Exercise 3.11 - Special Relativity. Woodhouse

García Parra Joel Alberto

Mayo 2019

1. Exercise 3.11

Show that for the constant acceleration wordline in this section, the proper acceleration is $cd\theta/d\tau$, where θ is the rapidity (Excercise 2.14).

2. Solution

- First of all we take the equation that gives the Exercise 2.14:

$$c\tanh\theta = v \tag{1}$$

  Note that this equation refers to a time t in the frame.

  Now, let's derivate this equation with respect of the propper time τ.

$$\frac{dv}{d\tau} = c\frac{d(\tanh\theta)}{d\tau} \tag{2}$$

- In **(2)**, we can multiplie by $1 = \frac{d\theta}{d\theta}$ and get:

$$\begin{aligned}\frac{d(\tanh\theta)}{d\tau}(1) &= c\frac{d(\tanh\theta)}{d\tau}\left(\frac{d\theta}{d\theta}\right)\\ &= c\frac{d\tanh\theta}{d\theta}\left(\frac{d\theta}{d\tau}\right)\end{aligned} \tag{3}$$

- We can take the part of $\frac{d(\tanh\theta)}{d\theta}$ and actually derivate it, but first we put $\tanh\theta$ in his exponential form to derivate it without using any trigonometric identity:

$$\tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} \tag{4}$$

  Now, deriving with respect of θ:

$$\begin{aligned}\frac{d}{d\theta}\left(\frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}\right) &= \frac{\left(e^{\theta} + e^{-\theta}\right)\left(e^{\theta} + e^{-\theta}\right) - \left(e^{\theta} - e^{-\theta}\right)\left(e^{\theta} - e^{-\theta}\right)}{\left(e^{\theta} + e^{-\theta}\right)^2}\\ &= \frac{\left(e^{\theta} + e^{-\theta}\right)^2 - \left(e^{\theta} - e^{-\theta}\right)^2}{\left(e^{\theta} + e^{-\theta}\right)^2}\\ &= 1 - \left(\frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}\right)^2\\ &\text{Remmember the (4) equation}\\ &= 1 - (\tanh\theta)^2\end{aligned} \tag{5}$$

- With this result, we can express $\frac{dv}{d\tau}$ in the next form:

$$\frac{dv}{d\tau} = c\frac{d\theta}{d\tau}\underbrace{\boxed{\left(1 - tanh^2\theta\right)}}_{dv/d\tau} \tag{6}$$

- Using the definition of propper acceleration in this frame, we remmember:

$$a = \frac{1}{1 - v^2/c^2}\frac{dv}{d\tau} \tag{7}$$

- But, we have defined v at the beginning of the problem, so, let's put that in the equation:

$$v = c\tanh\theta$$

Where if we separate the term v/c and square:

$$v^2/c^2 = \tanh^2\theta \tag{8}$$

- We have another look to the propper acceleration:

$$a = \frac{1}{1 - \underbrace{\boxed{\tanh^2\theta}}_{v^2/c^2}}\frac{dv}{d\tau}$$

- Note that we already have $\frac{dv}{d\theta}$, so let's put it in:

$$a = \frac{1}{1 - \tanh^2\theta}\underbrace{\boxed{\left(c\frac{d\theta}{d\tau}\left(1 - \tanh^2\theta\right)\right)}}_{dv/d\tau} \tag{9}$$

This equation easily become in to what we have been looking for:

$$\boxed{a = c\frac{d\theta}{d\tau}} \tag{10}$$