Problem 1. (18 points total). This problem has three parts. Consider the following augmented matrix

$$A = \begin{array}{c} \begin{array}{ccccc} & t & & s & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \end{array}$$

(i) Is the matrix A above in reduced row echelon form? Why or why not?

Yes, it is.

Because: ① Each pivot is to the right of the pivot above its row

② The all zero rows are on the last row of the matrix

③ On all the pivots position, there is a leading one

④ On each pivot column, the pivot position has the only none zero one.

⑤ Before all the pivots are zero.

(ii) Write down the linear system of equations in the variables $x_1, ..., x_5$ for which A is the associated augmented matrix.

$$\because A = \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

$$\therefore \begin{cases} x_1 + x_2 = 1 \\ x_3 + 2x_4 = 2 \\ x_5 = 3 \end{cases}$$

(iii) Which of the following ~~are~~ solutions to the system of equations for which A is the associated augmented matrix. **Circle your answer(s).** There may be more than one correct answer, and options are listed in the form (a, b, c, d, e) which means $x_1 = a, x_2 = b, x_3 = c, x_4 = d, x_5 = e$. **Each incorrect circled answer will result in deduction of 2 points for this problem.**

- $(1, 0, 2, 0, 3)$
- $(1, 1, 2, 0, 3)$
- $(0, 1, 2, 0, 3)$
- $(1, 2, 3)$
- $(3, 0, 2, 0, 1)$
- $(1, 1, 2, 1, 3)$
- $(1, 0, 0, 1, 3)$
- $(1, 0, 4, -1, 3)$
- $(1, 4, -1, 3)$
- $(0, 1, 0, 1, 3)$
- $(5, 4, 3, 2, 1)$
- None of the above

Problem 2. (17 points total). Suppose A is a 5×4 matrix with 3 pivot positions.

(i) (2 points). How many solutions are there to $A\mathbf{x} = \mathbf{0}$? Justify your answer.

∵ $Ax=0$

∴ it is consistent

∵ A has 3 pivot positions

∴ A has 1 free variables

∴ $Ax=0$ has infinitely many solutions

(ii) (5 points). Does $A\mathbf{x} = \mathbf{b}$ always have a solution? Justify your answer.

No.

When $b=0$, it is alway consistent, it always have a solution.

When after row reduction to RREF, if the constant column exists $\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}$ $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}$, $t_4 \neq 0$ or $t_5 \neq 0$.

$Ax=b$ is inconsistent, it does not have a solution.

(iii) (5 points). What's the dimension of $Nul(A)$ the nullspace of A?

∵ $\text{rank}A = 3$

2∵ $\dim(Nul(A)) + \text{rank}A = n$

∴ $\dim(Nul(A)) = 1$

Score: 5.0/5.0

5x4

(iv) (5 points). What if A has 4 pivots instead of 3. Is $A\mathbf{x} = \mathbf{b}$ always consistent? Justify your answer.

No.

When after row reduction to RREF, the constant column

exists $\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix}$, $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}$, $t_5 \neq 0$,

$Ax=b$ is inconsistent.

If $b=0$, it is always consistent.

Problem 3. (15 points). Let $A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 2 & 2 & 3 \end{bmatrix}$. Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Put your answers in the blank matrices provided.

$$P = \begin{bmatrix} -\frac{3}{2} & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$C_A(\lambda) = \det(A - \lambda I) = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 2 & 3 \\ 2 & 2-\lambda & 3 \\ 2 & 2 & 3-\lambda \end{pmatrix} = \det \begin{pmatrix} 2-\lambda & 2 & 3 \\ 2 & 2-\lambda & 3 \\ 0 & \lambda & -\lambda \end{pmatrix}$$

Set $C_A(\lambda) = 0$

$$\therefore C_A(\lambda) = (2-\lambda) \cdot (-1)^{1+1} \cdot \det \begin{pmatrix} 2-\lambda & 3 \\ \lambda & -\lambda \end{pmatrix} + 2 \cdot (-1)^{2+1} \det \begin{pmatrix} 2 & 3 \\ \lambda & -\lambda \end{pmatrix}$$

$$= (2-\lambda)[(2-\lambda)(-\lambda) - 3\lambda] - 2 \cdot (-2\lambda - 3\lambda)$$

$$= (2-\lambda) \cdot \lambda \cdot (\lambda - 5) + 10\lambda$$

$$= -\lambda^2 (\lambda - 7)$$

$\therefore \lambda = 0$ with $m=2$ or $\lambda = 7$ with $m = 1$

(extra paper for Problem 3)

$$E_{\lambda=0} = \text{null}\begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} = \text{null}\begin{pmatrix} 2 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$= \text{null}(A - 0\cdot I)$$

$$E_{\lambda=7} = \text{null}\begin{pmatrix} -5 & 2 & 3 \\ 2 & -5 & 3 \\ 2 & 2 & -4 \end{pmatrix} = \text{null}\begin{pmatrix} -5 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$= \text{null}(A - 7I)$$

$$\therefore P = \begin{pmatrix} -\frac{3}{2} & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Score: 15.0/15.0

Problem 4. (15 points total). Find the orthogonal complement in $\mathbb{R}^4$ of the subspace $S = \text{span}\{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}3\\4\\4\\5\end{bmatrix}\}$. Show all your work.

$$\text{Set } S^\perp = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = 0 \text{ and } \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\begin{pmatrix}3\\4\\4\\5\end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{matrix} x_1+2x_2+3x_3+4x_4=0 \\ 3x_1+4x_2+4x_3+5x_4=0 \end{matrix} \right\}$$

$\therefore \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 4 & 5 & 0 \end{array}\right)$ is equivalent to the equation of $S^\perp$

$$\therefore \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 3 & 4 & 4 & 5 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -2 & -5 & -7 & 0 \end{array}\right)$$

$$\therefore \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}2t+3s\\-\frac{5}{2}t-\frac{7}{2}s\\t\\s\end{pmatrix} = t\begin{pmatrix}2\\-\frac{5}{2}\\1\\0\end{pmatrix} + s\begin{pmatrix}3\\-\frac{7}{2}\\0\\1\end{pmatrix} \quad t, s \in \mathbb{R}$$

$\because \begin{pmatrix}2\\-\frac{5}{2}\\1\\0\end{pmatrix}$ and $\begin{pmatrix}3\\-\frac{7}{2}\\0\\1\end{pmatrix}$ are linearly independent, they are a basis for $S^\perp$

$$\therefore S^\perp = \left\{\begin{pmatrix}2\\-\frac{5}{2}\\1\\0\end{pmatrix}, \begin{pmatrix}3\\-\frac{7}{2}\\0\\1\end{pmatrix}\right\}$$

Score: 10.0/10.0

Problem 5. (35 points). Parts (i)–(iv) of this problem are about a matrix A which admits an LU factorization $A = LU$ with

$$U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

and the unique solution to $L\mathbf{y} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$ is $\mathbf{y} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$.

(i) (10 points). Solve $A\mathbf{x} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$ using the LU decomposition.

$\because A = LU$

$\therefore LUx = \begin{pmatrix} 3 \\ 13 \\ 4 \end{pmatrix}$

$\therefore L(Ux) = \begin{pmatrix} 3 \\ 13 \\ 4 \end{pmatrix}$

$\because Ly = \begin{pmatrix} 3 \\ 13 \\ 4 \end{pmatrix}$

$\therefore Ux = y = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$

$$\therefore \left(\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

(ii) (5 points). Calculate $rank(A)$. Justify your answer.

$\because Ax = \begin{pmatrix} 3 \\ 13 \\ 4 \end{pmatrix}$ has unique solution $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ A does not have free variables.

$\because Ly = \begin{pmatrix} 3 \\ 13 \\ 4 \end{pmatrix}$ $\therefore$ size(L) = 3x3 $\therefore$ rank A = 3

$\downarrow$ 3x1 $\to$ 3x1 $\because$ size(U) = 3x3, A = LU

$\therefore$ size(A) = 3x3

(iii) (5 points). What is the dimension of $Nul(A^T)$? Justify your answer.

$\because rank(A^T) = rank(A) = 3$

size(A) = 3x3

$\because \dim(Nul(A^T)) + rank(A^T) = n = 3$

$\therefore \dim(Nul(A^T)) = 0$

(iv) (5 points). Give a basis for $row(A)$. Justify your answer.

$\because$ rank A = 3 = n

size(A) = 3x3

$\therefore$ A is invertible

$\therefore$ basis for row(A) = {(1 0 0), (0 1 0), (0 0 1)}.

(1 0 0), (0 1 0), (0 0 1) are linearly independent.

(v) (5 points). Give a basis for $col(A)$. Justify your answer.

According to (iv), A is invertible.

$\therefore col(A) = span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is basis for col(A) and it is linearly independent.

(vi) (5 points). Does the matrix $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ have an LU decomposition? If yes, find L and U. If no, explain why not.

No. Because LU factorization cannot interchange rows. B has a zero on its pivot position

Score: 10.0/18.0

Problem 6. (18 points total). Consider the set S, defined by

$$S = \{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5 \mid \det(\begin{bmatrix} 1 & 5 & x_1 & 6 & 2 \\ 2 & 4 & x_2 & 7 & 2 \\ 3 & 3 & x_3 & 8 & 3 \\ 4 & 2 & x_4 & 7 & 2 \\ 5 & 1 & x_5 & 6 & 2 \end{bmatrix}) = 0\}$$

Is S a subspace of $\mathbb{R}^5$? If so, prove it. If not, explain why not.

when $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\det\begin{pmatrix} 1 & 5 & 0 & 6 & 2 \\ 2 & 4 & 0 & 7 & 2 \\ 3 & 3 & 0 & 8 & 3 \\ 4 & 2 & 0 & 7 & 2 \\ 5 & 1 & 0 & 6 & 2 \end{pmatrix} = 0$

$\therefore 0 \in S$

Set $t = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix} \in S$ and $k = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{pmatrix} \in S$, $c \in \mathbb{R}$

WTS: $ct \in S$
and $(t+k) \in S$

$\because$ change all the t to k, we get equation ② $= 0$

change all the t to $(t+k)$ we get equation ③

$\because$ ③ = ① + ② = 0

$\therefore (t+k) \in S$

$\therefore S$ is a subspace

$\because t_1 \cdot \det\begin{pmatrix} 2 & 4 & 7 & 2 \\ 3 & 3 & 8 & 3 \\ 4 & 2 & 7 & 2 \\ 5 & 1 & 6 & 2 \end{pmatrix} - t_2 \begin{pmatrix} 1 & 5 & 6 & 2 \\ 3 & 3 & 8 & 3 \\ 4 & 2 & 7 & 2 \\ 5 & 1 & 6 & 2 \end{pmatrix} + t_3 \begin{pmatrix} 1 & 5 & 6 & 2 \\ 2 & 4 & 7 & 2 \\ 4 & 2 & 7 & 2 \\ 5 & 1 & 6 & 2 \end{pmatrix} - t_4 \cdot \begin{pmatrix} 1 & 5 & 6 & 2 \\ 2 & 4 & 7 & 2 \\ 3 & 3 & 8 & 3 \\ 5 & 1 & 6 & 2 \end{pmatrix} +$

$t_5 \begin{pmatrix} 1 & 5 & 6 & 2 \\ 2 & 4 & 7 & 2 \\ 3 & 3 & 8 & 3 \\ 4 & 2 & 7 & 2 \end{pmatrix} \cdots ① = 0$

multiply all t by c, $\det\begin{pmatrix} 1 & 5 & ct_1 & 6 & 2 \\ 2 & 4 & ct_2 & 7 & 2 \\ 3 & 3 & ct_3 & 8 & 3 \\ 4 & 2 & ct_4 & 7 & 2 \\ 5 & 1 & ct_5 & 6 & 2 \end{pmatrix} = ① \cdot c = 0c = 0$

$\therefore ct \in S$

Score: 2.0/20.0

Problem 7. (20 points total). Suppose that $S \subset \mathbb{R}^8$ with $\dim S = 5$. Is there a subspace $V \subset \mathbb{R}^8$ with $\dim V = 2$ and $V \cap S = \{\mathbf{0}\}$? Justify your answer.

Yes, there is.

$\because S \subset \mathbb{R}^8$ with $\dim S = 5$

$\therefore$ Set $S = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} t_1 + \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} t_2 + \begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} t_3 + \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix} t_4 + \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix} t_5 +$

$\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix} t_6 + \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix} t_7 + \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix} t_8$ and randomly pick 3 t equals 0, for example $t_1 = t_3 = t_8 = 0$, the rest of $t \neq 0$ now S is a subspace of $\mathbb{R}^8$ with $\dim S = 5$

Similarly, Set a $V \subset \mathbb{R}^8$ with $\dim V = 2$

$V = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} k_1 + \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} k_2 + \begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} k_3 + \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix} k_4 + \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix} k_5 + \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix} k_6 + \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix} k_7 + \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix} k_8$

and randomly pick $2k \neq 0$, for example $k_1 \neq 0$ and $k_3 \neq 0$. the rest of $k = 0$

now $S \cap V = \{0\}$

As long as pick 3 t_i in S equals 0 ($i \in \mathbb{R}$), and pick one or 2 of the same "i"s as t_i in k. $V \cap S = \{0\}$.

Score: 14.0/14.0

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Problem 8. (14 points total). This problem has two parts.

(i) (7 points). Prove that if $A\mathbf{x} \in Nul(A^T)$ then $A\mathbf{x} = \mathbf{0}$. *Hint: it may help to recall that $S \cap S^{\perp} = \{\mathbf{0}\}$ always holds for a subspace S.*

$\because Ax \in Nul(A^T)$

$Nul(A^T) = (Col(A))^{\perp}$

$\therefore Ax \in (Col A)^{\perp}$

$\because$ Ax is a linear combination of colA and x,

$\therefore Ax \in Col A$.

$\because (Col(A)) \cap (Col(A))^{\perp} = \{0\}$

$\therefore Ax = 0$

(ii) (7 points). Use part (i) above to show that $Nul(A) = Nul(A^TA)$ holds for all matrices A.

WTS: $Nul(A) = Nul(A^TA)$ which is to show that if ① $x \in Nul(A)$ then $x \in Nul(A^TA)$ and ② if $x \in Nul(A^TA)$, then $x \in Nul(A)$.

①: $\because$ if $x \in Nul(A)$

$Ax = 0$

$\therefore A^T \cdot (Ax) = A^T \cdot 0 = 0$

$\therefore (A^T \cdot A)x = 0$

$\therefore x \in Nul(A^T \cdot A)$

② $\because$ if $x \in Nul(A^T \cdot A)$,

$\therefore (A^T \cdot A)x = 0$

$\therefore A^T \cdot (Ax) = 0$

$\therefore Ax \in Nul(A^T)$

According to (i)

$\therefore Ax = 0$

$\therefore x \in Nul(A)$.

$\therefore Nul(A) = Nul(A^T \cdot A)$.

Problem 9. (18 points total, 3 points each) For the following questions, answer using the word "True" or the word "False". A statement is "True" when it's *always true* and otherwise is false. You **don't need to justify your answer** to receive full credit. However, **to discourage guessing we will deduct 2 points for each incorrect answer given.**

(a) Let A be a matrix. The pivot columns of the reduced row echelon form of A are a basis for the row space of the transpose of A.

False

(b) If A is invertible then $rank(A) = rank(A^{-1})$.

True

(c) The rank of a matrix must be at least as big as the dimension of its nullspace.

False

(d) A nonzero element of $Nul(A)$, for a square matrix A, must be an eigenvector of A.

True

(e) If every entry of a square matrix A is positive then $\det(A) \neq 0$.

n×n

False

(f) Let A be $m \times k$ and B be $k \times n$ matrices which have reduced row echelon matrices given by matrices M and N respectively. Then the reduced row echelon matrix of AB is given by the matrix MN.

False

(*extra paper*)

Not graded.

(*extra paper*)

$\cancel{col(1\ 2)}$

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix} \quad (1\ 2)$$

$$RREF = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$L(A\,x) = b.$

$\downarrow$

y

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

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(*extra paper*)

$x_1 - \underline{5t} - 7s + \underline{3t} + 4s$

$x_1 + \quad -2t - 3s.$

$x_2 = \dfrac{5t + 7s}{-2}$

$-2x_2 = 5t + 7s$

$-2x_2 - 5t - 7s = 0$

$2t + 3s.$

$-\frac{5}{2}t - \frac{7}{2}s$

t

s

$-3 \; -6 \; -9 \; -12$

$1 \; 2 \; 3 \; 4$

$0 \; -2 \; -5 \; -7$