = \mathbb{L}^{n+2}.$$

In the case of $\mathbb{S}^n \times \mathbb{R}$ we set $\kappa = 1$ and $\mathbb{E}^{n+2} = \mathbb{R}^{n+2}$. In the case of $\mathbb{H}^n \times \mathbb{R}$ we set $\kappa = -1$ and $\mathbb{E}^{n+2} = \mathbb{L}^{n+2}$.

We denote by ∇, $\bar{\nabla}$ and $\bar{\bar{\nabla}}$ the connections of $\mathcal{V}$, $\mathbb{M}^n \times \mathbb{R}$ and $\mathbb{E}^{n+2}$ respectively, by $\bar{N}(x)$ the normal to $\mathbb{M}^n \times \mathbb{R}$ in $\mathbb{E}^{n+2}$ at a point $x \in \mathbb{M}^n \times \mathbb{R}$, i.e.,

$$\bar{N}(x) = (x^0, \ldots, x^n, 0),$$

and by $N(x)$ the normal to $\mathcal{V}$ in $\mathbb{M}^n \times \mathbb{R}$ at a point $x \in \mathcal{V}$. We denote by S the shape operator of $\mathcal{V}$ in $\mathbb{M}^n \times \mathbb{R}$. The shape operator of $\mathbb{M}^n \times \mathbb{R}$ is $\bar{\mathrm{S}}X = -\kappa \mathrm{d}\bar{N}(X) = \kappa\left(-X + \left\langle X, \frac{\partial}{\partial t}\right\rangle \frac{\partial}{\partial t}\right)$. We should be careful with the sign convention in the definition of the shape operator: here we have chosen

$$\bar{\bar{\nabla}}_X Y = \bar{\nabla}_X Y + \langle \bar{\mathrm{S}}X, Y\rangle \bar{N},$$

i.e.,

$$\langle \bar{\mathrm{S}}X, Y\rangle = \kappa\langle \bar{\bar{\nabla}}_X Y, \bar{N}\rangle,$$

because in the case of $\mathbb{S}^n \times \mathbb{R}$ we have $\langle \bar{N}, \bar{N}\rangle = 1$, whereas in the case of $\mathbb{H}^n \times \mathbb{R}$ we have $\langle \bar{N}, \bar{N}\rangle = -1$.

Let $(e_1, \dots, e_n)$ be a local orthonormal frame on $\mathcal{V}$, $e_{n+1} = N$ and $e_0 = \bar{N}$ (on $\mathcal{V}$). We define the forms ω_j^i, ω_j^{n+1}, ω_{n+1}^i and ω_{n+1}^{n+1} as in Section 2.2. Moreover we set

$$\omega_\gamma^0(e_k) = \langle \bar{\mathrm{S}} e_k, e_\gamma \rangle = -\kappa \langle e_k, e_\gamma \rangle + \kappa \left\langle e_k, \frac{\partial}{\partial t} \right\rangle \left\langle e_\gamma, \frac{\partial}{\partial t} \right\rangle,$$
$$\omega_0^\gamma = -\kappa \omega_\gamma^0.$$

With these definitions we have

$$\bar{\bar{\nabla}}_{e_k} e_\beta = \sum_\alpha \omega_\beta^\alpha(e_k) e_\alpha.$$

Let $(E_0, \dots, E_{n+1})$ be the canonical frame of $\mathbb{E}^{n+2}$ (with $\langle E_0, E_0 \rangle = \kappa$ and $E_{n+1} = \frac{\partial}{\partial t}$). Let $A \in \mathcal{M}_{n+2}(\mathbb{R})$ be the matrix (the indices going from 0 to $n+1$) whose columns are the coordinates of the e_β in the frame (E_α), i.e.,

$$e_\beta = \sum_\alpha A_\beta^\alpha E_\alpha.$$

Then, on the one hand we have

$$\bar{\bar{\nabla}}_{e_k} e_\beta = \sum_\alpha \mathrm{d}A_\beta^\alpha(e_k) E_\alpha,$$

and on the other hand we have

$$\bar{\bar{\nabla}}_{e_k} e_\beta = \sum_\alpha \sum_\gamma \omega_\beta^\gamma(e_k) A_\gamma^\alpha E_\alpha.$$

Thus we have

$$A^{-1} \mathrm{d}A = \Omega$$

with $\Omega = (\omega_\beta^\alpha) \in \mathcal{M}_{n+2}(\mathbb{R})$, the indices going from 0 to $n+1$.

Setting $G = \mathrm{diag}(\kappa, 1, \dots, 1) \in \mathcal{M}_{n+2}(\mathbb{R})$, we have

$$A \in \mathrm{SO}^+(\mathbb{E}^{n+2}), \quad \Omega \in \mathfrak{so}(\mathbb{E}^{n+2}),$$

where $\mathrm{SO}^+(\mathbb{E}^{n+2})$ is the connected component of I_{n+2} in

$$\mathrm{SO}(\mathbb{E}^{n+2}) = \{Z \in \mathcal{M}_{n+2}(\mathbb{R}); {}^{\mathrm{t}}ZGZ = G, \det Z = 1\}$$

and where

$$\mathfrak{so}(\mathbb{E}^{n+2}) = \{H \in \mathcal{M}_{n+2}(\mathbb{R}); {}^{\mathrm{t}}HG + GH = 0\}.$$

In the case of $\mathbb{S}^n \times \mathbb{R}$ we have $\mathrm{SO}^+(\mathbb{E}^{n+2}) = \mathrm{SO}(\mathbb{R}^{n+2})$.

3. Isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$

3.1. **The compatibility equations.** We consider a simply connected Riemannian manifold $\mathcal{V}$ of dimension n. Let $\mathrm{d}s^2$ be the metric on $\mathcal{V}$ (we will also denote it by $\langle \cdot, \cdot \rangle$), ∇ the Riemannian connection of $\mathcal{V}$ and R its Riemann curvature tensor. Let S be a field of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\mathcal{V} \to \mathrm{T}_y\mathcal{V}$, T a vector field on $\mathcal{V}$ such that $||T|| \leqslant 1$ and ν a smooth function on $\mathcal{V}$ such that $\nu^2 \leqslant 1$.

The compatibility equations for hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ established in Section 2.1 suggest we introduce the following definition.

Definition 3.1. We say that $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ satisfies the compatibility equations respectively for $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ if

$$||T||^2 + \nu^2 = 1$$

and, for all $X, Y, Z \in \mathfrak{X}(\mathcal{V})$,

$$\begin{aligned} \mathrm{R}(X,Y)Z \;=\; & \langle \mathrm{S}X, Z\rangle \mathrm{S}Y - \langle \mathrm{S}Y, Z\rangle \mathrm{S}X \\ & +\kappa(\langle X,Z\rangle Y - \langle Y,Z\rangle X - \langle Y,T\rangle\langle X,Z\rangle T \\ & -\langle X,T\rangle\langle Z,T\rangle Y + \langle X,T\rangle\langle Y,Z\rangle T + \langle Y,T\rangle\langle Z,T\rangle X), \end{aligned} \tag{7}$$

$$\nabla_X \mathrm{S}Y - \nabla_Y \mathrm{S}X - \mathrm{S}[X,Y] = \kappa\nu(\langle Y,T\rangle X - \langle X,T\rangle Y), \tag{8}$$

$$\nabla_X T = \nu \mathrm{S}X, \tag{9}$$

$$\mathrm{d}\nu(X) = -\langle \mathrm{S}X, T\rangle, \tag{10}$$

where $\kappa = 1$ and $\kappa = -1$ for $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ respectively.

Remark 3.2. We notice that (9) implies (10) except when $\nu = 0$ (by differentiating the identity $\langle T, T\rangle + \nu^2 = 1$ with respect to X).

3.2. **Codimension 1 isometric immersions into $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$.** In this section we will prove the following theorem.

Theorem 3.3. *Let $\mathcal{V}$ be a simply connected Riemannian manifold of dimension n, $\mathrm{d}s^2$ its metric and ∇ its Riemannian connection. Let S be a field of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\mathcal{V} \to \mathrm{T}_y\mathcal{V}$, T a vector field on $\mathcal{V}$ and ν a smooth function on $\mathcal{V}$ such that $||T||^2 + \nu^2 = 1$.*

Let $\mathbb{M}^n = \mathbb{S}^n$ or $\mathbb{M}^n = \mathbb{H}^n$. Assume that $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ satisfies the compatibility equations for $\mathbb{M}^n \times \mathbb{R}$. Then there exists an isometric immersion $f : \mathcal{V} \to \mathbb{M}^n \times \mathbb{R}$ such that the shape operator with respect to the normal N associated to f is

$$\mathrm{d}f \circ \mathrm{S} \circ \mathrm{d}f^{-1}$$

and such that

$$\frac{\partial}{\partial t} = \mathrm{d}f(T) + \nu N.$$

Moreover the immersion is unique up to a global isometry of $\mathbb{M}^n \times \mathbb{R}$ preserving the orientations of both $\mathbb{M}^n$ and $\mathbb{R}$.

To prove this theorem, we consider a local orthonormal frame $(e_1, \dots, e_n)$ on $\mathcal{V}$ and the forms ω^i, ω^{n+1}, ω_j^i, ω_j^{n+1}, ω_{n+1}^i and ω_{n+1}^{n+1} as in Section 2.2. We set $\mathbb{E}^{n+2} = \mathbb{R}^{n+2}$ or $\mathbb{E}^{n+2} = \mathbb{L}^{n+2}$ (according to $\mathbb{M}^n$). We denote by $(E_0, \dots, E_{n+1})$ the canonical frame of $\mathbb{E}^{n+2}$ (with $\langle E_0, E_o\rangle = -1$ in the case of $\mathbb{L}^{n+2}$); in particular we have $E_{n+1} = \frac{\partial}{\partial t}$. We set

$$T^k = \langle T, e_k\rangle, \quad T^{n+1} = \nu, \quad T^0 = 0.$$

Moreover we set

$$\omega_j^0(e_k) = \kappa(T^j T^k - \delta_j^k), \quad \omega_{n+1}^0(e_k) = \kappa\nu T^k,$$
$$\omega_0^i = -\kappa\omega_i^0, \quad \omega_0^{n+1} = -\kappa\omega_{n+1}^0, \quad \omega_0^0 = 0.$$

We define the one-form η on $\mathcal{V}$ by

$$\eta(X) = \langle T, X\rangle.$$

In the frame $(e_1, \ldots, e_n)$ we have $\eta = \sum_k T^k \omega^k$. Finaly we define the following matrix of one-forms:

$$\Omega = (\omega^\alpha_\beta) \in \mathcal{M}_{n+2}(\mathbb{R}),$$

the indices going from 0 to $n+1$.

From now on we assume that the hypotheses of Theorem 3.3 are satisfied. We first prove some technical lemmas that are consequences of the compatibility equations.

Lemma 3.4. *We have*

$$\mathrm{d}\eta = 0.$$

Proof. We have

$$\begin{aligned} \mathrm{d}\eta(X,Y) &= X \cdot \eta(Y) - Y \cdot \eta(X) - \eta([X,Y]) \\ &= \langle \nabla_X T, Y\rangle - \langle \nabla_Y T, X\rangle \\ &= \langle \nu \mathrm{S} X, Y\rangle - \langle \nu \mathrm{S} Y, X\rangle \\ &= 0, \end{aligned}$$

where we have used condition (9). □

Lemma 3.5. *We have*

$$\mathrm{d}T^\alpha = \sum_\gamma T^\gamma \omega^\gamma_\alpha.$$

Proof. This is a consequence of condition (9) for $\alpha = j$, of condition (10) for $\alpha = n+1$, and of the definitions for $\alpha = 0$. □

Lemma 3.6. *We have*

$$\mathrm{d}\Omega + \Omega \wedge \Omega = 0.$$

Proof. We set $\Psi = \mathrm{d}\Omega + \Omega \wedge \Omega$ and $R^i_{klj} = \langle \mathrm{R}(e_k, e_l)e_j, e_i\rangle$.

By Proposition 2.4 we have

$$\Psi^i_j = -\frac{1}{2}\sum_k \sum_l R^i_{klj}\omega^k \wedge \omega^l + \omega^i_{n+1} \wedge \omega^{n+1}_j + \omega^i_0 \wedge \omega^0_j.$$

Since the Gauss equation (7) is satisfied, we have

$$R^i_{klj} = \bar{R}^i_{klj} + \omega^{n+1}_j \wedge \omega^{n+1}_i (e_k, e_l)$$

with

$$\bar{R}^i_{klj} = \kappa(\delta^k_j \delta^l_i - \delta^l_j \delta^k_i - T^l T^i \delta^k_j - T^k T^j \delta^l_i + T^k T^i \delta^l_j + T^l T^j \delta^k_i).$$

On the other hand, a computation shows that $\omega^i_0 \wedge \omega^0_j (e_k, e_l) = \bar{R}^i_{klj}$. Thus we have $R^i_{klj} = \omega^i_{n+1} \wedge \omega^{n+1}_j (e_k, e_l) + \omega^i_0 \wedge \omega^0_j (e_k, e_l)$. We conclude that $\Psi^i_j = 0$.

By Proposition 2.4 we have

$$\Psi^{n+1}_j = \frac{1}{2}\sum_k \sum_l \langle \nabla_{e_k} \mathrm{S} e_l - \nabla_{e_l} \mathrm{S} e_k - \mathrm{S}[e_k, e_l], e_j\rangle \omega^k \wedge \omega^l + \omega^{n+1}_0 \wedge \omega^0_j.$$

Since the Codazzi equation (8) is satisfied, we have

$$\langle \nabla_{e_k} \mathrm{S} e_l - \nabla_{e_l} \mathrm{S} e_k - \mathrm{S}[e_k, e_l], e_j\rangle = \kappa(T^l T^{n+1} \delta^k_j - T^k T^{n+1} \delta^l_j).$$

On the other hand, a computation shows that

$$\omega^{n+1}_0 \wedge \omega^0_j (e_k, e_l) = \kappa(T^k T^{n+1} \delta^l_j - T^l T^{n+1} \delta^k_j).$$

We conclude that $\Psi^{n+1}_j = 0$.

We have $\omega_j^0 = \kappa(T^j\eta - \omega^j)$. Since $\mathrm{d}\eta = 0$ (by Lemma 3.4) we get

$$\mathrm{d}\omega_j^0 = \kappa(\mathrm{d}T^j \wedge \eta - \mathrm{d}\omega^j) = \kappa \mathrm{d}T^j \wedge \eta + \kappa \sum_k \omega_k^j \wedge \omega^k$$

by Proposition 2.4. Thus by a straightforward computation we get

$$\begin{aligned}
\Psi_j^0(e_p, e_q) &= \mathrm{d}\omega_j^0(e_p, e_q) + \sum_k \omega_k^0 \wedge \omega_j^k(e_p, e_q) + \omega_{n+1}^0 \wedge \omega_j^{n+1}(e_p, e_q) \\
&= \kappa(\mathrm{d}T^j(e_p)\eta(e_q) - \mathrm{d}T^j(e_q)\eta(e_p) + \omega_q^j(e_p) - \omega_p^j(e_q)) \\
&\quad +\kappa \left(T^p \sum_k T^k \omega_j^k(e_q) - T^q \sum_k T^k \omega_j^k(e_p) - \omega_j^p(e_q) + \omega_j^q(e_p) \right) \\
&\quad +\kappa \left(T^p T^{n+1} \omega_j^{n+1}(e_q) - T^q T^{n+1} \omega_j^{n+1}(e_p) \right).
\end{aligned}$$

Using the definition of η and Lemma 3.5 for $\alpha = j$,we conclude that $\hat{\Psi}_j^{n+2} = 0$.

We have $\omega_{n+1}^0 = \kappa T^{n+1}\eta$, and so $\mathrm{d}\omega_{n+1}^0 = \kappa \mathrm{d}T^{n+1} \wedge \eta$ by Lemma 3.4. Thus by a straightforward computation we get

$$\begin{aligned}
\Psi_{n+1}^0(e_p, e_q) &= \mathrm{d}\omega_{n+1}^0(e_p, e_q) + \sum_k \omega_k^0 \wedge \omega_{n+1}^k(e_p, e_q) \\
&= \kappa(T^q \mathrm{d}T^{n+1}(e_p) - T^p \mathrm{d}T^{n+1}(e_q)) \\
&\quad +\kappa \left(T^p \sum_k T^k \omega_{n+1}^k(e_q) - T^q \sum_k T^k \omega_{n+1}^k(e_p) \right) \\
&\quad +\kappa(-\omega_{n+1}^p(e_q) + \omega_{n+1}^q(e_p)).
\end{aligned}$$

The last two terms cancel because S is symmetric. Using Lemma 3.5 for $\alpha = n+1$, we conclude that $\Psi_{n+1}^0 = 0$.

The fact that $\Psi_0^0 = 0$ and $\Psi_{n+1}^{n+1} = 0$ is clear. We conclude by noticing that $\Psi_{n+1}^i = -\Psi_i^{n+1} = 0$. □

For $y \in \mathcal{V}$, let $\mathcal{Z}(y)$ be the set of matrices $Z \in \mathrm{SO}^+(\mathbb{E}^{n+2})$ such that the coefficients of the last line of Z are the $T^\beta(y)$. It is a manifold of dimension $\frac{n(n+1)}{2}$ (since the map $F : \mathrm{SO}^+(\mathbb{E}^{n+2}) \to \mathbb{S}(\mathbb{E}^{n+2}), Z \mapsto (Z_\beta^{n+1})_\beta$ (i.e., $F(Z)$ is the last line of Z), where $\mathbb{S}(\mathbb{E}^{n+2}) = \{x \in \mathbb{E}^{n+2}; \langle E, E \rangle = 1\}$ is a submersion).

We now prove the following proposition.

Proposition 3.7. *Assume that the compatibility equations for $\mathbb{M}^n \times \mathbb{R}$ are satisfied. Let $y_0 \in \mathcal{V}$ and $A_0 \in \mathcal{Z}(y_0)$. Then there exist a neighbourhood U_1 of y_0 in $\mathcal{V}$ and a unique map $A : U_1 \to \mathrm{SO}^+(\mathbb{E}^{n+2})$ such that*

$$A^{-1}\mathrm{d}A = \Omega,$$
$$\forall y \in U_1, \quad A(y) \in \mathcal{Z}(y),$$
$$A(y_0) = A_0.$$

Proof. Let U be a coordinate neighbourhood in $\mathcal{V}$. The set

$$\mathcal{F} = \{(y, Z) \in U \times \mathrm{SO}^+(\mathbb{E}^{n+2}); Z \in \mathcal{Z}(y)\}$$

is a manifold of dimension $n + \frac{n(n+1)}{2}$, and

$$\mathrm{T}_{(y,Z)}\mathcal{F} = \{(u, \zeta) \in \mathrm{T}_y U \oplus \mathrm{T}_Z \mathrm{SO}^+(\mathbb{E}^{n+2}); \zeta_\beta^{n+1} = (\mathrm{d}T^\beta)_y(u)\}.$$

Indeed, in the neighbourhood of point of U there exists a map $y \mapsto M(y) \in \mathrm{SO}^+(\mathbb{E}^{n+2})$ such that the last line of $M(y)$ is $(T^\beta(y))_\beta$, and we have $Z \in \mathcal{Z}(y)$ if and only if

$$ZM(y)^{-1} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$$

for some $B \in \mathrm{SO}^+(\mathbb{E}^{n+1})$. Then, if φ is a local parametrization of the set of such matrices, the map $(y, v) \mapsto (y, \varphi(v)M(y))$ is a local parametrization of $\mathcal{F}$.

Let Z denote the projection $U \times \mathrm{SO}^+(\mathbb{E}^{n+2}) \to \mathrm{SO}^+(\mathbb{E}^{n+2}) \subset \mathcal{M}_{n+2}(\mathbb{R})$. We consider on $\mathcal{F}$ the following matrix of 1-forms:

$$\Theta = Z^{-1}\mathrm{d}Z - \Omega;$$

namely for $(y, Z) \in \mathcal{F}$ we have

$$\Theta_{(y,Z)} : \mathrm{T}_{(y,Z)}\mathcal{F} \to \mathcal{M}_{n+2}(\mathbb{R}),$$
$$\Theta_{(y,Z)}(u, \zeta) = Z^{-1}\zeta - \Omega_y(u).$$

We claim that, for each $(y, Z) \in \mathcal{F}$, the space

$$\mathcal{D}(y, Z) = \ker \Theta_{(y,Z)}$$

has dimension n. We first notice that the matrix Θ belongs to $\mathfrak{so}(\mathbb{E}^{n+2})$ since Ω and $Z^{-1}\mathrm{d}Z$ do as well. Moreover we have

$$(Z\Theta)^{n+1}_\beta = \mathrm{d}Z^{n+1}_\beta - \sum_\gamma Z^{n+1}_\gamma \omega^\gamma_\beta = \mathrm{d}T^\beta - \sum_\gamma T^\gamma \omega^\gamma_\beta = 0$$

by Lemma 3.5. Thus the values of $\Theta_{(y,Z)}$ lie in the space

$$\mathcal{H} = \{H \in \mathfrak{so}^+(\mathbb{E}^{n+2}); (ZH)^{n+1}_\beta = 0\},$$

which has dimension $\frac{n(n+1)}{2}$ (indeed, the map $F : \mathrm{SO}^+(\mathbb{E}^{n+2}) \to \mathbb{S}(\mathbb{E}^{n+2}), Z \mapsto (Z^{n+1}_\beta)_\beta$ is a submersion, and we have $H \in \mathcal{H}$ if and only if $ZH \in \ker(\mathrm{d}F)_Z$). Moreover, the space $\mathrm{T}_{(y,Z)}\mathcal{F}$ contains the subspace $\{(0, ZH); H \in \mathcal{H}\}$, and the restriction of $\Theta_{(y,Z)}$ on this subspace is the map $(0, ZH) \mapsto H$. Thus $\Theta_{(y,Z)}$ is onto $\mathcal{H}$, and consequently the linear map $\Theta_{(y,Z)}$ has rank $\frac{n(n+1)}{2}$. This finishes proving the claim.

We now prove that the distribution $\mathcal{D}$ is involutive. Using Lemma 3.6 we get

$$\begin{aligned} \mathrm{d}\Theta &= -Z^{-1}\mathrm{d}Z \wedge Z^{-1}\mathrm{d}Z - \mathrm{d}\Omega \\ &= -(\Theta + \Omega) \wedge (\Theta + \Omega) - \mathrm{d}\Omega \\ &= -\Theta \wedge \Theta - \Theta \wedge \Omega - \Omega \wedge \Theta. \end{aligned}$$

From this formula we deduce that if $\xi_1, \xi_2 \in \mathcal{D}$, then $\mathrm{d}\Theta(\xi_1, \xi_2) = 0$, and so $\Theta([\xi_1, \xi_2]) = \xi_1 \cdot \Theta(\xi_2) - \xi_2 \cdot \Theta(\xi_1) - \mathrm{d}\Theta(\xi_1, \xi_2) = 0$, i.e., $[\xi_1, \xi_2] \in \mathcal{D}$. Thus the distribution $\mathcal{D}$ is involutive, and so, by the theorem of Frobenius, it is integrable.

Let $\mathcal{A}$ be the integral manifold through (y_0, A_0). If $\zeta \in \mathrm{T}_{A_0}\mathrm{SO}^+(\mathbb{E}^{n+2})$ is such that $(0, \zeta) \in \mathrm{T}_{(y_0,A_0)}\mathcal{A} = \mathcal{D}(y_0, A_0)$, then we have $0 = \Theta_{(y_0,A_0)}(0, \zeta) = A_0^{-1}\zeta$. This proves that

$$\mathrm{T}_{(y_0,A_0)}\mathcal{A} \cap \left(\{0\} \times \mathrm{T}_{A_0}\mathrm{SO}^+(\mathbb{E}^{n+2})\right) = \{0\}.$$

Thus the manifold $\mathcal{A}$ is locally the graph of a function $A : U_1 \to \mathrm{SO}^+(\mathbb{E}^{n+2})$, where U_1 is a neighbourhood of y_0 in U. By construction, this map satisfies the properties of Proposition 3.7 and is unique. □

We now prove the theorem.

Proof of Theorem 3.3. Let $y_0 \in \mathcal{V}$, $A \in \mathcal{Z}(y_0)$ and $t_0 \in \mathbb{R}$. We consider on $\mathcal{V}$ a local orthonormal frame $(e_1, \ldots, e_n)$ in the neighbourhood of y_0, and we keep the same notation. Then by Proposition 3.7 there exists a unique map $A : U_1 \to \mathrm{SO}^+(\mathbb{E}^{n+2})$ such that

$$A^{-1}\mathrm{d}A = \Omega,$$
$$\forall y \in U_1, \quad A(y) \in \mathcal{Z}(y),$$
$$A(y_0) = A_0,$$

where U_1 is a neighbourhood of y_0, which we can assume is simply connected.

We set $f^0 = A_0^0$, $f^i = A_0^i$ and we call f^{n+1} the unique function on U_1 such that $\mathrm{d}f^{n+1} = \eta$ and $f^{n+1}(y_0) = t_0$ (this function exists since U_1 is simply connected and $\mathrm{d}\eta = 0$). Thus we defined a map $f : U_1 \to \mathbb{E}^{n+2}$. Since $A_0^{n+1} = T^0 = 0$ and $A \in \mathrm{SO}^+(\mathbb{E}^{n+2})$, in the case of $\mathbb{S}^n \times \mathbb{R}$ we have $(f^0)^2 + \sum_i (f^i)^2 = \sum_\alpha (A_0^\alpha)^2 = 1$, and in the case of $\mathbb{H}^n \times \mathbb{R}$ we have $-(f^0)^2 + \sum_i (f^i)^2 = -(A_0^0)^2 + \sum_i (A_0^i)^2 + (A_0^{n+1})^2 = -1$ and $f^0 = A_0^0 > 0$. Thus in both cases we have $(f^0, \ldots, f^n) \in \mathbb{M}^n$, i.e., the values of f lie in $\mathbb{M}^n \times \mathbb{R}$.

Since $\mathrm{d}A = A\Omega$, we have, for $\alpha < n+1$,

$$\begin{aligned} \mathrm{d}f^\alpha(e_k) &= \sum_j A_j^\alpha \omega_0^j(e_k) + A_{n+1}^\alpha \omega_0^{n+1}(e_k) \\ &= \sum_j A_j^\alpha (\delta_j^k - T^j T^k) - A_{n+1}^\alpha T^{n+1} T^k \\ &= A_k^\alpha - T^k \sum_\beta A_\beta^\alpha A_\beta^{n+1} \\ &= A_k^\alpha \end{aligned}$$

and

$$\mathrm{d}f^{n+1}(e_k) = \eta(e_k) = T^k = A_k^{n+1}.$$

This means that $\mathrm{d}f(e_k)$ is given by the column k of the matrix A.

Since A is an invertible matrix, $\mathrm{d}f$ has rank n, and so f is an immersion. Also, since $A \in \mathrm{SO}^+(\mathbb{E}^{n+2})$, we have $\langle \mathrm{d}f(e_p), \mathrm{d}f(e_q) \rangle = \delta_q^p$, and so f is an isometry.

The columns of $A(y)$ form a direct orthonormal frame of $\mathbb{E}^{n+2}$. Columns 1 to n form a direct orthonormal frame of $\mathrm{T}_{f(y)} f(\mathcal{V})$ and column 0 is the projection of $f(y)$ on $\mathbb{M}^n \times \{0\}$, i.e., the unit normal $\bar{N}(f(y))$ to $\mathbb{M}^n \times \mathbb{R}$ at the point $f(y)$. Thus column $(n+1)$ is the unit normal $N(f(y))$ to $f(\mathcal{V})$ in $\mathbb{M}^n \times \mathbb{R}$ at the point $f(y)$.

We set $X_j = \mathrm{d}f(e_j)$. Then we have

$$\begin{aligned} \langle \mathrm{d}X_j(X_k), N \rangle &= \sum_\alpha \mathrm{d}A_j^\alpha(e_k) A_{n+1}^\alpha = \sum_\alpha \sum_\gamma A_\gamma^\alpha A_{n+1}^\alpha \omega_j^\gamma(e_k) \\ &= \omega_j^{n+1}(e_k) = \langle \mathrm{S}e_k, e_j \rangle. \end{aligned}$$

This means that the shape operator of $f(\mathcal{V})$ in $\mathbb{M}^n \times \mathbb{R}$ is $\mathrm{d}f \circ \mathrm{S} \circ \mathrm{d}f^{-1}$.

Finally, the coefficients of the vertical vector $\frac{\partial}{\partial t} = E_{n+1}$ in the orthonormal frame $(\bar{N}, X_1, \ldots, X_n, N)$ are given by the last line of A. Since $A(y) \in \mathcal{Z}(y)$ for all $y \in U_2$ we get

$$\frac{\partial}{\partial t} = \sum_j T^j X_j + T^{n+1} N = \mathrm{d}f(T) + \nu N.$$

We now prove that the local immersion is unique up to a global isometry of $\mathbb{M}^n \times \mathbb{R}$. Let $\tilde{f} : U_3 \to \mathbb{M}^n \times \mathbb{R}$ be another immersion satisfying the conclusion of the theorem, where U_3 is a simply connected neighbourhood of y_0 included in U_1, let $(\tilde{X}_\beta)$ be the associated frame (i.e., $\tilde{X}_j = \mathrm{d}\tilde{f}(e_j)$, $\tilde{X}_{n+1}$ is the normal of $\tilde{f}(\mathcal{V})$ in $\mathbb{M}^n \times \mathbb{R}$ and $\tilde{X}_0$ is the normal to $\mathbb{M}^n \times \mathbb{R}$ in $\mathbb{E}^{n+2}$) and let $\tilde{A}$ be the matrix of the coordinates of the frame $(\tilde{X}_\beta)$ in the frame (E_α). Up to a direct isometry of $\mathbb{M}^n \times \mathbb{R}$, we can assume that $f(y_0) = \tilde{f}(y_0)$ and that the frames $(X_\beta(y_0))$ and $(\tilde{X}_\beta(y_0))$ coincide, i.e., $A(y_0) = \tilde{A}(y_0)$. We notice that this isometry necessarily fixes $\frac{\partial}{\partial t}$ since the T^α are the same for x and $\tilde{x}$. The matrices A and $\tilde{A}$ satisfy $A^{-1}\mathrm{d}A = \Omega$ and $\tilde{A}^{-1}\mathrm{d}\tilde{A} = \Omega$ (see Section 2.3), $A(y), \tilde{A}(y) \in \mathcal{Z}(y)$ and $A(y_0) = \tilde{A}(y_0)$. Thus by the uniqueness of the solution of the equation in Proposition 3.7 we get $A(y) = \tilde{A}(y)$. Considering the columns 0 of these matrices, we get $f^i = \tilde{f}^i$ and $f^0 = \tilde{f}^0$. Finally we have $\mathrm{d}f^{n+1} = \eta = \mathrm{d}\tilde{f}^{n+1}$ and $f^{n+1}(y_0) = \tilde{f}^{n+1}(y_0)$; thus we have $f^{n+1} = \tilde{f}^{n+1}$. This finishes proving that $f = \tilde{f}$ on U_3.

Finally we prove that this local immersion f can be extended to $\mathcal{V}$ in a unique way. Let $y_1 \in \mathcal{V}$. Then there exists a curve $\Gamma : [0,1] \to \mathcal{V}$ such that $\Gamma(0) = y_0$ and $\Gamma(1) = y_1$. Each point of Γ has a neighbourhood such that there exists an isometric immersion (unique up to an isometry of $\mathbb{M}^n \times \mathbb{R}$ preserving the orientations of $\mathbb{M}^n$ and $\mathbb{R}$) of this neighbourhood satisfying the properties of the theorem. From this family of neighbourhoods we can extract a finite family $(W_1, \dots, W_p)$ covering Γ with $W_1 = U_1$. Then the above uniqueness argument shows that we can extend successively the immersion f to the W_k in a unique way. In particular $f(y_1)$ is defined. Moreover, this value $f(y_1)$ does not depend on the choice of the curve Γ joining y_0 to y_1 because $\mathcal{V}$ is simply connected. □

Proposition 3.8. *If* $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ *satisfies the compatibility equations and corresponds to an immerion* $f : \Sigma \to \mathbb{M}^n \times \mathbb{R}$, *then* $(\mathrm{d}s^2, -\mathrm{S}, T, -\nu)$, $(\mathrm{d}s^2, -\mathrm{S}, -T, \nu)$ *and* $(\mathrm{d}s^2, \mathrm{S}, -T, -\nu)$ *also satisfy the compatibility equations and correspond to the immersion* $\sigma \circ f$ *where* σ *is an isometry of* $\mathbb{M}^n \times \mathbb{R}$:

1. *reversing the orientation of* $\mathbb{M}^n$ *and preserving the orientation of* $\mathbb{R}$ *in the case of* $(\mathrm{d}s^2, -\mathrm{S}, T, -\nu)$,
2. *preserving the orientation of* $\mathbb{M}^n$ *and reversing the orientation of* $\mathbb{R}$ *in the case of* $(\mathrm{d}s^2, -\mathrm{S}, -T, \nu)$,
3. *reversing the orientations of both* $\mathbb{M}^n$ *and* $\mathbb{R}$ *in the case of* $(\mathrm{d}s^2, \mathrm{S}, -T, -\nu)$.

Proof. We deal with the first case (the two others are similar). Let $\hat{f} = \sigma \circ f$. Then the normal to $\mathbb{M}^n \times \mathbb{R}$ is $\sigma \circ \bar{N}$, and since σ reverses the orientation of $\mathbb{M}^n \times \mathbb{R}$ the normal to $\hat{f}(\mathcal{V})$ in $\mathbb{M}^n \times \mathbb{R}$ is $\hat{N} = -\sigma \circ N$. From this we deduce that $\hat{\mathrm{S}} = -\mathrm{S}$. Moreover we have $\frac{\partial}{\partial t} = \mathrm{d}f(T) + \nu N$, and so, since σ preserves the orientation of $\mathbb{R}$, we have

$$\frac{\partial}{\partial t} = \sigma \circ \mathrm{d}f(T) + \nu\sigma \circ N = \mathrm{d}\hat{f}(T) - \nu\hat{N}.$$

We conclude that $\hat{T} = T$ and $\hat{\nu} = -\nu$. □

3.3. **Remark: Another proof in the case of $\mathbb{H}^n \times \mathbb{R}$.** In this section we outline another proof of Theorem 3.3 in the case of $\mathbb{H}^n \times \mathbb{R}$ that does not involve the Lorentz space. Greek letters will denote indices between 1 and $n+1$.

We first consider an orientable hypersurface $\mathcal{V}$ of an $(n+1)$-dimensionnal Riemannian manifold $\bar{\mathcal{V}}$. Let $(e_1, \dots, e_n)$ be a local orthonormal frame on $\mathcal{V}$, e_{n+1} the normal to $\mathcal{V}$, and $(E_1, \dots, E_{n+1})$ a local orthonormal frame on $\bar{\mathcal{V}}$. We denote by ∇ and $\bar{\nabla}$ the Riemannian connections on $\mathcal{V}$ and $\bar{\mathcal{V}}$ respectively, and by S the shape operator of $\mathcal{V}$ (with respect to the normal e_{n+1}). We define the forms ω^α, ω^α_β on $\mathcal{V}$ as in Section 2.2. Then we have

$$\bar{\nabla}_{e_k} e_\beta = \sum_\gamma \omega^\gamma_\beta(e_k) e_\gamma.$$

Let $A \in \mathrm{SO}_{n+1}(\mathbb{R})$ be the matrix whose columns are the coordinates of the e_β in the frame (E_α), namely $A^\alpha_\beta = \langle e_\beta, E_\alpha \rangle$. Let $\Omega = (\omega^\alpha_\beta) \in \mathcal{M}_{n+1}(\mathbb{R})$. The matrix A satisfies the following equation:

$$A^{-1}\mathrm{d}A = \Omega + L(A)$$

with

$$L(A)^\alpha_\beta = \sum_k \left(\sum_{\gamma,\delta,\varepsilon} A^\varepsilon_\alpha A^\gamma_k A^\delta_\beta \bar{\Gamma}^\delta_{\gamma\alpha} \right) \omega^k,$$

where the $\bar{\Gamma}^\delta_{\gamma\alpha}$ are the Christoffel symbols of the frame (E_α). Notice that these matrices have size $n+1$, whereas those of Section 2.3 have size $n+2$.

We now assume that $\bar{\mathcal{V}} = \mathbb{H}^n \times \mathbb{R}$ and that $\mathcal{V}$ is a Riemannian manifold of dimension n endowed with S, T, ν satisfying the compatibility equations for $\mathbb{H}^n \times \mathbb{R}$. We consider a local orthonormal frame $(e_1, \dots, e_n)$ on $U \subset \mathcal{V}$, the associated one-forms ω^α, ω^α_β and the matrix of one-forms $\Omega \in \mathcal{M}_{n+1}(\mathbb{R})$.

We use the fact that there exists an orthonormal frame on $\mathbb{H}^n$ whose Christoffel symbols are constant. More precisely, we can choose the frame (E_α) on $\mathbb{H}^n \times \mathbb{R}$ such that $\bar{\Gamma}^i_{ij} = -\bar{\Gamma}^j_{ii} = \frac{1}{\sqrt{n}}$ for $i \neq j$, $i, j \leqslant n$ and all the other Christoffel symbols vanish.

The first step is to prove the following proposition, which is analogous to Proposition 3.7.

Proposition 3.9. *Let $y_0 \in \mathcal{V}$ and $A_0 \in \mathcal{Z}(y_0)$. Then there exist a neighbourhood U_1 of y_0 in $\mathcal{V}$ and a unique map $A : U_1 \to \mathrm{SO}_{n+1}(\mathbb{R})$ such that*

$$A^{-1}\mathrm{d}A = \Omega + L(A),$$
$$\forall y \in U_1, \quad A(y) \in \mathcal{Z}(y),$$
$$A(y_0) = A_0,$$

where $\mathcal{Z}(y)$ is defined in a way analogous to that of Section 3.2.

To prove this proposition, we introduce the form $\Theta = Z^{-1}\mathrm{d}Z - \Omega - L(Z)$ on $\mathcal{F} = \{(y, Z) \in U \times \mathrm{SO}_{n+1}(\mathbb{R}); Z \in \mathcal{Z}(y)\}$; this is well defined since the Christoffel symbols are constant. A long calculation shows that the distribution $\mathcal{D}(y, Z) = \ker \Theta_{(y,Z)}$ is involutive. We conclude as in the proof of Proposition 3.7.

The second step is to prove the following proposition.

Proposition 3.10. *Let $x_0 \in \mathbb{H}^n \times \mathbb{R}$. There exist a neighbourhood U_2 of y_0 contained in U_1 and a function $f : U_2 \to \mathbb{H}^n \times \mathbb{R}$ such that*

$$\mathrm{d}f = (B \circ f) A\omega,$$
$$f(y_0) = x_0,$$

where ω is the column $(\omega^1, \dots, \omega^n, 0)$ and, for $x \in \mathbb{H}^n \times \mathbb{R}$, $B(x) \in \mathcal{M}_{n+1}(\mathbb{R})$ is the matrix of the coordinates of the frame $(E_\alpha(x))$ in the frame $(\frac{\partial}{\partial x^\alpha})$ (we choose the upper half-space model for $\mathbb{H}^n$).

To prove it, we consider the form $B^{-1}\mathrm{d}x - A\omega$ on $U_1 \times \bar{\mathcal{V}}$, and we show that its kernel again defines an involutive distribution.

The last step is to check that this map f satisfies the conclusions of Theorem 3.3.

4. Applications to minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$

4.1. **The associate family.** Let $\mathbb{M}^2 = \mathbb{S}^2$ or $\mathbb{M}^2 = \mathbb{H}^2$. Let Σ be a Riemann surface with a metric $\mathrm{d}s^2$ (which we also denote by $\langle \cdot, \cdot \rangle$), ∇ its Riemannian connection, and J the rotation of angle $\frac{\pi}{2}$ on $\mathrm{T}\Sigma$. Let S be a field of symmetric operators $\mathrm{S}_y : \mathrm{T}_y\Sigma \to \mathrm{T}_y\Sigma$. Let T be a vector field on Σ and ν a smooth function on Σ such that $||T||^2 + \nu^2 = 1$.

Proposition 4.1. *Assume that S is trace-free and that $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ satisfies the compatibility equations for $\mathbb{M}^2 \times \mathbb{R}$. For $\theta \in \mathbb{R}$ we set*

$$\mathrm{S}_\theta = e^{\theta \mathrm{J}}\mathrm{S} = (\cos\theta)\mathrm{S} + (\sin\theta)\mathrm{JS},$$
$$T_\theta = e^{\theta \mathrm{J}}T = (\cos\theta)T + (\sin\theta)\mathrm{J}T,$$

i.e., S_θ and T_θ are obtained by rotating S and T by the angle θ.

Then S_θ is symmetric and trace-free, $||T_\theta||^2 + \nu^2 = 1$ and $(\mathrm{d}s^2, \mathrm{S}_\theta, T_\theta, \nu)$ satisfies the compatibility equations for $\mathbb{M}^2 \times \mathbb{R}$.

Proof. The fact that S_θ is symmetric and trace-free comes from an elementary computation. Moreover we have $||T_\theta|| = ||T||$. We notice that, since $\dim\Sigma = 2$, the Gauss equation (7) is equivalent to

$$K = \det \mathrm{S} + \kappa(1 - ||T||^2),$$

where K is the Gauss curvature of $\mathrm{d}s^2$. Since $\det(e^{\theta\mathrm{J}}) = 1$, we have $\det \mathrm{S}_\theta = \det \mathrm{S}$, and so the Gauss equation is satisfied for $(\mathrm{d}s^2, \mathrm{S}_\theta, T_\theta, \nu)$.

Since $e^{\theta\mathrm{J}}$ commutes with ∇_X (see [AR04], section 3.2) and preserves the metric, equations (9) and (10) are also satisfied for $(\mathrm{d}s^2, \mathrm{S}_\theta, T_\theta, \nu)$.

To prove that the Codazzi equation (8) is satisfied by $(\mathrm{d}s^2, \mathrm{S}_\theta, T_\theta, \nu)$, we first notice that, since

$$\nabla_X e^{\theta\mathrm{J}}\mathrm{S}Y - \nabla_Y e^{\theta\mathrm{J}}\mathrm{S}X - e^{\theta\mathrm{J}}\mathrm{S}[X,Y] = e^{\theta\mathrm{J}}(\nabla_X \mathrm{S}Y - \nabla_Y \mathrm{S}X - \mathrm{S}[X,Y]),$$

it suffices to prove that

$$\langle e^{\theta\mathrm{J}}T, Y\rangle X - \langle e^{\theta\mathrm{J}}T, X\rangle Y = e^{\theta\mathrm{J}}(\langle T, Y\rangle X - \langle T, X\rangle Y).$$

This is obvious at a point where $X = 0$. At a point where $X \neq 0$, we can write $Y = \lambda X + \mu \mathrm{J}X$, and a computation shows that both expressions are equal to $\mu\cos\theta\langle T, \mathrm{J}X\rangle X + \mu\sin\theta\langle T, X\rangle X - \mu\cos\theta\langle T, X\rangle \mathrm{J}X + \mu\sin\theta\langle T, \mathrm{J}X\rangle \mathrm{J}X$. $\square$

Theorem 4.2. *Let Σ be a simply connected Riemann surface and $x : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ a conformal minimal immersion. Let N be the induced normal. Let S be the symmetric operator on Σ induced by the shape operator of $x(\Sigma)$. Also, let T be the vector field on Σ such that $\mathrm{d}x(T)$ is the projection of $\frac{\partial}{\partial t}$ onto $\mathrm{T}(x(\Sigma))$ and let $\nu = \langle N, \frac{\partial}{\partial t}\rangle$.*

Let $z_0 \in \Sigma$. Then there exists a unique family $(x_\theta)_{\theta\in\mathbb{R}}$ of conformal minimal immersions $x_\theta : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ such that:

(1) $x_\theta(z_0) = x(z_0)$ *and* $(\mathrm{d}x_\theta)_{z_0} = (\mathrm{d}x)_{z_0}$,
(2) *the metrics induced on Σ by x and x_θ are the same,*
(3) *the symmetric operator on Σ induced by the shape operator of $x_\theta(\Sigma)$ is $e^{\theta \mathrm{J}}\mathrm{S}$,*
(4) $\frac{\partial}{\partial t} = \mathrm{d}x_\theta(e^{\theta\mathrm{J}}T) + \nu N_\theta$, *where N_θ is the unit normal to x_θ.*

Moreover we have $x_0 = x$, and the family (x_θ) is continuous with respect to θ.

The family of immersions $(x_\theta)_{\theta\in\mathbb{R}}$ is called the associate family of the immersion x, and the immersion $x_{\frac{\pi}{2}}$ is called the conjugate immersion of the immersion x, and the immersion x_π is called the opposite immersion of the immersion x.

Proof. Let $\mathrm{d}s^2$ be the metric on Σ induced by x. Then $(\mathrm{d}s^2, \mathrm{S}, T, \nu)$ satisfies the compatibility equations for $\mathbb{M}^2 \times \mathbb{R}$. Thus, by Proposition 4.1, $(\mathrm{d}s^2, e^{\theta\mathrm{J}}\mathrm{S}, e^{\theta\mathrm{J}}T, \nu)$ does as well. Thus by Theorem 3.3 there exists a unique immersion x_θ satisfying the properties of the theorem. The fact that $x_0 = x$ is clear.

Finally, $(\mathrm{d}s^2, e^{\theta\mathrm{J}}\mathrm{S}, e^{\theta\mathrm{J}}T, \nu)$ defines a matrix of one-forms Ω_θ and a matrix of functions A_θ satisfying $A_\theta^{-1}\mathrm{d}A_\theta = \Omega_\theta$ (by Proposition 3.7). By continuity of Ω_θ with respect to θ, we obtain the continuity of A_θ with respect to θ and then the continuity of x_θ with respect to θ. □

Remark 4.3. Let $\tau : \Sigma' \to \Sigma$ be a conformal diffeomorphism. If τ preserves the orientation, then $(x\circ\tau)_\theta = x_\theta\circ\tau$; if τ reverses the orientation, then $(x\circ\tau)_\theta = x_{-\theta}\circ\tau$.

In the sequel, we will speak of associate and conjugate immersions even if condition 1 is not satisfied; i.e., we will consider these notions up to isometries of $\mathbb{M}^2 \times \mathbb{R}$ preserving the orientations of both $\mathbb{M}^2$ and $\mathbb{R}$.

Remark 4.4. The opposite immersion is $x_\pi = \sigma \circ x$, where σ is an isometry of $\mathbb{M}^2 \times \mathbb{R}$ preserving the orientation of $\mathbb{M}^2$ and reversing the orientation of $\mathbb{R}$ (see Proposition 3.8, case (2)).

Remark 4.5. This associate family for minimal immersions in $\mathbb{M}^2 \times \mathbb{R}$ is analogous to the associate family for minimal immersions in $\mathbb{R}^3$. Conformal minimal immersions in $\mathbb{R}^3$ are given by the Weierstrass representation

$$x(z) = x(z_0) + \mathrm{Re}\int_{z_0}^{z} (1-g^2, i(1+g^2), 2g)\omega,$$

where g is a meromorphic function on Σ (the Gauss map) and ω a holomorphic one-form. Then the associate immersions are

$$x_\theta(z) = x(z_0) + \mathrm{Re}\int_{z_0}^{z} (1-g^2, i(1+g^2), 2g)e^{-i\theta}\omega.$$

Let $x = (\varphi, h) : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ be a conformal minimal immersion. Then h is a real harmonic function and φ is a harmonic map to $\mathbb{M}^2$. We set

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

The Hopf differential of φ is the following 2-form (see [Ros02b]):

$$\mathrm{Q}\varphi = 4\left\langle \frac{\partial\varphi}{\partial z}, \frac{\partial\varphi}{\partial z}\right\rangle \mathrm{d}z^2 = \left(\left\|\frac{\partial\varphi}{\partial u}\right\|^2 - \left\|\frac{\partial\varphi}{\partial v}\right\|^2 - 2i\left\langle \frac{\partial\varphi}{\partial u}, \frac{\partial\varphi}{\partial v}\right\rangle\right)\mathrm{d}z^2.$$

It is a holomorphic 2-form on Σ, and since x is conformal we have

$$\mathrm{Q}\varphi = -4\left(\frac{\partial h}{\partial z}\right)^2 \mathrm{d}z^2 = -(\mathrm{d}(h+ih^*))^2 = -4\left\langle T, \frac{\partial x}{\partial z}\right\rangle \mathrm{d}z^2,$$

where h^* is the harmonic conjugate function of h (i.e., $\frac{\partial h^*}{\partial u} = -\frac{\partial h}{\partial v}$ and $\frac{\partial h^*}{\partial v} = \frac{\partial h}{\partial u}$). The reader can refer to [SY97] for harmonic maps.

Proposition 4.6. *Let $x = (\varphi, h) : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ be a conformal minimal immersion, and $(x_\theta) = ((\varphi_\theta, h_\theta))$ its associate family of conformal minimal immersions. Let h^* be the harmonic conjugate of h. Then we have*

$$h_\theta = (\cos\theta)h + (\sin\theta)h^*, \quad \mathrm{Q}\varphi_\theta = e^{-2i\theta}\mathrm{Q}\varphi.$$

Proof. We have

$$\begin{aligned} \frac{\partial h_\theta}{\partial u} = \left\langle \frac{\partial x_\theta}{\partial u}, \frac{\partial}{\partial t}\right\rangle = \left\langle \frac{\partial}{\partial u}, T_\theta\right\rangle &= \cos\theta\left\langle \frac{\partial}{\partial u}, T\right\rangle + \sin\theta\left\langle \frac{\partial}{\partial u}, \mathrm{J}T\right\rangle \\ &= \cos\theta\left\langle \frac{\partial}{\partial u}, T\right\rangle - \sin\theta\left\langle \frac{\partial}{\partial v}, T\right\rangle \\ &= \cos\theta\frac{\partial h}{\partial u} - \sin\theta\frac{\partial h}{\partial v}. \end{aligned}$$

In the same way we have $\frac{\partial h_\theta}{\partial v} = \cos\theta\left\langle\frac{\partial}{\partial v}, T\right\rangle + \sin\theta\left\langle\frac{\partial}{\partial v}, \mathrm{J}T\right\rangle = \cos\theta\frac{\partial h}{\partial v} + \sin\theta\frac{\partial h}{\partial u}$. This proves that $h_\theta = (\cos\theta)h + (\sin\theta)h^*$. The expression of $\mathrm{Q}\varphi_\theta$ follows immediately. □

Remark 4.7. Recently, Hauswirth, Sá Earp and Toubiana ([HSET08]) defined the following notion of associated immersions in $\mathbb{H}^2 \times \mathbb{R}$: two isometric conformal minimal immersions in $\mathbb{H}^2 \times \mathbb{R}$ are said to be associated if their Hopf differential differ by the multiplication by some constant $e^{i\theta}$. Moreover, they proved that two isometric conformal minimal immersions in $\mathbb{H}^2 \times \mathbb{R}$ having the same Hopf differential are equal up to an isometry of $\mathbb{H}^2 \times \mathbb{R}$. Thus the notions of associated immersions in the sense of this paper and in the sense of [HSET08] are equivalent.

In [SET05], Sá Earp and Toubiana ask the following question: if two conformal minimal immersions $x, \tilde{x} : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ are isometric, then are they associated? (This result holds for $\mathbb{R}^3$.)

Remark 4.8. Abresch and Rosenberg ([AR04]) defined a holomorphic Hopf differential for constant mean curvature surfaces in $\mathbb{M}^2 \times \mathbb{R}$. For minimal surfaces in $\mathbb{M}^2 \times \mathbb{R}$, this Hopf differential is

$$\begin{aligned} Q(X,Y) &= -\frac{\kappa}{2}(\langle T, X\rangle\langle T, Y\rangle - \langle T, \mathrm{J}X\rangle\langle T, \mathrm{J}Y\rangle) \\ &\quad + i\frac{\kappa}{2}(\langle T, \mathrm{J}X\rangle\langle T, Y\rangle + \langle T, X\rangle\langle T, \mathrm{J}Y\rangle). \end{aligned}$$

A computation shows that

$$Q = \frac{\kappa}{2}\mathrm{Q}\varphi.$$

Proposition 4.9. *Let $x : \Sigma \to \mathbb{M}^2 \times \mathbb{R}$ be a conformal minimal immersion. If x does not define a horizontal $\mathbb{M}^2 \times \{t\}$, then the zeros of T are isolated.*

Proof. The height function $h = \langle x, \frac{\partial}{\partial t}\rangle$ satisfies $\mathrm{d}h(X) = \langle T, X\rangle$; thus the zeroes of T are the zeroes of $\mathrm{d}h$. Since h is harmonic, either the zeroes of $\mathrm{d}h$ are isolated or h is constant. The latter case is excluded by hypothesis. □

Remark 4.10. Umbilic points (i.e., zeroes of the shape operator) may be non-isolated: for example, helicoids and unduloids in $\mathbb{S}^2 \times \mathbb{R}$ have curves of umbilic points (see Section 4.2).

We now give some geometric properties of conjugate surfaces.

The transformation $\mathrm{S} \mapsto \mathrm{JS}$ implies that curvature lines and asymptotic lines are exchanged by conjugation (as in $\mathbb{R}^3$). (More generally the normal curvature and the normal torsion of a curve are swapped up to a sign.) The reader can refer to [Kar05] for geometric properties of conjugate surfaces in $\mathbb{R}^3$.

Moreover, the transformation $T \mapsto JT$ implies the following transformation: a horizontal curve γ along which the surface is vertical (i.e., $\nu = 0$ along γ and γ' is orthogonal to T) is mapped to a vertical curve (i.e., $\nu = 0$ along γ and γ' is proportional to T), and vice versa. We also notice that a minimal surface cannot be horizontal along a horizontal curve unless the minimal surface is a horizontal surface $\mathbb{M}^2 \times \{t\}$ (indeed, this would imply that $T = 0$ along this curve).

Hence conjugation swaps two pairs of Schwarz reflections:

(1) the symmetry with respect to a vertical plane containing a curvature line becomes the rotation with respect to a horizontal geodesic of $\mathbb{M}^2$, and vice versa,
(2) the symmetry with respect to a horizontal plane containing a curvature line becomes the rotation with respect to a vertical straight line, and vice versa.

The first case is illustrated by a generatrix curve of an unduloid or a catenoid and a horizontal line of a helicoid; the second case is illustrated by the waist circle of an unduloid or a catenoid and the axis of a helicoid. These examples are detailed in Sections 4.2 and 4.3.

4.2. **Helicoids and unduloids in** $\mathbb{S}^2 \times \mathbb{R}$**.** Apart from the horizontal spheres $\mathbb{S}^2 \times \{t\}$ and the vertical cylinders $\mathbb{S}^1 \times \mathbb{R}$ ($\mathbb{S}^1$ being a great circle in $\mathbb{S}^2$), the most simple examples of minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ are helicoids and unduloids. These surfaces are described in [PR99] and [Ros02b]. They are properly embedded and foliated by circles. Unduloids are rotational and vertically periodic; helicoids are invariant by a screw motion.

Helicoids. For $\beta \neq 0$, the helicoid $\mathcal{H}_\beta$ is given by the following conformal immersion:

$$x(u,v) = \begin{pmatrix} \sin\varphi(u)\cos\beta v \\ \sin\varphi(u)\sin\beta v \\ \cos\varphi(u) \\ v \end{pmatrix},$$

where the function φ satisfies

$$\varphi'(u)^2 = 1 + \beta^2\sin^2\varphi(u), \quad \varphi''(u) = \beta^2 \sin\varphi(u)\cos\varphi(u). \tag{11}$$

We can assume that $\varphi(0) = 0$ and $\varphi'(u) > 0$. When $\beta > 0$ we say that $\mathcal{H}_\beta$ is a right helicoid; when $\beta < 0$ we say that $\mathcal{H}_\beta$ is a left helicoid.

The normal to $\mathbb{S}^2 \times \mathbb{R}$ in $\mathbb{R}^4$ is

$$\bar{N}(u,v) = \begin{pmatrix} \sin\varphi(u)\cos\beta v \\ \sin\varphi(u)\sin\beta v \\ \cos\varphi(u) \\ 0 \end{pmatrix}.$$

The normal to $\mathcal{H}_\beta$ in $\mathbb{S}^2 \times \mathbb{R}$ is

$$N(u,v) = \frac{1}{\varphi'(u)} \begin{pmatrix} \sin\beta v \\ -\cos\beta v \\ 0 \\ \beta\sin\varphi(u) \end{pmatrix}.$$

We compute

$$\left\langle \frac{\partial^2 x}{\partial u^2}, N \right\rangle = \left\langle \frac{\partial^2 x}{\partial v^2}, N \right\rangle = 0, \quad \left\langle \frac{\partial^2 x}{\partial u \partial v}, N \right\rangle = -\beta\cos\varphi(u).$$

Using the fact that $\langle \mathrm{S}X, Y\rangle = \langle \mathrm{d}Y(X), N\rangle$, we compute that the matrix of S in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\beta\cos\varphi(u)}{\varphi'(u)^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular the points where $\cos\varphi(u) = 0$ are umbilic points. We also have

$$T = \frac{1}{\varphi'(u)^2}\frac{\partial}{\partial v}, \quad \nu = \frac{\beta\sin\varphi(u)}{\varphi'(u)}.$$

Remark 4.11. When $\beta = 0$, the formula defines a vertical cylinder $\mathbb{S}^1 \times \mathbb{R}$. When $\beta \to \infty$, the surface converges to the foliation by horizontal spheres $\mathbb{S}^2 \times \{t\}$.

Unduloids. For $\alpha > 1$ or $\alpha < -1$, the unduloid $\mathcal{U}_\alpha$ is given by the following conformal immersion:

$$x(u,v) = \begin{pmatrix} \sin\psi(u)\cos\alpha v \\ \sin\psi(u)\sin\alpha v \\ \cos\psi(u) \\ u \end{pmatrix},$$

where the function ψ satisfies

$$1 + \psi'(u)^2 = \alpha^2\sin^2\psi(u), \quad \psi''(u) = \alpha^2\sin\psi(u)\cos\psi(u). \tag{12}$$

We can assume that $\psi'(0) = 0$, $\psi(u) \in (0,\pi)$ and $\cos\psi(0) > 0$.

The normal to $\mathcal{U}_\alpha$ in $\mathbb{S}^2 \times \mathbb{R}$ is

$$N(u,v) = \frac{1}{\alpha\sin\psi(u)} \begin{pmatrix} -\cos\psi(u)\cos\alpha v \\ -\cos\psi(u)\sin\alpha v \\ \sin\psi(u) \\ \psi'(u) \end{pmatrix}.$$

We compute that the matrix of S in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\alpha\cos\psi(u)}{1+\psi'(u)^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular the points where $\cos\psi(u) = 0$ are umbilic points. We also have

$$T = \frac{1}{1+\psi'(u)^2}\frac{\partial}{\partial u}, \quad \nu = \frac{\psi'(u)}{\alpha\sin\psi(u)}.$$

Remark 4.12. When $\alpha = \pm 1$, the formula defines a vertical cylinder $\mathbb{S}^1 \times \mathbb{R}$. When $\alpha \to \infty$, the surface converges to the foliation by horizontal spheres $\mathbb{S}^2 \times \{t\}$.

Proposition 4.13. *The conjugate surface of the unduloid $\mathcal{U}_\alpha$ is the helicoid $\mathcal{H}_\beta$ with $\alpha^2 = 1 + \beta^2$ and α, β having the same sign.*

Proof. We set $y_1(u) = \alpha \cos\psi(u)$ and $y_2(u) = \beta\cos\varphi(u)$. A computation shows that both y_1 and y_2 are solutions of the equation

$$(y')^2 = (y^2 - \alpha^2)(y^2 - \beta^2),$$

and hence of the equation

$$y'' = y(2y^2 - \alpha^2 - \beta^2).$$

We have $\psi'(0) = 0$, and so by (12) we have $y_1(0)^2 = \beta^2$ and thus $y_1'(0) = 0$; also, $\varphi(0) = 0$, so $y_2(0) = \beta$ and thus $y_2'(0) = 0$. Moreover, $\cos\psi(0) > 0$, so $y_1(0)$ has the sign of α; since α and β have the same sign, we have $y_1(0) = \beta$. By the Cauchy-Lipschitz theorem we conclude that $y_1 = y_2$. From this we deduce using (12) and (11) that $\varphi'(u)^2 = 1 + \psi'(u)^2$; thus $\mathcal{U}_\alpha$ and $\mathcal{H}_\beta$ are locally isometric, and $\mathrm{S}_{\mathcal{H}_\beta} = \mathrm{J}\mathrm{S}_{\mathcal{U}_\alpha}$ and $T_{\mathcal{H}_\beta} = \mathrm{J}T_{\mathcal{U}_\alpha}$. Finally we have $\nu_{\mathcal{U}_\alpha} = -\frac{y_1'}{\alpha^2 - y_1^2}$ and $\nu_{\mathcal{H}_\beta} = -\frac{y_2'}{\alpha^2 - y_2^2}$, so we get $\nu_{\mathcal{H}_\beta} = \nu_{\mathcal{U}_\alpha}$. □

Remark 4.14. The vertical cylinder $\mathbb{S}^1 \times \mathbb{R}$ is globally invariant by conjugation, but the vertical lines and the horizontal circles are exchanged. For example, a rectangle of height t and whose basis is an arc of angle θ becomes a rectangle of height θ and whose basis is an arc of angle t.

The horizontal sphere $\mathbb{S}^2 \times \{0\}$ is pointwise invariant by conjugation (since it satisfies $\mathrm{S} = 0$ and $T = 0$).

Remark 4.15. The horizontal projections of helicoids and unduloids are the Gauss maps of constant mean curvature Delaunay surfaces in $\mathbb{R}^3$: helicoids in $\mathbb{S}^2 \times \mathbb{R}$ come from nodoids in $\mathbb{R}^3$, and unduloids in $\mathbb{S}^2 \times \mathbb{R}$ come from unduloids in $\mathbb{R}^3$. This correspondence is described in [Ros03].

4.3. **Helicoids and generalized catenoids in $\mathbb{H}^2 \times \mathbb{R}$.** Apart from the horizontal planes $\mathbb{H}^2 \times \{t\}$ and the vertical planes $\mathbb{H}^1 \times \mathbb{R}$ ($\mathbb{H}^1$ being a geodesic of $\mathbb{H}^2$), the most simple examples of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ are helicoids and catenoids. These surfaces are described in [PR99] and [NR02]. They are properly embedded. Catenoids are rotational; helicoids are invariant by a screw motion and foliated by geodesics of $\mathbb{H}^2$.

More generally, Hauswirth classified minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ foliated by horizontal curves of constant curvature in $\mathbb{H}^2$ ([Hau06]). These surfaces form a two-parameter family. This family includes, among others, catenoids, helicoids and Riemann-type examples. All the surfaces described in this section belong to the Hauswirth family.

Helicoids. For $\beta \neq 0$, the helicoid $\mathcal{H}_\beta$ is given by the following conformal immersion:

$$x(u,v) = \begin{pmatrix} \cosh\varphi(u) \\ \sinh\varphi(u)\cos\beta v \\ \sinh\varphi(u)\sin\beta v \\ v \end{pmatrix},$$

where the function φ satisfies

$$\varphi'(u)^2 = 1 + \beta^2\sinh^2\varphi(u), \quad \varphi''(u) = \beta^2\sinh\varphi(u)\cosh\varphi(u). \tag{13}$$

We can assume that $\varphi(0) = 0$ and $\varphi'(u) > 0$. The function φ is defined on a bounded interval. When $\beta > 0$ we say that $\mathcal{H}_\beta$ is a right helicoid; when $\beta < 0$ we say that $\mathcal{H}_\beta$ is a left helicoid.

The normal to $\mathcal{H}_\beta$ in $\mathbb{H}^2 \times \mathbb{R}$ is

$$N(u,v) = \frac{1}{\varphi'(u)} \begin{pmatrix} 0 \\ \sin \beta v \\ -\cos \beta v \\ \beta \sinh \varphi(u) \end{pmatrix}.$$

Now $\beta > 0$. we compute that the matrix of S in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\beta \cosh \varphi(u)}{\varphi'(u)^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We also have

$$T = \frac{1}{\varphi'(u)^2} \frac{\partial}{\partial v}, \quad \nu = \frac{\beta \sinh \varphi(u)}{\varphi'(u)}.$$

Remark 4.16. When $\beta = 0$, the fomula defines a vertical plane $\mathbb{H}^1 \times \mathbb{R}$. When $\beta \to \infty$, the surface converges to the foliation by horizontal planes $\mathbb{H}^2 \times \{t\}$.

Catenoids. For $\alpha \neq 0$, the catenoid $\mathcal{C}_\alpha$ is given by the following conformal immersion:

$$x(u,v) = \begin{pmatrix} \cosh \psi(u) \\ \sinh \psi(u) \cos \alpha v \\ \sinh \psi(u) \sin \alpha v \\ u \end{pmatrix},$$

where the function ψ satisfies

$$(14) \qquad 1 + \psi'(u)^2 = \alpha^2 \sinh^2 \psi(u), \quad \psi''(u) = \alpha^2 \sinh \psi(u) \cosh \psi(u).$$

We can assume that $\psi'(0) = 0$ and $\psi(u) > 0$. The function ψ is defined on the interval $(-u_0, u_0)$ with

$$u_0 = \int_{\psi(0)}^{\infty} \frac{\mathrm{d}\psi}{\sqrt{\alpha^2 \sinh^2 \psi - 1}} = \int_1^{\infty} \frac{\mathrm{d}x}{\sqrt{(x^2+\alpha^2)(x^2-1)}}.$$

Thus we have

$$u_0 < \int_1^{\infty} \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = \frac{\pi}{2}.$$

This proves that the height of the catenoid $\mathcal{C}_\alpha$ is smaller than π; moreover the height tends to 0 when $\alpha \to \infty$ and to π when $\alpha \to 0$ (theorem 1 in [NR02] holds for $t \in (0, \frac{\pi}{2})$). The function ψ is decreasing on $(-u_0, 0)$ and increasing on $(0, u_0)$. The waist circle is given by $u = 0$.

The normal to $\mathcal{C}_\alpha$ in $\mathbb{H}^2 \times \mathbb{R}$ is

$$N(u,v) = \frac{1}{\alpha \sinh \psi(u)} \begin{pmatrix} -\sinh \psi(u) \\ -\cosh \psi(u) \cos \alpha v \\ -\cosh \psi(u) \sin \alpha v \\ \psi'(u) \end{pmatrix}.$$

We compute that the matrix of S in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\alpha \cosh \psi(u)}{1 + \psi'(u)^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also have

$$T = \frac{1}{1+\psi'(u)^2} \frac{\partial}{\partial u}, \quad \nu = \frac{\psi'(u)}{\alpha \sinh \psi(u)}.$$

A minimal surface foliated by horocycles. We search a minimal surface such that each horizontal curve is a horocycle in $\mathbb{H}^2$ and such that all the horocycles have the same asymptotic point. Such a surface can be parametrized in the following way:

$$x(u,v) = \begin{pmatrix} \frac{\lambda(u)}{2} + \frac{1+f(u,v)^2}{2\lambda(u)} \\ f(u,v) \\ -\frac{\lambda(u)}{2} + \frac{1+f(u,v)^2}{2\lambda(u)} \\ u \end{pmatrix}$$

with $\lambda > 0$ and $\frac{\partial f}{\partial v} > 0$. This immersion is conformal if and only if

$$\frac{\partial f}{\partial u} = \frac{f\lambda'}{\lambda}, \quad \left(\frac{\partial f}{\partial v}\right)^2 = 1 + \left(\frac{\lambda'}{\lambda}\right)^2.$$

We deduce from the second relation that $\frac{\partial^2 f}{\partial v^2} = 0$, and so

$$f(u,v) = \alpha(u)v + \beta(u).$$

Reporting in the first relation we get

$$\frac{\alpha'}{\alpha} = \frac{\beta'}{\beta} = \frac{\lambda'}{\lambda}.$$

The immersion is minimal if and only if Δx is proportional to the normal $\bar{N}$ to $\mathbb{H}^2 \times \mathbb{R}$; a computation shows that this happens if and only if $(\lambda')^2 + \alpha^2\lambda^2 = \lambda\lambda''$, i.e., if and only if $2(\lambda')^2 + \lambda^2 = \lambda\lambda''$, or, equivalently,

$$\left(\frac{1}{\lambda}\right)'' = -\frac{1}{\lambda}.$$

Up to a reparametrization and an isometry of $\mathbb{H}^2$ we can choose $\lambda(u) = \alpha(u) = \frac{1}{\cos u}$ for $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\beta(u) = 0$. Thus we get the following proposition.

Proposition 4.17. *The map*

$$x(u,v) = \begin{pmatrix} \frac{v^2+1}{2\cos u} + \frac{\cos u}{2} \\ \frac{v}{\cos u} \\ \frac{v^2-1}{2\cos u} + \frac{\cos u}{2} \\ u \end{pmatrix}$$

defined for $(u,v) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}$ is a conformal minimal embedding such that the curves $u = u_0$ are horocycles in $\mathbb{H}^2$ having the same asymptotic point. We will denote this surface by $\mathcal{C}_0$.

Moreover, the surface $\mathcal{C}_0$ is the unique one (up to isometries of $\mathbb{H}^2 \times \mathbb{R}$) having this property.

In the upper half-plane model for $\mathbb{H}^2$, the curve at height u of $\mathcal{C}_0$ is the horizontal Euclidean line $x_2 = \cos u$. Figure 1 is a picture of $\mathcal{C}_0$ (in this picture the model for $\mathbb{H}^2$ is the Poincaré unit disk model). The surface $\mathcal{C}_0$ has height π. It is symmetric with respect to the horizontal plane $\mathbb{H}^2 \times \{0\}$, and it is invariant by a one-parameter family of horizontal parabolic isometries.

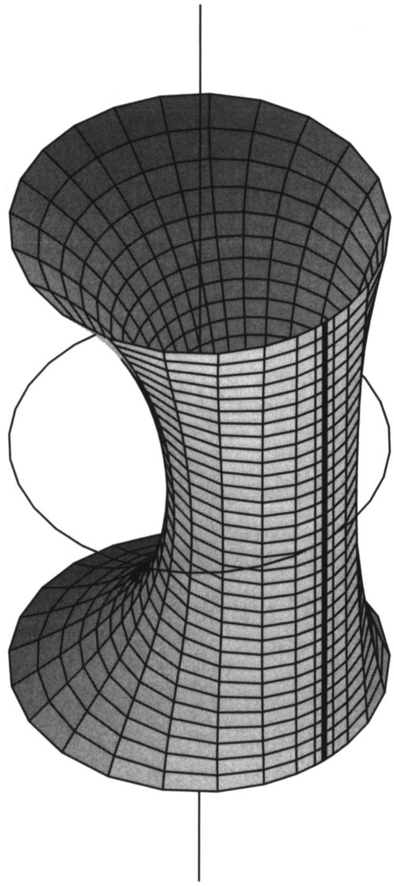

FIGURE 1. A minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ foliated by horocycles

The normal to $\mathcal{C}_0$ in $\mathbb{H}^2 \times \mathbb{R}$ is

$$N(u,v) = \begin{pmatrix} -\frac{v^2+1}{2} + \frac{\cos^2 u}{2} \\ -v \\ \frac{1-v^2}{2} + \frac{\cos^2 u}{2} \\ \sin u \end{pmatrix}.$$

We compute that the matrix of S in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\cos u \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also have

$$T = \cos^2 u \frac{\partial}{\partial u}, \quad \nu = \sin u.$$

Minimal surfaces foliated by equidistants. For $\gamma \in (0,1)$ or $\gamma \in (-1,0)$, we consider the following immersion:

$$x(u,v) = \begin{pmatrix} \cosh\chi(u)\cosh\gamma v \\ \sinh\chi(u) \\ \cosh\chi(u)\sinh\gamma v \\ u \end{pmatrix}$$

with

$$1 + \chi'(u)^2 = \gamma^2\cosh^2\chi(u), \quad \chi''(u) = \gamma^2\cosh\chi(u)\sinh\chi(u). \tag{15}$$

It is a conformal minimal immersion.

We choose χ such that $\chi'(0) = 0$ and $\chi(u) > 0$. The function χ is defined on the interval $(-u_0, u_0)$ with

$$u_0 = \int_{\chi(0)}^{\infty} \frac{\mathrm{d}\chi}{\sqrt{\gamma^2\cosh^2\chi - 1}} = \int_1^{\infty} \frac{\mathrm{d}x}{\sqrt{(x^2-\gamma^2)(x^2-1)}}.$$

Thus we have

$$u_0 > \int_1^{\infty} \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = \frac{\pi}{2}.$$

We have defined a minimal surface $\mathcal{G}_\gamma$, which we call a generalized catenoid. Its height is greater than π, and tends to π when $\gamma \to 0$ and to $+\infty$ when $\gamma \to 1$. The function χ is decreasing on $(-u_0, 0)$ and increasing on $(0, u_0)$. The surface is symmetric with respect to the horizontal plane $\mathbb{H}^2 \times \{0\}$, and it is invariant by a one-parameter family of horizontal hyperbolic isometries. The horizontal curves are equidistants to a geodesic in $\mathbb{H}^2$.

The normal to $\mathcal{G}_\gamma$ in $\mathbb{H}^2 \times \mathbb{R}$ is

$$N(u,v) = -\frac{1}{\gamma\cosh\chi(u)} \begin{pmatrix} \sinh\chi(u)\cosh\gamma v \\ \cosh\chi(u) \\ \sinh\chi(u)\sinh\gamma v \\ -\chi'(u) \end{pmatrix}.$$

We compute that the matrix of S in the frame $(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is the following:

$$-\frac{\gamma\sinh\chi(u)}{1+\chi'(u)^2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also have

$$T = \frac{1}{1+\chi'(u)^2}\frac{\partial}{\partial u}, \quad \nu = \frac{\chi'(u)}{\gamma\cosh\chi(u)}.$$

Remark 4.18. When $\gamma = \pm 1$, the formula defines a vertical plane $\mathbb{H}^1 \times \mathbb{R}$.

Proposition 4.19. *The conjugate surface of the catenoid $\mathcal{C}_\alpha$ is the helicoid $\mathcal{H}_\beta$ with $\beta^2 = 1 + \alpha^2$ and α, β having the same sign.*

Proof. We set $y_1(u) = \alpha\cosh\psi(u)$ and $y_2(u) = \beta\cosh\varphi(u)$. A computation shows that both y_1 and y_2 are solutions of the equation

$$(y')^2 = (y^2-\alpha^2)(y^2-\beta^2),$$

and hence of the equation

$$y'' = y(2y^2 - \alpha^2 - \beta^2).$$

We have $\psi'(0) = 0$, and so by (14) we have $y_1(0)^2 = \beta^2$ and thus $y_1'(0) = 0$; also, $\varphi(0) = 0$, so $y_2(0) = \beta$ and thus $y_2'(0) = 0$. Moreover, $y_1(0)$ has the sign of α, i.e., the sign of β, so we get $y_1(0) = \beta$. By the Cauchy-Lipschitz theorem we conclude that $y_1 = y_2$ (and in particular they have the same domain of definition). From this we deduce using (14) and (13) that $\varphi'(u)^2 = 1 + \psi'(u)^2$, and thus $\mathcal{C}_\alpha$ and $\mathcal{H}_\beta$ are locally isometric, $\mathrm{S}_{\mathcal{H}_\beta} = \mathrm{JS}_{\mathcal{C}_\alpha}$ and $T_{\mathcal{H}_\beta} = \mathrm{J}T_{\mathcal{C}_\alpha}$. Finally we have $\nu_{\mathcal{C}_\alpha} = \frac{y_1'}{y_1^2-\alpha^2}$ and $\nu_{\mathcal{H}_\beta} = \frac{y_2'}{y_2^2-\alpha^2}$, so we get $\nu_{\mathcal{H}_\beta} = \nu_{\mathcal{C}_\alpha}$. □

Proposition 4.20. *The conjugate surface of the surface $\mathcal{C}_0$ is the helicoid $\mathcal{H}_1$.*

Proof. In the case where $\beta = 1$, the function φ satisfies $\varphi' = \cosh\varphi$, and thus we have $\varphi(u) = \ln(\tan(\frac{u}{2} + \frac{\pi}{4}))$, $\varphi'(u) = \frac{1}{\cos u}$ and $\sinh\varphi(u) = \tan u$. Then, using the above calculations, we easily check that $\mathcal{C}_0$ and $\mathcal{H}_1$ are locally isometric, and that $\mathrm{S}_{\mathcal{H}_1} = \mathrm{JS}_{\mathcal{C}_0}$, $T_{\mathcal{H}_1} = \mathrm{J}T_{\mathcal{C}_0}$, $\nu_{\mathcal{H}_1} = \nu_{\mathcal{C}_0}$. □

Remark 4.21. The conjugate surface of the surface $\mathcal{C}_0$ with the opposite orientation is the helicoid $\mathcal{H}_{-1}$.

Proposition 4.22. *The conjugate surface of the generalized catenoid $\mathcal{G}_\gamma$ is the helicoid $\mathcal{H}_\beta$ with $\beta^2 + \gamma^2 = 1$ and β, γ having the same sign.*

Proof. We set $y_1(u) = \gamma\sinh\chi(u)$ and $y_2(u) = \beta\cosh\varphi(u)$. A computation shows that both y_1 and y_2 are solutions of the equation

$$(y')^2 = (y^2 + \gamma^2)(y^2 - \beta^2),$$

and hence of the equation

$$y'' = y(2y^2 + \gamma^2 - \beta^2).$$

We have $\chi'(0) = 0$, and so by (15) we have $y_1(0)^2 = \beta^2$ and thus $y_1'(0) = 0$; also, $\varphi(0) = 0$, so $y_2(0) = \beta$ and thus $y_2'(0) = 0$. Moreover, $y_1(0)$ has the sign of γ, i.e., the sign of β, so we get $y_1(0) = \beta$. By the Cauchy-Lipschitz theorem we conclude that $y_1 = y_2$ (and in particular they have the same domain of definition). From this we deduce using (15) and (13) that $\varphi'(u)^2 = 1 + \chi'(u)^2$, and thus $\mathcal{G}_\gamma$ and $\mathcal{H}_\beta$ are locally isometric, $\mathrm{S}_{\mathcal{H}_\beta} = \mathrm{JS}_{\mathcal{G}_\gamma}$ and $T_{\mathcal{H}_\beta} = \mathrm{J}T_{\mathcal{G}_\gamma}$. Finally we have $\nu_{\mathcal{G}_\gamma} = \frac{y_1'}{y_1^2+\gamma^2}$ and $\nu_{\mathcal{H}_\beta} = \frac{y_2'}{y_2^2+\gamma^2}$, so we get $\nu_{\mathcal{H}_\beta} = \nu_{\mathcal{G}_\gamma}$. □

Remark 4.23. This study shows that there are three types of helicoid conjugates according to the parameter of the screw-motion associated to the helicoid: the first type ones are the catenoids, which are rotational surfaces, the second type one is $\mathcal{C}_0$, which is invariant by a one-parameter family of horizontal parabolic isometries and which corresponds to a critical value of the parameter, the third type ones are the generalized catenoids, which are invariant by a one-parameter family of horizontal hyperbolic isometries.

This phenomenon is very similar to what happens to the conjugate cousins in $\mathbb{H}^3$ of the helicoids in $\mathbb{R}^3$. There exists an isometric correspondence between minimal surfaces in $\mathbb{R}^3$ and constant mean curvature one surfaces in $\mathbb{H}^3$ called the cousin relation (see [Bry87] and [UY93]). Starting from a helicoid in $\mathbb{R}^3$, we consider its conjugate surface, which is a catenoid in $\mathbb{R}^3$, and then the cousin surface in $\mathbb{H}^3$, which is a catenoid cousin. Catenoid cousins are of three types according to the parameter of the minimal helicoid: some are rotational surfaces, one is invariant by a one-parameter family of parabolic isometries (and corresponds to a critical value of the parameter), and some are invariant by a one-parameter family of hyperbolic isometries. These surfaces are described in detail in [SET01] and [Ros02a].

Remark 4.24. All the above surfaces belong to the Hauswirth family: with the notation of [Hau06], helicoids correspond to $d = 0$, $c > 0$, $c \neq 1$; catenoids correspond to $c = 0$, $d > 1$; the surface $\mathcal{C}_0$ corresponds to $c = 0$, $d = 1$; the surfaces $\mathcal{G}_\gamma$ correspond to $c = 0$, $d \in (0, 1)$.

Remark 4.25. The vertical plane $\mathbb{H}^1 \times \mathbb{R}$ is globally invariant by conjugation, but the vertical lines and the horizontal geodesics of $\mathbb{H}^2$ are exchanged. The horizontal plane $\mathbb{H}^2 \times \{0\}$ is pointwise invariant by conjugation (since it satisfies $\mathrm{S} = 0$ and $T = 0$). This is similar to what happens in $\mathbb{S}^2 \times \mathbb{R}$.

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