Problem 1. (20 points total). This problem has six parts in total. Consider the following linear system

$$4x_1 + x_2 + 6x_3 = 2$$
$$2x_1 + 2x_2 + 3x_3 = 1$$

(i) (1 point). Write down the associated augmented matrix for the above system.

$$\left(\begin{array}{ccc|c} 4 & 1 & 6 & 2 \\ 2 & 2 & 3 & 1 \end{array}\right)$$

(ii) (7 points). Use the reduction algorithm to solve the above system. Note: *no other method will receive credit, you must use the reduction algorithm.*

$$\left(\begin{array}{ccc|c} 4 & 1 & 6 & 2 \\ 2 & 2 & 3 & 1 \end{array}\right) \xrightarrow{-2r_2+r_1} \left(\begin{array}{ccc|c} 0 & -3 & 0 & 0 \\ 2 & 2 & 3 & 1 \end{array}\right) \xrightarrow[r_2 \leftrightarrow r_1]{\frac{1}{2}r_2} \left(\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{-r_2+r_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + \frac{3}{2}x_3 = \frac{1}{2} \\ x_2 = 1 \end{cases}$$

let $x_3 = t$, $t \in \mathbb{R}$

$\therefore x_1 + \frac{3}{2}x_3 = \frac{1}{2}$

$x_1 + \frac{3}{2}t = \frac{1}{2}$

$x_1 = \frac{1}{2} - \frac{3}{2}t$

$$\begin{cases} x_1 = \frac{1}{2} - \frac{3}{2}t \\ x_2 = 0 \\ x_3 = t, \ t \in \mathbb{R} \end{cases}$$

Score: 12.0/12.0

(iii) (2 points). Is the original system consistent or inconsistent?

The original system is consistent, because it does not have the form of $(0\ 0\ 0 \mid c)$, ~~st~~ $c \neq 0$.

(iv) (3 points). How many variables are basic and how many variables are free?

There are 2 basic variables and 1 free variables.

(v) (5 points). Set $\mathbf{a}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Is $\mathbf{b}$ a linear combination of $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$? If not, explain. If yes, write $\mathbf{b}$ as a linear combination of $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$.

WTS: if $\exists c_1, c_2, c_3$, $c_1, c_2, c_3 \in \mathbb{R}$ s.t. $c_1 \cdot a_1 + c_2 a_2 + c_3 a_3$ is consistent, which means $c_1\begin{pmatrix} 4 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_3\begin{pmatrix} 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is consistent.

$\cancel{\left(\begin{array}{ccc|c} 4 & 1 & 6 & 2 \\ 2 & 2 & 3 & 1 \end{array}\right)}$ which is ~~to~~ equivalent to show if the original system is consistent.

According to question iii, b is a linear combination of a_1, a_2, and a_3.

let $t = 0$, $b = \frac{1}{2} \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

(vi) (2 points). Is $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? Why or why not?

WTS: if $\exists c_1, c_2, c_3$, $c_1, c_2, c_3 \in \mathbb{R}$ s.t. $b = a_1 \cdot c_1 + a_2 \cdot c_2 + a_3 c_3$

according to question v. $b \in \text{Span}\{a_1, a_2, a_3\}$

Score: 10.0/10.0

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Problem 2. (10 points). Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

(i) (5 points). Does $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$ have a unique solution? Justify your answer.

~~WTS~~ $c_1\begin{pmatrix}2\\-1\end{pmatrix} + c_2\begin{pmatrix}2\\1\end{pmatrix} = 0$

$$\left(\begin{array}{cc|c} 2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right) \underset{\frac{1}{4}r_1}{\overset{-2r_1+r_2}{\sim}} \left(\begin{array}{cc|c} 0 & 4 & 0 \\ 1 & -1 & 0 \end{array}\right) \overset{r_1+r_2}{\sim} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

$\therefore \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases}$ $\therefore c_1u + c_2v = 0$ have a unique solution.

(ii) (5 points). For which $h, k \in \mathbb{R}$ is $\begin{bmatrix} h \\ k \end{bmatrix} \in \text{span}\{\mathbf{u}, \mathbf{v}\}$? Justify your answer.

~~WTS:~~ $\begin{pmatrix}2\\-1\end{pmatrix}c_1 + \begin{pmatrix}2\\1\end{pmatrix}\cdot c_2 = \begin{pmatrix}h\\k\end{pmatrix}$ $c_1, c_2 \in \mathbb{R}$ is consistent.

$$\left(\begin{array}{cc|c} 2 & 2 & h \\ -1 & 1 & k \end{array}\right) \underset{-r_2}{\overset{2r_2+r_1}{\sim}} \left(\begin{array}{cc|c} 0 & 4 & 2k+h \\ 1 & -1 & -k \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -1 & -k \\ 0 & 4 & 2k+h \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & -1 & -k \\ 0 & 1 & \frac{2k+h}{4} \end{array}\right) \overset{r_2+r_1}{\sim} \left(\begin{array}{cc|c} 1 & 0 & \frac{h-2k}{4} \\ 0 & 1 & \frac{2k+h}{4} \end{array}\right)$$ it's always consistent

So $\forall h, k \in \mathbb{R}$, $\begin{pmatrix}h\\k\end{pmatrix} \in \text{span}\{u, v\}$

Problem 3. (12 points). Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Prove that $||\mathbf{v}+\mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

Proof: $\cancel{\|V+W}$ $\|V\| + \|W\| \geq |V+W|$

$|V+W| \geq 0$, $\|V\| + \|W\| \geq 0$

so $\|V+W\|^2 \leq (\|V\| + \|W\|)^2$

$= \|W\|^2 + \|V\|^2 + 2\|V\| \cdot \|W\|$

when $V \cdot W \neq 0$:

$2\|V\| \cdot \|W\| \geq 0$

so, $\|V+W\|^2 \leq \|V\|^2 + \|W\|^2$

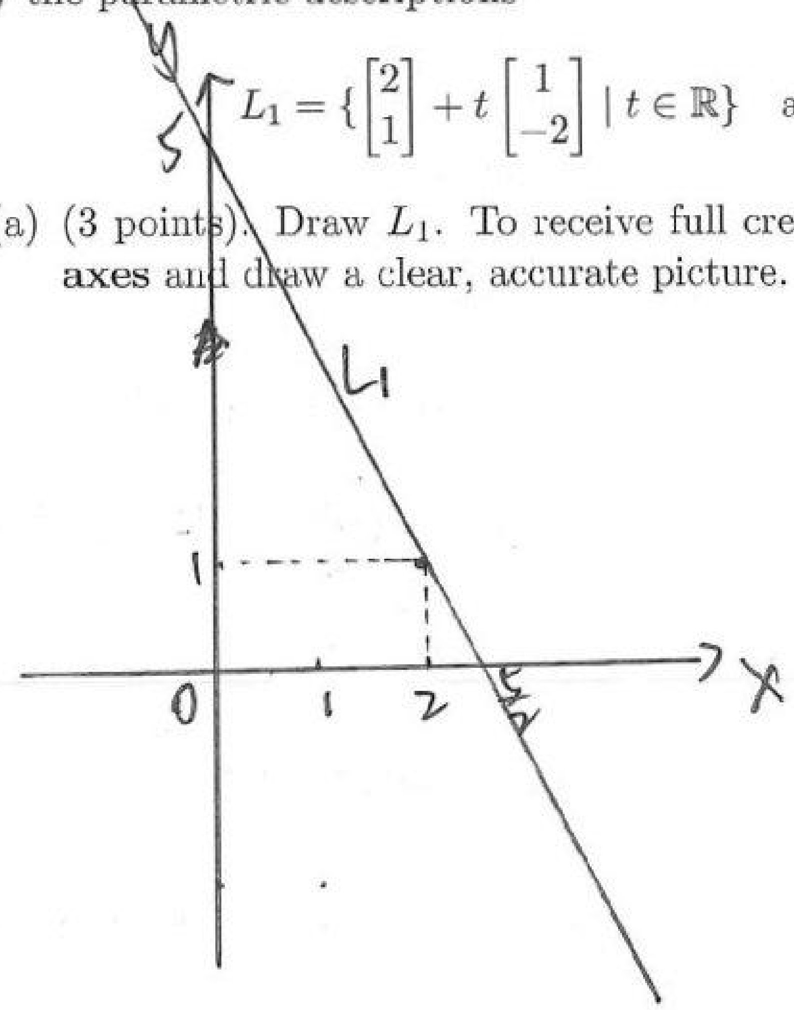
So, $V \cdot W = 0$, $V \neq W$

So, V and W are orthogonal

Problem 4. (12 points total). This problem has 3 parts total. Consider two lines in $\mathbb{R}^2$ given by the parametric descriptions

$$L_1 = \{\begin{bmatrix}2\\1\end{bmatrix} + t\begin{bmatrix}1\\-2\end{bmatrix} \mid t \in \mathbb{R}\} \quad \text{and} \quad L_2 = \{\begin{bmatrix}5\\4\end{bmatrix} + t\begin{bmatrix}1\\2\end{bmatrix} \mid t \in \mathbb{R}\}$$

(a) (3 points). Draw L_1. To receive full credit you must **clearly mark and label your axes** and draw a clear, accurate picture.



(b) (3 points). Write down an equation of *any line* which intersects L_1 at a right angle.

~~$L = t\binom{2}{1} + c,\ c \in \mathbb{R}$~~

Let L intersects L_1 at a right angle,

$L = \{ c + t\binom{2}{1} \mid t \in \mathbb{R}, c \in \mathbb{R}\}$

~~L that~~ $\binom{2}{1} \cdot \binom{1}{-2} = 0$

Score: 6.0/6.0

(c) (6 points). Do L_1 and L_2 intersect? If so, find the point of intersection.

~~$L_1: y = -2x+5$~~

~~L_1 is parallel to~~

L_1 has the same direction as $\begin{pmatrix}1\\-2\end{pmatrix}$

also, L_1 passes through the point $\begin{pmatrix}2\\1\end{pmatrix}$

$L_1: \quad y = -2x+5$

Similarly, L_2 has the same direction as $\begin{pmatrix}1\\2\end{pmatrix}$ and

L_2 passes through the point $\begin{pmatrix}5\\4\end{pmatrix}$

$L_2: \quad y = 2x-6$

$$2x-6 = -2x+5$$

$$4x = 11$$

$$x = \frac{11}{4}$$

$$y = 2 \times \frac{11}{4} - 6 = -\frac{1}{2}$$

So, the point of intersection is ~~$(\frac{11}{4}, -\frac{1}{2})$~~ $\begin{pmatrix}\frac{11}{4}\\-\frac{1}{2}\end{pmatrix}$

Score: 18.0/21.0

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Problem 5. (21 points total, 3 points each) For the following questions, answer using the word "True" or the word "False". You **don't need to justify your answer** to receive full credit. There's no partial credit.

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n+1   A

n $\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \end{pmatrix}\begin{pmatrix}\vdots\end{pmatrix} = 0$

(a) True/False: If A is of size $n \times (n+1)$ and $\mathbf{x} \in \mathbb{R}^{n+1}$ then $A\mathbf{x} = \mathbf{0}$ is solvable.

True

(b) True/False: A linear system with two basic variables cannot be consistent.

False

(c) True/False: Pivot columns cannot correspond to free variables.

True

(d) True/False: If $\mathbf{u} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2\}$.

True

(e) True/False: If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then $|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||^2\, ||\mathbf{v}||^2$. $|u \cdot v| \leq ||u||^2 \cdot ||v||$

True

(f) True/False: If $\mathbf{v}, \mathbf{w}$ are nonzero vectors in $\mathbb{R}^n$ then $\mathbf{proj}_{\mathbf{v}}\mathbf{w} = \mathbf{proj}_{\mathbf{w}}\mathbf{v}$.

False

(g) True/False: If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$ then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^m$.

True

(Nothing on this page below here will be graded but is extra space if you want it)

Not graded.

(*extra paper*)

Not graded.

(*extra paper*)