

## Chapter 1 Pure Mathematics

### Introduction

#### Euclidean geometry as pure mathematics

"Euclidean geometry": Greece, ~600 and 300 B.C.

codified by Euclid in The Elements

→ definitions

→ axioms (assumptions)

→ single deductive chain of 465 theorems

Self-sufficiency is the hallmark of pure mathematics

### Games

- Games = ① Objects to play with Chessboard & Chessmen  
② An opening arrangement Initial game setup  
③ Rules How the pieces move  
④ A goal "Checkmate"

e.g. Euclidean Geometry

→ A plane, some points & lines

→ List of Axioms

→ Rules of formal logic

→ Prove profound & interesting theorems

\* Games similar to Pure Mathematics:

the objects with which a game is played have no meaning outside the context of a game

the essence is its abstract structure; the objects is a visual aid or a fairy tale

& Euclidean never define plane, lines, points!

### Why study pure mathematics?

- ① Pure math is applicable freedom of interpretation  
→ applied mathematics  
② Pure math is a culture clue logic → "correct" from Western rationality  
→ study of deduction  
③ Pure math is fun the world's best game

## Chapter 2 Graphs

### Sets

### Paradox

The Pythagorean Paradox (~600 B.C.) =  $\sqrt{2}$  ↗ intuition: rational / logical: must not be rational ✓

"Rigor": theorems discovered by intuition  
demonstrated by logic

Russell's Paradox (1902): ordinary set =  $A = \{1, 2, 3, 4\}$

"extraordinary" set =  $A = \{1, 2, 3, 4, A\}$

"theory of types": set = exclude collections that are elements of themselves

Not a "set" anymore,  
call it a Class

### Graphs

Graph = {Vertex set, Edge set}

↓

finite non-empty

↓

empty / two-element subset of the vertex set

If  $\{X, Y\}$  is an edge =  $\{X, Y\}$  is incident to each of  $X$  and  $Y$  ↔ vice versa



Y no incident edges  
isolated vertex

### Graph Diagrams

## Graph Programs

Multi-graph = allow "skeins" (several edges joining the same pair of vertices)

Pseudograph = Multi-graph + allow "loops" (vertices joining themselves)

Digraph = edges have directions

## Common Graphs

Cyclic graph on  $v$  vertices:  $C_v$

Null graph on  $v$  vertices:  $N_v$

Complete graph on  $v$  vertices:  $K_v$

Utility graph:

Complement of  $G$ :

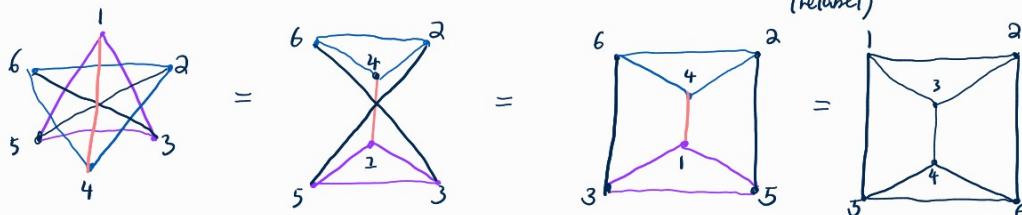
Subgraph:

## Isomorphism

Two graphs are isomorphic if  $\exists$  a 1:1 correspondance between their vertex sets s.t. adjacency of vertices is preserved.

## Recognizing Isomorphic Graphs

Method of Steel balls and Rubber bands



✓ Same number of vertices

✓ same number of edges

✓ same distribution of degrees

✓ same number of components

Isomorphic  $\rightarrow$  four conditions ✓



Equal  $\neq$  Isomorphic (equal: vertex set, exactly the same inc. label)

"A is B" commonly mean "A is isomorphic to B"

Polya Enumeration Theorem

Compute the number of graphs given the vertex count

## Chapter 3 Planar Graphs

### Introduction

A graph is planar if it is isomorphic to a graph drawn in-plane ("no edges cross")

without edge-crossings. Otherwise a graph is non-planar.

Planarity

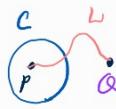
## UG, K<sub>5</sub>, and the Jordan Curve Theorem

Jordan Curve Theorem no points repeated except start point ↗ some endpoints

If  $C$  is a continuous simple closed curve in a plane,

then  $C$  divides the rest of the plane into two regions having  $C$  as the common boundary.

If a point  $P$  in one of these regions is joined to a point  $Q$  in the other region by a continuous curve  $L$  in the plane, then  $L$  intersects  $C$ .

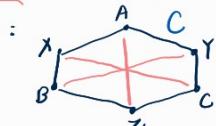


Corollary

If two points of  $C$  is joined by a continuous curve  $L$  in the plane having no other points common with  $C$ , then except for its endpoints,  $L$  is completely contained in one of the regions determined by  $C$ .

Thm UG is non-planar

Proof:



{A, Z} {B, Y} {C, Z}

two of them must be in one region divided by  $C$   
(pigeonhole principle)



Thm K<sub>5</sub> is non-planar (with the smallest number of vertices)

Thm Subgraph of a planar graph is planar. erasing vertices/edges cannot create edge-crossings.

Thm Supergraph of a non-planar graph is non-planar. suppose planar; contradiction

Subgraph = selective erasing ; Supergraph = selective augmenting

Are there more non-planar graphs?

Examples of non-planar graphs (infinite)

that are not supergraph of UG or K<sub>5</sub>



(1) prove not supergraph

(2) prove non-planar: assume planar.  
remove vertex and  
joining the edges become K<sub>5</sub>/UG  $\rightarrow$  <sup>non-</sup>planar  
contradiction.



Expansions

Def If some new vertices of degree 2 are added to some edges of a graph, the resulting graph is an expansion of the original graph.

Expansions  $\neq$  Supergraphs

both are augmentations of a graph, but by different procedures.

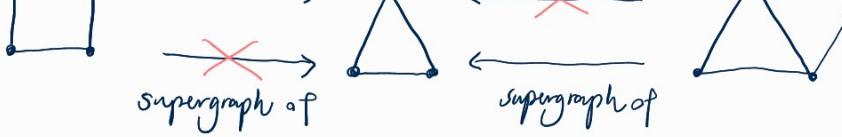
"Some new vertices of degree 2 are added to some edges"

the only allowed operation  
for Expansions

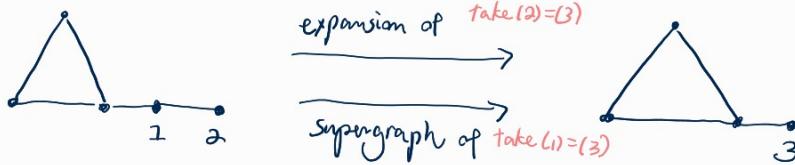
the only forbidden operation  
for Supergraphs

e.g.





e.g.



Thm Expansion of  $K_3$  or  $K_5$  is non-planar.

### Kuratowski's Theorem

Every non-planar graph is a supergraph of an expansion of  $K_3$  or  $K_5$ .

Corollary The set of all planar graphs is equal to  
the set of all graphs that are not supergraphs of expansions of  $K_3$  and  $K_5$ .

### Determining whether a graph is planar or non-planar

non-planar:  $\exists$  a subgraph that is isomorphic to a  $K_3$  or  $K_5$   
or the expansion of  $K_3$  or  $K_5$

## Chapter 4 Euler's Formula

### Introduction

Leonhard Euler "the Seven Bridges of Königsberg" (1736)  
→ earliest known work of graph theory

Def A walk in a graph:

a sequence  $A_1 A_2 A_3 \dots A_n$  of (not necessarily distinct) vertices,  $A_i$  joined by an edge to  $A_{i+1}, \dots$

The walk  $A_1 A_2 A_3 \dots A_n$  joins  $A_1$  and  $A_n$ .

Def A graph is connected if every pair of vertices is joined by a walk (otherwise: disconnected)

e.g. every cyclic / complete graph is connected

e.g. every null graph is disconnected except for  $N_1$ .

Def A planar graph (drawn in plane without edge-crossing) cuts the plane into regions called faces of the graph.  
(generalization of Jordan Curve Theorem)  
(number of faces is independent of the drawing)

only planar graph has faces

only define with cross-free drawings

Def A graph is polygonal if it is

\* planar

\* connected

\* every edge borders on two different faces

e.g.



since two edges only border on face #1

not polygonal since disconnected

non-planar  $\rightarrow$  not polygonal since no faces

## Euler's Formula

For planar connected graph,  $v+f-e=2$ .

### Mathematical Induction

Proof: ① Euler's formula on polygonal graphs

② Euler's formula on planar connected graphs that are not polygonal.

### Some consequences of Euler's formula

Thm If  $G$  is planar and connected with  $v \geq 3$ ,

$$\text{then } \frac{3}{2}f \leq e \leq 3v-6$$

#### Proof

Case ①:  $G$  has a face bounded by  $< 3$  edges.

$\rightarrow G$  must be  $P_3$  (path graph)  $f=1, e=2, v=3$

Case ②: Every face of  $G$  is bounded by  $\geq 3$  edges.

$\rightarrow \sum_i 3 \leq \sum_i \text{num. of edges bounding face } i \leq 2e$  equal if  $G$  is polygonal

$$\rightarrow 3f \leq 2e \rightarrow \frac{3}{2}f \leq e$$

$$\rightarrow v+f-e=2 \Rightarrow 2 \leq v+\frac{3}{2}e-e \Rightarrow e \leq 3v-6$$

Thm If  $G$  is planar and connected with  $v \geq 3$

and  $G$  is not a supergraph of  $K_3$ .

$$\text{then } 2f \leq e \leq 2v-4$$

Thm If  $G$  is planar (and connected)

then  $G$  has a vertex of degree  $\leq 5$

### Algebraic Topology

Euclidean geometry = connected with the "metric" properties of figures

Topology = properties of figures preserved by "continuous deformations"

e.g. Jordan Curve Theorem on "continuous simple closed curve"

Henri Poincaré = "qualitative" subject

Algebra = studies sets where "operations" are defined.

high school algebra: on Real numbers of  $\oplus \ominus \times \div$

Algebraic Topology = algebraic methods applied to topological problems

Convert → Take a T problem, convert to A problem  
Solve → try to solve the A problem  
reconvert → Recanvert the A solution into T terms  
result: a solution to the T problem

e.g. analytical geometry

Means of convert: associate every geometry point the "coordinates"

Straight lines  $\rightarrow$  equations  $\rightarrow$  system of equations, ...

# Enter's Formulas

↳ means of convert → graph theory (topology)  $K_5$  is non-planar  
 ↓  
 high school algebra (algebra) Proof:  $K_5$  connected,  
 "disadvantage": less conducive to understanding ← but  $e \neq 3v-6$ ,  
 contradiction

## Exercises

Def A component of a graph is a connected subgraph that is not contained in a larger connected subgraph.

## Generalization of Euler's Formula

If  $G$  is planar,  $v+f-e = 1+p$  where  $p$  is the # of components  
 When  $G$  is connected,  $p=1 \Rightarrow$  reduces to  $v+f-e=2$

Def The connectivity of a graph  $c$  is the smallest number of vertices whose removal (and incident edges) results in either  $K_1$  or a disconnected graph.

Def A bridge in a graph is an edge whose removal increases # of components.

Thm. If the connectivity is  $\geq 6$ , the graph is non-planar.

# Chapter 5 Platonic Graphs

## Introduction

- ① historical : "Platonic Solids" (after Plato)
  - ② heuristic : spectacular warning if overindulge the natural tendency
  - ③ pedagogical : power of Euler's formula

Def A graph is regular if all vertices have the same degree  
"regular of degree d"

e.g.  $C_7$  (cyclic) = regular of degree 2

$K_v$  (complete) : regular of degree  $v-1$

$Nv$  (null) = regular of degree 0

$UG$  = regular of degree 3

Def A graph is planar if it is

## ★ polygons

- \* regular

✓ regular  
✓ all f

- planar
- connected
- every edge borders on two different faces

{ isomorphic to a graph without edge-crossing }

All faces bounded by some number of edges

e.g.  $C_4$  (cycle) = planar

$K_4$  (complete) = only  $K_1$ ,  $K_3$ ,  $K_4$  planar



$N_v(\text{null})$  = only  $N_1$

$\text{UG}$  = not planar, since not planar

Theorem Other than ①  $K_1$ , ② Cyclic graphs, there are only 5 planar graphs. Lemma If  $G$  regular of degree  $d$ , then

$$e = \frac{d \cdot v}{2} \quad f = \frac{d \cdot v}{n}$$

solve  $(n-2)(d-2) < 4$

$$e = \frac{d \cdot v}{2}$$

$$f = \frac{d \cdot v}{n}$$

solve  $(n-2)(d-2) < 4$

Proof ... algebra with Euler's formula ...

$$v + f - e = 2 \Rightarrow v + \frac{dv}{2} - \frac{dv}{n} = 2 \Rightarrow v(2n + d - nd) = 4n$$

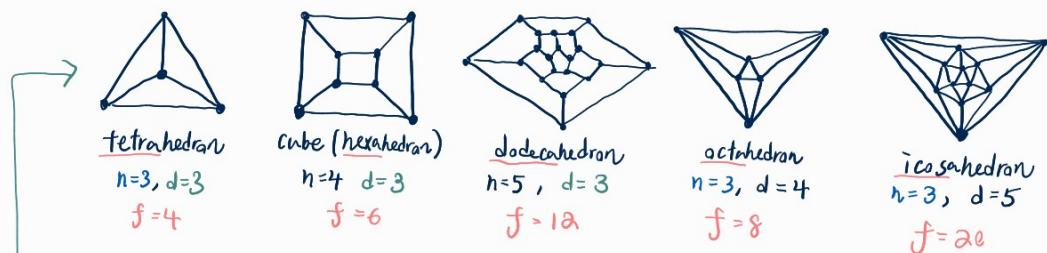
then compute  $v = \frac{4n}{2n + d - nd}$

$$(n-2)(d-2) < 4 \quad \text{where } n: \# \text{ edges bounding a face of a polygonal graph}$$

$\Rightarrow$  only five solutions!

$d$ : the degree of each vertex of  $G$

... more algebra on  $v, e, f, \dots$



Histam

Terminology: "regular polygons" = edges same length, angle same size infinitely many

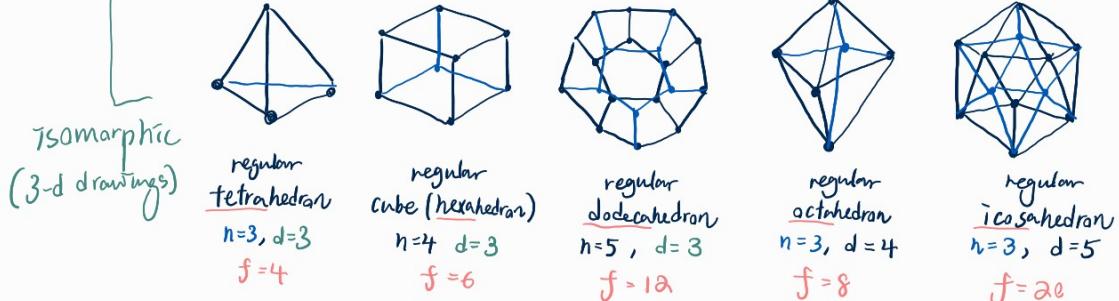
"regular polyhedron" = congruent regular polygons for faces, corner angles same size

Only 5 regular polyhedra

every planar graph 1:1 corresponds to a regular polygon or regular polyhedron

(cyclic graph) (5 planar graph)

except  $K_4$   
(trivial)



Isomorphic  
(3-d drawings)

regular tetrahedron  
 $n=4$ ,  $d=3$   
 $f=4$

regular cube (hexahedron)  
 $n=8$ ,  $d=3$   
 $f=6$

regular dodecahedron  
 $n=20$ ,  $d=3$   
 $f=12$

regular octahedron  
 $n=6$ ,  $d=4$   
 $f=8$

regular icosahedron  
 $n=12$ ,  $d=5$   
 $f=20$

Pythagoreans >

universe (dodeca)

+ {fire, earth, air, water}

Kepler = fire regular polyhedra  $\leftrightarrow$  six known planets

Exercises

Def A dual graph of a planar graph is formed by:

- ① taking a cross-free drawing of the planar graph
- ② place a dot inside each face
- ③ joining two dots whenever

the borders of two faces have one or more edge in common

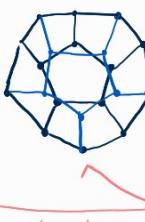
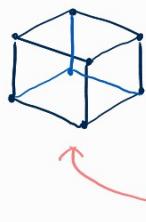
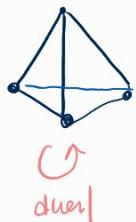
Only a planar graph has a dual

- Only a planar graph have a dual
- A dual is always planar and connected
- A dual is not always unique (from different cross-free drawings)

\* Planar graph: dual is unique

$K_1$  = dual is itself

$C_v$  = dual is always  $K_2$



## Chapter 6 Colorings

### Chromatic number

Def. A graph is colored if adjacent vertices have been assigned with different colors.

Def. Chromatic number  $\chi$  = the smallest number of colors which a graph can be colored.

### Coloring planar graphs

The Four Color Conjecture (Theorem) : Every planar graph has  $\chi \leq 4$ .

The Five Color Theorem ← easier to prove

"every planar graph has at least one vertex with degree less than or equal to 5"

the version for 4 is false e.g.  
Icosahedron ( $\delta=5$ )

### Coloring maps

Map coloring problem : find the smallest number  $m$  s.t.

the faces of every planar graph can be colored with  $\leq m$  colors  
s.t. faces sharing a border have different colors.

## Chapter 7 The Genus of a Graph

### The genus of a graph

Planar = ① no edge-crossing    ② drawn in a plane

Genus =   
 include this condition

generalize this by considering other surface

Infinite family of surface

\* hollow \* of negligible thickness



$S_0$



$S_1$



$S_2$

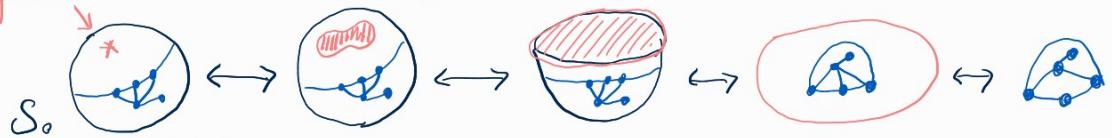


$S_3$

## Generalize Planar Graph (Chapter 3)

Def The genus of a graph  $g$  is the subscript of the first surface among  $S_0, S_1, S_2, \dots$  on which the graph can be drawn without edge-crossing.  
 ( $\rightarrow$  every graph has a genus!.)

Thm The set of all planar graphs = the set of all graphs with  $g = 0$



\* difference = the punctured face on  $S_0$  became the infinite face in planar

### Examples

$K_4, C_7$  has  $g = 0$  because they are planar.

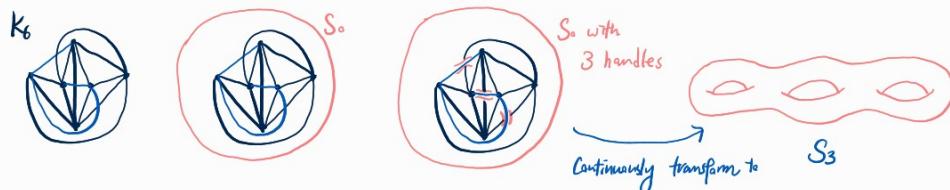
$UG, K_5$  has  $g = 1$



Thm Every graph has a genus

① transfer the drawing to  $S_0$

② add to  $S_0$  enough tubular "handles" to serve as "overpasses"



Thm If a graph has genus  $g$ .

then  $G$  can be drawn without edge-crossing on every surface  $S_n$  where  $g \leq n$

Def The faces of  $G$  are the regions divided by the edges and vertices of  $G$  on the surface  $S_g$ , where  $g$  is the genus of  $G$

## Generalize Euler's formula (Chapter 4)



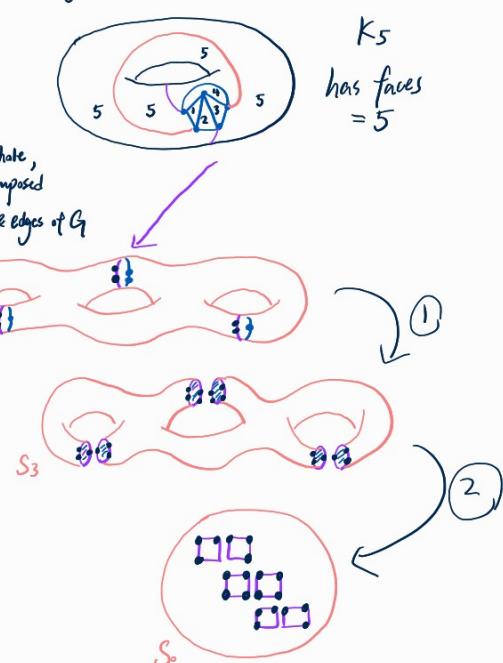
### Euler's Second Formula

$$v + f - e = 2 - 2g$$

### Proof

① Split each ring into two rings, forming the rim of an open-ended tube (total  $2g$  rim)

② Cover each rim with disc-shaped surface, inflate and continuously deform into  $S_0$



③ New graph  $H$  = crossing-free  
 connected planar }  $\rightarrow$  Euler's first formula applies  
 $v_H + f_H - e_H = 2$

④  $v_H - v_G = e_H - e_G$  (rings are Cyclic graph)  
 $f_H = f_G + 2g$  (new faces are created only for the rings)  
 $\therefore v_H + f_H - e_H = (v_H - e_H + e_G) + (f_H - 2g) - e_G$   
 $= v_H - e_H + f_H - 2g$   
 $= 2 - 2g$

### Some consequences

Lemma If a connected graph has  $v \geq 3$ , then  $3f \leq 2e$

Thm If  $G$  is connected with  $v \geq 3$  and genus  $g$ , then

$$g \geq \frac{1}{6}e - \frac{1}{2}(v-2)$$

\* provides a lower bound of  $g$  (even only knowing  $v, e$ )

Thm If  $v \geq 3$ , the complete graph  $K_v$  has genus

$$g = \lceil \frac{(v-3)(v-4)}{12} \rceil \quad \begin{array}{l} \text{for "}" \\ \text{for "}" \end{array} \begin{array}{l} \text{proof} \\ \text{difficult to prove until 1968} \end{array}$$

Corollary If  $G$  has  $v \geq 3$  and genus  $g$ , then

prove by drawing  $K_v$  without edge-crossing  
on  $S_n$  where  $n = \frac{(v-3)(v-4)}{12}$

$$g \leq \lceil \frac{(v-3)(v-4)}{12} \rceil \quad \text{Proof: } G \text{ subgraph of } K_5 \Rightarrow g_G \leq g_{K_5} \quad (\text{Lemma})$$

Corollary If  $G$  has  $v \geq 3$  and genus  $g$ , then

$$\text{lower } \lceil \frac{1}{6}e - \frac{1}{2}(v-2) \rceil \leq g \leq \lceil \frac{(v-3)(v-4)}{12} \rceil \text{ upper}$$

### Estimating the genus of a connected graph

the estimation becomes better when  $e$  is closer to its max  $\frac{1}{2}v(v-1)$

Thm If  $G$  is a connected incomplete graph with  $v \geq 3$ ,

then the upper and lower bound of  $g$  is either same or consecutive integers

$$\text{when } e \geq \frac{1}{2}v(v-1) - 5$$

Generalize  
planar graphs  
(Chapter 5)

### $g$ -Planar graphs

Def A graph is  $g$ -planar if

- (+ planar  
= polygonal) [ \* connected  
\* regular graph of genus  $g$   
\* every edge borders two faces  
\* every face bounded by same number of edges

Notation  $d$  = degree of each vertex

$n$  : # edges bounding each face

$$P(d, n) = \frac{(4g-4)n}{nd - 2d - dn} \quad \begin{array}{l} \text{(in Chapter 5:} \\ \text{use to compute } v \text{ when } g=0 \end{array}$$

Lemma If  $G$  is  $g$ -planar,  $e = \frac{d \cdot v}{2}$ ,  $f = \frac{d \cdot v}{n}$

Lemma(a) If  $G$  is  $g$ -planar and  $g > 0$ , then  $n \geq 3$  and  $d \geq 3$

Lemma (b) Let  $g > 1$ .  $(n-2)(d-2) > 4 \Leftrightarrow P(d, n) > 0$

Lemma (c) If  $g > 1$ ,  $(n-2)(d-2) > 4$ , then  
 $d' \geq d \geq 3 \Rightarrow P(d', n') \leq P(d, n)$   
 $n' \geq n \geq 3$

Chapter 5 → Infinitely many 0-planar graphs

Except 5 of them, either (trivial)  $K_1$  or (uninteresting)  $C_v$

Thm For each  $\underline{g > 1}$ , there are at most a finite number of  $g$ -planar graphs.

Proof  $v + f - e = 2 - 2g \Rightarrow v = \frac{(4g-4)n}{nd-2d-2n} = P(d, n)$

Case ①:  $n=3 \parallel P(d, n) > 0 \Rightarrow (n-2)(d-2) > 4$  so  $d \geq 7 \stackrel{(c)}{\Rightarrow} v = P(d, n) \leq P(7, 3)$

Case ②:  $n=4 \parallel \dots \Rightarrow v \leq P(5, 4)$

Case ③:  $n=5 \parallel \dots \Rightarrow v \leq P(4, 5)$

Case ④:  $n=6 \parallel \dots \Rightarrow v \leq P(4, 6)$

Case ⑤:  $n \geq 7 \parallel$  Any  $d \geq 3$  is admissible so  $v \leq P(3, 7)$  the largest among above numbers

In all cases,  $v \leq P(3, 7) = \frac{(4g-4) \cdot 7}{21-6-14} = 28(g-1)$   $\Rightarrow$  2-planar  $\Rightarrow v \leq 28$

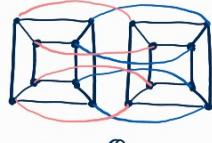
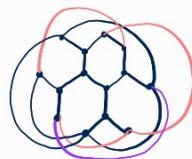
Since for each value of  $g > 1$ ,

there are only a finite number of graphs with at most  $P(3, 7)$  vertices

⇒ only a finite number of  $g$ -planar graphs.

Thm All 1-planar graphs have either: (infinitely many)

$d=3, n=6$        $d=4, n=4$        $d=6, n=3$



e.g.  
complete graph  $K_7$

Generalize:

Coloring  
(Chapter 6)

## The Heawood Coloring Theorem

Def If  $n \geq 0$  is an integer, then (That this is true ( $X(S_n)$  exists) for every  $n \geq 0 \Rightarrow$  the Heawood Coloring Theorem) the chromatic number of the surface  $S_n$ , denoted by  $X(S_n)$ , is the largest among the chromatic numbers of graphs having  $g \leq n$ .

Thm  $X(S_n)$  = the number of colors sufficient to color graphs with  $g \leq n$

Thm  $X(S_n)$  = the smallest number of colors sufficient to color every graph that can be drawn on  $S_n$  without edge-crossings.

The Four Color Conjecture (Theorem)  $\Rightarrow X(S_0) = 4$

Heawood Coloring Theorem If  $n > 0$  then

$$X(S_n) = \left\lceil \frac{7 + \sqrt{1 + 48n}}{2} \right\rceil$$

Proof  $X(S_n) \leq t \rightarrow$  long proof

$X(S_n) \geq t \rightarrow$  sufficient to find a graph  $g \leq n$ ,  $X=t$

$X$  has  $X \geq t$   $\Rightarrow$   $t$  is a lower bound for  $X$

$K_t$  has  $n = t \rightarrow$  obvious complete graph  $K_t$  has vertices =  $t$

$K_t$  has  $g \leq n \rightarrow$  consider  $t \geq 3$  ( $v \geq 3$ )

$$g = \lceil \frac{(t-3)(t-4)}{12} \rceil = \left\lceil \left( \lfloor \frac{7+1+48n}{2} \rfloor - 3 \right) \left( \lfloor \frac{7+1+48n}{2} \rfloor - 4 \right) \right\rceil \quad (t=x)$$

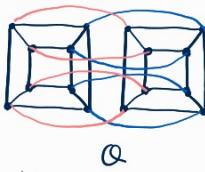
$$\leq \left\lceil \left\lfloor \frac{\frac{-1+1+48n}{4}}{12} \right\rfloor \right\rceil = \left\lceil \frac{\lfloor 12n \rfloor}{12} \right\rceil = \left\lceil \frac{12n}{12} \right\rceil = \lceil n \rceil = n$$

→ Tells us that  $S_0$  is the most "complicated" surface in coloring where  $X(S_0)$  is not known exactly. — until the Four Color Conjecture is proved!

$$X(S_0) = \left\lceil \frac{7+\pi}{2} \right\rceil = \left\lceil \frac{6}{2} \right\rceil = 4$$

### Exercises

$$d=4, n=4$$



→ skeleton of a "toroidal polyhedron"



a polyhedron that will look like  $S_1$  when inflated (square hole through a cube)

→  $K_7$  also the skeleton of a toroidal polyhedron → "Császár polyhedron"

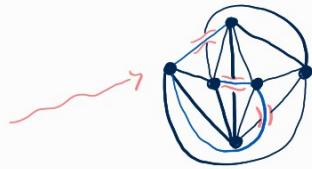
Def The crossing number of graph  $k$

is the min. number of edge-crossings that a graph can be drawn on plane.

e.g. All planar graph =  $k=0$

$K_5$  and  $UG$  =  $k=1$

$K_6 \leftarrow$  no formula for  $K_v$  =  $k=3$



## Chapter 8 Euler Walks and Hamilton Walks

### Introduction

Two of the oldest problems: "When does a graph have an Euler/Hamilton walk?"

Chapter 3-7: graphical properties defined in terms of

the surfaces on which the graphs can be drawn

this Chapter: graphical properties that are unrelated to

the surfaces on which the graphs can be drawn

### Euler walks

Def A walk  $A_1, A_2, \dots, A_{n-1}, A_n$  in a graph is closed if  $A_1 = A_n$ , otherwise the walk is open

Def An euler walk is a walk that uses every edge in the graph exactly once.

Thm For a connected graph,

closed euler walk  $\Leftrightarrow$  every vertex (has degree) even

open euler walk  $\Leftrightarrow$  has exactly two odd vertices

"the problem of the highway inspector"

### Hamilton walks

Ireland one famous mathematician: William Rowan Hamilton

Def An open hamilton walk = a walk that uses every vertex exactly once

A closed hamilton walk = except for the initial vertex that uses exactly twice "the problem of the travelling salesman"

Theorem If the sum of the degrees of every pair of vertices of a graph  $G$  is at least  $v-1$

Not insightful/practical  
 "if a graph has a lot of edges and if they are evenly distributed, hamilton walk exists"  
 is at least  $v$   
 A much more complicated problem than euler walks  
 any two vertices sum is  $4 < 6-1=5$

every pair of vertices are either adjacent to each other or to a common 3rd vertex  
 $\Rightarrow G$  connected

$\Rightarrow G$  has an open hamilton walk  
 $\Rightarrow G$  has a closed hamilton walk

Converse is not true e.g.  $C_6$   
 ✓ open/closed hamilton walk



## Multigraphs

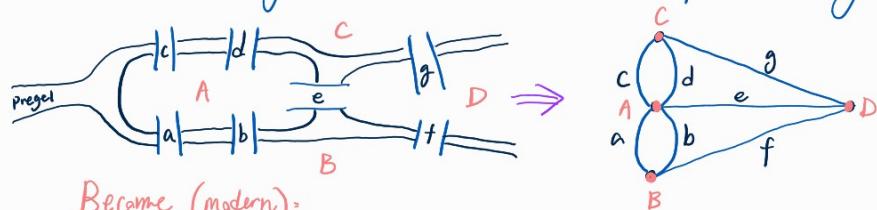
Def. A skein is an object of two dots connecting by two or more lines



Def. If some of the edges (and incident vertices) of a graph  $G$  are replaced by skeins, the result is a multigraph  $M(G)$  generated by  $G$ .  
 every graph is a multi-graph; a single graph generates  $M_1(G)$ ,  $M_2(G)$ , ..., infinitely many  
 Most concepts (complement, isomorphism, coloring, genus, etc.) can be defined to include Multigraphs

## The Königsberg Bridge Problem

Euler's epochal 1736 paper "the seven bridges of Königsberg"  
 ⇒ the birth notice of "the geometry of position" (now called topology!)  
 (Machine-building is a lot more fun than problem-solving)



Became (modern):

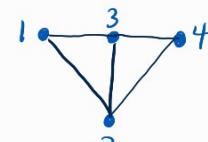
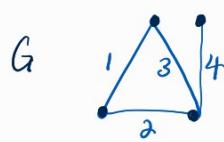
Does the multi-graph have an Euler walk?

## Exercises

Def. If  $G$  is a graph with  $e \neq 0$ ,

then the line graph  $L(G)$  is the graph having  
 # one vertex for each edge in  $G$

# two vertices of  $L(G)$  have an edge, if the corresponding edges in  $G$  share a vertex



$L(G)$

$K_4$



$L(K_4)$

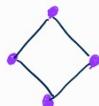


Prove

If  $G$  has a closed euler walk,  
then  $L(G)$  has both a closed euler walk  
and a closed hamilton walk

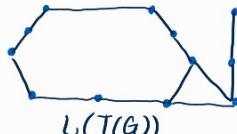
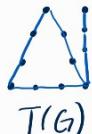
If  $G$  has a closed hamilton walk,  
then  $L(G)$  also has a closed hamilton walk

Cyclic graphs are isomorphic to their line graphs,  
and are the only graphs having this property.



Def

If  $G$  is a graph with  $e \neq 0$ ,  
then the trisection graph  $T(G)$  is the expansion of  $G$   
formed by slicing every edge of  $G$  by two vertices of degree 2.



Afterward

Appel and Haken (1976) : prove the Four Color Conjecture  
finding the unavoidable set of reducible configurations (total 1482)