Optimal Bayesian Estimation

1) Probabilistic framework

$$\begin{tikzcd} {} \arrow[r] & x_{k-1} \arrow[r] \arrow[d] & x_k \arrow[r] \arrow[d] & x_{k+1} \arrow[d] & \text{hidden} \\ & y_{k-1} & y_k & y_{k+1} & \text{measurements} \end{tikzcd}$$

$\longrightarrow$ stochastique

HMM
hidden Markov Model

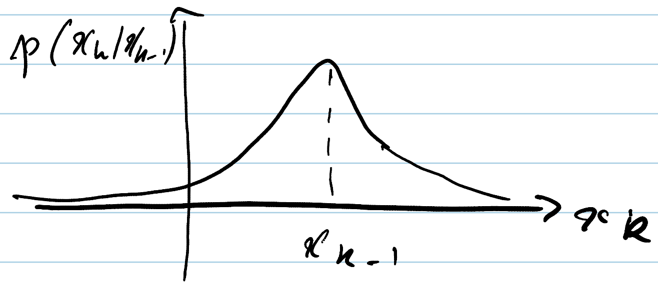
ex Marche aléatoire

$$\boxed{x_k = x_{k-1} + w_{k-1}} \qquad w_k \sim N(0, q)$$

moyenne

variance

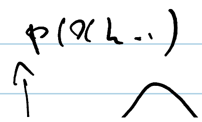
$$p(x_k \mid x_{k-1}) = \frac{1}{\sqrt{2\pi q}} \exp\left[-\frac{1}{2q}(x_k - x_{k-1})^2\right]$$



Model: $x_k \sim p(x_k \mid x_{k-1})$

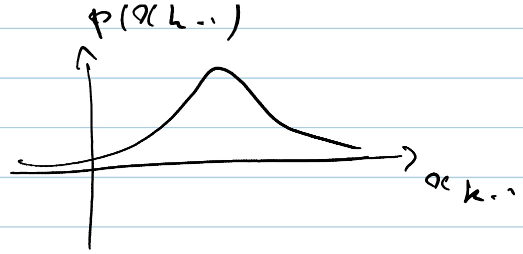
$y_k \sim p(y_k \mid x_k)$

2) Prediction



2) Prediction

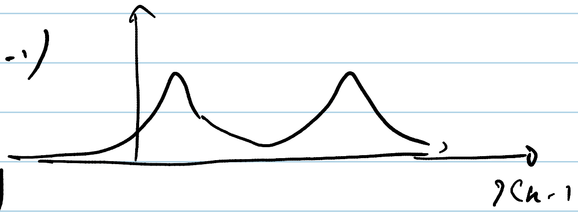
$p(x_{k-1})$ known



$$p(x_k, x_{k-1}) = p(x_k \mid x_{k-1}) \cdot p(x_{k-1})$$

Conditional probability

$$P(A, B) = P(A \mid B)\, P(B)$$



$$\boxed{p(x_k) = \int \underbrace{p(x_k \mid x_{k-1})}_{\text{model}} \cdot \underbrace{p(x_{k-1})}_{\text{known}}\, dx_{k-1}}$$

example

$$p(x_{k-1}) \sim N(m, P)$$

$P \rightarrow$ covariance MESURE

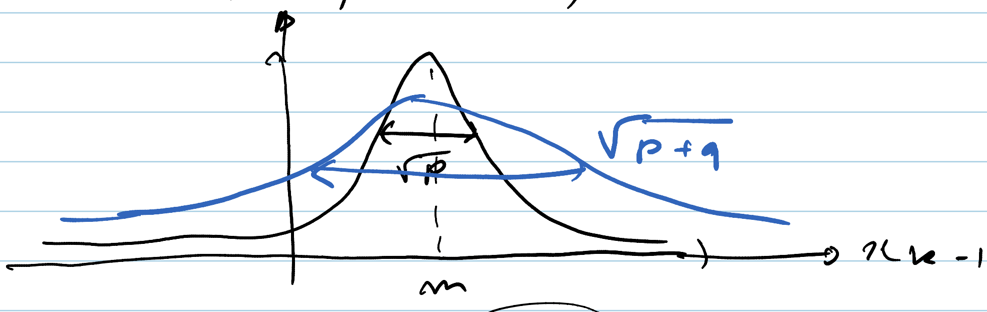
$m \rightarrow m$

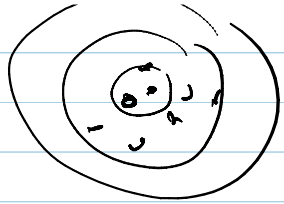
$$p(x_k \mid x_{k-1}) \sim N(0, Q)$$

$Q \rightarrow$ covariance ERROR

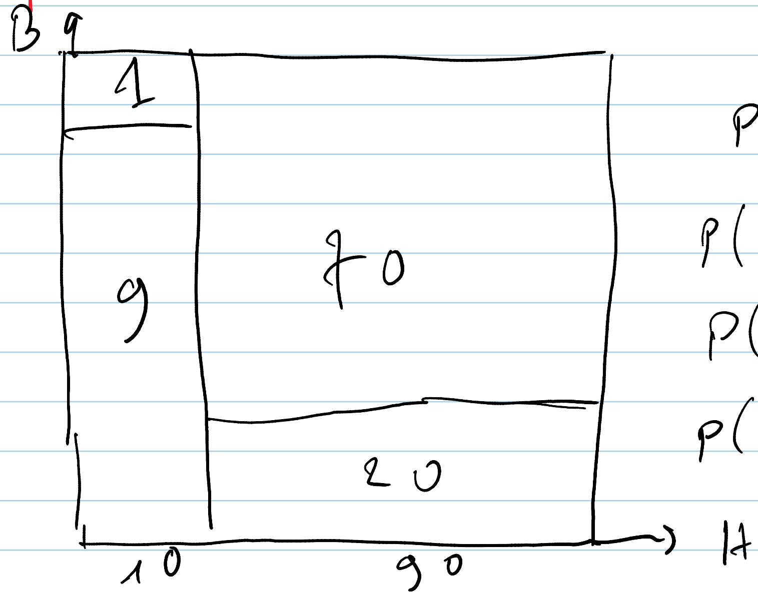
$$p(x_k) = \int \frac{1}{\sqrt{\cdots}} \exp\left(-\frac{1}{2q}(x_k - x_{k-1})^2\right) \cdot \frac{1}{\sqrt{\cdots}} \exp\left(-\frac{1}{2P}(x_{k-1} - m)^2\right) dx_{k-1}$$

$$\rightsquigarrow \; = \cdots \exp\left(-\frac{1}{2(p+q)}(x_k - m)^2\right)$$





c) Update



$P(B,H) = 9/100$

$p(B|H) = 9/10$

$P(B,H) = P(B|H)\cdot p(H)$

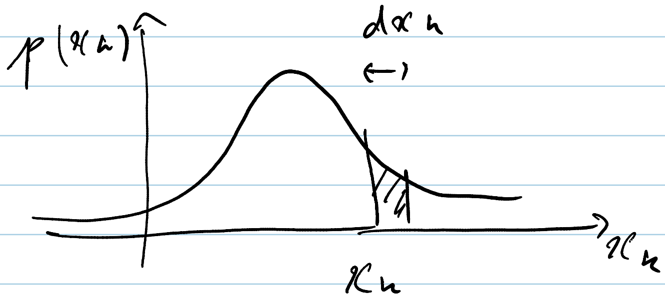
$p(H|B)$

$$P(B,H) = P(B|H)\,p(H) = P(H|B)\,P(B)$$

$$\boxed{P(H|B) = \frac{P(B|H)\,P(H)}{P(B)}} = \frac{9/100 \;\; 10/100}{29/100} \simeq 30\,\%$$

$p(x_k)$ known

$p(x_k|y_k)$?



$$P(x_k \in [x_k, x_k + dx_k]) = p(x_k)\,dx_k$$

$$\overbrace{p(x_k|y_k)\,dx_k}^{P} \cdot \overbrace{p(y_k)\,dy_k}^{P}$$

$$= \underbrace{p(y_k|x_k)\,dy_k}_{P} \cdot \underbrace{p(x_k)\,dx_k}_{P}$$

model

$p(y_k|x_k) \cdot p(x_k)$

a priori

$$\boxed{p(x_k|y_k) = \frac{p(y_k|x_k)\cdot p(x_k)}{p(y_k)}}$$

a priori

PRIOR

$$\boxed{p(x_k|y_k) \propto p(y_k|x_k)\, p(x_k)}$$

ex $\quad y_k = x_k + v_k \qquad v_k \sim N(0, r)$

hyp $\quad p(x_k) \sim N(m, p+q)$

$$p(x_k|y_k) \sim N(m, a)$$

$$a = p+q - \frac{(p+q)^2}{r+p+q} < p+q$$

4) Conclusion

$p(x_0$

PREDICT $\quad p(x_k|y_{k-1}) = \int p(x_k|x_{k-1}) \cdot p(x_{k-1}|y_{k-1})\, dx_{k-1}$

UPDATE: $\quad p(x_k|y_k) \propto p(y_k|x_k)\, p(x_k|y_{k-1})$