Problem 1. (13 points total). Consider the following linear system

$$\begin{aligned} x_1 &= 2 \\ 2x_1 + 2x_2 &= 6 \\ 3x_1 + 3x_2 + 3x_3 &= 3 \end{aligned}$$

(i) (8 points). Find the inverse of the associated coefficient matrix for the above system.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix}$$

$$\therefore (A|I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 3 & 3 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 0 & -\frac{3}{2} & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{3} \end{array}\right) \qquad \therefore A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

(ii) (5 points). Use your answer to part (i) to solve the linear system. *No other method of solution will be graded - you MUST use the inverse of the associated coefficient matrix.*

$$\because A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} \quad Ax = b$$

$$\therefore A^{-1}Ax = A^{-1}b$$

$$\therefore x = A^{-1}b = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \qquad \therefore \begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = -2 \end{cases}$$

Problem 2. (14 points, 2 points each). Let A be an $n \times n$ matrix. List seven separate conditions equivalent to A being invertible. *Note: if you write more than seven your grade is only based on the first seven so there's no point in writing more.*

(i) There exists B be an nxn matrix, such that AB= I_n.

(ii) For Ax=b, there is only only unique solution for any $b \in \mathbb{R}^n$.

(iii) For Ax=0, there ~~are~~ is ~~noly~~ only trivial solution.

(iv) The RREF form of A is an identity matrix.

(v) The columns of A are independent matrices.

(vi) A can be written as the product of elementary matrices.

(vii) The rows of A are independent matrices.

Score: 15.0/15.0

Problem 3. (15 points total). This questions has 3 parts. *Note: your answers have to be completely precise and correct to receive credit since there is no partial credit for incorrect definitions.*

(i) (5 points). Let $S \subset \mathbb{R}^n$. *Define* what it means for S to be a subspace of $\mathbb{R}^n$.

S is a subspace of R^n if and only if the following features are satisfied:

1. zero matrix belongs to S
2. For $u, v \in S$, $(u+v) \in S$
3. For $u \in S$, $c \in R$, $cu \in S$

(ii) (5 points). *Define* the null space $Nul(A)$ of an $m \times n$ matrix A.

Nul(A) of an mxn matrix A is the set of ~~All~~ all the solutions $X \in R^n$ which makes ~~的~~ $AX=0$.

(iii) (5 points). Using your definitions above, prove that the nullspace, $Nul(A)$ of an $m \times n$ matrix A is a subspace.

~~$\because AX=0$~~ $\because Nul(A) = \{X \mid AX = 0\}$

$\because X=0$, $AX=0$

$\therefore 0 \in Nul(A)$

$\because$ Take $X_1, X_2 \in Nul(A)$

$\therefore AX_1 = 0$

$AX_2 = 0$

$\therefore A(X_1+X_2) = AX_1 + AX_2 = 0 + 0 = 0$

$\therefore (X_1+X_2) \in Nul(A)$

$\because$ Take $X \in Nul(A)$, $c \in R$

$\because AX = 0$

$A(cX) = c(AX) = c \times 0 = 0$

$\therefore cX \in NulA$

$\therefore$ Nul(A) of an mxn matrix A is a subspace.

Score: 10.0/10.0

Problem 4. (26 points total). This problem has 3 parts in total. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation satisfying $T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}3\\5\end{bmatrix}$ and $T(\begin{bmatrix}-1\\2\end{bmatrix}) = \begin{bmatrix}0\\1\end{bmatrix}$ and let $S : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $S(\begin{bmatrix}x_1\\x_2\end{bmatrix}) = \begin{bmatrix}x_1+x_2\\2x_1-x_2\\x_1+2x_2\end{bmatrix}$.

(a) (10 points). Find the standard matrices for T and S.

$\because T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation

$$T\begin{pmatrix}0\\1\end{pmatrix} = T\left(\frac{1}{3}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-1\\2\end{pmatrix}\right) = \frac{1}{3}T\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{3}T\begin{pmatrix}-1\\2\end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix}3\\5\end{pmatrix} + \frac{1}{3}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\end{pmatrix}$$

$$T\begin{pmatrix}1\\0\end{pmatrix} = T\left(\frac{2}{3}\begin{pmatrix}1\\1\end{pmatrix} + \left(-\frac{1}{3}\right)\begin{pmatrix}-1\\2\end{pmatrix}\right) = \frac{2}{3}T\begin{pmatrix}1\\1\end{pmatrix} + \left(-\frac{1}{3}\right)T\begin{pmatrix}-1\\2\end{pmatrix}$$

$$= \frac{2}{3}\begin{pmatrix}3\\5\end{pmatrix} + \left(-\frac{1}{3}\right)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}2\\3\end{pmatrix}$$

$\therefore$ ~~A~~ $T(x) = A_1 X$

$$A_1 = \begin{pmatrix}2 & 1\\3 & 2\end{pmatrix}$$

$$\because S(x) = A_2 X = \begin{pmatrix}X_1 + X_2\\2X_1 - X_2\\X_1 + 2X_2\end{pmatrix} = \begin{pmatrix}1 & 1\\2 & -1\\1 & 2\end{pmatrix}\begin{pmatrix}X_1\\X_2\end{pmatrix}$$

$$\therefore A_2 = \begin{pmatrix}1 & 1\\2 & -1\\1 & 2\end{pmatrix}$$

$\therefore$ The standard matrix for T is $\begin{pmatrix}2 & 1\\3 & 2\end{pmatrix}$, and the standard matrix for S is $\begin{pmatrix}1 & 1\\2 & -1\\1 & 2\end{pmatrix}$.

(b) (4 points). Find the standard matrix for the transformation defined by $S \circ T : \mathbb{R}^2 \to \mathbb{R}^3$.

$\because S \circ T(x) = A_2 A_1 x$

$\therefore A_2 \cdot A_1 = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 1 & 0 \\ 8 & 5 \end{pmatrix}$

$\therefore$ The standard matrix for $S \circ T : \mathbb{R}^2 \to \mathbb{R}^3$ is $\begin{pmatrix} 5 & 3 \\ 1 & 0 \\ 8 & 5 \end{pmatrix}$

(c) (4 points). Is $S \circ T$ one-to-one? Justify your answer.

$\because A_2 A_1 = \begin{pmatrix} 5 & 3 \\ 1 & 0 \\ 8 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 5 & 3 \\ 8 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

It has no free variables, the columns of $A_2 A_1$ is linearly independent, so $S \circ T$ is one-to-one.

(d) (4 points). Is $S \circ T$ onto? Justify your answer.

$\because$ The number of pivots of $A_2 A_1$ is 2,

the number of row is 3.

$2 \neq 3$

$\therefore S \circ T$ is not onto.

(e) (4 points). Is $S \circ T$ invertible? Justify your answer.

$S \circ T$ is not invertible, because ~~it is not~~ the size of it is not $n \times n$,

it is not a square matrix.

Score: 9.0/12.0

Problem 5. (12 points total, 3 points each) For the following questions, answer using the word "True" or the word "False". You **don't need to justify your answer** to receive full credit. There's no partial credit.

(a) True/False: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear transformation satisfying $T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}3\\5\end{bmatrix}$. Then $T^{-1}(\begin{bmatrix}6\\10\end{bmatrix}) = \begin{bmatrix}2\\2\end{bmatrix}$.

True

(b) True/False: If A and B are two matrices such that AB and BA are defined then A and B are both square.

False

(c) True/False: The sum of two elementary matrices of the same size must also be an elementary matrix.

False

(d) True/False: If a matrix is elementary then it must have an LU factorization.

True

(Nothing on this page below here will be graded but is extra space if you want it)

(*extra paper*)

$4-3=1.$

$2-2=0.$

$2+7.$ $2+6=8$

$1+4=5$

$1+2=3.$

Not graded.

(*extra paper*)

$\frac{1}{3} - \frac{1}{3} = 0.$

$\frac{1}{3} + \frac{2}{3} = 1$

$b(5-c) - 10b$

$\frac{30 - bc - 10b}{(3-b)(5-c) - bc}$

$1 + 0 = 1.$

$\frac{3}{5} + \frac{1}{3}.$

$$\begin{pmatrix} \frac{1}{ad-bc}(bd-10b) \\ \frac{1}{ad-bc}(bc-10a) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}?$$

$\frac{5}{3} + \frac{1}{3} = \frac{6}{3} = 2.$

$\overline{3}$

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} b \\ 10 \end{pmatrix} = ?$$

$\frac{2}{3} + \frac{1}{3} = 1.$

$\frac{2}{3} - \frac{2}{3} = 0.$

$\frac{10}{3} - \frac{1}{3}.$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$\overline{5}$

$a + b = 3$ $\qquad \frac{}{ad-bc}$ $\qquad a = 3 - b$

$c + d = 5$ $\qquad\qquad\qquad d = 5 - c$

Not graded.

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(*extra paper*)