

Problem 1. (13 points total). Consider the following linear system

$$x_1 = 2$$

$$2x_1 + 2x_2 = 6$$

$$3x_1 + 3x_2 + 3x_3 = 3$$

(i) (8 points). Find the inverse of the associated coefficient matrix for the above system.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix}$$

$$\therefore (A|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 3 & 3 & -3 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 0 & -\frac{3}{2} & 1 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{3} \end{array} \right)$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

(ii) (5 points). Use your answer to part (i) to solve the linear system. *No other method of solution will be graded - you MUST use the inverse of the associated coefficient matrix.*

$$\therefore A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} \quad Ax = b$$

$$\therefore A^{-1}Ax = A^{-1}b$$

$$\therefore x = A^{-1}b = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\therefore \begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = -2 \end{cases}$$



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Problem 2. (14 points, 2 points each). Let A be an $n \times n$ matrix. List seven separate conditions equivalent to A being invertible. Note: if you write more than seven your grade is only based on the first seven so there's no point in writing more.

- (i) There exists B be an $n \times n$ matrix, such that $AB = I_n$.
- (ii) For $Ax = b$, there is only one unique solution for any $b \in \mathbb{R}^n$.
- (iii) For $Ax = 0$, there ~~are~~ ^{is} ~~only~~ ^{only} trivial solution.
- (iv) The RREF form of A is an identity matrix.
- (v) The columns of A are independent matrices.
- (vi) A can be written as the product of elementary matrices.
- (vii) The rows of A are independent matrices.



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Problem 3. (15 points total). This questions has 3 parts. *Note: your answers have to be completely precise and correct to receive credit since there is no partial credit for incorrect definitions.*

(i) (5 points). Let $S \subset \mathbb{R}^n$. Define what it means for S to be a subspace of \mathbb{R}^n .

S is a subspace of \mathbb{R}^n if and only if the following features are satisfied:

1. zero matrix belongs to S

2. For $u, v \in S$, $(u+v) \in S$

3. For $u \in S$, $c \in \mathbb{R}$, $cu \in S$

(ii) (5 points). Define the null space $Nul(A)$ of an $m \times n$ matrix A .

$Nul(A)$ of an $m \times n$ matrix A is the set of ~~all~~ the solutions ^{$x \in \mathbb{R}^n$} which
 makes ~~of~~ $AX=0$.

(iii) (5 points). Using your definitions above, prove that the nullspace, $Nul(A)$ of an $m \times n$ matrix A is a subspace.

$$\therefore \cancel{AX=0} \quad \therefore Nul(A) = \{x \mid Ax=0\}$$

$$\therefore x=0, Ax=0$$

$$\therefore 0 \in Nul(A)$$

$$\therefore x_1, x_2 \in Nul(A)$$

$$\text{Take } \therefore Ax_1=0$$

$$Ax_2=0$$

$$\therefore A(x_1+x_2) = Ax_1 + Ax_2 = 0+0=0$$

$$\therefore (x_1+x_2) \in Nul(A)$$

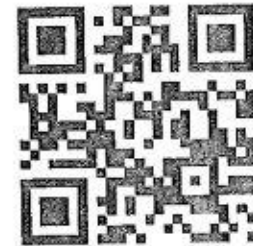
$$\therefore \text{Take } x \in Nul(A), c \in \mathbb{R}$$

$$\therefore Ax=0$$

$$A(cx) = c(Ax) = c \cdot 0 = 0$$

$$\therefore cx \in Nul(A)$$

$\therefore Nul(A)$ of an $m \times n$ matrix A is a subspace.



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Problem 4. (26 points total). This problem has 3 parts in total. Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation satisfying $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and let $S: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the linear transformation defined by $S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 - x_2 \\ x_1 + 2x_2 \end{bmatrix}$.

(a) (10 points). Find the standard matrices for T and S .

$$\begin{aligned} &\because T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ is a linear transformation} \\ T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= T\left(\frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \frac{1}{3}T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \frac{1}{3}T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) \\ &= \frac{1}{3}\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= T\left(\frac{2}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(-\frac{1}{3}\right)\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \frac{2}{3}T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \left(-\frac{1}{3}\right)T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) \\ &= \frac{2}{3}\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \left(-\frac{1}{3}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{aligned}$$

$$\therefore T(x) = A_1 x$$

$$A_1 = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$\therefore S(x) = A_2 x = \begin{pmatrix} x_1 + x_2 \\ 2x_1 - x_2 \\ x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\therefore A_2 = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}$$

\therefore The standard matrix for T is $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, and the standard matrix for S is $\begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}$.



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(b) (4 points). Find the standard matrix for the transformation defined by $S \circ T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

$$\therefore S \circ T = A_2 A_1 X$$

$$\therefore A_2 A_1 = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 1 & 0 \\ 8 & 5 \end{pmatrix}$$

\therefore The standard matrix for $S \circ T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is $\begin{pmatrix} 5 & 3 \\ 1 & 0 \\ 8 & 5 \end{pmatrix}$

(c) (4 points). Is $S \circ T$ one-to-one? Justify your answer.

$$\therefore A_2 A_1 = \begin{pmatrix} 5 & 3 \\ 1 & 0 \\ 8 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 5 & 3 \\ 8 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

It has no free variables, the columns of $A_2 A_1$ is linearly independent, so $S \circ T$ is one-to-one.

(d) (4 points). Is $S \circ T$ onto? Justify your answer.

\therefore The number of pivots of $A_2 A_1$ is 2,
the number of row is 3.
 $2 \neq 3$

$\therefore S \circ T$ is not onto.

(e) (4 points). Is $S \circ T$ invertible? Justify your answer.

$S \circ T$ is not invertible, because ~~it is not~~ the size of it is not $n \times n$,
it is not a square matrix.



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Problem 5. (12 points total, 3 points each) For the following questions, answer using the word "True" or the word "False". You **don't need to justify your answer** to receive full credit. There's no partial credit.

(a) True/False: Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an invertible linear transformation satisfying $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Then $T^{-1}\left(\begin{bmatrix} 6 \\ 10 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. True

(b) True/False: If A and B are two matrices such that AB and BA are defined then A and B are both square.

False

(c) True/False: The sum of two elementary matrices of the same size must also be an elementary matrix.

False

(d) True/False: If a matrix is elementary then it must have an LU factorization.

True

(Nothing on this page below here will be graded but is extra space if you want it)



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(extra paper)

$4-3=1$
 $2-2=0$
 $2+3$ $2+6=8$
 $1+4=5$
 $1+2=3$



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(extra paper)

$$\frac{1}{3} - \frac{1}{3} = 0.$$

$$\frac{1}{3} + \frac{2}{3} = 1$$

$$\frac{b(5-c) - 10b}{(3-b)(5-c) - bc}$$

$$1 + 0 = 1.$$

$$\frac{3}{5} + \frac{1}{3}.$$

$$\begin{pmatrix} \frac{1}{ad-bc} (bd-10b) \\ \frac{1}{ad-bc} (bc-10a) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix},$$

$$\frac{1}{ad-bc} \begin{pmatrix} d-b \\ -c \ a \end{pmatrix} \begin{pmatrix} b \\ 10 \end{pmatrix} = ?$$

$$\frac{5}{3} + \frac{1}{3} = \frac{6}{3} = 2.$$

$$\frac{2}{3} + \frac{1}{3} = 1.$$

$$\frac{2}{3} - \frac{2}{3} = 0.$$

$$\frac{1}{3}$$

$$\frac{10}{3} - \frac{1}{3}.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$a + b = 3$$

$$c + d = 5$$

$$\overline{ad-bc}$$

$$a = 3 - b$$

$$d = 5 - c$$

$$\frac{1}{5}$$



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(extra paper)