

Chapter ① Preliminaries

- 1.1 Sets and Functions
- 1.2 Mathematical Induction
- 1.3 Finite and Infinite Sets

Cantor's Theorem If A is any set,
then there is no surjection of A onto the set $P(A)$
of all subsets of A .

Chapter ② The Real Numbers

2.1 The Algebraic and Order Properties of \mathbb{R}

Trichotomy Property of \mathbb{R}

If $a \in \mathbb{R}$, then exactly one of the followings hold:
 $a \in \mathbb{R}$, $a=0$, $-a \in \mathbb{P}$

Arithmetic-Geometric Mean Inequality

$$(a_1, a_2, \dots, a_n)^{\frac{1}{n}} \leq \frac{1}{n}(a_1 + a_2 + \dots + a_n)$$

Bernoulli's Inequality

If $x > -1$, then

$$(1+x)^n \geq 1+nx \quad \forall n \in \mathbb{N}$$

2.2 Absolute Value and the Real Line

Triangle Inequality

If $a, b \in \mathbb{R}$, then $|a+b| \leq |a| + |b|$

2.3 The Completeness property of \mathbb{R}

Completeness Property of \mathbb{R} (also: Supremum Property)

Every non-empty set of real numbers that has an upper bound
also has a supremum in \mathbb{R} (least upper bound)

Archimedean Property of \mathbb{R}

If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N}$ s.t. $x \leq n_x$.

Corollary

- (1) If $S := \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$ then $\inf S = 0$
- (2) If $t > 0$, $\exists n_t \in \mathbb{N}$ s.t. $0 < \frac{1}{n_t} < t$
- (3) If $y > 0$, $\exists n_y \in \mathbb{N}$ s.t. $n_y - 1 \leq y \leq n_y$

The Density Theorem

If $x, y \in \mathbb{R}$ and $x < y$, \exists rational number $r \in \mathbb{Q}$
s.t. $x < r < y$.

2.4 Intervals

Chapter (3) Sequence and Series

3.1 Sequence and their limits

Def A sequence $X = (x_n)$ in \mathbb{R} converges to $x \in \mathbb{R}$,
 or x is a limit of (x_n) , if $\forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N}$
 s.t. $\forall n \geq K(\varepsilon)$, the term x_n satisfy $|x_n - x| < \varepsilon$.

3.2 Limit Theorems

Thm A convergent sequence of real numbers is bounded.

3.3 Monotone Sequences

Monotone Convergence Theorem

A monotone sequence of real numbers is convergent iff it is bounded.
 e.g. $\epsilon_n := (1 + \frac{1}{n})^n \forall n \in \mathbb{N}$ is bounded and increasing;
 it is convergent and converge to the Euler number $e \approx 2.718\dots$

3.4 Subsequences and the Bolzano-Weierstrass Theorem

Monotone Subsequence Theorem

If $X = (x_n)$ is a sequence of real numbers,
 then \exists a subsequence of X that is monotone.

Bolzano-Weierstrass Theorem

A bounded sequence of real numbers has a convergent subsequence.

3.5 The Cauchy Criterion

Def A sequence $X = (x_n)$ of real numbers is a Cauchy sequence
 if $\forall \varepsilon > 0, \exists H(\varepsilon) \in \mathbb{N}$ s.t. $\forall n, m \geq H(\varepsilon)$,
 x_n, x_m satisfy $|x_n - x_m| < \varepsilon$

Cauchy Convergence Criterion

A sequence of real number is convergent iff it is a Cauchy sequence.

3.6 Property Divergent Sequences

Def $\lim(x_n) = -\infty \quad \forall \alpha \in \mathbb{R}$
 $\exists K(\alpha) \in \mathbb{N}$ s.t. if $n \geq K(\alpha)$, $x_n < \alpha$.
 $\geq K(\beta) \quad < \beta$.

3.7 Introduction to Infinite Series

Def If $X = (x_n)$ is a sequence in \mathbb{R} ,
 $\sum x_n$ is the (infinite) series generated by X ,
 which is a sequence $S = (s_k)$ where
 $s_1 := x_1$ $s_k := x_1 + x_2 + \dots + x_k$ partial sum of the series

Geometric Series

$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots + r^n + \dots$ generated by the sequence $X = (r^n)_{n=0}^{\infty}$
 diverges if $|r| \geq 1$

Harmonic Series

$\sum_{n=0}^{\infty} \frac{1}{n}$ diverges

n-th term test

If the series $\sum x_n$ converges, then the limit of the sequence (x_n) is 0.

Cauchy Criterion for Series

$\sum x_n$ converges iff $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\sum_{k=n+1}^{\infty} |x_k| < \varepsilon$

$\lim_{n \rightarrow \infty} x_n = c$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |x_n - c| < \epsilon$,

$$|\sum_{n=1}^m x_n - \sum_{n=1}^m c| = |\sum_{n=1}^m (x_n - c)| < \epsilon$$

Chapter (4): Limits

4.1 Limits of Functions

Def Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point of A if $\forall \delta > 0$, $\exists x \in A$, $x \neq c$ s.t. $|x - c| < \delta$.

Def Let $A \subseteq \mathbb{R}$, c be a cluster point of A .

For a function $f: A \rightarrow \mathbb{R}$, a real number L is a limit of f at c ,

If $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ s.t. if $x \in A$ and $0 < |x - c| < \delta(\epsilon)$, then $|f(x) - L| < \epsilon$.

Also says that f converges to L at c .

4.2 Limit Theorems

4.3 Some Extensions of the Limit Concept

One-sided Limits = $\begin{cases} 0 < x - c < \delta(\epsilon) \\ 0 < c - x < \delta(\epsilon) \end{cases}$

Chapter (5): Continuous Functions

5.1 Continuous Functions

f is continuous at c if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Sequential Criterion for Continuity

A function $f: A \rightarrow \mathbb{R}$ is continuous at a point $c \in A$

iff \forall sequence (x_n) in A that converges to c , Discontinuity: if $\exists (x_n)$ s.t. $\lim_n (x_n) = c$ but $\lim_n f(x_n) \neq f(c)$.

Dirichlet's function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases} \quad \text{discontinuous}$$

Let $\lim_n (x_n) = c$ rational, Density Thm asserts but (x_n) a sequence of irrational numbers, this must exist.

Then $f(x_n) = 0 \neq f(c) = 1$

Thomae's function

$$\begin{aligned} h: \{x \in \mathbb{R} : x > 0\} \rightarrow \mathbb{R} \\ = \begin{cases} 0 & \text{for irrational number } x > 0 \\ \frac{1}{n} & \text{for rational number } \frac{m}{n}, m, n \text{ no common factor except 1} \\ 1 & \text{for } x = 0 \end{cases} \end{aligned}$$

h is continuous at every irrational number in A
discontinuous at every rational number in A

Let (x_n) a seq. of rational numbers converge to an irrational number b . $f((x_n)) = \frac{1}{n} \rightarrow 0$ (Proof by Arithmetical Property)
Let (x_n) a seq. of irrational numbers converge to a rational number c . Then $f(x_n) = 0 \neq f(c)$

5.2 Combinations of Continuous Functions

5.3 Continuous Functions on Intervals

Bolzano's Intermediate Value Theorem

Let I be an interval and let $f: I \rightarrow \mathbb{R}$ be continuous on I .

If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$,

then $\exists c \in I$ between a and b s.t. $f(c) = k$.

5.4 Uniform Continuity

Uniform Continuity Theorem

Let I be a closed bounded interval and let $f: I \rightarrow \mathbb{R}$ be continuous on I .
Then f is uniformly continuous on I .

Def Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. c is a cluster point of A if $\forall \delta > 0$, $\exists x \in A$, $x \neq c$ s.t. $|x - c| < \delta$.

Def for $A \subseteq \mathbb{R}$ and $f: A \rightarrow \mathbb{R}$. f is a Lipschitz function if

$$\frac{|f(x) - f(u)|}{|x-u|} \leq K \text{ for some constant } K > 0$$

all slopes of f are bounded

Thm If $f: A \rightarrow \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A .
Note: Jx is uniformly continuous, but not a Lipschitz function on $[0, 2]$.

Approximation

$\hookrightarrow \exists$ step function, piece-wise linear functions, polynomial functions s.t.

$$\forall \varepsilon > 0, |f(x) - S_\varepsilon(x)| < \varepsilon \quad \forall x \in I.$$

Weierstrass Approximation Theorem

5.5

Continuity and Cauchy

5.6

Monotone and Inverse Functions

Thm Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \rightarrow \mathbb{R}$ be monotone on I .

Then the set of points $D \subseteq I$ at which f is discontinuous is a countable set.

Continuous Inverse Theorem

Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \rightarrow \mathbb{R}$ be strictly monotone and continuous on I .

Then the function g inverse to f is strictly monotone and continuous on $J := f(I)$

Chapter 6 = Differentiation6.1 Derivative

Def Let $I \subseteq \mathbb{R}$ be an interval, let $f: I \rightarrow \mathbb{R}$, $c \in I$.

A real number L is the derivative of f at c

If $\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ s.t. if $x \in I$ satisfies $0 < |x-c| < \delta(\varepsilon)$,

then
$$\left| \frac{f(x) - f(c)}{x-c} - L \right| < \varepsilon \quad f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x-c}$$

f is differentiable at c and $f'(c) = L$.

given by the limit

Thm If f is differentiable at c , f is continuous at c .

Converse not true = e.g. $f(x) = |x|$, f is continuous at 0 but not differentiable at 0.

since $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1, \lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$

Weierstrass Function

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(3^n x)$$

continuous at every point, but derivative does not exist anywhere.

Chain Rule

Let I, J be intervals in \mathbb{R} , $g: I \rightarrow \mathbb{R}$ and $f: J \rightarrow \mathbb{R}$ s.t. $f(J) \subseteq I$. let $c \in J$.
If f is differentiable at c and g is differentiable at $f(c)$,
then $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

6.2 The Mean Value Theorem

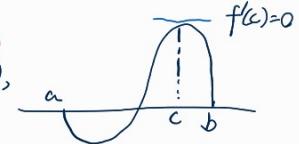
Interior Extremum Theorem

Let c be an interior point of the interval I
at which $f: I \rightarrow \mathbb{R}$ has a relative extremum.

If the derivatives of f at c exists, then $f'(c)=0$.

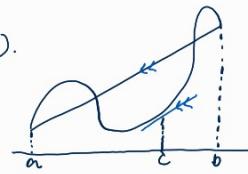
e.g. $f(x) = |x|$, $x=0$ minimum but $f'(0)$ not exists.Rolle's Theorem

Suppose f continuous on $I := [a, b]$, and $f'(x)$ exists $\forall x \in (a, b)$, and that $f(a) = f(b) = 0$. Then $\exists c \in (a, b)$ s.t. $f'(c) = 0$.

Mean Value Theorem

Suppose f continuous on $I := [a, b]$, and $f'(x)$ exists $\forall x \in (a, b)$. Then $\exists c \in (a, b)$ s.t.

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Darboux's Theorem

If f is differentiable on $I = [a, b]$, and if k between $f'(a)$ and $f'(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = k$. f' has the Intermediate Value Property.

6.3 L'Hospital's RulesCauchy Mean Value Theorem

Let f, g continuous on $[a, b]$ and differentiable on (a, b) . Assume $g'(x) \neq 0 \quad \forall x \in (a, b)$. Then $\exists c \in (a, b)$ s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

L'Hospital's Rule, I & II

Let $-\infty \leq a < b < \infty$, f, g differentiable on (a, b)

s.t. $g'(x) \neq 0 \quad \forall x \in (a, b)$. Suppose that

$$\textcircled{I} \quad \lim_{x \rightarrow a^+} f(x) = 0 = \lim_{x \rightarrow a^+} g(x) \quad \left| \quad \lim_{x \rightarrow a^+} g(x) = \pm\infty \textcircled{II} \right.$$

(a) If $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$

(b) If $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, then $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$

6.4 Taylor's TheoremTaylor's Theorem

Let $n \in \mathbb{N}$, let $I := [a, b]$, $f: I \rightarrow \mathbb{R}$ s.t. $f, f', f'', \dots, f^{(n)}$ are continuous on I and that $f^{(n+1)}$ exists on (a, b) .

If $x_0 \in I$, then $\forall x \in I$, \exists a point c between x and x_0 s.t.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \\ &= P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \end{aligned}$$

*Lagrange form / derivative form
of the remainder*

Convex Functions

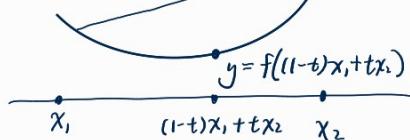
Let $I \subset \mathbb{R}$ be an interval. $f: I \rightarrow \mathbb{R}$ is convex on I

If $\forall t$ satisfying $0 \leq t \leq 1$ and any points $x_1, x_2 \in I$, we have

$$f((1-t)x_1 + x_2) \leq (1-t)f(x_1) + tf(x_2)$$

let $f: I \rightarrow \mathbb{R}$ have a second derivative.
Then f convex iff $f''(x) \geq 0 \quad \forall x \in I$.

$$y = (1-t)f(x_1) + tf(x_2)$$



Newton's Method

Let $I := [a, b]$ and $f: I \rightarrow \mathbb{R}$ be twice differentiable on I . Suppose that $f(a)f(b) < 0$ and there are constants m, M s.t.

$|f'(x)| \geq m > 0$ and $|f''(x)| \leq M \quad \forall x \in I$

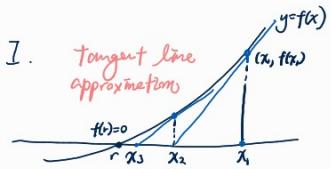
Let $K = \frac{M}{2m}$. Then \exists a subinterval I^* containing a zero r of f s.t.

$\forall x_n \in I^*$ the sequence (x_n) defined by

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \quad \forall n \in \mathbb{N}$$

belongs to I^* and (x_n) converges to r .

Moreover, $|x_{n+1} - r| \leq K|x_n - r|^2 \quad \forall n \in \mathbb{N}$.



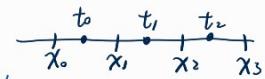
Approximate x_n s.t.
 $f(x_n) \approx 0$
 $(x_n) \rightarrow r$

Chapter 7 : Riemann Integral

7.1 Riemann Integral

Def $\dot{P} = \{(x_{i-1}, x_i], t_i\}_{i=1}^n$ tagged partition

$\|\dot{P}\| = \max\{x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}\}$ norm of partition



Riemann Sum

$$S(f; \dot{P}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

Def $f: [a, b] \rightarrow \mathbb{R}$ is Riemann Integrable on $[a, b]$

If $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ s.t.

If \dot{P} is any tagged partition of $[a, b]$ with $\|\dot{P}\| < \delta(\varepsilon)$, then

$$|S(f; \dot{P}) - L| < \varepsilon$$

The set of all Riemann Integrable functions: $R[a, b]$.

L is uniquely determined and $L = \int_a^b f$ or $\int_a^b f(x) dx$

Thm if $g \in R[a, b]$ and $f(x) = g(x)$ except for a finite number of points in $[a, b]$,

then $f \in R[a, b]$ and $\int_a^b f = \int_a^b g$.

Boundedness Theorem

If $f \in R[a, b]$, then f is bounded on $[a, b]$.

Thomae's function

$$h: \{x \in \mathbb{R} : x > 0\} \rightarrow \mathbb{R}$$

$$= \begin{cases} 0 & \text{for irrational number } x > 0 \\ \frac{1}{n} & \text{for rational number } \frac{m}{n}, m, n \text{ no common factor except 1} \\ 1 & \text{for } x = 0 \end{cases}$$

h is continuous at every irrational number in A

discontinuous at every rational number in A , not monotone

$\#$ is Riemann Integrable with integral = 0

7.2 Riemann Integrable Functions

Cauchy Criterion,

$f: [a, b] \rightarrow \mathbb{R}$ belongs to $\mathcal{R}[a, b]$

iff $\forall \varepsilon > 0, \exists \eta_\varepsilon > 0$ s.t. If $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any tagged partitions of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ and $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon$, then

$$|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})| < \varepsilon$$

Squeeze Theorem

$f: [a, b] \rightarrow \mathbb{R}$ belongs to $\mathcal{R}[a, b]$ Find partitions to prove
Riemann Integrable is hard!

if $\forall \varepsilon > 0, \exists$ functions α_ε and $\omega_\varepsilon \in \mathcal{R}[a, b]$ with Squeeze is easier

$$\alpha_\varepsilon(x) \leq f(x) \leq \omega_\varepsilon(x) \quad \forall x \in [a, b]$$

and s.t. $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$

Thm If f is a step function, then $f \in \mathcal{R}[a, b]$

$\left\{ \begin{array}{l} \text{a continuous function} \\ \text{a monotone function} \end{array} \right.$

7.3 The Fundamental Theorem

The Fundamental Theorem of Calculus (First Form)

Suppose \exists a finite set $E \subset [a, b]$ and functions $f, F: [a, b] \rightarrow \mathbb{R}$ s.t.

(a) F is continuous on $[a, b]$ ✓ if F is differentiable at every point

(b) $F'(x) = f(x) \quad \forall x \in [a, b] \setminus E$ ✓ even if f is not defined at some point

(c) $f \in \mathcal{R}[a, b]$! even if F is differentiable at every point, may fail

Then

$$\int_a^b f = F(b) - F(a)$$

e.g. $K(x) = \begin{cases} x^2 \cos(\frac{1}{x^2}) & \text{for } x \in (0, 1] \\ 0 & \text{for } x = 0 \end{cases}$

$K'(x) = \begin{cases} 2x \cos(\frac{1}{x^2}) + (\frac{1}{x^3}) \sin(\frac{1}{x^2}) & \text{for } x \in (0, 1] \\ 0 & \text{for } x = 0 \end{cases}$

So K is continuous and differentiable on $[a, b]$ but K' is unbounded, so $K' \notin \mathcal{R}[a, b]$ and the theorem does not apply.

The Fundamental Theorem of Calculus (Second Form)

Let $f \in \mathcal{R}[a, b]$, f continuous at $c \in [a, b]$.

Then the indefinite integral

$$F(z) = \int_a^z f \quad \text{for } z \in [a, b]$$

is differentiable at c and $F'(c) = f(c)$.

if f is continuous on $[a, b]$,
then F is the anti-derivative of f .

Def $Z \subset \mathbb{R}$ is a null set if $\forall \varepsilon > 0$,

\exists a countable collection $\{(a_k, b_k)\}_{k=1}^\infty$ of open intervals s.t.

$$Z \subseteq \bigcup_{k=1}^\infty (a_k, b_k) \quad \text{and} \quad \sum_{k=1}^\infty (b_k - a_k) < \varepsilon$$

Def $\mathcal{Q}(x)$ holds almost everywhere on I ,

if \exists a null set $Z \subset I$ s.t. $\mathcal{Q}(x)$ holds $\forall x \in I \setminus Z$

" $\mathcal{Q}(x)$ for a.e. $x \in I$ "

E.g. \mathcal{Q}_r of rational numbers on $[0, 1]$ is a null set.

Lebesgue's Integrability Criterion

A bounded function $f: [a, b] \rightarrow \mathbb{R}$ is Riemann Integrable

iff f is continuous almost everywhere on $[a, b]$.

E.g. Dirichlet's function is discontinuous on every point on $[0, 1]$, so it is NOT Riemann Integrable.

E.g. Thomae's function is discontinuous on rational points i.e. null set, so it is Riemann Integrable.

Dirichlet's function
 $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

discontinuous

1.1 The Darboux Integral

Def Lower integral of f on I :

$$L(f) := \sup \{ L(f; P) : P \in \mathcal{P}(I) \} = \sum_{k=1}^m \left\{ \inf_{x \in [x_{k-1}, x_k]} \{ f(x) \} \right\} \cdot (x_k - x_{k-1})$$

Upper integral of f of I :

$$U(f) := \inf \{ U(f; P) : P \in \mathcal{P}(I) \} = \sum_{k=1}^m \left\{ \sup_{x \in [x_{k-1}, x_k]} \{ f(x) \} \right\} \cdot (x_k - x_{k-1})$$

Darboux Integral

Let $I := [a, b]$ and $f: I \rightarrow \mathbb{R}$ be bounded.

Then f is Darboux Integrable on I if $L(f) = U(f)$,
and the Darboux Integral is $L(f) = U(f)$.

Equivalence Theorem

A function f on $I = [a, b]$ is Darboux Integrable iff it is Riemann Integrable.

7.5 Approximate Integration

Equal Partitions

$$T_n(f) := \frac{1}{2} (L_n(f) + R_n(f)) \quad P_n: \text{the partition } a < a_1 h_n < a_2 h_n < \dots < a_n h_n = b$$

$$= h_n \left(\frac{1}{2} f(a) + \sum_{k=1}^{n-1} f(a + kh_n) + \frac{1}{2} f(b) \right)$$

Let f monotone on $[a, b]$. Then

$$\left| \int_a^b f - T_n(f) \right| \leq |f(b) - f(a)| \cdot \frac{(b-a)}{2h}$$

Trapezoid's Rule

Midpoint Rule \rightarrow by piecewise linear functions

Simpson's Rule by parabolic arcs

Chapter 8 Sequences of Functions

8.1 Pointwise and Uniform Convergence

Pointwise Convergence

A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to \mathbb{R}

converges to a function $f: A_0 \rightarrow \mathbb{R}$ iff $\forall \varepsilon > 0$ and each $x \in A_0$,

$\exists K(\varepsilon, x) \in \mathbb{N}$ s.t. $\forall N > K(\varepsilon, x)$, then

$$|f_n(x) - f(x)| < \varepsilon.$$

f : limit on A_0 of the sequence (f_n) $f = \lim (f_n)$ on A_0 .

(f_n) : convergent on A_0 / converges pointwise on A_0 $f_n \rightarrow f$ on A_0

E.g. Let $f_n := \frac{x}{n}$. Then $\lim (f_n) = x \lim \frac{1}{n} = x \cdot 0 = 0$.



Uniform Convergence

A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to \mathbb{R}
 converges uniformly on $A_0 \subseteq A$ to a function $f: A_0 \rightarrow \mathbb{R}$
 if $\forall \varepsilon > 0, \exists K(\varepsilon) \in \mathbb{N}$ s.t. if $n \geq K(\varepsilon)$, then

$$|f_n(x) - f(x)| < \varepsilon.$$

(f_n) : uniformly convergent on A_0 $f_n \xrightarrow{\text{def}} f$ on A_0 .

E.g. $f_n := \frac{x}{n}, f = 0$. Let $x_k = n_k = 1$, then $|f_{n_k}(x_k) - f(x_k)| = 1 > \varepsilon$.
 → not uniformly convergent.

Def If $A \subseteq \mathbb{R}$ and $\varphi: A \rightarrow \mathbb{R}$ is a function, then φ is bounded on A
 if the set $\varphi(A)$ is a bounded subset of \mathbb{R} . If φ is bounded, define
 the uniform norm of φ on A as

$$\|\varphi\|_A := \sup \{|\varphi(x)| : x \in A\}$$

Lemma A sequence (f_n) of bounded functions converges uniformly on A to f
 iff $\|f_n - f\|_A \rightarrow 0$.

E.g. $f_n := \frac{x}{n}$ cannot be applied since f_n is unbounded.

Cauchy Criterion for Uniform Convergence

Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$.

Then it converges uniformly on A to a bounded function f

iff $\forall \varepsilon > 0, \exists H(\varepsilon) \in \mathbb{N}$ s.t. $\forall n, m \geq H(\varepsilon)$,

$$\|f_m - f_n\|_A \leq \varepsilon$$

§.2 Interchange of Limits

Thm (f_n) sequence of continuous functions on $A \subseteq \mathbb{R}$

$(f_n) \xrightarrow{\text{def}} f$ on A . Then f is continuous on A .

Thm (f_n) sequence of functions on J bounded interval on \mathbb{R} .

$\exists x_0 \in J$ s.t. $f_n(x_0)$ converges

(f'_n) sequence of derivatives exists on J .
 $(f'_n) \xrightarrow{\text{def}} g$ on J

Note: (f_n) differentiable

$(f_n) \xrightarrow{\text{def}} f \not\Rightarrow f$ differentiable.

E.g. Weierstrass function

If considering $(f_n) = \text{partial sums} \rightarrow$ differentiable
 but $f \rightarrow$ not differentiable.

Then $(f_n) \xrightarrow{\text{def}} f$ on J that has a derivative at every point in J ,
 and $f' = g$.

Thm (f_n) sequence of functions $\in R[a, b]$,

$(f_n) \xrightarrow{\text{def}} f$ on $[a, b]$. Then $f \in R[a, b]$ and $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$.

Bounded Convergence Theorem

(f_n) sequence of functions $\in R[a, b]$.

$(f_n) \xrightarrow{\text{def}} f$ a function $f \in R[a, b]$. Converges pointwise

$\exists B > 0$ s.t. $|f_n(x)| \leq B \quad \forall x \in [a, b]$. Then $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$.

Dini's Theorem

(f_n) monotone sequence of continuous functions on $I^- = [a, b]$ closed & bounded
 $(f_n) \rightarrow$ a continuous function f
 Then the convergence of the sequence is uniform.

8.3 The Exponential and Logarithmic Functions.

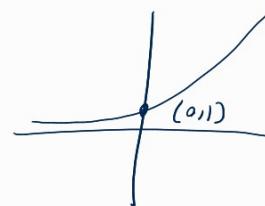
Exponential Function

The unique function $E: \mathbb{R} \rightarrow \mathbb{R}$ s.t.

$$E'(x) = E(x) \quad \forall x \in \mathbb{R}$$

$$E(0) = 1$$

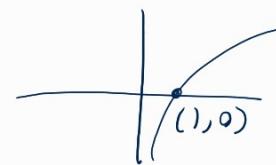
$e := E(1)$ is the Euler number. $\lim_{x \rightarrow -\infty} E(x) = 0$



$$\lim_{x \rightarrow \infty} E(x) = \infty$$

Inverse function: Logarithm

$$\lim_{x \rightarrow -\infty} L(x) = -\infty$$



$$\lim_{x \rightarrow \infty} L(x) = \infty$$

Power Function

If $\alpha \in \mathbb{R}$ and $x > 0$, then extend from rational power to arbitrary real number

$$x^\alpha := e^{\alpha \ln x} = E(\alpha L(x))$$

8.4 Trigonometric Functions

Thm \exists functions $C: \mathbb{R} \rightarrow \mathbb{R}$ s.t.

Defn Define (C_n) (S_n)

$$(i) C''(x) = -C(x) \text{ and}$$

$$(ii) C_1(x) = 1$$

$$S''(x) = -S(x) \quad \forall x \in \mathbb{R}$$

$$S_1(x) = x$$

$$(iii) C(0) = 1 \quad S(0) = 0$$

$$C'(0) = 0 \quad S'(0) = 1$$

$$C'(0) = 0 \quad S'(0) = 1$$

$$(iv) S_n(x) = \int_0^x C_n(t) dt$$

$$(v) C_{n+1}(x) = 1 - \int_0^x S_n(t) dt \quad \forall n \in \mathbb{N} \quad x \in \mathbb{R}$$

$$\Rightarrow S'_n(x) = C_n(x) \quad \Rightarrow \quad C_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{By Induction}$$

$$C'_{n+1}(x) = -S_n(x)$$

$$S_{n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\text{Define } S(x) := \lim S_n(x) \quad \text{Let } \cos x := C(x)$$

$$C(x) := \lim C_n(x) \quad \sin x := S(x)$$

$$\text{Then } C'(x) = -S(x), \quad S'(x) = C(x) \quad \forall x \in \mathbb{R}$$

$$(C(x))^2 + (S(x))^2 = 1 \quad \forall x \in \mathbb{R}$$

Thm If $f: \mathbb{R} \rightarrow \mathbb{R}$ s.t. $f''(x) = -f(x)$,

then $\exists \alpha, \beta \in \mathbb{R}$ s.t.

$$f(x) = \alpha C(x) + \beta S(x)$$

Thm C is even : $C(-x) = C(x)$

S is odd : $S(-x) = -S(x)$

Addition Formulas

$$C(x+y) = C(x)C(y) - S(x)S(y)$$

$$S(x+y) = S(x)C(y) + C(x)S(y)$$

Chapter 9 Infinite Series

9.1 Absolute Convergence

Def Let $X := (X_n)$ be a sequence in \mathbb{R} .

$\sum X_n$ is absolutely convergent if $\sum |X_n|$ is convergent in \mathbb{R} .

A series is conditionally convergent if it is convergent but not absolutely convergent.

E.g. Alternating Harmonic Series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent

since Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent because of the subtraction

Thm If a series in \mathbb{R} is absolutely convergent, it is convergent.

Thm If a series $\sum X_n$ is convergent, then any series obtained from grouping it also converges to the same value.

Rearrangement Theorem

If a series $\sum X_n$ is absolutely convergent, then any rearrangement $\sum Y_k$ of $\sum X_n$ also converges to the same value.

9.2 Tests for Absolute Convergence

Limit Comparison Test II

Suppose (X_n) (Y_n) non-zero real sequences and the limit exists in \mathbb{R} .

$$r := \lim \frac{|X_n|}{|Y_n|}$$

(a) If $r \neq 0$, then $\sum X_n$ is abs. convergent iff $\sum Y_n$ is abs. convergent

(b) If $r = 0$, if $\sum Y_n$ is abs. convergent, then $\sum X_n$ is abs. convergent

Root Test

(a) If $\exists r \in \mathbb{R}$ with $r < 1$ and $K \in \mathbb{N}$ s.t.

$$|X_n|^{\frac{1}{n}} < r \quad \forall n \geq K$$

then $\sum X_n$ is abs. convergent.

(b) If $\exists K \in \mathbb{N}$ s.t.

$$|X_n|^{\frac{1}{n}} \geq 1 \quad \forall n \geq K$$

then $\sum X_n$ is divergent.

Suppose the limit

$$r := \lim |X_n|^{\frac{1}{n}}$$

exists. Then

$$r = \begin{cases} < 1, & \sum X_n \text{ abs. convergent} \\ > 1, & \sum X_n \text{ divergent} \end{cases}$$

Ratio Test

Suppose $X := (X_n)$ a sequence of non-zero real numbers.

(a) If $\exists r \in \mathbb{R}$ with $0 < r < 1$ and $K \in \mathbb{N}$ s.t.

$$\left| \frac{X_{n+1}}{X_n} \right| < r \quad \forall n \geq K$$

then $\sum X_n$ is abs. convergent.

(b) If $\exists K \in \mathbb{N}$ s.t.

$$\left| \frac{X_{n+1}}{X_n} \right| \geq 1 \quad \forall n \geq K$$

then $\sum X_n$ is divergent.

Let (X_n) non-zero sequence in \mathbb{R}

Suppose the limit

$$r := \lim \left| \frac{X_{n+1}}{X_n} \right|$$

exists in \mathbb{R} . Then

$$r = \begin{cases} < 1 & \sum X_n \text{ abs. convergent} \\ > 1 & \sum X_n \text{ divergent} \end{cases}$$

Integral Test

Let f be a positive, decreasing function on $\{t : t \geq 1\}$.

Then $\sum_{k=1}^{\infty} f(k)$ converges iff the Improper integral exists.

$$\int_1^{\infty} f(t) dt = \lim_{b \rightarrow \infty} \int_1^b f(t) dt$$

Raabe's Test

Let $X := (X_n)$ be a sequence of non-zero real numbers.

—(1)—

(a) If $\exists a > 1$ and $K \in \mathbb{N}$ s.t.

$$\left| \frac{X_{n+1}}{X_n} \right| \leq 1 - \frac{a}{n} \quad \text{for } n \geq K$$

then $\sum X_n$ is abs. convergent.

(b) If $\exists a \leq 1$ and $K \in \mathbb{N}$ s.t.

Let

$$a := \lim \left(n \cdot \left(1 - \left| \frac{X_{n+1}}{X_n} \right| \right) \right)$$

whenever the limit exists. Then

$$a = \begin{cases} > 1 & \sum X_n \text{ abs. convergent} \\ \leq 1 & \sum X_n \text{ divergent} \end{cases}$$

$$\left| \frac{a_{n+1}}{x_n} \right| > 1 - \frac{a}{n} \quad \text{for } n \geq K$$

$\sum x_n$ not abs. convergent.

then $\sum x_n$ is not abs. convergent.

Euler's Constant

$$C_n := 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$$

is convergent. $\approx 0.5772 \dots$ Unsolved: rational or irrational

9.3 Tests for Non-Absolute Convergence

Alternating Series Test

Dirichlet's Test

Abel's Test

9.4 Series of Functions

Def. If (f_n) a sequence of functions defined on a subset $D \subseteq \mathbb{R}$ with values in \mathbb{R} ,

the sequence of partial sums (S_n) of the infinite series $\sum f_n$ is defined for $x \in D$ by

$$\begin{aligned} S_1(x) &:= f_1(x) \\ S_2(x) &:= f_1(x) + f_2(x) \\ &\vdots \\ S_{n+1}(x) &:= S_n(x) + f_{n+1}(x) \end{aligned}$$

If the sequence (S_n) converges to a function f on D ,

the infinite series of functions $\sum f_n$ converges to f on D . $\sum f_n$ denote the series/the limit function when it exists.

Def. If $\sum |f_n(x)|$ converges $\forall x \in D \Rightarrow \sum f_n$ is abs. convergent on D .

If (S_n) of partial sums is uniformly convergent on D to f ,

$\sum f_n$ is uniformly convergent on D .

Thm. If f_n is continuous on $D \subseteq \mathbb{R}$ to \mathbb{R} $\forall n \in \mathbb{N}$ and

$\sum f_n \rightarrow f$ on D , then f is continuous.

Thm. Suppose the real-valued $f_n \in R[a,b]$.

$\sum f_n \rightarrow f$ on $J = [a,b]$, then $f \in R[a,b]$ and $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$

Thm. (f_n) sequence of functions on J bounded interval on \mathbb{R} .

$\exists x_0 \in J$ s.t. $\sum f_n(x_0)$ converges

$\sum (f'_n)$ sequence of derivatives exists on J .

$\sum (f'_n)$ converges uniformly on J

Then $\sum (f'_n) \rightarrow f'$ a function on J that has a derivative at every point in J ,
and $f' = \sum (f'_n)$

Cauchy Criterion

Weierstrass M-Test

Tests for Uniform Convergence of Series

Let (M_n) be a sequence of positive real numbers s.t. $|f_n(x)| \leq M_n$ for $x \in D$, $n \in \mathbb{N}$.

If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D

Power Series

$\sum f_n$ is a power series around $x=c$ if f_n has the form

$$f_n(x) = a_n(x-c)^n \quad \text{for simplicity, } c=0 \Rightarrow \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \dots + a_n x^n$$

for $a_n, c \in \mathbb{R}$, $n \in \mathbb{N}$

Radius of Convergence

Let $\sum a_n x^n$ be If the sequence $(|a_n|^{\frac{1}{n}})$ is bounded, set power series.

$$r := \limsup (|a_n|^{\frac{1}{n}})$$

Then the radius of convergence of $\sum a_n x^n$ is

$$R := \begin{cases} 0 & \text{if } p = +\infty \\ \frac{1}{p} & \text{if } 0 < p < +\infty \\ +\infty & \text{if } p = 0 \end{cases}$$

The interval of convergence is $(-R, R)$

Cauchy - Hadamard Theorem

If R is the radius of convergence of the power series $\sum a_n x^n$, then if $|x| < R \rightarrow$ the series is abs. convergent.
 $|x| > R \rightarrow$ the series is divergent.

Chapter 10 The Generalized Riemann Integral

10.1 Definitions and Main Properties

Gauge on $[a, b]$: a strictly positive function $\mathfrak{J} : [a, b] \rightarrow (0, \infty)$

\mathfrak{J} -fine tagged partition $\dot{P} : t_i \in I_i \subseteq [t_{i-1}, \mathfrak{J}(t_i)], t_i + \mathfrak{J}(t_i)]$ for $i=1, \dots, n$

Generalized Riemann Integral

A function $f : [a, b] \rightarrow \mathbb{R}$ is generalized Riemann Integrable on $[a, b]$

$\exists L \in \mathbb{R}$ st. $\forall \epsilon > 0$, \exists a gauge \mathfrak{J}_ϵ on $[a, b]$ s.t.

\dot{P} is any \mathfrak{J}_ϵ -fine partition of $[a, b]$, then

$$|S(f, \dot{P}) - L| < \epsilon$$

Collection of generalized Riemann integrable functions: \mathbb{R}^*

Gauges more useful than norm $\|\dot{P}\|$:

More dedicated control: length of the largest subintervals

use of small subintervals

large subintervals

when function vary rapidly / remain constant

E.g. Dirichlet's function $\notin R[a, b]$ but $\in R^*[a, b]$, with integral = 0 set $\mathfrak{J}_\epsilon = \begin{cases} \frac{\epsilon}{2^{n+2}} & \text{when } x \text{ rational} \\ 1 & \text{when } x \text{ irrational} \end{cases}$

~ Cauchy Criterion

~ Squeeze Theorem \Rightarrow similar to that of Riemann Integral

The Fundamental Theorem of Calculus (stranger)

The Fundamental Theorem of Calculus (First Form)

Suppose \exists a ^{countable} finite set $E \subset [a, b]$ and functions $f, F : [a, b] \rightarrow \mathbb{R}$ s.t.

(a) F is continuous on $[a, b]$ ✓ if F is differentiable at every point

(b) $F'(x) = f(x) \quad \forall x \in [a, b] \setminus E$ ✓ even if f is not defined at some point

Then (f) $f \in R^*[a, b]$

Then and $\int_a^b f = F(b) - F(a)$

The Fundamental Theorem of Calculus (Second Form)

Let $f \in R^*[a, b]$, f continuous at $c \in [a, b]$.

Then the indefinite integral

$$F(z) = \int_a^z f \quad \text{for } z \in [a, b]$$

if f is continuous on $[a, b]$, then F is the anti-derivative of f .

(a) F is continuous on $[a, b]$

(b) \exists a null set Z_1 st. if $x \in (a, b) \setminus Z_1$, then

F is differentiable at x and $F'(x) = f(x)$.

(c) If f is continuous at $c \in [a, b]$, then $F'(c) = f(c)$

The Multiplication Theorem

If $f, g \in R[a, b]$, then $f \cdot g \in R[a, b]$.

If $f \in R^*[a, b]$ and g a monotone function on $[a, b]$, then $f \cdot g \in R^*[a, b]$

10.2 Improper and Lebesgue Integral

Hake's Theorem

If $f: [a,b] \rightarrow \mathbb{R}$, then $f \in R^*[a,b]$ iff $\forall \epsilon > 0$,
the restriction of f to $[a, \epsilon]$ $\in R^*[a, \epsilon]$ and

$$\lim_{\epsilon \rightarrow b^-} \int_a^\epsilon f = A \in \mathbb{R} \quad * \text{The generalized Riemann Integral does not need to be extended by taking such limit}$$

In this case $\int_a^b f = A$

Fact Generalized Riemann Integrable functions are not "absolute integrals";

\exists generalized Riemann integrable functions whose absolute values is not generalized Riemann integrable.

Lebesgue Integrable Functions

$f \in R^*[a,b]$ is Lebesgue integrable on $[a,b]$ if $|f| \in R^*[a,b]$, denoted by $L[a,b]$

Def If $f \in L[a,b]$, define seminorm $\|f\| := \int_a^b |f|$.

If $f, g \in L[a,b]$, define $\text{dist}(f, g) = \|f - g\| = \int_a^b |f - g|$

Completeness Theorem

A sequence (f_n) of functions on $L[a,b]$ converges to a function $f \in L[a,b]$

iff $\forall \epsilon > 0$, $\exists H(\epsilon)$ s.t. if $m, n \geq H(\epsilon)$, then

$$\|f_m - f_n\| = \text{dist}(f_m, f_n) < \epsilon.$$

10.3 Infinite Intervals

Generalized Riemann Integrable
Lebesgue Integrable
Hake's Theorem
Fundamental Theorem

} generalize to unbounded closed intervals
 $[a, \infty)$, $(-\infty, b]$, $(-\infty, \infty)$

10.4 Convergence Theorems

Thm (f_n) sequence of functions $\in R^*[a,b]$,

$(f_n) \xrightarrow{*} f$ on $[a,b]$. Then $f \in R^*[a,b]$ and $\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$.

Def (f_k) sequence in $R^*(I)$ is equi-integrable

if $\forall \epsilon > 0$, \exists gauge \mathcal{G}_ϵ on I s.t. if \tilde{P} is any \mathcal{G}_ϵ -fine partition of I and $k \in \mathbb{N}$,

then

$$|S(f_k; \tilde{P}) - \int_I f_k| < \epsilon$$

Equi-integrability Theorem

If $(f_k) \in R^*(I)$ is equi-integrable on I and if $f(x) = \lim f_k(x) \forall x \in I$,

then $f \in R^*(I)$ and $\int_I f = \lim_{k \rightarrow \infty} \int_I f_k$

Monotone Convergence Theorem

Let $(f_k) \in R^*(I)$ be monotone and $f(x) = \lim f_k(x)$ almost everywhere on I .

Then $f \in R^*(I)$ iff the sequence of integrals $(\int_I f_k)$ is bounded in \mathbb{R} ,

in which case

$$\int_I f = \lim_{k \rightarrow \infty} \int_I f_k.$$

(Lebesgue) Dominated Convergence Theorem

Let (f_k) a sequence in $R^*(I)$ and let $f(x) = \lim f_k(x)$ almost everywhere on I .

If \exists functions d, w in $R^*(I)$ s.t.

$$d(x) \leq f_k(x) \leq w(x) \quad \text{for almost every } x \in I,$$

then $f \in R^*(I)$ and $\int_I f = \lim_{k \rightarrow \infty} \int_I f_k$.

Moreover, if $d, w \in L(I)$, then f_k and $f \in L(I)$ and

$$\|f_k - f\| = \int_I |f_k - f| \rightarrow 0$$

Thm $f: [a,b] \rightarrow \mathbb{R}$ is (Lebesgue) measurable if \exists a sequence (s_k) of step functions on $[a,b]$ s.t.

Thm: holds for (s_k) sequence of non-increasing functions

$$f(x) = \lim_{k \rightarrow \infty} s_k(x) \text{ for almost every } x \in [a, b]$$

The collection of all measurable functions: $\mathcal{M}[a, b]$.

E.g. $\begin{cases} \text{Step functions} \\ \text{Monotone functions} \\ \text{Continuous functions} \end{cases}$ are measurable functions

E.g. $\begin{cases} \text{Dirichlet function} \\ \text{Thomae's function} \end{cases}$ are measurable functions

Since $\mathbb{Q} \cap [0, 1]$ is a null set,
take $(s_k) = 0$ -function.

Then $s_k(x) \rightarrow f(x)$ for $x \in [0, 1] \setminus \mathbb{Q}$

Measurability Theorem

If $f \in R^*[a, b]$, then $f \in \mathcal{M}[a, b]$ Converse not true:

Integrability Theorem measurable does not imply integrable

Let $f \in \mathcal{M}[a, b]$. Then $f \in R^*[a, b]$ iff $\exists \alpha, \omega \in R^*[a, b]$ st.

$\alpha(x) \leq f(x) \leq \omega(x)$ for almost every $x \in [a, b]$.

Moreover if either $\alpha, \omega \in \mathcal{L}[a, b]$, then $f \in \mathcal{L}[a, b]$.

Chapter (II) A Glimpse into Topology

II.1 Open and Closed Sets in \mathbb{R}

Cantor Set

The Cantor set \bar{F} is the intersection of the sets F_n , $n \in \mathbb{N}$, obtained by successive removal of open middle thirds, starting with $[0, 1]$.



(i) Total length of the removed intervals is 1

$$L = \frac{1}{3} + \frac{2}{3^2} + \dots + \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1 - \frac{2}{3}}{1 - \frac{2}{3}} = 1 \quad \Rightarrow \bar{F} \text{ has length 0}$$

(ii) \bar{F} contains no non-empty open intervals as a subset

(iii) \bar{F} has infinitely (even uncountably) many points.

Ininitely = \bar{F} contains all endpoints of intervals in the form $\frac{2^k}{3^n}$ for $k \in \mathbb{N}$

uncountably = More points than $\frac{2^k}{3^n}$; prove by contradiction (otherwise $[0, 1]$ is countable) which is false

II.2 Compact Sets every open cover of K has a finite subcover

Heine-Borel Theorem

A subset K of \mathbb{R} is compact iff it is closed and bounded.

Thm. A subset K of \mathbb{R} is compact iff every sequence in K has a subsequence that converges to a point in K . Bolzano-Weierstrass: every bounded sequence has a converging subsequence.

II.3 Continuous Functions

II.4 Metric Spaces

Thm. A metric space (S, d) is complete if each Cauchy sequence converges to a point of S

E.g. \mathbb{Q} is not complete, e.g. (x_n) rational converges to $\sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$

C with metric $d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}$ is complete

C with metric $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$ is not complete.

Semimetric space

A space with a semimetric d where $d(x, y) = 0$ if $x = y$

d might not converge to a unique limit but $d(x, y) = 0 \nrightarrow x = y$
identification \rightarrow equivalent classes.

Appendix E

A continuous but nowhere differentiable function

Let $f_0 : \mathbb{R} \rightarrow \mathbb{R} = \text{dist}(x, \mathbb{Z})$ A "sawtooth" function with slope ± 1
 $= \inf\{|x - k|, k \in \mathbb{Z}\}$ on $[\frac{k}{2}, \frac{(k+1)}{2}]$

$f_m := (\frac{1}{4^m}) f_0(4^m x)$ with $0 \leq f_m(x) \leq \frac{1}{2 \cdot 4^m}$ for $m > 0$ Also "sawtooth"

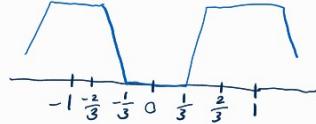


(Let $g := \sum g_m(x)$. Weierstrass-M implies g is continuous.)

A space-filling curve

Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be the continuous, even function with period 2.

$$\varphi(t) := \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{1}{3} \\ 3t-1 & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3} \\ 1 & \text{for } \frac{2}{3} \leq t \leq 1 \end{cases}$$



For $t \in [0, 1]$, define

$$f(t) := \sum_{k=0}^{\infty} \frac{\varphi(3^k t)}{2^{k+1}} \quad g(t) := \sum_{k=0}^{\infty} \frac{\varphi(3^{2k+1} t)}{2^{k+1}}$$

Since $0 \leq \varphi(t) \leq 1$, Weierstrass M test implies f, g are continuous on $[0, 1]$.

An arbitrary point (x_0, y_0) in $[0, 1] \times [0, 1]$ is the image under (f, g) of some point $t_0 \in [0, 1]$

* Let x_0, y_0 have a binary (base 2) representation:

$$x_0 = \frac{a_0}{2} + \frac{a_1}{2^2} + \frac{a_2}{2^3} + \dots \quad y_0 = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \dots$$

Let t_0 have ternary (base 3) representation:

$$t_0 = \sum_{k=0}^{\infty} \frac{2a_k}{3^{k+1}} = \frac{2a_0}{3} + \frac{2a_1}{3^2} + \frac{2a_2}{3^3} + \dots$$

$$\text{Note if } a_0 = \begin{cases} 0, & \Rightarrow 0 \leq t_0 \leq \frac{1}{3} \Rightarrow \varphi(t_0) = 0 \\ 1, & \Rightarrow \frac{2}{3} \leq t_0 \leq 1 \Rightarrow \varphi(t_0) = 1 \end{cases} \Rightarrow \varphi(a_0) = x_0$$

Similarly $\forall n \in \mathbb{N} \exists m_n \in \mathbb{N}$ st.

$$3^n t_0 = 2m_n + \frac{2a_n}{3} + \frac{2a_{n+1}}{3^2} + \dots$$

and $\varphi(3^n t_0) = a_n$ since φ has period 2

$$f(t_0) = \sum \frac{\varphi(3^k t_0)}{2^{k+1}} = \sum \frac{a_{2k}}{2^{k+1}} = \widehat{x_0}$$

$$g(t_0) = \sum \frac{\varphi(3^{2k+1} t_0)}{2^{k+1}} = \sum \frac{a_{2k+1}}{2^{k+1}} = \widehat{y_0}$$

Author: kiking