# NYCU CS 515501 2024

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## Homework Problems #4

1.

a) $a_n = 3^n$

$$\sum_{k=0}^{\infty} a^k x^k = \frac{1}{1-ax}$$

$$\Rightarrow \sum_{k=0}^{\infty} 3^k x^k = \frac{1}{1-3x} \#$$

b) $a_n = 2n+3$

$$\sum_{k=0}^{\infty} (k+1) x^k = 1 + 2x + 3x^2 + \cdots$$

$$\sum_{k=0}^{\infty} 2(k+1) x^k = 2 + 4x + 6x^2 + \cdots$$

$$\sum_{k=0}^{\infty} \left[2(k+1)+1\right] x^k = 3 + 5x + 7x^2 + \cdots$$

$$= 2 \cdot \frac{1}{(1-x)^2} + \frac{1}{1-x} = \frac{3-x}{(1-x)^2} \#$$

c) $a_n = \binom{8}{n}$

$$(1+x)^n = \sum_{k=0}^{n} C_k^n x^k$$

$$= 1 + C_1^n x + C_2^n x^2 + \cdots + x^n$$

$$\sum_{k=0}^{\infty} C_k^8 x^k = (1+x)^8 \#$$

↑ don't need to worry about $k>8$ because $C_k^8 = 0$ if $k>8$

2. power series of $\frac{x^3}{(1+4x)^2}$

$$\frac{1}{(1-ax)^n} = 1 + C_1^n ax + C_2^{n+1} a^2 x^2 + \dots$$

$$\frac{x^3}{[1-(-4)x]^2} = x^3 + C_1^2 (-4)^1 x^4 + \dots C_9^{10} (-4)^9 x^{12}$$

$$C_9^{10} \cdot (-4)^9 = 10 \cdot (-262144) = -2621440 \quad \#$$

3. show that $\frac{1}{(1-x)(1-x^2)(1-x^3)\dots} = \sum_{n=0}^{\infty} p(n) x^n$

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k}, \quad \frac{1}{1-x^k} = 1 + x^k + x^{2k} + x^{3k} + \dots$$

$$\Rightarrow \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \prod_{k=1}^{\infty} (1 + x^k + x^{2k} + x^{3k} + \dots)$$

To determine the coefficient of $x^n$, we need to consider all ways to pick terms s.t. their exponents sum to $n$. This corresponds to partition of $n$.

ex. $n=3$

1. $x^3 \to k=3$
2. $x^2 \cdot x^1 \to k=2,1$
3. $x^1 \cdot x^1 \cdot x^1 \to k=1$ ( $k=1$ 的 $x^3$ )

4. let $P(A)$ = the probability of event $A$

by the principle of Inclusion-exolusion,

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{1\le i\le n} P(A_i) - \sum_{1\le i<j\le n} P(A_i \cap A_j) + \sum_{1\le i<j<k\le n} P(A_i \cap A_j \cap A_k) - \dots$$

$$+ (-1)^{n+1} P(A_1 \cap A_2 \cap \dots \cap A_n) \quad \#$$

5. a) pairwise disjoint → $A_1 \cap A_2 = A_2 \cap A_3 = A_1 \cap A_3 = \{\emptyset\}$

$100 + 100 + 100 - 0 - 0 - 0 + 0 = 300$ #

b) $100 + 100 + 100 - 50 - 50 - 50 + 0 = 150$ #

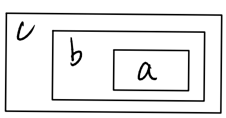
6.

a) reflexive: visited a → visited a ✓

symmetric: visited a → visited b,
visited b doesn't imply visited a ✗

antisymmetric: if everyone who has visited a also visited b,
everyone who has visited b also visited a,
doesn't mean $a = b$ ✗

transitive: if everyone who has visited a also visited b,
everyone who has visited c also visited c,
everyone who has visited a also visited c ✓




b) reflexive: a can have common link of a (if exist link on a) ✗

symmetric: if a and b have no common link, b and a don't have ✓

antisymmetric: if a and b have no common link, b and a don't
have common link, doesn't mean $a = b$. ✗

transitive: if a only have $\boxed{\text{Link1}}$, b only have $\boxed{\text{Link2}}$, c only have
$\boxed{\text{Link3}}$, then $(a, b) \in R$, $(b, c) \in R$ but $(a, c) \notin R$ ✗

6.

c)

reflexive: if there's no link on a, then $(a,a) \notin R$ X

symmetric: if $(a,b) \in R$, then there is also at least one common link on b and a. $(b,a) \in R$ V

antisymmetric: if $(a,b) \in R$ and $(b,a) \in R$, a doesn't need to $= b$ X

transitive: if a only have $\boxed{\text{Link 1}}$, b have $\boxed{\text{Link 1}}$ and $\boxed{\text{Link 2}}$, c only have $\boxed{\text{Link 2}}$, then $(a,b) \in R$, $(b,c) \in R$ but $(a,c) \notin R$ X

d)

reflexive: there's no Webpage has link to a $\rightarrow$ $(a,a) \notin R$ X

symmetric: $(a,b) \in R \rightarrow (b,a) \in R$ V

antisymmetric: $(a,b) \in R$ && $(b,a) \in R \not\rightarrow a = b$ X

transitive: $\boxed{\text{page 1}}$ includes links to a and b,

$\boxed{\text{page 2}}$ includes links to b and c,

but can not guarantee $(a,c) \in R$ X

7.

a) $R^{-1} = \{(a,b) \mid b \text{ divides } a\}$

b) $\overline{R} = \{(a,b) \mid a \text{ can't divides } b\}$ #

8.

a) $R^{-1} \Rightarrow$ switch A & B $\Rightarrow M_{R^{-1}} = (M_R)^T$

$$M_{R^{-1}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$ #

b) $R^2$

$$M_{R^2} = (M_R)^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \#$$

c) $\overline{R}$ → complement every element

$$M_{\overline{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \#$$

9.

equivalance relation: reflexive, symmetric, transitive

reflexive: $(a,b)$ and $(a,b)$. $a+b=a+b$, $((a,b),(a,b)) \in R$ ✓

symmetric: if $((a,b),(c,d)) \in R$, $a+d=b+c$, then $((c,d),(a,b)) \in R$

$\because c+b=d+a$ ✓

transitive: if $((a,b),(c,d)) \in R$, $((c,d),(e,f)) \in R$

$a+d=b+c$, $c+f=d+e$

$a+f=a+(d+e-c)=a+[(b+c)-a+e-c]=b+e$

then $((a,b),(e,f)) \in R$ ✓

⇒ R is an equivalance relation because R is reflexive, symmetric and transitive. (Q.E.D.) #

10.

a) reflexive: a is not taller than a ✗ ⇒ no #

b) reflexive: $a=a$ ✓

antisymmetric: if $(a,b) \in R$ && $(b,a) \in R$
then $a=b$ ✓

transitive: if $(a,b) \in R$ && $(b,c) \in R \Rightarrow (a,c) \in R$

10. b) four situation

1. $a=b=c \rightarrow (a,c) \in R$ $(a=c)$
2. $a=b$, $b$ is an ancestor of $c \rightarrow (a,c) \in R$ ($a$ is an ancestor of $c$)
3. $b=c$, $a$ is an ancestor of $b \rightarrow (a,c) \in R$ ($a$ is an ancestor of $c$)
4. $a$ is an ancestor of $b$, $b$ is an ancestor of $c$
   $\rightarrow (a,c) \in R$ ($a$ is an ancestor of $c$)

$\Rightarrow$ yes, $(S,R)$ is a poset #

11. a) reflexive: $a=a$, $(a,a) \notin Z \Rightarrow \times$ No #

b) reflexive: $a \geq a$, $(a,a) \in Z$ ✓

antisymmetric: $a \geq b$ and $b \geq a \Rightarrow a=b$ ✓ $\Rightarrow$ Yes #

transitive: $a \geq b$ && $b \geq c \Rightarrow a \geq c$ ✓

c) reflexive: $a=a$, $(a,a) \notin R \Rightarrow \times$ No #

12. The maximal elements are all values in the top row. maximal: $l, m$
The minimal elements are all values in the bottom row. minimal: $a, b, c$ #

only 1 maximal elements $\rightarrow$ greatest element exist
$\Rightarrow$ greatest element DNE in this Hasse diagram #

only 1 minimal elements $\rightarrow$ least element exist
$\Rightarrow$ least element DNE in this Hasse diagram #

13. 25 identical donuts, 4 police officers, $3 \leq$ ◎ $\leq 7$

give all police officers 3 donuts first $\rightarrow$ remain 13 donuts

$13 \rightarrow$ 4 4 4 1 $\rightarrow C^4_1 = 4$
4 4 3 2 $\rightarrow C^4_1 C^3_1 = 12$
4 3 3 3 $\rightarrow C^4_1 = 4$

$4+12+4=20 \Rightarrow$ 20 ways #