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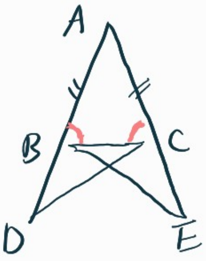
Chapter 1. General

1800s: wave equation → 1864: Maxwell → 1888: Hertz confirmed → 1896: Marconi radio transmission

$\underbrace{\text{Pure Mathematicians} \to \text{Theoretician} \to \text{Experimental Scientist} \to \text{Practical man}}_{\text{150 yrs}}$

Power of mathematics: Intuition and Rigour

Chapter 2. Motion



Proving Isosceles Triangles's base angles are equal

1. Euclid's proof: congruent triangles
2. Motion: flipping over → Zeno's paradox (movement)
3. Rigid motion via Transformations

Chapter 3. Arithmetics

$b \equiv c \pmod{a}$ b and c are congruent to the modulus a

arithmetics mod n: addition ✓ subtraction ✓ multiplication ✓
division: prime modulus ✓ ✗ composite modulus

Fermat's number: $2^{2^n}+1 \to$ prime?
✓: 3, 5, 17, 257, 65537
✗: $2^{32}+1 \equiv 0 \pmod{641}$

Fermat's Theorem: $x^{p-1} \equiv 1 \pmod{p}$ $3^{5-1} = 9 \times 9 = 81 \equiv 1 \pmod{5}$

$(1 \cdot x)(2 \cdot x) \cdots ((p-1) \cdot x)$
$\equiv x^{p-1}(1 \cdot 2 \cdots (p-1))$
$\equiv 1 \cdot 2 \cdots \cdot (p-1)$ cancel $\pmod{p}$ } prime multiplication table: all nums appear once

Wilson's Theorem: $1 \cdot 2 \cdot \cdots \cdot (p-1) \equiv -1 \pmod{p}$
(test for prime)

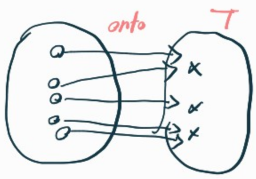
$1 \cdot (2 \cdot 4) \cdot (3 \cdot 5) \cdot 6$
$\equiv 1 \cdot 1 \cdot 1 \cdot (-1) \equiv -1$ ← pair off with reciprocal

Chapter 4. Set

Euclidean Geometry as a part of Set Theory
e.g. a point lies on a line, geometrically
if it's a set-theoretic member of a line

Chapter 5. Function

$f: D \to T$ Domain Target



f(domain): range of f
↑
equal T: onto T
f is a surjection

surjection + injection = bijection = inverse exists

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for each point in T

Chapter 6

Abstract Algebra

Abstract from Arithmetics <

Rings:

6 rules

(1) Associative Law of Addition $(a+b)+c = a+(b+c)$

(2) Commutative Law of Addition $a+b=b+a$

(3) Existence of Zero $a+0=a=0+a$

(4) Existence of Additive Inverse $a+(-a)=0=(-a)+a$

(5) Associative Law of Multiplication $(ab)c=a(bc)$

Commutative Ring with Unity

(6) Commutative Law of Multiplication $ab=ba$

(7) Existence of Unity $1a=a=a1$

(8) Distributive Laws $a(b+c)=ab+ac$, $(a+b)c=ac+bc$

Fields:

Division <

10 Rules

(9) Existence of Multiplicative Inverse for non-zero element $aa^{-1}=1=a^{-1}a$

(10) $0\neq 1$ $\quad 0\neq 1$

Examples of Fields / Rings

→ The Set of constructible lengths by Platonic restriction ie. Ruler, Compasses

$r+s,\ r-s,\ rs,\ \frac{r}{s},\ p+q\sqrt{r}$

Field

⇒ $\sqrt[3]{2}$ is not constructible

→ Congruence classes ($+$, $\cdot$ are well-defined)

⇒ (1)-(8) are satisfied → Commutative Ring with Unity.

⇒ prime modulus → Field

→ Polynomial Rings : $R[x]$ coefficients are real numbers — Ring

→ Complex numbers $ai+b$ — Field

⇒ Congruences classes of Polynomials to the modulus x^2+1

$x^2 \equiv -1 \pmod{x^2+1}$

all poly $\equiv ax+b \pmod{x^2+1}$, $a,b\in\mathbb{R}$

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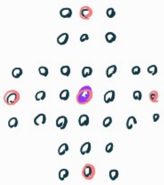
$\Rightarrow (ax+b)(-ax+b) \equiv a^2+b^2 \pmod{x^2+1}$

Inverse can be defined !

Game of Solitaire



Prove finite final positions for end of game left with one peg

Init : all holes except centre contains a peg

Rule : jump a peg over another horizontally / vertically adjacent peg (not diagonally) into an empty hole

$\Rightarrow$ remove the jumped-over peg

(board size doesn't matter)

Win : Remove all peg but one (+ end in centre)

Explain from the endings!

Define : ① Integer coordinates on the board

② Situation : a set of pegs on the board

Solve : ③ $A(S)$: Σp^{k+l} for adding p^{k+l} $\forall (k,l)$ pegs on the board

$B(S)$: Σp^{k-l} for adding p^{k-l} —"—

where $+, \times$ of p is defined by a field with 4 elements

+	0	1	p	q
0	0	1	p	q
1	1	0	q	p
p	p	q	0	1
q	q	p	1	0

×	0	1	p	q
0	0	0	0	0
1	0	1	p	q
p	0	p	q	1
q	0	q	1	p

Then each legal move from S to $T \Rightarrow A(S) = A(T)$

e.g. $p^{k+l} \Rightarrow$ a move to the left : $p^{k-2+l} - (p^{k+l} + p^{k+l+l}) = 0$

similarly , $B(S) = B(T)$ for any legal move

Note : $(A(S), B(S))$ has 16 pairs (4 elements in a field)

$\Rightarrow$ Separate the board positions into 16 sets , s.t. a series of moves does not change which set the positions belong to

Init : $A(S) = B(S) = 1$

Final : One single peg on board $\Rightarrow (p^{k+l}, p^{k-l}) = (1,1)$

$\Rightarrow k+l,\ k-l$ multiples of 3

$\Rightarrow (0,0)$ is one of the valid soln.

Chapter 7. (Symmetry) Group

Group : 5 rules

① A set G

② An operation " $*$ " $\forall x, y \in G$ s.t. $x * y \in G$

Define an operation " $*$ "

③ Associative : $x*(y*z) = (x*y)*z$

Author: kiking $*: G \times G \to G$

4. Existence of Identity $I \in G$ s.t.
$$I * x = x = x * I$$
5. Existence of inverse $x' \; \forall x \in G$ s.t.
$$x * x' = I = x' * x$$

Example: $(\mathbb{Z}, +)$, Identity: 0

$(\mathbb{R}, +)$, Identity: 0

$(\mathbb{Q}/\{0\}, \times)$, Identity: 1

(Symmetries of $S \in \mathbb{R}^2$, $\times$), Identity: I

Symmetry Group ‖

Symmetry: a rigid motion $f: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\forall$ points $x \in S$, $f(x) \in S$

(a rigid motion of the plane which leaves S in the same place)

rule 1: symmetries is a set
2: symmetry is closed under multiplication
3: true for function
4: Identity symmetry is defined
5: Inverse of symmetry is defined

Non-Example: $(z \in \mathbb{Z}$ s.t. $z \in (-10, 10), +)$ violate (2)

$(\mathbb{Z}_{>1}, +)$ violate (4)

$(\mathbb{Z}, -)$ violate (3)

$(\mathbb{Q}, \times)$ violate (5) since $0 \times 0 \neq 1$ but 1 is the only possible identity

Group & Subgroups: Subgroup: subset $H \subseteq G$ s.t. H forms a group

Order of a Group: # elements in a finite Group

Numerical relation of Groups

Lagrange's Theorem: Order of subgroups must divide order of Group

Proof: Consider a subgroup $J \subseteq K$ ‖
if multiply J with each element in K,
will result in some distinct cosets s.t.
$\to$ one of them is J $\to$ no distinct cosets share common elements
$\to$ every elements of K lie in some coset $\to$ each coset has equal size

Sylow's Theorem: If h divides the Order of a Group G, and h is a power of prime $\Rightarrow$ then G has a subgroup of Order h

Isomorphism:

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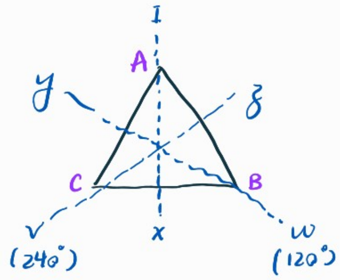
Isomorphism:

Given two groups G and H,
they are isomorphic if $\exists$ a bijection $f: G \to H$
s.t. $f(\alpha * \beta) = f(\alpha) * f(\beta)$

Example

✓ Symmetries of a Triangle
is isomorphic to



✓ Permutations of a 3-element Set

e.g. each reflection/rotation symmetry of a triangle is a permutation of rearranging the vertices A, B, C

X	I	w	v	x	y	z
I	I	w	v	x	y	z
w	w	v	I	z	x	y
v	v	I	w	y	z	x
x	x	y	z	I	w	v
y	y	z	x	v	I	w
z	z	x	y	w	v	I

Multiplication Table

x * w
↓ ↓
A B
B C C A

Symmetries

X	I	w	v	x	y	z
A	A	B	C	A	C	B
B	B	C	A	C	B	A
C	C	A	B	B	A	C

Permutations

Same:
Order of 6

Difference
① 6 subgroups vs 4 subgroups
② Addition mod 6 satisfies Commutative Law (Not for symmetry groups)

Non-Examples

× (Integers mod 6, +)

17 wallpaper groups (from Arabian works)

→ 17 plane symmetry groups p1, p2, pm, pg, cm, pmm, p4g, p4m, p4, cmm, pmg, pgg, p3, p31m, p3m1, p6, p6m
→ tilings with translational symmetries in 2 independent directions
→ frieze patterns: in 1 direction
→ 230 symmetry groups in 3D

Chapter 8

|| Child Learning: number → ring $\mathbb{Z}$ → whole classes of rings → more classes

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Axiomatics

→ deduct properties → ...

Define terms → List laws → axioms + axiomatic systems can be divorced from reality
/assumptions

e.g. Ring, Field, Group

"Are the group axioms true?" is an non-sense question

Euclid Axioms:

(6) Given any line, and any point not on the line, ∃ exactly one line // to the first line and passing the point

⇒ Not provable from other axioms

⇒ "Not true" for Earth/Round surfaces → only means that Earth doesn't fit in Euclidean Geometry

⇒ may be that's why Euclid states this axiom explicitly

An axiomatic theory must be consistent (David Hilbert)

If we can prove two contradictory theorems, then we can prove anything.

Prove by Contradiction

Prove anything p is True:

1. Assume p is false
2. Observe a and b are contradictory, deducted from the axioms.
3. So p must not be false

⇒ p is True, QED.

Example: Field

Law(9): if change to Existence of Multiplicative Inverse ∀ elements

⇒ inconsistent,

since $(0 \cdot 0) \cdot 0^{-1} = 0 \cdot 0^{-1} = 1$
$0 \cdot (0 \cdot 0^{-1}) = 0 \cdot 1 = 0$ ⟩ $0 \neq 1$ Law(10) but Associative Law Law(5)

Consistency, Completeness, Independence (David Hibert)

→ Have enough axioms to prove the truth/falsity of any conceivable statements of the system

→ Cannot add extra axioms anymore: either deductible or pointless

→ No axioms can be deducted from others

Model = example of an axiom system

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Model: example of an axiom system

e.g. Coordinate Geometry: algebraic model of the axioms of Euclidean Geometry

To prove Independence:

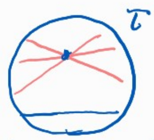
Construct a model that satisfy some but not all axioms of the axiom system

⇒ if exists, that means the failed axioms are not deductible from other axioms ⇒ independent, QED.

Euclidean Geometry.

∃ exactly one line parallel to the given line and pass through the given point

Hyperbolic Geometry



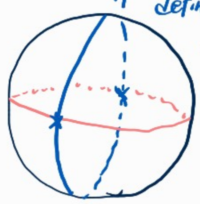
if define line, point lie inside τ

∃ infinitely many lines parallel (i.e. not meeting) the given line

Elliptic Geometry

∃ no line parallel to the given line

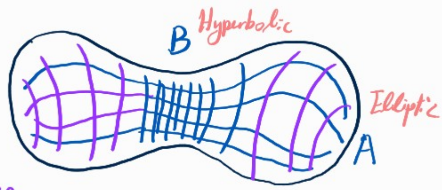
(i.e. two lines always meet-up)



if define line: circle passing through centre

Riemann: Curved Geometry

Near A: Ellipse Near B: Hyperbola ⇒



3D: Curved Space ← Einstein: Space-Time

curvature <> gravitational attraction of matters

Elliptic Time? Get back before set out!

E.g. Radio Stars at diametrically opposite points in sky

⇒ could be just one star seen from two opposite directions

Chapter 9

Counting

Finite & Infinite

Cardinals (Transfinite numbers)

Galileo: ∃ bijection between a set and a smaller subset of itself

e.g. $f(n) = n^2$ → bijection of $\mathbb{N}$

⇒ Same Cardinal of $\mathbb{N}$

infinite cardinals

$\aleph_0$ alpha-zero: Cardinal of $\mathbb{N}$

Examples of $\aleph_0$: odd / even numbers (both infinite)

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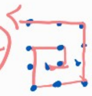
$\aleph_0$

$\aleph_0 + \aleph_0 = \aleph_0$ (e.g. {odd} $\cup$ {even} $= \mathbb{N}$)

$\mathbb{Q}$: Rational Number is also $\aleph_0$ ($\frac{p}{q}$: nth distinct rational number) 

$\ldots \frac{1}{2}, \frac{1}{3}, \ldots, \frac{2}{2}, \frac{2}{3}, \ldots$

larger cardinals

$\mathbb{R}$: Real Number has a larger Cardinal!

Assume $\exists$ bijection

$0 \to A.a_1a_2a_3\ldots$

$1 \to B.b_1b_2b_3\ldots$

$\vdots$

can always construct a number not in the R.H.S! Contradiction.

Set of all subsets of any A

$\to$ always have larger cardinal than A

$\to$ Proof: Assume such bijection $h(x)$ exists.

Injection : easy to see

Assume Surjective.

sub set T $\left|\boxed{\{x \mid x \notin h(x)\}}\right|$

these elements are $h(t)$ for some t

$$\begin{cases} \text{if } t \in T \\ \quad \to \text{violate } T \\ \text{if } t \notin T \\ \quad \to \text{should be in } T \end{cases}$$

Contradicts. Such bijection h doesn't exist. $\to$ larger cardinals

$\mathbb{R}$. real numbers

$\to$ algebraic numbers

satisfy polynomial equations

$a_n x^{n-1} + a_{n-1} x^{n-2} + \ldots + a_0 = 0$, $a_i \in \mathbb{Q}$ rational numbers

larger cardinal $\to$ transcendental numbers

e.g. π, e (not satisfy above polynomial equations)

Proof of Existence (Cantor's Theorem)

① Algebraic numbers are countable

$\to$ Has Cardinal $\aleph_0$

define $f(m)$

m^{th} distinct algebraic numbers along the sequence

$\to$ observe for each polynomial of degree d, only a finite number of polynomials has height h

Finite: height 1 $\{1, -1\}$ height 2 $\{2, -2, x, -x\}$ height 3 $\{2x, -2x, x+1, x-1, 1-x, -x-1\}$

Count: $\underbrace{p_1(x)}$ $\underbrace{p_2(x)}$ $\underbrace{p_3(x)}$ $\underbrace{p_4(x)}$ $\ldots \longrightarrow$

Count: $\beta_1, \beta_2, \ldots$ Roots Roots Roots $\longrightarrow$

Finite Roots of $p_1(x) = 0$

By definition, each root is an algebraic number

② Real numbers are uncountable

$\Rightarrow$ Transcendental numbers exist

$\Rightarrow$ They have larger cardinal than $\aleph_0$

since Cardinal of $\mathbb{R} - \aleph_0 > \aleph_0$

Chapter 10

Topological Space

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Ch... Topology

Topological spaces

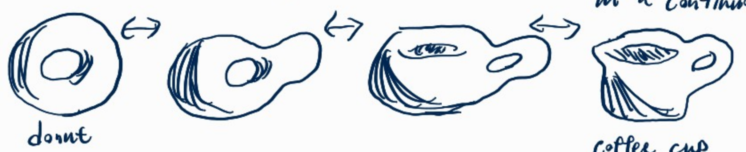
Two topological spaces are topologically equivalent if

$\exists f: A \to B$ s.t. (i) f is a bijection

(ii) f is continuous

(iii) f^{-1} is continuous

} pass to one another in a continuous way, and also came back in a continuous way.



Example of unidirectional continuous

Continuous (points originally close to each other remain so)

NOT continuous (points around dividing line → further apart)

Mobius Trip, Klein Bottle

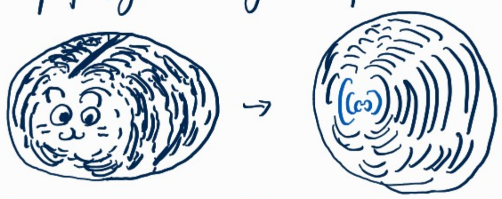
Non-orientable ("one-sideness") ← Topological Property

Gluing Directions

↖ Cannot embed in 3D without crossing

Hairy Ball Theorem

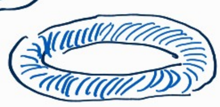
No perfectly smooth system of hairs exists



Earth Wind System

→ Cyclone must exist

But if a planet is a Torus, smooth wind system exists!



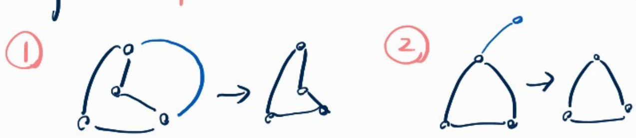
Chapter 11

Network

Euler's Formula $V-E+F=1$

On sphere, $V-E+F=2$ (Considering the region outside the graph)

Proof: Collapse



$E-1, F-1$ $\quad$ $V-1, E-1$

Both Collapses doesn't alter $V-E+F$

⇒ Sequences of Collapses lead to a single vertex

⇒ $V=1$ ✱ ($V=F=1$ on sphere)

Non-Planar Graph

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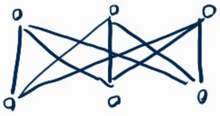


Any non-planar network must contain within it one of these two

Proof: $V=6$, $E=9$

Assume $F=4$ (+1 for outside face)

Notice all closed loop must have either 4 or 6 edges.

5 faces of ≥ 4 edges $\Rightarrow$ 20 double-counted edges $\Rightarrow \geq 10$ E Contradict.

5 vs 4 coloring Theorem

Proof of 5-color Theorem (A.B. Kempe, 1879)

For any given map,

reduce its faces such that if it is 5-colorable, M'

then the original map is 5-colorable, M

Reduce faces until the final map contains ≤ 5 faces

$\rightarrow$ Original map is 5-colorable

① Old map: keep same color

M

Reduce vertices with ≥ 3 faces

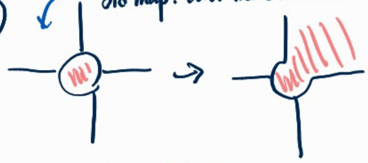
② Old map: choose color $\neq$ other 3 faces

M

Remove 3-sided faces

③ old map: color the 5th color

M



① $\rightarrow$ Each V connects 3 edges

Each E lies on 2 Faces

$3V = 2E = aF$, a: avg # sides of Faces

$\Rightarrow a = 6 - \frac{12}{F} < 6$

$\Rightarrow$ Every face has ≥ 5 sides

$\Rightarrow$ At least one face has 5 side

Color Q T the same

M

Original map is 5-colorable

5-Colorable for a face with exactly 5 face $\uparrow$ with face reduction

4-Color Theorem

$\rightarrow$ 1972, Appel-Haken Proof

Define Minimal Criminal: the map having least regions that is NOT 4-colorable

Define Unavoidable Configurations: every map must contain this configuration

Define Reducible Configuration: Configuration not appear in minimal criminal

Proof: Find an Unavoidable set of reducible configuration

$\Rightarrow \geq 1800$ configurations! Computational intensive

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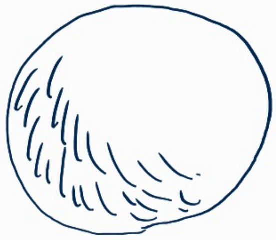
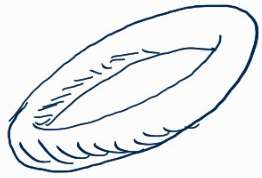
Chapter 12

Topological Invariants





Difficult to prove topologically inequivalent spaces

Hole is not in the torus, it's in the surrounding space

⇒ One way: find topological property differences

! every closed curve on sphere divide it in two

but ∃ closed curve on torus that don't divide

Surface on a Topological Space

(i) Triangulable 

(ii) Connected (in one piece)

(iii) Without edges

✓ Sphere, Torus, Klein Bottle, Projective Plane

✗ Mobius Trip

Euler Characteristics of a Surface

V-E+F invariant of any map on the surface

→ invariant for topologically equal space

→ Topological Invariant

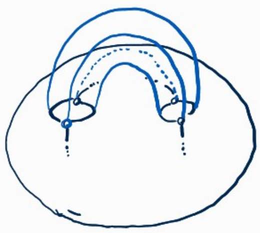
2 topological invariants

S	$\chi(S)$	Orientable?
sphere	2	yes
torus	0	yes
double torus	-2	yes
projective plane	1	no
Klein Bottle	0	no

Standard Surface

Std Orientable Surface

Genus n: Glue n handles



Sphere: $\chi(S) = 2$

Torus: V: equal

E: +2

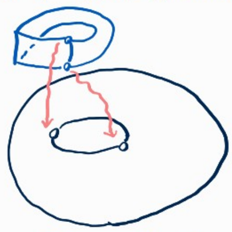
F: -2+2 → equal

⇒ $\chi(S) = 0$

$= 2 - 2n$

Std Non-orientable Surface

Genus n: Glue n Mobius strip



Sphere: $\chi(S) = 2$

V: equal

E: +1

F: -1+1 → equal

→ $\chi(S) = 1$

$= 2 - n$

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Euler Characteristics + Orientability

show that all standard surfaces are topologically distinct.

Any Surface is topologically equivalent to a Standard Surface.

Proof:

On any surface, apply a "surgery": draw a curve on it that don't divide the surface into two. Remove a narrow strip around the curve.

If the strip is a cylinder, put back two discs to fill the hole, keeping the arrow on the disc to remind the original direction, $+2\ \chi(S)$

If the strip is a mobius strip, put back only one disc, $+1\ \chi(S)$

Assertion: Any surface has $\chi(S) \leq 2$.

maximal dual tree; Original Network + Edges not meet M

$$\chi(S) = \underbrace{\chi(M)}_{1} + \underbrace{\chi(C)}_{\leq 1} \leq 2$$

Then the sequence of surgeries must stop. But it only stops when we no longer find the closed curve.

Assertion: A surface that is disconnected by any closed curve on it is topologically equivalent to a sphere.

$\chi(S) = \chi(M) + \chi(C)$

if $\chi(S) \neq 2$, $\chi(C)$ is not tree

→ closed curve C

→ divides S

→ M must cut through C

→ contradiction

M, C disjoint

So $\chi(S) = 2$, $\chi(C)$ a tree

M, C a flatten disc, meet along edges

⇒ a sphere

So Surgery Stop = We obtain a sphere.

→ A continuous transformation to a sphere.

→ Now sewing back: either sewing back a handle or a Mobius Strip. Also a continuous transformation to a standard surface.

∴ Any surface is topologically equivalent to a standard surface.

Color-Mapping of Any Surface with $\chi(S) = n$:

$$\frac{1}{2}\left(7 + \sqrt{49 - 24n}\right)$$

colors suffices.

Not exact for Klein Bottle

→ need only 6

formula gives 7

Chapter 13 Algebraic Topology

Associate a group with a space, s.t.

Topologically equivalent spaces have Isomorphic groups > invariants

★ Fundamental group of space S: (Poincaré)

$\pi(S)$

= (Set of Homotopy classes of Loops, Continuously deformation

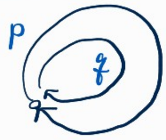
Operation * of Composition)

path composition

Author: kiking

At time t, point at $p(t)$. From $A \to B \to C \to p*q$
where p defined at interval $t_0 \le x \le t_1$, q $t_2 \le x \le t_3$,
and $p*q \to t_0 \le x \le t_1 + t_3 - t_2$

Group Rule : Multiplicative Inverse I



$p*q \ne$ trivial loop $\to$ time pass (I)
(but still a loop $\to$ closed)

But $[p]*[q] = [p*q] \to$ is homotophic to the trivial loop (Run faster)

so $[p]^{-1}$ is well-defined

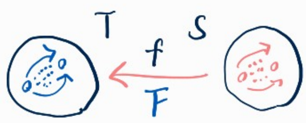
If two spaces S and T
are topologically equivalent
$\Rightarrow$ the fundamental groups $\pi(S)$ and $\pi(T)$
are isomophic groups

① S, T topologically equiv. $\to$ $f: S \to T$ continuous, f^{-1} cont.

② f defines a function F s.t.

$[f(p)] = F([p])$

Homophy Class of a loop on T $\to$ Homophy Class of ori. loops on S, get transformed by F



defined by f

③ $F(\overset{loop}{[p]} * \overset{loop}{[q]}) = F(\underset{loop}{[p]}) * F(\underset{loop}{[q]})$ $\leftarrow$ Bijection exists
$\to [p], [q]$ are Isomorphic

Order of the group $\pi(S)$: (# elements in a finite group)

$\to$ When S is a disc, $\mathbb{R}$, $\mathbb{R}^2$, ball....

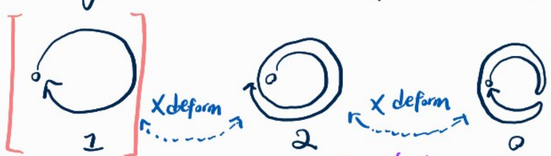
only have trivial element I. (order = 1)

All loops can be deformed to a trivial loop (point)

$\to$ When S is a Circle, order = order of $\mathbb{Z}$

Winding Number of the loop (directional)

Each cannot be deformed to each order



Consider fitting the loop to a spring

Author: kiking

Each spring distinguished by the Same Endpoint

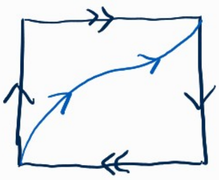
A loop with winding number n + A loop with winding number m → A loop with winding number $n+m$

(bijection exists) ←

isomaphic to integers group $\mathbb{Z}$! under Addition

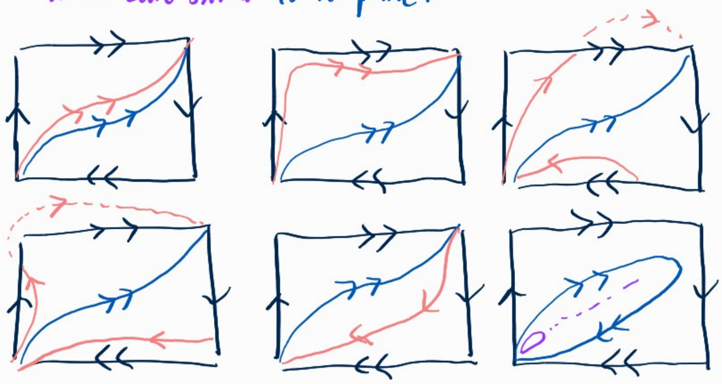
→ When S is a Projective Plane, order $=2$

	I	r
I	I	r
r	r	I



r: Homotopy Class of this path (loop)

Alone cannot shrink to a point, but going around it twice can shrink to a point.



Going around once get twisted, going twice back to normal

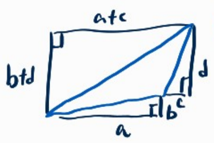
Chapter 14 Hyperspace

Distance: (i) between any two points is positive

(ii) between any two points, same distance in either direction

(iii) distance from $A \to B$ is $\leq (A \to C) + (C \to B)$

$\Rightarrow$ important algebraic inequality.



Need to prove: $\sqrt{a^2+b^2} + \sqrt{c^2+d^2} \geq \sqrt{(a+c)^2+(b+d)^2}$

$(a^2+b^2)(c^2+d^2) - (ac+bd)^2 \geq 0$

$(ad-bc)^2 \geq 0$ true for square number

2-space: Polygon

3-space: Polyhedra:

5 Regular
- Tetrahedron
- Cube
- Octahedron
- Dodecahedron
- Icosahedron

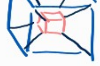
4-space: Polytope.

solid: the "faces", which are 3-d regular polyhedron (in regular polyhedra, the faces are 2-d regular polygons)

6 Regular
- Simplex: 5 tetrahedron ← solids count
- Hypercube: 8 cube Tesseract
- 16-cell: 16 tetrahedron

Author: kiking

16-cell: 16 tetrahedron
24-cell: 24 octahedron
120-cell: 120 dodecahedron
600-cell: 600 tetrahedron

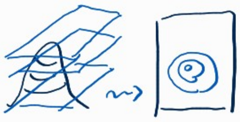


5-space, 6-space, ...

all has 3 regular polytopes only

Projections

2-space creatures understand 3-space:
a series of cross-sections



3-space creature understand 4-space:
cross-sections as 3-d objects, stacking to see full

E.g. Hypercube project to 3-space: always a Cube

Stacking Sections:

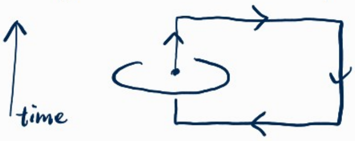
Appear, Growing, Shrinking, Disappear

Hypersphere:

Hypercube: Appear, Stay as a cube for a while, Disappear

More thought Exercises:

untying a knot in 4-space



forward in time
sideway in space
backward in time to before the circle
forward in time to close loop

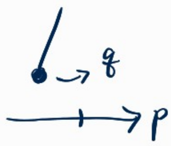
linking of two circles in 3-space

⇒ In 4-d, link a sphere with a circle

⇒ or knot a sphere (same way as knot a circle in 3d)

Swinging Pendulum:

p: position, q: velocity



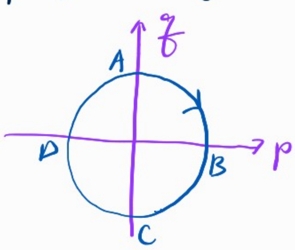


diagram: phase diagram

(p, q)-plane: phase space

2-d since the state is determined by 2 numbers (position coordinate, velocity)

Any dynamical system has phase space, one dim for each position var.
one dim for each velocity var.

⇒ Sun, Earth, Moon + Gravitational Attraction

⇒ 18-dim for the phase space

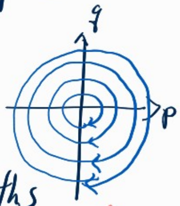
State of the system at any time → a point in this 18-dim phase space

Author: kiking

As System develop:

a point describe a path

diff initial points → a family of paths



Fluid : flow along the paths

Newton's Law of Conservation of Energy

this imaginary fluid behaves exactly like a genuine fluid.

⇒ General theory of Fluid Dynamics

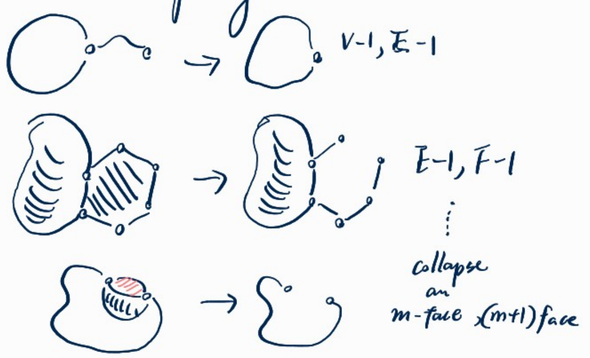
→ applied to the general theory of Dynamical Systems

N-dim of Euler's Formula: (Euler-Poincaré formula)

$$F_0 - F_1 + F_2 - \cdots \pm F_n = 1$$

Proof: similar to 2-d,

by observing that the eq. doesn't change when "collapsing"



Algebraic Topology generalized to higher dim

Path from a line segment ↕ n-dim Hypercube

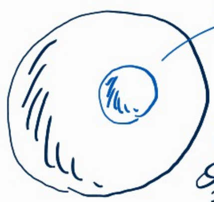
Joining end to end ↕ joining face to face

A group: hypercube whose boundary is squashed up to a single point

Elements: Homotopy classes of n-dim "path"s

n-th homotopy group $\pi_n(S)$ of the space S

Fundamental group: $\pi_1(S)$, 1st of a whole series of algebraic invariant



→ remove a spherical hole from a solid ball

$\pi_1(S)$ cannot detect it: any path can still shrink to a point

$\pi_2(S)$ can detect it: if put a square surrounding it

Author: kiking

Higher Homotopy group can detect the differences $\Leftarrow$ ... whose boundary is squashed to a point (paper bag around the hole) then this square cannot be shrunk to a point in S

Poincaré conjecture:

if S has the same sequence of πs as an n-sphere, then S is an n-sphere up to topological equivalence

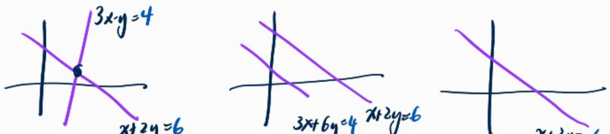
- When $n=2 \to$ proven true in Ch.12 Topo. Invariant Assertion 2
- When $n \geq 5 \to$ true
- When $n=4 \to$ true, proven in 1982 Michael Freeman, Field's Medal 1986
- When $n=3 \to$ true, proven in 2003 Grigori Perelman offered but declined Field's Medal 2006

Chapter 15 Linear Algebra

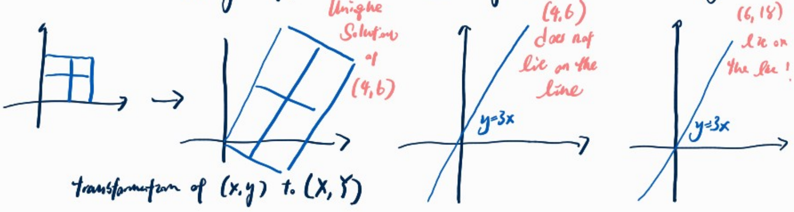
A proper understanding of linear algebra requires a synthesis of 3 points of views:

Textbooks tend to focus on only one aspect

(i) the underlying geometrical motivation



(1) Single Solution	(2) No Solution	(3) Inf. many solutions.
		$x+2y=6$, $3x+6y=18$
$\Downarrow$	$\Downarrow$	$\Downarrow$
$x+2y=X$, $3x-y=Y$.	$x+2y=X$, $3x+6y=Y$	$x+2y=6$, $3x+6y=18$



transformation of (x,y) to (X,Y)

Solutions depend on geometric properties of linear transformation

$T(x,y) = \begin{pmatrix} x+2y, \\ 3x-y \end{pmatrix}$

$S(x,y) = \begin{pmatrix} x+2y, \\ 3x+6y \end{pmatrix}$

Range of T = plane
Solution Space = point

Range of S = line
Solution Space = line (may not exist)

3 Eq. in 3 unknown:

	Range	Solution Space	
$\mathbb{R}^3$	Solid	point	$\mathbb{R}^0$
$\mathbb{R}^2$	Plane	Line	$\mathbb{R}^1$
$\mathbb{R}^1$	Line	Plane	$\mathbb{R}^2$
$\mathbb{R}^0$	Point	Solid.	$\mathbb{R}^3$

Trivial Transformation
$Z(x,y) = \begin{pmatrix} 0x, \\ 0y \end{pmatrix}$

Range of Z = point
Solution Space = plane

(2 Eq. in 2 unknown)

$\Rightarrow$ Dimension of the Range + Dimension of the Solution Space $= n$ (on $\mathbb{R}^n$)

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(ii) the matrix - theoretic technique

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

Several transformation:

if $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix}$

i.e. $UT(x,y) = U(X,Y) = (\bar{X},\bar{Y})$

Then $\begin{pmatrix} Aa+Bc & Ab+Bd \\ Ca+Dc & Cb+Dd \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix}$

↳ Suggests us to define the product of Matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} a & b \\ c & b \end{pmatrix} = \begin{pmatrix} Aa+Bc & Ab+Bd \\ Ca+Dc & Cb+Dd \end{pmatrix}$$

With this we could compute a series of linear transformation

Eg. Rotate by θ then by ϕ

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -\cos\phi\sin\theta - \sin\theta\cos\phi \\ \sin\phi\cos\theta + \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix}$$

Trigonometry "add'l formula"

$\cos(\phi+\theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$

$\sin(\phi+\theta) = \sin\phi\cos\theta + \cos\phi\sin\theta$

(iii) Abstract formulation in Algebra

(attempts to avoid using coordinates in theory)

Characterization of Linear Transformation:

functions $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\forall p,q,r,s,a$,

looks like Isomorphism in Group Theory under addition → $T((p,q)+(r,s)) = T(p,q)+T(r,s)$ Addition

$T(a(p,q)) = aT(p,q)$ Multiplication

Vector Space over $\mathbb{R}$: 7 Rules

a set V with 2 operations: Addition, Scalar Multiplication

$\forall\, u,v \in V,\ a \in \mathbb{R} \xrightarrow{\text{denote}} u+v \in V \qquad av \in V$

(1) V: a Commutative group under Addition, with identity element 0

(2) $a0 = 0,\ \forall a \in \mathbb{R}$

(3) $0v = 0,\ \forall v \in V$

(4) $1v = v,\ \forall v \in V$

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(5) $(\alpha+\beta)v = \alpha v + \beta v, \ \forall \alpha, \beta \in \mathbb{R}, v \in V$
(6) $\alpha(v+w) = \alpha v + \alpha w, \ \alpha \in \mathbb{R}, \forall v, w \in V$
(7) $\alpha\beta v = \alpha(\beta v), \ \forall \alpha, \beta \in \mathbb{R}, v \in V$

Standard Vector Spaces: $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots$

Others: Polynomial Ring $\mathbb{R}[x], \mathbb{R}[x,y], \mathbb{R}[x,y,z], \ldots$

$\Rightarrow$ arise in
- the solution of differential equations,
- certain parts of group theory,
- modern formulation of the calculus

Linear Transformation **Definition**:

A function $T: V \to W$ where V, W arbitrary Vector Spaces,
s.t. $T(u+v) = T(u) + T(v)$
$T(au) = aT(u)$
$\forall u, v \in V, a \in \mathbb{R}$

✓ One can prove all desired theorems about linear transformation

✓ No particular coordinates are involved $\to$ proof are very clean and direct

✗ To perform calculations in particular cases $\to$ use matrix notations

Chapter 16
Real Analysis

3 Cornerstones of modern maths:
Algebra, Topology, Analysis

(Mathematical Logic: hold the bricks together)

Analysis: Study of infinite processes
- infinite series
- limits
- differentiation
- integration

Example

Infinite Addition:
$S = 1-1+1-1+1-1+\cdots$

Then
$S = (1-1)+(1-1)+\cdots = 0$
$S = 1-(1-1)-(1-1)-\cdots = 1$

without a meaning of infinite addition,

Author: kiking

$1-S = 1-(1-1+1-1+\cdots)$
$= 1-1+1-1+\cdots$
$= S \Rightarrow S = \frac{1}{2}$

all are correct!

Consider the limit of the subsegment sums,

Series $= a_1 + a_2 + a_3 + \cdots$

Approx. Sum $=$ $b_1 = a_1$
$b_2 = a_1 + a_2$
$b_3 = a_1 + a_2 + a_3$
$\vdots$

$\Rightarrow$ A sequence b_n tends to the limit L if the difference
$$b_n - L$$
can be as small as we please by taking n sufficiently large.

$\rightarrow$ S does not have a limit,
but e.g. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$, $b_n = 2-\frac{1}{2^{n-1}}$ has a limit 2,

* A sequence tend to a limit is convergent..
* An infinite series $a_1+a_2+a_3+\cdots$ is the limit L of the sequence b_n of approx. sums, provided that limit exists.

However, from sum to law of algebra $\rightarrow$ not always possible.

Example, convergent series

$$K = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}\cdots\cdots$$

has limit $\log_e 2$.

But $2K = 2-\frac{2}{2}+\frac{2}{3}-\frac{2}{4}+\frac{2}{5}-\frac{2}{6}\cdots$
$= (2-1)-\frac{1}{2}+(\frac{2}{3}-\frac{1}{3})-\frac{1}{4}+\cdots$
$= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$
$= K$!

Problem: We have to guess the limit L to prove convergent.

Consider the error terms: if $\exists k>0$ s.t.
$$-k \leq a_{n+1} + \cdots + a_{n+m} \leq k$$
for every m, the the error terms is small
$\Rightarrow$ can expect the series to converge.

$\Rightarrow$ no guessing of L, no infinite sum.

The completeness axiom (of real numbers):

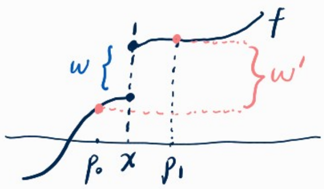
[Rational numbers have gaps (irrational numbers in between, but not real numbers)]

Author: kiking

Real numbers = limit of a given sequence of which the error terms become arbitrary small.
(infinite series of rational numbers)

Continuity :

"no jumps"



f is cont. at a point x

if by choosing p_0 and p_1 sufficiently close to x,
we can make $f(p_0)$ and $f(p_1)$ as close together as we please.

f is continuous if it is cont. at all point x.

e.g. $f(x) = x^2$

Why continuous at 0 :

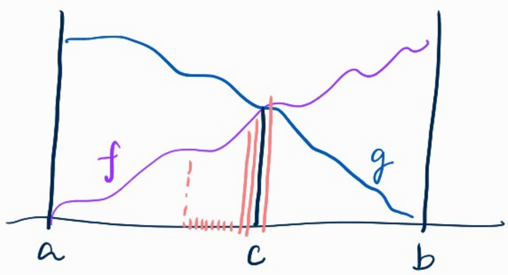
by choosing $-k < p_0 < 0$ and $0 < p_1 < k$,

we have $-2k^2 < p_0^2 - p_1^2 < 2k^2$

By choosing k small enough, we have $2k^2$ as small as we please

(no function can be dist. cont. at all irrational points
but continuous at rational points)

Theorem



if we have two continuous function f and g on the real line, s.t.

$f(a) < g(a) \quad f(b) > g(b)$

then $\exists$ some points c s.t.

$f(c) = g(c)$

Proof :

divided the interval into 10 parts.

For the 1st point where f is larger than g,
divide it up to 10 parts again.

The end point of the intervals form a sequence
$p_1, p_2, p_3, \ldots$

Complete Axiom $\Rightarrow$ the sequence converge to a real number.

Definition of continuity $\Rightarrow f(p) = g(p)$

Chapter 17. The theory of Probability

- Combinatorial Probability

Author: kiking

- Independence
- Paradoxical Dice
- Binomial Bias
- Random Walks

Chapter 18 Computers & their Uses

- Binary Notation
- A Ball-bearing Computer
- The Structure of Computers
- Writing a Program
- The Use of Computers

Chapter 19 Application of Modern Mathematics

- How to Maximize Profits (Linear Programming)
- The Eightfold Way (Atomic Theory
  → Representation of Groups)
- Catastrophe Theory (Discontinuous Theory
  e.g. Zeeman Castrophe
  Machine)

Chapter 20 Foundations

Frege's Naive Set Theory
- Russell's Paradox (e.g. Village Barber)
  $\{P \quad \text{not-}P\}$
- $C = \{x \mid x \text{ has property } P\}$
  C is the class of all sets x which have property P
  if x has property P, does not deduce $x \in C$

Author: kiking

unless x is a set

- Thoses classes which are not sets are called proper classes.

- Hilbert Programme (whether every problem can be solved)

  vs

  Gödel Theorem

  (1) if axiometic set theory is consistent,
  ∃ theorems which can neither be proved or disproved.

  (2) there is no constructive procedure which will prove axiometic set theory to be consistent

Undecidability

From Gödel Theorem (1), in ordinary arthimetic there exist statements P s.t. neither P nor not-P can be proved.

⇒ Undecidable statements

E.g.1. Whether Diophantine equation has a solution ⇒ undecidable

(Matijasevič's proof) → "a formula of prime"

E.g.2 Continuum Hypothesis

Chapter 9 → is the cardinal c of the real numbers the next cardinal after $\aleph_0$?

A: Yes & No.

→ Independent of other axioms of set theory.

ie. Add an axiom saying its true, will not make set theory inconsistent.

20-th century version of non-Euclidean Geometry

Appendix

- The Four-Color Theorem
- Polynomials & Primes
- Chaos (Dynamical systems

Author: kiking

e.g. $x_{t+1} = kx_t(1-x_t)$ (Rabbit Birth)

$0 \le k \le 4$, $0 \le x_t \le 1$

deterministic but random)

→ fixed point $f(x) = x$

→ periodical point $f^{(n)}(x) = x$

→ orbit : $x, f(x), f^{(2)}(x), \ldots, f^{(n)}(x), \ldots$

→ attractor, repellor

General → Motion (Geometry) → Arithmetics (Modular)

Chp. 4-5 → Set → Function

Chp. 6-7 → Abstract Algebra → (Symmetry) Group

Chp. 8-9 → Axiomatics → Counting (Cardinal)

Chp. 10-12 → Topology → Network (Graph) → Topological Invariants

Chp. 13-14 → Algebraic Topology → Hyperspace

Chp. 15-16 → Linear Algebra → Real Analysis

Chp. 17-19 → Probability → Computer → Modern Applications

Chp. 20 → Foundations (Naive Set Theory, Gödel Theorem)

→ Appendix

Author: kiking