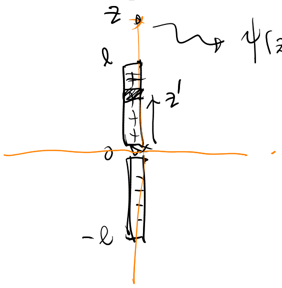
1. Considere una carga lineal ubicada en el eje z de la siguiente forma

$$\lambda(z) = \begin{cases} \lambda_0 & \text{para} \quad 0 < z < l \\ -\lambda_0 & \text{para} \quad -l < z < 0 \end{cases}$$

   (a) 20% Determine el potencial eléctrico para $z > l$, extrapole el potencial para todo el espacio haciendo uso de los polinomios de Legendre.

   (b) 20% Encuentre el campo eléctrico E_θ para $\theta = \pi/2$ en función de r.

   (c) 5% Calcule el momento dipolar de la distribución de carga y compare con el resultado del potencial[1]



$$dq = \lambda\, dz'$$

$$\psi(z) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|z-z'|}$$

$$\int_{-l}^{l} \frac{\lambda(z')\,dz'}{z(1-z'/z)}$$

$$\frac{1}{z}\int_{-l}^{l} \lambda(z')\sum_{n=0}^{\infty}\left(\frac{z'}{z}\right)^n dz'$$

$$\lambda(z) = \begin{cases} \lambda_0 & z \in (0,l] \\ -\lambda_0 & z \in [-l,0) \end{cases}$$

$$\lambda_0 \sum_{n=0}^{\infty}\frac{1}{z^n}\left[\int_0^l z'^n\,dz' - \int_{-l}^0 z'^n\,dz'\right]$$

$$\left.\frac{z'^{n+1}}{n+1}\right|_0^l - \left.\frac{z'^{n+1}}{(n+1)}\right|_{-l}^0$$

$$\frac{l^{n+1}}{n+1} + \frac{(-l)^{n+1}}{n+1}$$

$$\frac{l^{n+1}}{n+1}\left(1+(-1)^{n+1}\right)$$

$(0,\ n\ \text{PAR};\ 2,\ n\ \text{IMPAR})$

$$\psi(z) = \frac{\lambda_0}{4\pi\epsilon_0}\sum_{\substack{n=0 \\ \text{IMPARES}}}^{\infty}\frac{1}{z^{n+1}}\frac{2l^{n+1}}{n+1} = \frac{2\lambda_0}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{l^{2n+2}}{z^{2n+2}(2n+2)}$$

$n \to 2n+1$

$z \to \infty \quad \sim \frac{1}{z^2}$ DIPOLAR

(a)

$$\psi(z) = \frac{2\lambda_0}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\left(\frac{l}{z}\right)^{2n+2}\frac{1}{(2n+2)}$$

$$\frac{1}{z^{2n+2}} \to \frac{1}{r^{2n+2}}P_{2n+1}(\cos\theta)$$

$$\Rightarrow \psi(r,\theta) = \frac{2\lambda_0}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\left(\frac{l}{r}\right)^{2n+2}\frac{P_{2n+1}(\cos\theta)}{2n+2}$$

b)

$$E_\theta\Big|_{\pi/2} = -\frac{1}{r}\frac{\partial}{\partial\theta}\psi(r,\theta)\Big|_{\pi/2} = -\frac{2\lambda_0}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{(2n+2)}\frac{l^{2n+2}}{r^{2n+3}}(-\operatorname{sen}\theta)\frac{d}{d\cos\theta}P_{2n+1}(\cos\theta)\Big|_{\theta=\pi/2}$$

$$P'_{2n+1}(\cos\theta)\Big|_{\theta=\pi/2}$$

$$(-1)\,P'_{2n+1}(0)$$

$$E_\theta\Big|_{\pi/2} = \frac{2\lambda_0}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{l^{2n+2}}{r^{2n+3}}\frac{(-1)^n(2n+1)!!}{(2n+2)!!}$$

$$P_n(x) = \sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}$$

$$P'_n(0) = \sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!(n-2k)x^{n-2k-1}}{2^n k!(n-k)!(n-2k)!}$$

$\Downarrow \quad 2k = n-1$

$n \to 2n+1$

$k \to n$

SOLO EXISTE ESTE TERMINO $\to k = \left(\frac{n-1}{2}\right)$

$$P'_{2n+1}(0) = \frac{(-1)^n(4n+2-2n)!\,\cancel{(2n+1-2n)}}{2^{2n+1}n!(n+1)!\cancel{(2n+1-2n)}!}$$

1

1

$$(2n+2)! = 2(n+1)(2n+1)!$$

$$= \frac{(-1)^n(2n+2)!}{2^{2n+1}n!(n+1)!}$$

$$= \frac{(-1)^n(2n+1)!!\cancel{(2n)!!}}{(2n)!!\cancel{(2n)!!}}$$

c)

$$\vec{P} = \int \vec{r}\lambda\,dz = \hat{z}\int_0^l z\lambda_0\,dz - \hat{z}\int_{-l}^0 z\lambda_0\,dz$$

MOMENTO DIPOLAR

$\vec{r} \to z\hat{z}$

$$= \hat{z}\lambda_0\left[\frac{l^2}{2}+\frac{l^2}{2}\right] = \hat{z}\lambda_0 l^2$$

$$V_{\text{dipolo}} = \frac{1}{4\pi\epsilon_0}\frac{\vec{P}\cdot\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{\lambda_0 l^2 z}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{\lambda_0 l^2\cos\theta}{r^2}$$

$z \to r\cos\theta$

DE LA RESP.

$$\psi(r,\theta) = \frac{2\lambda_0}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\left(\frac{l}{r}\right)^{2n+2}\frac{P_{2n+1}(\cos\theta)}{2n+2} \xrightarrow{n=0} \psi(r,\theta) = \frac{2\lambda_0}{4\pi\epsilon_0}\left(\frac{l}{r}\right)^2\frac{P_1(\cos\theta)}{2}$$

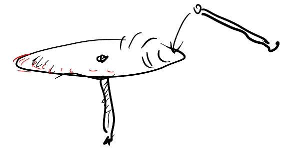
2. Considere la ecuación de ondas con velocidad v en un platillo delgado de radio a con el centro fijo.

(a) 20% Muestre que una solución particular es $\psi(\rho,\phi,t) = J_n(k\rho)e^{in\phi}e^{i\omega t}$, encuentre los posibles valores de k y ω compatibles con la condición de borde libre.

(b) 15% ¿Cuáles son las 3 primeras frecuencias normales de oscilación?

(c) 20% Si en $t=0$ se golpea el borde del disco en reposo $\to \psi|_{t=0}=0$ con un golpe seco modelado como un delta de Dirac $\dot\psi = C\delta(\rho-a)\delta(\phi)$, ¿cuál es la solución particular que satisface estas condiciones iniciales (junto con las de borde)?

(d) 5% Cuál es la relación de amplitudes en el borde entre las 2 primeras frecuencias.



EC ONDAS $\Rightarrow$ $\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi$

a)

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\psi\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{\rho}{\rho^2}\frac{\partial}{\partial\phi}\psi\right) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \Rightarrow \text{SEA } \psi = J_n(k\rho)e^{in\varphi}e^{i\omega t}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\psi\right) - \frac{1}{\rho^2}n^2\psi = \frac{1}{v^2}(-\omega^2)\psi \Rightarrow \text{CANCELANDO } e^{in\varphi}e^{i\omega t} \Rightarrow \psi \to J_n(k\rho)$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}J_n(k\rho)\right) - \frac{n^2}{\rho^2}J_n(k\rho) + \frac{\omega^2}{v^2}J_n(k\rho) = 0 \to \text{SE MUESTRA QUE } J_n e^{in\varphi}e^{i\omega t} \text{ ES SOLUCION CON } \boxed{\omega = kv}$$

$$\Rightarrow k^2 = \frac{\omega^2}{v^2}$$

CONDICIONES DE BORDE $\to$ NEWMAN $\left.\frac{\partial\psi}{\partial\rho}\right|_{\rho=a} = 0 \Rightarrow \left.J_n'(k\rho)\right|_{\rho=a} = 0 \quad J_n'(ka) = 0$

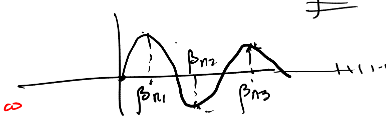
DISCO BORDE LIBRE

$\Rightarrow ka = \beta_{nm} \to$ EMESIMO CERO DE J_n'

VALORES DE k y ω COMPATIBLES CON LAS COND. DE BORDE $\to$

$$k = \frac{\beta_{nm}}{a}$$

$$\omega = \frac{\beta_{nm}v}{a} \equiv \omega_{nm} \quad n = 1,\dots\infty;\ m = 1,\dots\infty$$



$\to$ EL CENTRO ES FIJO $\Rightarrow$ $n \neq 0$.

b) 3 FREC $\to$

$$\omega_{11} = \beta_{11}\frac{v}{a} = 1.8412\frac{v}{a}$$
$$\omega_{21} = \beta_{21}\frac{v}{a} = 3.0542\frac{v}{a}$$
$$\omega_{31} = \beta_{31}\frac{v}{a} = 4.2012\frac{v}{a}$$

TRES PRIMEROS VALORES.

	$J_0'(x)^a$	$J_1'(x)$	$J_2'(x)$	$J_3'(x)$
1	3.8317	1.8412	3.0542	4.2012
2	7.0156	5.3314	6.7061	8.0152
3	10.1735	8.5363	9.9695	11.3459

$\to$ CEROS DE J_n' | TOMADO DEL LIBRO (ARFKEN

$\beta_{11} = 1.8412$
$\beta_{21} = 3.0542$
$\beta_{31} = 4.2012$

c) $\dot\psi = C\delta(\rho-a)\delta(\phi)$ $\Rightarrow$ CONDICIONES INICIALES

$$\psi|_{t=0} = 0 \Rightarrow T(t) \sim \{e^{i\omega t}, e^{-i\omega t}\} \to \text{SEN}\,\omega t$$

$$\left.\frac{\partial\psi}{\partial t}\right|_{t=0} = C\delta(\rho-a)\delta(\varphi) \to$$

COMBINAC. DE SOL. PARTICULARES

$$\psi = \sum_{nm} A_{nm} J_n(\beta_{nm}\rho/a)e^{in\varphi}\,\text{SEN}(\omega_{nm}t) \to \text{SATIS. } \psi|_{t=0} = 0$$

$$\dot\psi = \frac{\partial\psi}{\partial t} = \sum_{nm} A_{nm} J_n(\beta_{nm}\rho/a)e^{in\varphi}\omega_{nm}\cancel{\cos(\omega_{nm}t)}^{\,1}\Big|_{t=0} = C\,\delta(\rho-a)\,\delta(\phi)$$

ORTOGONALIDAD.

$$\int_0^a J_n(\beta_{nm}\rho/a)J_n(\beta_{nm'}\rho/a)\rho\,d\rho = N_{nm}\delta_{mm'}$$

CON $N_{nm} = a^2\left(1-n^2/\beta_{nm}^2\right)J_n^2(\beta_{nm})$

$$\int_0^{2\pi} e^{in\varphi}e^{-in'\varphi} = 2\pi\,\delta_{nn'}$$

DEL TEXTO.

$$\Rightarrow A_{nm} = \frac{C}{2\pi N_{nm}\omega_{nm}}\int\delta(\rho-a)\delta(\varphi)e^{-in\varphi}J_n(\beta_{nm}\rho/a)\rho\,d\rho\,d\varphi$$

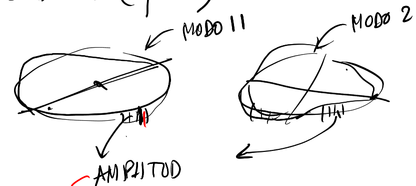
$$a\,J_n(\beta_{nm})$$

$$A_{nm} = \frac{C\,\cancel{a}\,\cancel{J_n(\beta_{nm})}}{2\pi a^{\cancel{2}}\left(1-n^2/\beta_{nm}^2\right)J_n^{\cancel{2}}(\beta_{nm})\,\beta_{nm}(v/\cancel{a})}$$

$$A_{nm} = \frac{c}{2\pi v}\frac{1}{\left(\beta_{nm} - n^2/\beta_{nm}\right) J_n(\beta_{nm})} \quad *$$

d) Relacion de amplitudes entre las 2 primeras frecuencias, en el borde ($\rho = a$)

$$\rightarrow \frac{A_{11} J_1(\beta_{11})}{A_{21} J_2(\beta_{21})} = \frac{\left(\beta_{21} - 4/\beta_{21}\right)}{\left(\beta_{11} - 1/\beta_{11}\right)} = \frac{3.0542 - 4/3.0542}{1.8412 - 1/1.8412} = \underline{1.3439} \checkmark$$



$$\underline{\underline{\Psi}}\Big|_{\substack{\rho = a \\ \varphi = 0 \\ t = T/2}} \rightarrow \text{Sen } \omega t = 1$$