

A diagrammatic approach to the

Rasmussen invariant via tangles & cobordisms.

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(Reference : [arxiv:2503.05414](https://arxiv.org/abs/2503.05414) / slides are available online)

Overview.

$$K = \bigcirc$$

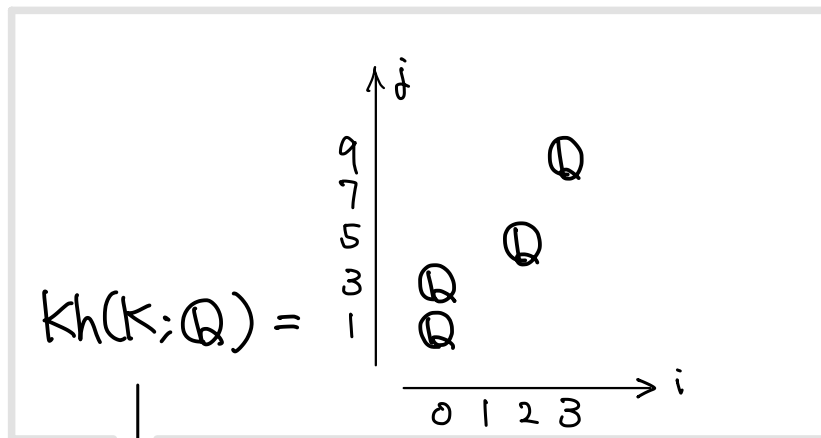
a knot / link in \mathbb{R}^3

$$\xrightarrow[\text{(Jones, 1984)}]{J}$$

$$J(K) = q^1 + q^3 + q^5 - q^9$$

— the Jones polynomial of K .

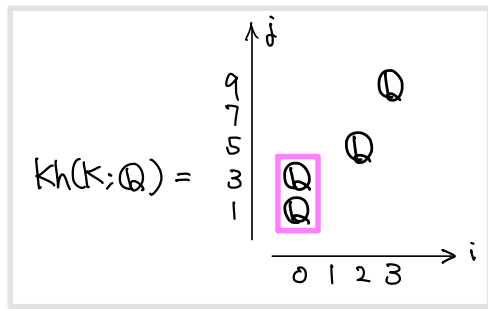
$$\xrightarrow[\text{Categorify}]{\text{Kh}} \text{(Khovanov, 2000)}$$



— the Khovanov homology of K

$$\hat{\chi} = \sum_{i,j} (-1)^i q^j \dim_{\mathbb{Q}}(-)$$

Overview.



\rightsquigarrow

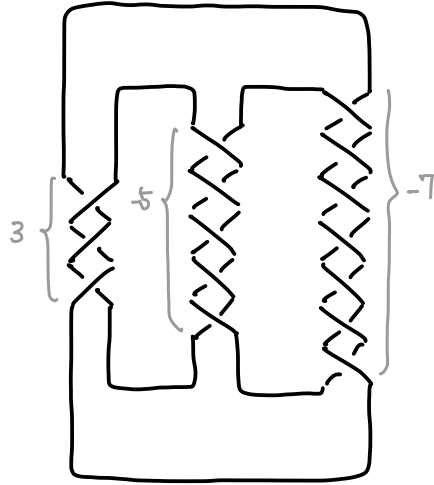
$$S(K) = 2$$

— the Rasmussen invariant of K .

⑤ is algorithmically computable,
but its geometric interpretation remains a mystery.

We give a "diagrammatic description" of ⑤ based on
Bar-Natan's reformulation of Kh. homology via tangles & cobordisms.

Overview. (Application)



$$P = P(3, -5, -7)$$

- $S(P) \neq 0.$
 $\Rightarrow P$ is NOT smoothly slice.
- $\Delta_P = 1$ (Alexander poly).
 $\Rightarrow P$ is topologically slice

computable by hand!

\rightsquigarrow Such knot gives rise to an exotic $4\mathbb{R}^4$.

Contents.

§1. $\text{Cob}_{\bullet/2}(B)$ — the category of dotted cobordisms.

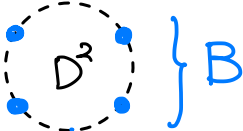
§2. Bar-Natan's reformulation of Kh. homology.

§3. Reformulation of the s -invariant.

§4. Computations.

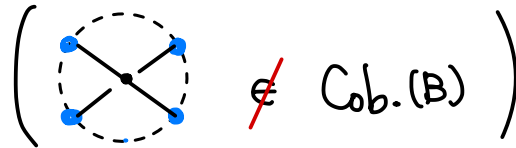
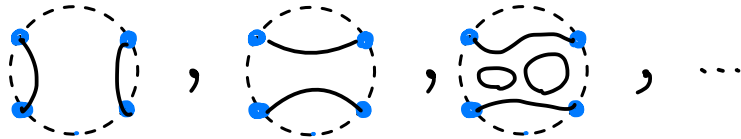


NEW!

§1. Fix $B \subset \partial D^2$, a set of even num. of pts.  $\} B$

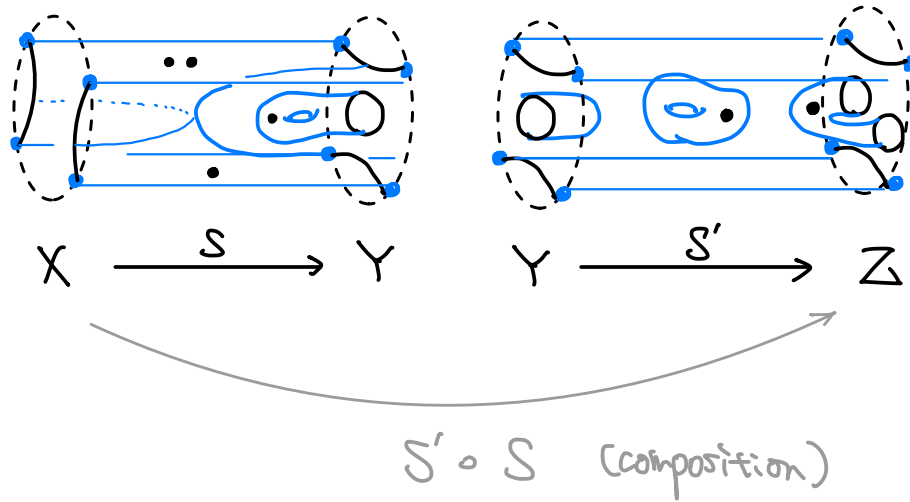
\rightsquigarrow Cob.(B) — the (preadditive) category of (1+1)-dotted cobordisms

objects ... $\{ \text{compact } \overset{\text{unoriented}}{1\text{-mfd}} X \subset D^2, \partial X = B \}$





morphisms ... (next page)

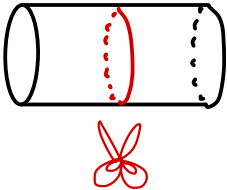
morphisms ... \mathbb{Z} $\left\{ \begin{array}{l} S: X \longrightarrow Y ; \\ \text{dotted surface } S \subset D^2 \times I, \\ \partial S = X \times \{0\} \cup B \times I \cup Y \times \{1\} \end{array} \right\} / \text{isotopy rel } \partial.$



Remark Objects of $\text{Cob}_\bullet(B)$ are NOT sets!

Local relations in $\text{Hom Cob.}(B)$

(S)  = 0 , (S.)  = 1 ,

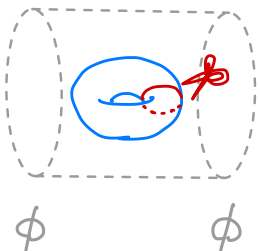
(NC)  = $\left(\begin{array}{c} \text{D} \\ \bullet \end{array} \right) \left(\begin{array}{c} \text{D} \\ \vdots \end{array} \right) + \left(\begin{array}{c} \text{D} \\ \vdots \end{array} \right) \left(\begin{array}{c} \text{D} \\ \bullet \end{array} \right) - \left(\begin{array}{c} \text{D} \\ \text{D} \end{array} \right) \left(\begin{array}{c} \text{D} \\ \text{D} \end{array} \right) \left(\begin{array}{c} \text{D} \\ \vdots \end{array} \right)$

$\text{Cob}_{\bullet/2}(B)$ \coloneqq $\text{Cob.}(B)$ modulo local relations.

Consider when $B = \emptyset$; $X, Y = \emptyset$.



$$\text{Hom}_{\text{Cob}_{\bullet/2}(\emptyset)}(\emptyset, \emptyset) = \mathbb{Z} \{ \text{closed dotted surface} \}$$

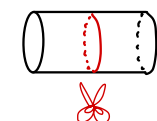
isotopy
& hc. rel.

e.g. 1)  $\stackrel{\text{CNC}}{=} \underbrace{\text{[diagram]}}_{=1} + \underbrace{\text{[diagram]}}_{=1} - \underbrace{\text{[diagram]}}_{=0} = 2 \cdot \emptyset$

2)  $= 2 \cdot \text{[diagram with two dots]}$

3)  $= \text{[dotted circle with a question mark]}$

(S)  $= 0$, (S.)  $= 1$,

CNC  $= \left(\text{[diagram 1]} \text{ [diagram 2]} \right) + \left(\text{[diagram 3]} \text{ [diagram 4]} \right) - \left(\text{[diagram 5]} \text{ [diagram 6]} \right)$

Lem

$$\begin{aligned}
 \boxed{\bullet \bullet} &= \text{diagram of a sphere with two dots and a red cut} \\
 &= \boxed{\bullet} \underbrace{\text{diagram of a sphere with two dots}}_{=: h} + \boxed{} \underbrace{\left(\text{diagram of a sphere with three dots} - \text{diagram of two stacked spheres with two dots each} \right)}_{=: t}
 \end{aligned}$$

Lem

$$\text{diagram of a sphere with dot } n \text{ and a dotted wall} = \text{diagram of a sphere with dot } n$$

(dotted spheres can pass through walls.)

Notation

$$\boxed{\circ} \stackrel{:=}{=} \boxed{\bullet} - \boxed{} \frac{\textcircled{\bullet\bullet}}{h}$$

hollow dot

$$\Rightarrow \left\{ \begin{array}{l} \textcircled{\circ} = 1, \quad \boxed{\bullet\circ} = t \boxed{} \\ \\ \text{cylinder with red ribbon} = \text{cup with } \bullet + \text{cup with } \circ + \text{cup with } \bullet + \text{cup with } \circ \\ \vdots \end{array} \right.$$

Notation

$$\boxed{\star} := \boxed{\bullet} + \boxed{\circ}$$

star

$$\Rightarrow \left\{ \begin{array}{l} \text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} = \text{Diagram 4} \\ \boxed{\star\star} = \boxed{} \frac{(h^2 + 4t)}{=: \mathcal{D}} \end{array} \right.$$

$$\left(\begin{array}{l} h = \text{Diagram 5} , \\ t = \text{Diagram 6} - \text{Diagram 7} \end{array} \right)$$

Now, we can easily compute ...

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \mathbb{D} \text{ Diagram 4} = 0.$$

Diagram 1: A genus-2 surface (torus with two holes).
 Diagram 2: A genus-1 surface (torus with one hole) and a star.
 Diagram 3: A circle containing two stars.
 Diagram 4: A circle.

$$\text{Diagram 5} = \text{Diagram 6} = \text{Diagram 7} = \mathbb{D} \text{ Diagram 8} = \mathbb{D}.$$

Diagram 5: A genus-2 surface with a dot on the right hole.
 Diagram 6: A genus-1 surface with a dot on the hole and a star.
 Diagram 7: A circle containing two stars and a dot.
 Diagram 8: A circle containing a dot.

...

Prop

$$\left. \begin{array}{l} \Sigma_{2g} = 0, \quad \Sigma_{2g+1} = 2\mathbb{D}^g \\ \Sigma_{2g}^\bullet = \mathbb{D}^g, \quad \Sigma_{2g+1}^\bullet = h\mathbb{D}^g \end{array} \right\} \in \mathbb{Z}[h,t]$$

Prop. The evaluation of closed surfaces give

$$\text{Hom}_{\text{Ob} \circ \mathbb{Z}}(\phi) (\phi, \phi) \cong \mathbb{Z}[h, t] \quad (=: R)$$

as graded rings. $(\deg S = \chi(S) - 2 \cdot (\text{number of dots}))$

$$\phi \longmapsto 1 \quad (\deg 0)$$

$$\textcircled{\bullet\bullet} \longmapsto h \quad (\deg -2)$$

$$\textcircled{\bullet\bullet\bullet} - \textcircled{\bullet\bullet} \longmapsto t \quad (\deg -4)$$

Prop.

$$\mathrm{Hom}_{\mathrm{cob}, \ell(\phi)}(\phi, \underline{O}) \cong R\{1\} \oplus R\{-1\} \quad (\doteq A)$$

grading shift
↙

$$\text{cup}_{(1)} \longmapsto (1, 0)$$

$$\text{cap}_{(-1)} \longmapsto (0, 1)$$

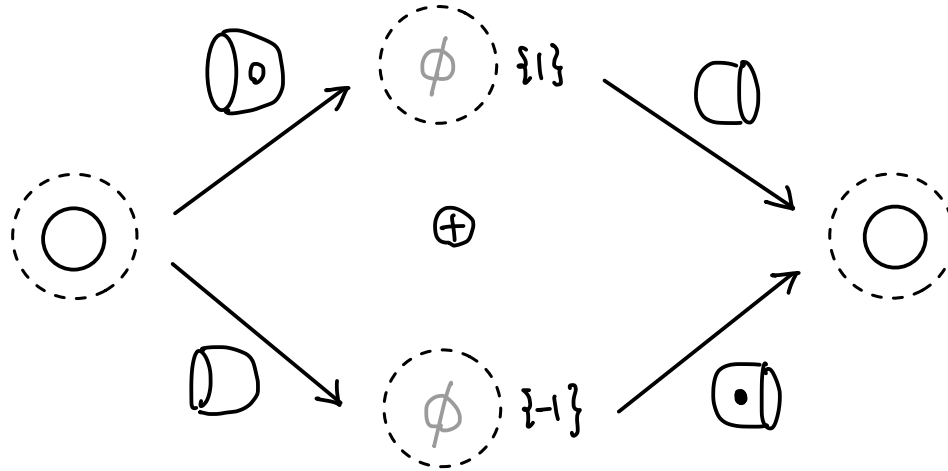
$$\mathrm{Hom}_{\mathrm{cob}, \ell(\phi)}(\phi, \underbrace{O \cdots O}_r) \cong \underbrace{A \otimes \cdots \otimes A}_r$$

To prove this ...

Key Prop 1. (DeLooping)

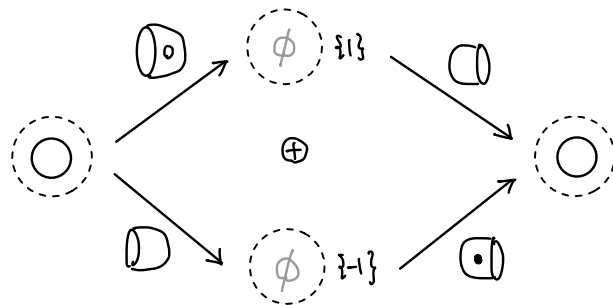
(Note Here, $\text{Cob} \circ \mathbb{Z}$ is turned into an additive ext. by taking its add. closure.)

In $\text{Cob} \circ \mathbb{Z}(B)$, the following isomorphisms hold.

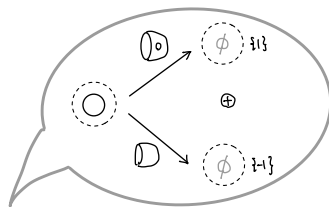


$$\begin{aligned} \therefore) \quad \text{cup} &= \text{cup}_+ + \text{cup}_- \\ &\stackrel{\text{def}}{=} \text{cylinder} = \text{id}_0 \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} \curvearrowright & \nearrow \\ \searrow & \curvearrowleft \end{pmatrix} &= \begin{pmatrix} \text{cap}_+ & \text{cap}_- \\ \text{cup}_- & \text{cup}_+ \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \text{id}_{\phi \oplus \phi} \end{aligned}$$



We have,



$$\text{Hom}_{\text{cob}, \mathbb{Z}/2}(\phi, \underline{O}) \cong \text{Hom}_{\text{cob}, \mathbb{Z}/2}(\phi, \phi)\{1\}$$

$$\oplus \text{Hom}_{\text{cob}, \mathbb{Z}/2}(\phi, \phi)\{-1\}$$

$$\cong \mathbb{R}\{1\} \oplus \mathbb{R}\{-1\}$$

$$\text{D}_{(1)} \longmapsto (\text{circle with dot}, \text{circle with line}) = (1, 0) \quad =: 1$$

$$\text{D}_{(-1)} \longmapsto (\text{circle with two dots}, \text{circle with dot and line}) = (0, 1) \quad =: X$$

$A = R\langle 1, x \rangle$ can be endowed a
(commutative) Frobenius algebra structure as follows:

1) R -alg. str. $(A = R[x]/(x^2 - hX - t))$

$$\begin{array}{ccc}
 m: A \otimes A & \longrightarrow & A \\
 \begin{array}{ccc}
 1 \otimes 1 & \longmapsto & 1 \\
 1 \otimes x, x \otimes 1 & \longmapsto & x \\
 x \otimes x & \longmapsto & hX + t1
 \end{array}
 & , &
 \begin{array}{ccc}
 \eta: R & \longrightarrow & A \\
 1 & \longmapsto & 1
 \end{array}
 \end{array}$$

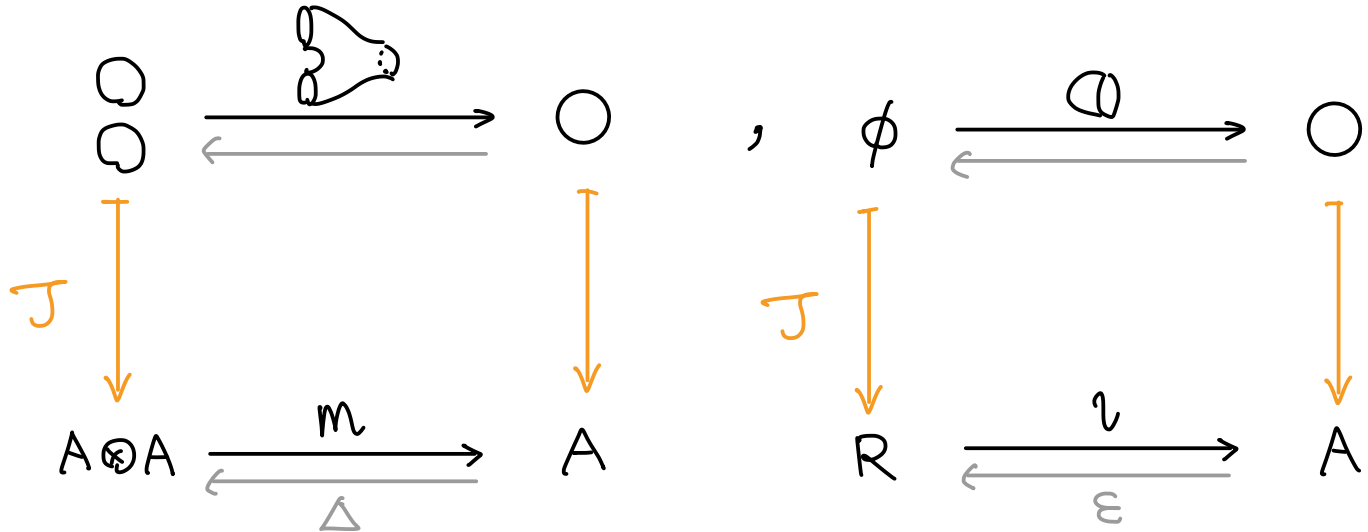
2) R -coalg. str.

$$\begin{array}{ccc}
 \Delta: A & \longrightarrow & A \otimes A \\
 \begin{array}{ccc}
 1 & \longmapsto & 1 \otimes x + x \otimes 1 - h(1 \otimes 1) \\
 x & \longmapsto & x \otimes x + t(1 \otimes 1)
 \end{array}
 & &
 \begin{array}{ccc}
 \varepsilon: A & \longrightarrow & R \\
 1 & \longmapsto & 0 \\
 x & \longmapsto & 1.
 \end{array}
 \end{array}$$

Prop. The "tautological functor" (representable functor)

$$\mathcal{J} = \text{Hom}_{\text{Cob}_{\text{or}}(\phi)}(\phi, -) : \text{Cob}_{\text{or}}(\phi) \longrightarrow \text{Mod}_R$$

coincides with the $(1+1)$ -TQFT \mathcal{F}_A
obtained from the Frobenius algebra A .



Conclusion

The algebraic ingredients used to construct Khovanov homology is recovered from the loc. relations !

$$A \cong R \langle \underbrace{\square_1}_{1}, \underbrace{\square_X}_X \rangle \quad \left(\underbrace{\square_Y}_Y := \square_1 - h \square_1 \right)$$

$$\left\{ \begin{array}{l} \square_{\bullet\bullet} = h \square_{\bullet} + t \square_{} \quad (X^2 = hX + t) \\ \square_{\circ\circ} = -h \square_{\circ} + t \square_{} \quad (Y^2 = -hY + t) \\ \square_{\bullet\circ} = t \square_{} \quad (XY = t) \end{array} \right.$$

$XY \qquad 1$

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§4. Computations.

§2. Diag(B) ... the cat. of tangle diagrams with $\partial = B$.

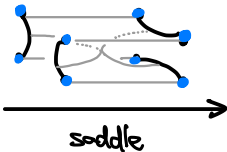
The formal Khovanov bracket

$$\begin{array}{ccc}
 [-] : \text{Diag}(B) & \longrightarrow & \text{Kom}(\text{Cobor}(B)) \\
 \Downarrow & & \Downarrow \\
 \text{[Diagram]} & \longrightarrow & [\text{Diagram}]
 \end{array}$$

The cat. of chain complexes
in $\text{Cobor}(B)$.

is defined as follows ...

$$\left[\begin{array}{c} \text{Diagram: A dashed circle with four blue dots at the corners. Inside, two black lines cross at a central black dot, forming an 'X' shape.} \end{array} \right] = \left\{ \begin{array}{c} \text{Diagram 0: A dashed circle with four blue dots. Two black lines connect the top dots to the bottom dots, forming two vertical-ish curves.} \\ \text{Diagram 1: A dashed circle with four blue dots. Two black lines connect the top dots to the bottom dots, forming two horizontal-ish curves.} \end{array} \right\}$$



 saddle

$$\left[\begin{array}{c} \text{Diagram: A dashed circle with four blue dots. Inside, a black line zig-zags between the dots, forming a path that visits all four dots.} \end{array} \right] = \left\{ \begin{array}{c} \text{Diagram } 00: \text{A dashed circle with four blue dots. Two black lines form a figure-eight shape in the center.} \\ \text{Diagram } 01: \text{A dashed circle with four blue dots. Two black lines form a more complex zig-zag pattern.} \\ \text{Diagram } 10: \text{A dashed circle with four blue dots. Two black lines form a different zig-zag pattern.} \\ \text{Diagram } 11: \text{A dashed circle with four blue dots. Two black lines form a yet another zig-zag pattern.} \end{array} \right\}$$

$\hookrightarrow d^2 = 0$

$$\left[\begin{array}{c} \text{Diagram: A dashed circle with four blue dots. In the center is a circle containing the letter 'n'.} \end{array} \right] = \left\{ \begin{array}{c} n\text{-dim. anti-commutative cube} \\ \text{in } \text{Cob}_{\%2}(B) \end{array} \right\} \hookrightarrow d^2 = 0$$

Remark.

The "chain complex" $[T]$ is NOT a set!

i.e. $x \in [T]$ does NOT make sense.

However, $d^2 = 0$ makes sense in $\text{Cobor}_2(B)$.

Furthermore,

- chain maps $(f \circ d = d \circ f)$
- chain homotopies $(f - g = d \circ h + h \circ d)$
- chain homotopy equivalences $(g \circ f \simeq \text{id}, f \circ g \simeq \text{id})$

do make sense!

Thm [Bar-Natan '05]

The chain homotopy type of $[T]$ is invariant under the Reidemeister moves.

In particular, when $B = \phi$, the composition

$$\begin{array}{ccccccc} \text{Drg}(\phi) & \xrightarrow{[-]} & \text{Kom}(\text{Cob}(\phi)) & \xrightarrow[\text{(TQFT)}]{\mathcal{F}_A} & \text{Kom}(\text{Mod}_R) & \xrightarrow[\text{(homology)}]{H} & \text{Mod}_R \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K & \longmapsto & [K] & \longmapsto & \text{Ch}(K) & \longmapsto & \text{Kh}(K) \end{array}$$

coincides with the Khovanov homology functor.

To prove this ...

Key Prop 2. (Gaussian elimination)

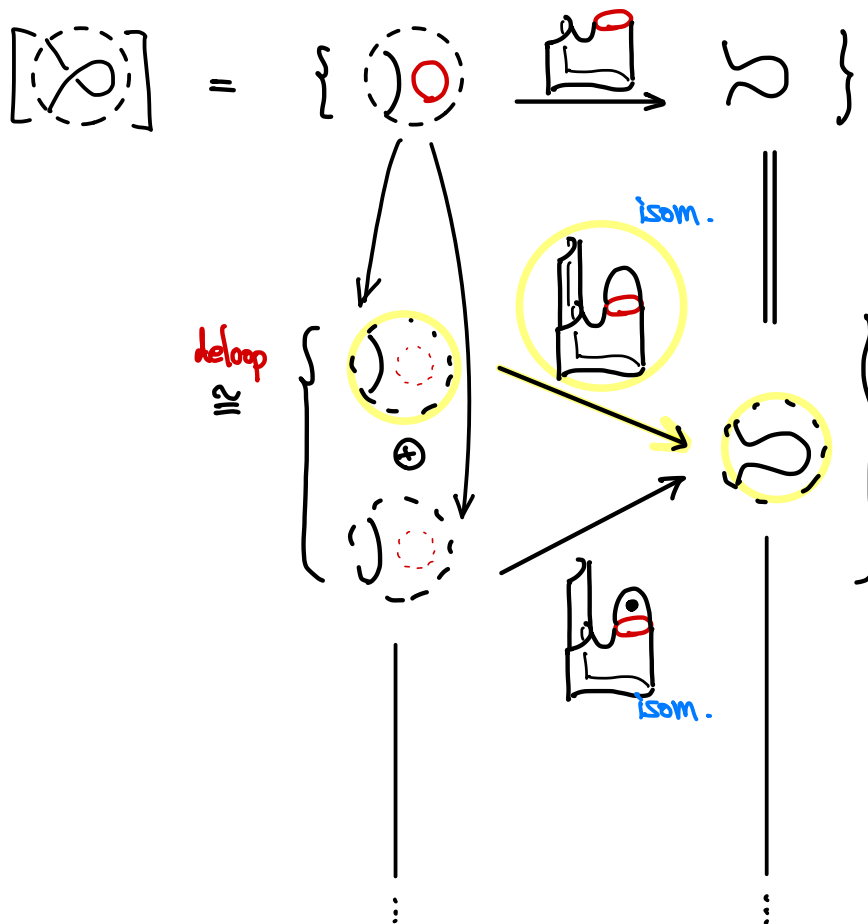
In $\text{Kom}(\text{Cob}_{\geq 2}(B))$, we have



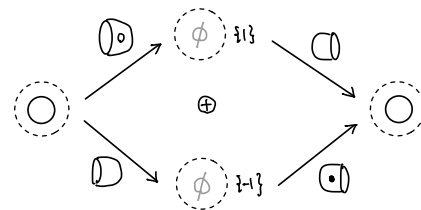
$$\begin{array}{c}
 \Omega = \{ \dots \rightarrow (X) \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix} \xrightarrow{\begin{pmatrix} z & w \end{pmatrix}} W \rightarrow \dots \} \\
 \downarrow \cong \text{isom.} \\
 \Omega' = \{ \dots \rightarrow (X) \xrightarrow{\begin{pmatrix} 0 \\ y \end{pmatrix}} \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix} \xrightarrow{\begin{pmatrix} a & 0 \\ 0 & s \end{pmatrix}} \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix} \xrightarrow{\begin{pmatrix} 0 & w \end{pmatrix}} W \rightarrow \dots \} \\
 \downarrow \cong \text{hpy eq.} \\
 \Omega'' = \{ \dots \rightarrow (X) \xrightarrow{y} (Y_i) \xrightarrow{s} (Z_i) \xrightarrow{w} W \rightarrow \dots \}
 \end{array}$$

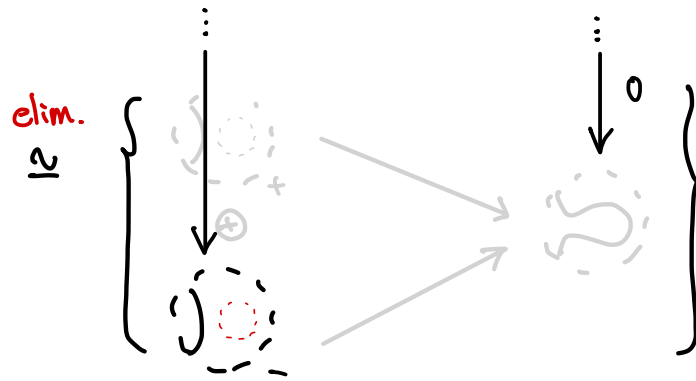
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\text{invertible}} \begin{pmatrix} a & 0 \\ 0 & s \end{pmatrix}$
 $\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \downarrow$
 $s = d - ca^{-1}b$

Proof of Thm (only for R1)



Delooping





$$\approx \left[\begin{array}{c} \text{ } \\ \text{ } \end{array} \right]$$

\therefore We get explicit ch. htpy equivalences

$$\left[\text{ } \right] \xrightleftharpoons{\approx} \left[\begin{array}{c} \text{ } \\ \text{ } \end{array} \right]$$

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Cheet Sheet.

Local relations

$$\text{circle} = 0, \quad \text{circle with dot} = \text{circle with hole} = 1,$$

$$\text{cylinder} = \text{circle with dot} \text{ circle} + \text{circle} \text{ circle with hole}$$

Notations

$$h = \text{circle with two dots}, \quad t = \text{circle with three dots} - \text{circle with two dots},$$

$$\square \circ := \square \bullet - h \square \quad (Y = X - h)$$

$$\square \star := \square \bullet + \square \circ \quad (\star = 2X - h)$$

Reductions

$$\square \bullet \bullet = h \square \bullet + t \square,$$

$$\square \circ \circ = -h \square \circ + t \square,$$

$$\square \bullet \circ = t \square, \quad \square \star \star = \text{circle with dot} \square$$

DeLooping

