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Chapter 1 Pure Mathematics

Introduction

Euclidean geometry as pure mathematics

"Euclidean geometry": Greece, ~ 600 and 300 B.C.
codified by Euclid in *The Elements*
→ definitions
→ axioms (assumptions)
→ single deductive chain of 465 theorems

Self-sufficiency is the hallmark of pure mathematics

Games

e.g. Euclidean Geometry

Games: ① Objects to play with — Chessboard & Chessmen → A plane, some points & lines
② An opening arrangement — Initial game setup → List of Axioms
③ Rules — How the pieces move → Rules of formal logic
④ A goal — "Checkmate" → Prove profound & interesting theorems

☆ Games similar to Pure Mathematics:
the objects with which a game is played have no meaning outside the context of a game
the essence is its abstract structure; the objects is a visual aid or a fairy tale
☆ Euclidean never define plane, lines, points!

Why study pure mathematics?

① Pure math is applicable — freedom of interpretation
→ applied mathematics
② Pure math is a culture clue — logic → "correct" from Western rationality
→ study of deduction
pure math → the only completely deductive study
③ Pure math is fun — the world's best game

Chapter 2 Graphs

Sets

Paradox

The Pythagorean Paradox (~600 B.C.): $\sqrt{2}$ → intuition: rational
→ logical: must not be rational ✓

"Rigor": theorems discovered by intuition
demonstrated by logic

Russell's Paradox (1902): ordinary set: A = {1, 2, 3, 4}
"~~extraordinary~~" set: A = {1, 2, 3, 4, A}
Not a "set" anymore, call it a Class

"theory of types": set: exclude collections that are elements of themselves

Graphs

Graph = {Vertex set, Edge set}
Vertex set ↓ finite non-empty
Edge set ↓ empty / two-element subset of the vertex set

If {X, Y} is an edge: {X, Y} is incident to each of X and Y ⟷ vice versa

Graph Diagrams

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Graph Diagrams

edges

Multi-graph : allow "skeins" (several edges joining the same pair of vertices)

Pseudograph : Multi-graph + allow "loops" (vertices joining themselves)

Digraph : edges have directions

Common Graphs

Path graph: except this

Cyclic graph on v vertices : C_v — vertex set $\{1,2,\dots,v\}$, edge set $\{\{1,2\},\{2,3\},\dots,\{v-1,v\},\{v,1\}\}$

C_3 C_4

Null graph on v vertices : N_v — vertex set $\{1,2,\dots,v\}$, edge set $\{\phi\}$

N_1 N_2

$C_3 = K_3$

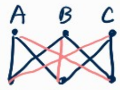
$K_1 = N_1$

Complete graph on v vertices : K_v — vertex set $\{1,2,\dots,v\}$, all possible edges

K_4 K_2

total edges : $\frac{1}{2}(v-1)(v)$

Utility graph: UG



exactly this graph

Complement of G : $\overline{G}$ — same vertex set as G, edge set : all 2-element subsets of the vertex set not already in edge set of G

$\overline{C_4}$ $\overline{C_5}$ $\overline{UG}$

Subgraph: vertex set, edge set are subsets of the original graph

if erase vertex, must erase all incident edges

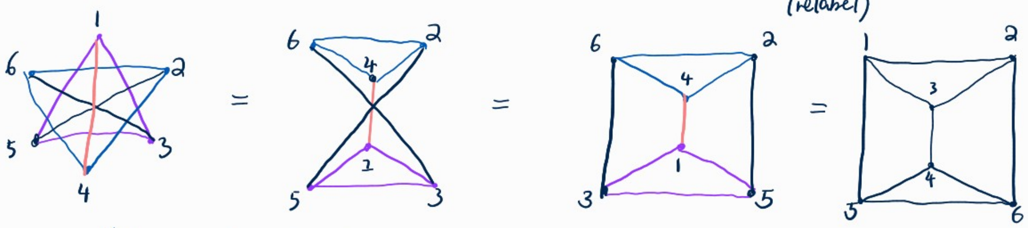
Isomorphism

Two graphs are isomorphic if $\exists$ a 1:1 correspondance between their vertex sets s.t. adjacency of vertices is preserved.

Recognizing Isomorphic Graphs

Method of Steel balls and Rubber bands

(relabel)



✓ same number of vertices

✓ same number of edges

✓ same distribution of degrees

✓ same number of components

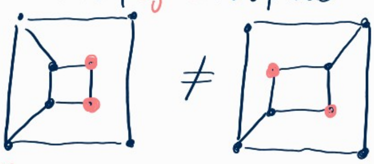
degree of a vertex = num of incident edges

✓ same distribution of subgraphs (ie. isomorphic subgraphs)

Isomorphic $\rightarrow$ four conditions ✓

$\cancel{\leftarrow}$

e.g. Vertices of degree 2 are separated



Equal ≠ Isomorphic (equal : vertex set, edge set exactly the same inc. label)

"A is B" commonly mean "A is isomorphic to B"

Polya Enumeration Theorem

Compute the number of graphs given the vertex count

Chapter 3 Planar Graphs

Introduction

A graph is planar if it is isomorphic to a graph drawn in-plane "Pl... "

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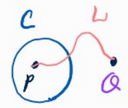
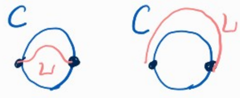
without edge-crossings. Otherwise a graph is non-planar. Planarity

UG, K_5, and the Jordan Curve Theorem

Jordan Curve Theorem no points repeated except start point / same endpoints



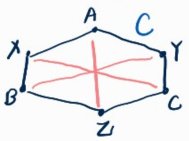
If C is a continuous simple closed curve in a plane, then C divides the rest of the plane into two regions having C as the common boundary.
If a point P in one of these regions is joined to a point Q in the other region by a continuous curve L in the plane, then L intersects C.



Corollary
If two points of C is joined by a continuous curve L in the plane having no other points common with C, then except for its endpoints, L is completely contained in one of the regions determined by C.

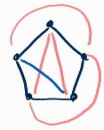
Thm UG is non-planar

Proof:



$\{A, Z\}$ $\{B, Y\}$ $\{C, Z\}$
two of them must be in one region divided by C (pigeonhole principle)

Thm K_5 is non-planar (with the smallest number of vertices)



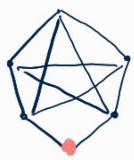
Thm Subgraph of a planar graph is planar. erasing vertices / edges cannot create edge-crossings.

Thm Supergraph of a non-planar graph is non-planar. suppose planar; contradiction

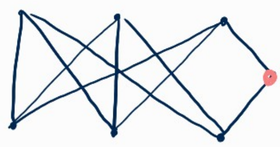
Subgraph: selective erasing; Supergraph: selective augmenting

Are there more non-planar graphs?

Examples of non-planar graphs (infinite)
that are not supergraph of UG or K_5



(1) prove not supergraph
(2) prove non-planar: assume planar. remove vertex and joining the edges become K_5/UG $\rightarrow$ non-planar contradiction.



Expansions

Def If some new vertices of degree 2 are added to some edges of a graph, the resulting graph is an expansion of the original graph.

Expansions $\neq$ Supergraphs
both are augmentations of a graph, but by different procedures.
"some new vertices of degree 2 are added to some edges"

the only allowed operation for Expansions

the only forbidden operation for Super-graphs

e.g. C_4 expansion of K_3 expansion of

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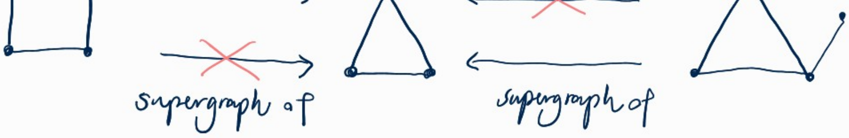
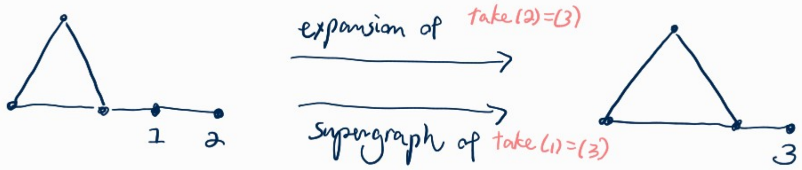


e.g.



Thm Expansion of UG or K_5 is non-planar.

Kuratowaki's Theorem

Every non-planar graph is a supergraph of an expansion of UG or K_5.

Corollary The set of all planar graph is equal to the set of all graphs that are not supergraphs of expansions of UG and K_5.

Determining whether a graph is planar or non-planar

non-planar: $\exists$ a subgraph that is isomorphic to a UG or K_5 or the expansion of UG or K_5

Chapter 4 Euler's Formula

Introduction

Leonhard Euler "the Seven Bridges of Königsberg" (1736)
→ earliest known work of graph theory

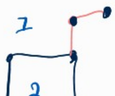
Def A walk in a graph:
a sequence $A_1 A_2 A_3 \dots A_n$ of (not necessarily distinct) vertices, A_1 joined by an edge to A_2, ...
The walk $A_1 A_2 A_3 \dots A_n$ join A_1 and A_n.

Def A graph is connected if every pair of vertices is joined by a walk (otherwise: disconnected)
e.g. every cyclic / complete graph is connected
e.g. every null graph is disconnected except for N_1.

Def A planar graph (drawn in plane without edge-crossing) cuts the plane into regions called faces of the graph.
← only planar graph has faces
← only define with cross-free drawings
(generalization of Jordan Curve Theorem)
(number of faces is independent of the drawing)

Def A graph is polygonal if it is
* planar
* connected
* every edge borders on two different faces

e.g.



not polygonal

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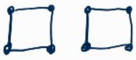
since two edges only border on face #1

 not polygonal since disconnected

non-planar → not polygonal since no faces

Euler's Formula

For planar connected graph, $v + f - e = 2$.

Mathematical Induction

Proof: ① Euler's formula on polygonal graphs
② Euler's formula on planar connected graphs that are not polygonal.

Some consequences of Euler's formula

Thm If G is planar and connected with $v \geq 3$,
then $\underbrace{\frac{3}{2}f \leq e}\underbrace{\leq 3v - 6}$

Proof

Case ①: G has a face bounded by < 3 edges.
→ G must be P_3 •—•—• (path graph) $f = 1, e = 2, v = 3$

Case ②: Every face of G is bounded by ≥ 3 edges.
→ $\sum_i 3 \leq \sum_i$ num. of edges bounding face $i \leq 2e$ equal if G is polygonal
→ $3f \leq 2e \Rightarrow \underbrace{\frac{3}{2}f \leq e}$
→ $v + f - e = 2 \Rightarrow 2 \leq v + \frac{3}{2}e - e \Rightarrow \underbrace{e \leq 3v - 6}$

Thm If G is planar and connected with $v \geq 3$
and G is not a supergraph of K_3.
then $2f \leq e \leq 2v - 4$

Thm If G is planar (and connected)
then G has a vertex of degree ≤ 5

Algebraic Topology

Euclidean geometry: connected with the "metric" properties of figures

Topology: properties of figures preserved by "continuous deformations"
e.g. Jordan Curve Theorem on "continuous simple closed curve"
Henri Poincaré: "qualitative" subject

Algebra: studies sets where "operations" are defined.
high school algebra: on Real numbers of ⊕ ⊖ ⊗ ⨸

Algebraic Topology: algebraic methods applied to topological problems

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convert → take a T problem, convert to A problem
solve → try to solve the A problem
reconvert → reconvert the A solution into T terms
result: a solution to the T problem

e.g. analytical geometry

means of convert: associate every geometry point the "coordinates"

Straight lines → equations → system of equations, ...

Euler's Formula

↳ means of convert: graph theory (topology) → high school algebra (algebra)

K_5 is non-planar

Proof: K_5 connected, but $e \nleq 3v-6$, contradiction

"disadvantage": less conductive to understanding

Exercises

Def A component of a graph is a connected subgraph that is not contained in a larger connected subgraph.

Generalization of Euler's Formula

If G is planar, $v+f-e=1+p$ where p is the # of components

When G is connected, $p=1$ ⇒ reduces to $v+f-e=2$

Def The connectivity of a graph c is the smallest number of vertices whose removal (+ incident edges) results in either K_1 or a disconnected graph.

Def A bridge in a graph is an edge whose removal increases # of components.

Thm If the connectivity is ≥ 6, the graph is non-planar.

Chapter 5 Platonic Graphs

Introduction

1. historical: "Platonic solids" (after Plato)
2. heuristic: spectacular warning if overindulge the natural tendency
3. pedagogical: power of Euler's formula

Def A graph is regular if all vertices have the same degree

"regular of degree d"

e.g. C_v (cyclic): regular of degree 2

K_v (complete): regular of degree $v-1$

N_v (null): regular of degree 0

UG: regular of degree 3

Def A graph is plantonic if it is

* polygonal — { * planar — { isomorphic to a graph without edge-crossing; * connected; * every edge borders on two different faces }
* regular

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all faces bounded by same number of edges

e.g. C_v (cyclic) : plantonic

K_v (complete) : only K_1, K_3, K_4 plantonic

N_v (null) : only N_1

UG : not plantonic, since not planar

<u>Thm</u> Other than ① K_1 ② Cyclic graphs, there are only 5 plantonic graphs.

Lemma If G regular of degree d, then $e = \frac{d \cdot v}{2}$ $f = \frac{d \cdot v}{n}$

$v + f - e = 2 \Rightarrow v + \frac{dv}{2} - \frac{dv}{n} = 2 \Rightarrow v(2n + 2d - nd) = 4n$ (solve $(n-2)(d-2) < 4$)

then compute $v = \frac{4n}{2n+2d-nd}$

<u>Proof</u> ... algebra with Euler's formula ...

$(n-2)(d-2) < 4$ where n: # edges bounding a face of a polygonal graph

$\Rightarrow$ only five solutions! d: the degree of each vertex of G

... more algebra on v, e, f...

tetrahedron	cube (hexahedron)	dodecahedron	octahedron	icosahedron
$n=3$, $d=3$	$n=4$ $d=3$	$n=5$, $d=3$	$n=3$, $d=4$	$n=3$, $d=5$
$f=4$	$f=6$	$f=12$	$f=8$	$f=20$

<u>History</u>

Terminology : "regular polygon" (not graph) = edges same length, angle same size — infinitely many

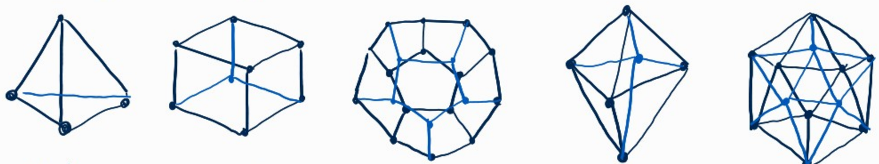
"regular polyhedron" = congruent regular polygons for faces, corner angles same size

Only 5 regular polyhedron

every plantonic graph 1:1 corresponds to a regular polygon (cyclic graph) or regular polyhedron (5 plantonic graph)

except K_1 (trival)

isomorphic (3-d drawings)



regular tetrahedron	regular cube (hexahedron)	regular dodecahedron	regular octahedron	regular icosahedron
$n=3$, $d=3$	$n=4$ $d=3$	$n=5$, $d=3$	$n=3$, $d=4$	$n=3$, $d=5$
$f=4$	$f=6$	$f=12$	$f=8$	$f=20$

Pythagoreams : universe (dodeca) + {fire, earth, air, water}

Kelper : five regular polyhedra $\longleftrightarrow$ six known planets

<u>Exercises</u>

<u>Def</u> A dual graph of a planar graph is formed by:

① taking a cross-free drawing of the planar graph

② place a dot inside each face

③ joining two dots whenever the borders of two faces have one or more edge in common

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→ Only a planar graph have a dual

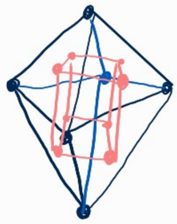
→ A dual is always planar and connected

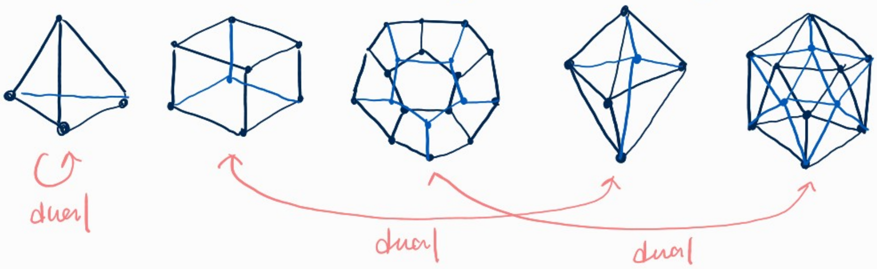
→ A dual is not always unique (from different cross-free drawings)

* Plantonic graph : dual is unique

K_1 : dual is itself

C_v : dual is always K_2





Chapter 6 Coloring

Chromatic number

Def. A graph is colored if adjacent vertices have been assigned with different colors.

Def Chromatic number X : the smallest number of colors which a graph can be colored.

Coloring planar graphs

The Four Color Conjecture (Theorem) : Every planar graph has $X \le 4$.

The Five Color Theorem ← easier to prove

"every planar graph has at least one vertex with degree less than or equal to 5"

the version for 4 is false e.g. Isocahedron ($d=5$)

Coloring maps

Map coloring problem : find the smallest number m s.t. the faces of every planar graph can be colored with $\le m$ colors s.t. faces sharing a harden have different colors.

Chapter 7 The Genus of a Graph

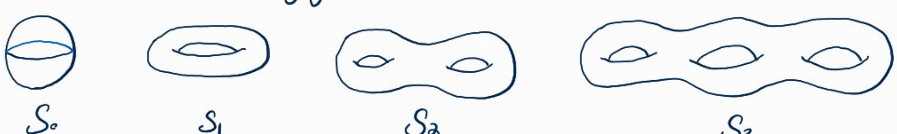
The genus of a graph

Planar = ① no edge-crossing ② drawn in a plane

Genus = $\underbrace{\text{①}}_{\text{include this condition}}$ $\underbrace{\text{②}}_{\text{generalize this by considering other surface}}$

Infinite family of surface

* hollow * of negligible thickness



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Generalize
Planar Graph
(Chapter 3)

Def The genus of a graph g is the subscript of the first surface among $S_0, S_1, S_2, \dots$ on which the graph can be drawn without edge-crossing.
($\rightarrow$ every graph has a genus!)

Thm The set of all planar graphs = the set of all graphs with $g = 0$

puncture

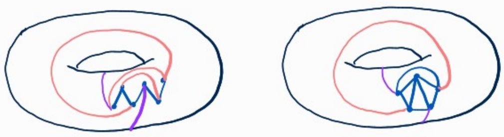
S_0

* difference : the punctured face on S_0 became the infinite face in planar

Examples

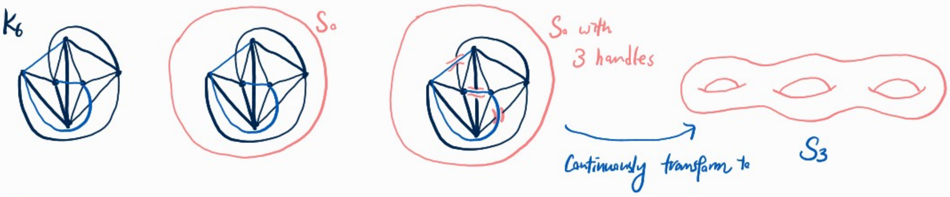
K_4, C_v has $g = 0$ because they are planar.

UG, K_5 has $g = 1$



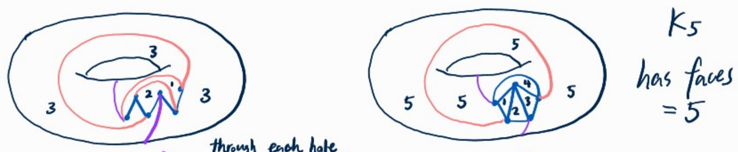
Thm Every graph has a genus

1. transfer the drawing to S_0
2. add to S_0 enough tubular "handles" to serve as "overpasses"



Thm If a graph has genus g, then G can be drawn without edge-crossing on every surface S_n where $g \le n$

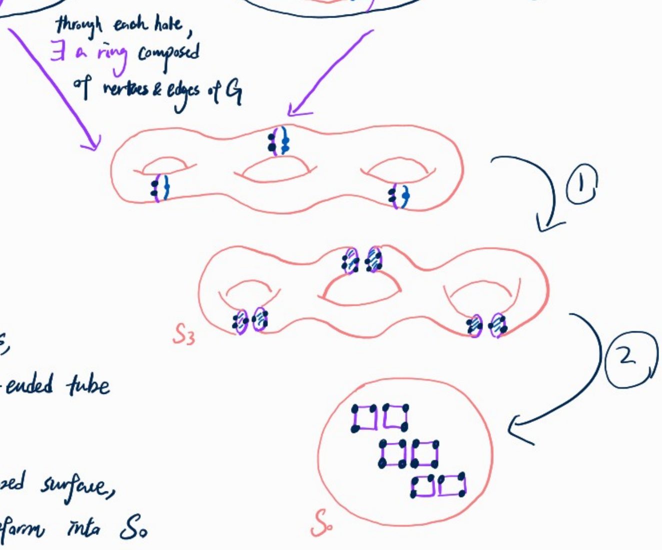
Def The faces of G are the regions divided by the edges and vertices of G on the surface S_g, where g is the genus of G



Generalize
Euler's formula
(Chapter 4)

Euler's Second Formula

$$v + f - e = 2 - 2g$$



Proof

1. Split each ring into two rings, forming the rim of an open-ended tube (total $2g$ rim)
2. Cover each rim with disc-shaped surface, inflate and continuously deform into S_0

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③ New graph H : crossing-free, connected, planar → Euler's first formula applies
$v_H + f_H - e_H = 2$

④ $v_H - v_G = e_H - e_G$ (rings are cyclic graph)
$f_H = f_G + 2g$ (new faces are created only for the rims)
$\therefore\ v_G + f_G - e_G = (v_H - e_H + e_G) + (f_H - 2g) - e_G$
$= v_H - e_H + f_H - 2g$
$= 2 - 2g$

Some consequences

Lemma If a connected graph has $v \geq 3$, then $3f \leq 2e$

Thm If G is connected with $v \geq 3$ and genus g, then
$$g \geq \frac{1}{6}e - \frac{1}{2}(v-2)$$
* provides a lower bound of g (even only knowing v, e)

Thm If $v \geq 3$, the complete graph K_v has genus
$$g = \left\lceil \frac{(v-3)(v-4)}{12} \right\rceil$$
for "$\geq$" proof
for "$\leq$" : difficult to prove until 1968
prove by drawing K_v without edge-crossing on S_n where $n = \frac{(v-3)(v-4)}{12}$

Corollary If G has $v \geq 3$ and genus g, then
$$g \leq \left\lceil \frac{(v-3)(v-4)}{12} \right\rceil$$
Proof : G subgraph of K_5 ⇒ $g_G \leq g_{K_5}$ (lemma)

Corollary If G has $v \geq 3$ and genus g, then
$$\underbrace{\left\lceil \frac{1}{6}e - \frac{1}{2}(v-2) \right\rceil}_{\text{lower}} \leq g \leq \underbrace{\left\lceil \frac{(v-3)(v-4)}{12} \right\rceil}_{\text{upper}}$$

Estimating the genus of a connected graph

the estimation becomes better when e is closer to its max $\frac{1}{2}v(v-1)$

Thm If G is a connected incomplete graph with $v \geq 3$, then the upper and lower bound of g is either same or consecutive integers when
$$e \geq \frac{1}{2}v(v-1) - 5$$

Generalize Plantonic graphs (Chapter 5)

g-Plantonic graphs

Def A graph is g-plantonic if
* connected
* regular
* every edge borders two faces
* every face bounded by same number of edges

graph of genus g

(+ planar = polygonal)

Notation d : degree of each vertex
n : # edges bounding each face

$$P(d,n) : \frac{(4g-4)n}{nd - 2d - 2n}$$
(in Chapter 5: use to compute v when $g=0$)

Lemma If G is g-plantonic, $e = \frac{d \cdot v}{2}$, $f = \frac{d \cdot v}{n}$

Lemma(a) If G is g-plantonic and $g > 0$, then $n \geq 3$ and $d \geq 3$

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Lemma (b) Let $g > 1$. $(n-2)(d-2) > 4 \iff P(d,n) > 0$

Lemma (c) If $g > 1$, $(n-2)(d-2) > 4$, $d' \geq d \geq 3$, $n' \geq n \geq 3$ then $\Longrightarrow P(d',n') \leq P(d,n)$

Chapter 5 $\longrightarrow$ Infinitely many 0-plantonic graph

Except 5 of them, either (trivial) K_1 or (uninteresting) C_v

Thm For each $g > 1$, there are at most a finite number of g-plantonic graphs.

Proof $v + f - e = 2 - 2g \Rightarrow v = \dfrac{(4g-4)n}{nd - 2d - 2n} = P(d,n)$

Case ① : $n = 3$ ‖ $P(d,n) > 0 \overset{(b)}{\Rightarrow} (n-2)(d-2) > 4$ so $d \geq 7 \overset{(c)}{\Rightarrow} v = P(d,n) \leq P(7,3)$

Case ② : $n = 4$ ‖ —〃— $v \leq P(5,4)$

Case ③ : $n = 5$ ‖ —〃— $v \leq P(4,5)$

Case ④ : $n = 6$ ‖ —〃— $v \leq P(4,6)$ the largest among above numbers

Case ⑤ : $n \geq 7$ ‖ Any $d \geq 3$ is admissible so $\boxed{v \leq P(3,7)}$

In all cases, $v \leq P(3,7) = \dfrac{(4g-4)\cdot 7}{21 - 6 - 14} = \boxed{28(g-1)} \Rightarrow$ 2-plantonic: $v \leq 28$, 3-plantonic: $v \leq 56$

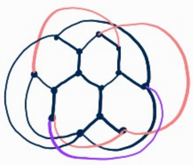
Since for each value of $g > 1$,

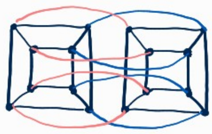
there are only a finite number of graphs with at most $P(3,7)$ vertices

$\Rightarrow$ only a finite number of g-plantonic graphs.

Thm All 1-plantonic graph have either: (infinitely many)

$d=3, n=6$ $\quad$ $d=4, n=4$ $\quad$ $d=6, n=3$





e.g. Q

e.g. Complete graph K_7

Generalize: Coloring (Chapter 6)

The Heawood Coloring Theorem

Def If $n \geq 0$ is an integer, then the chromatic number of the surface S_n, denoted by $X(S_n)$, is the largest among the chromatic numbers of graphs having $g \leq n$.

(That this is true ($X(S_n)$ exists) for every $n \geq 0 \Rightarrow$ the Heawood Coloring Theorem)

Thm $X(S_n)$: the number of colors sufficient to color graphs with $g \leq n$

Thm $X(S_n)$: the smallest number of colors sufficient to color every graph that can be drawn on S_n without edge-crossings.

The Four Color Conjecture (Theorem) $\Rightarrow X(S_0) = 4$

Heawood Coloring Theorem If $n > 0$ then

$$X(S_n) = \left\lfloor \frac{7 + \sqrt{1 + 48n}}{2} \right\rfloor$$

Proof $X(S_n) \leq t \rightarrow$ long proof

$X(S_n) \geq t \rightarrow$ sufficient to find a graph $g \leq n$, $X = t$

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F_t has $\chi = t$ → obvious   Complete graph K_t has vertices $= t$

K_t has $g \le n$ → consider $t \ge 3$ $(v \ge 3)$

$$g = \left\lceil \frac{(t-3)(t-4)}{12} \right\rceil = \left\lceil \frac{\left(\left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor - 3\right)\left(\left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor - 4\right)}{12} \right\rceil \quad (t = \chi)$$

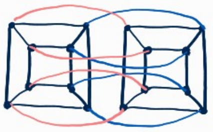
$$\le \left\lceil \frac{\left\lfloor \frac{-1+1+48n}{4} \right\rfloor}{12} \right\rceil = \left\lceil \frac{\lfloor 12n \rfloor}{12} \right\rceil = \left\lceil \frac{12n}{12} \right\rceil = \lceil n \rceil = n$$

→ Tells us that S_0 is the most "complicated" surface in coloring where $\chi(S_0)$ is not known exactly. —— until the Four Color Conjecture is proved!

$\chi(S_0) = \left\lfloor \frac{7+\sqrt{1}}{2} \right\rfloor = \left\lfloor \frac{8}{2} \right\rfloor = 4$

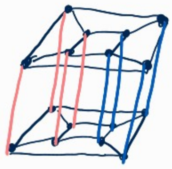
Exercises

$d=4,\ n=4$



Q

→ skeleton of a "toroidal polyhedron"



a polyhedron that will look like S_1 when inflated (square hole through a cube)

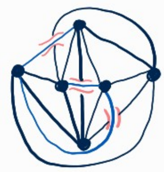
→ K_7 also the skeleton of a toroidal polyhedron → "Császár polyhedron"

Def The crossing number of graph k is the min. number of edge-crossings that a graph can be drawn on plane.

e.g. All planar graph : $k=0$

K_5 and UG : $k=1$

K_6 ← no formula for K_v : $k=3$



Chapter 8 Euler Walks and Hamilton Walks

Introduction

Two of the oldest problems : "When does a graph have an Euler/Hamilton walk?"

Chapter 3-7 : graphical properties defined in terms of the surfaces on which the graphs can be drawn

this Chapter : graphical properties that are unrelated to the surfaces on which the graphs can be drawn

Euler walks

Def A walk $A_1A_2 \cdots A_{n-1}A_n$ in a graph is closed if $A_1 = A_n$, otherwise the walk is open

Def An euler walk is a walk that uses every edge in the graph exactly once.

Thm For a connected graph,

closed euler walk ⟺ every vertex (has degree) even

open euler walk ⟺ has exactly two odd vertices

" the problem of the highway inspector "

Hamilton walks

Ireland one famous mathematician : William Rowan Hamilton

Def An open hamilton walk : a walk that uses every vertex exactly once

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A closed hamilton walk : except for the initial vertex that uses exactly twice

" the problem of the travelling salesman "

Thm If the sum of the degrees of every pair of vertices of a graph G

Not insightful/ practical

is at least $v-1$ → every pair of vertices are either adjacent to each other or to a common 3rd vertex → G connected (assume false, then sum < $v-1$) ⇒ G has an open hamilton walk

" if a graph has a lot of edges and if they are evenly distributed, hamilton walk exists "

is at least v ⇒ G has a closed hamilton walk

A much more complicated problem than euler walks

any two vertices sum is $4 < 6-1=5$ ~~⇐~~ Converse is not true e.g. C_6  ✓ open/closed hamilton walk

Multigraphs

Def A skein is an object of two dots connecting by two or more lines

Def If some of the edges (and incident vertices) of a graph G are replaced by skeins, the result is a multigraph $M(G)$ generated by G.

every graph is a multi-graph ; a single graph generates $M_1(G)$ $M_2(G)$ .... infinitely many

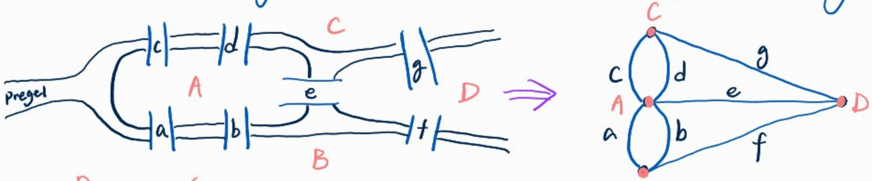
Most concepts (complement, isomorphism, coloring, genus, etc) can be defined to include Multigraphs

The Königsberg Bridge Problem

Euler's epochal 1736 paper " the seven bridges of Königsberg "

⇒ the birth notice of " the geometry of position " (now called topology!)

(Machine-building is a lot more fun than problem-solving)



Became (modern):

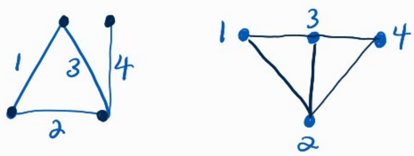
Does the multi-graph have an Euler walk ?

Exercises

Def If G is a graph with $e \neq 0$,

then the line graph $L(G)$ is the graph having

* one vertex for each edge in G
* two vertices of $L(G)$ have an edge, if the corresponding edges in G share a vertex

G  $L(G)$

K_4  $L(K_4)$

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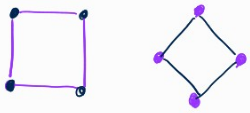
Prove

If G has a closed euler walk,
then $L(G)$ has both a closed euler walk
and a closed hamiton walk,

If G has a closed hamiton walk,
then $L(G)$ also has a closed hamilton walk

Cyclic graphs are isomorphic to their line graphs,
and are the only graphs having this property.



Def

If G is a graph with $e \neq 0$,
then the trisection graph $T(G)$ is the expansion of G
formed by slicing every edge of G by two vertices of degree 2.

G $T(G)$ $L(T(G))$

Afterword

Appel and Haken (1976) : prove the Four Color Conjecture
finding the unavoidable set of reducible configurations (total 1482)