Author: kiking

Chapter I. Theory of Sets

Topology: study of collections of objects that posses a mathematical structure.

Similar Example: Peano's Axioms for Natural Numbers

The set of Natural Numbers is a collection of objects N on which there is defined a function s, called the successor function, satisfying

(1) For each object $x \in N$, $\exists$ one and only one object $y \in N$ s.t. $y = s(x)$;

(2) If $s(x) = s(y)$, the $x = y$

(3) $\exists$ one and only one object $\in N$, denoted by 1, which is not the successor of an object $\in N$, i.e. $1 \neq s(x)\ \forall x \in N$

(4) Given a collection T of objects in N s.t.
1 is in T
$\forall x \in T$, $s(x) \in T$
then $T = N$
} Principle of Mathematical Induction

Concept:

Natural numbers: a collection of objects N with an additional mathematical structure, i.e. the successor functions

Topological space:

a collection of "subcollection of points" = open sets with a structure that endows this collection of points with some coherance, in the sense that we may speak of nearby points or points that in some sense are close to each other.

---

② Sets and Subsets

subsets
contains
proper subsets, improper subsets (itself, ϕ)
2^A: members are the subsets of A

③ Sets Operations

Author: kiking

③ Set Operations

union, intersection, Venn diagrams, complement

DeMorgan's laws $C(A\cup B) = C(A)\cap C(B)$

$C(A\cap B) = C(A)\cup C(B)$

④ Indexed Family of Sets

$\{A_\alpha\}_{\alpha\in I}$

⑤ Products of Sets

$A\times B$, Cartesian products of A and B

Direct Product $\prod_{i=1}^{n} A_i$

An n-tuple : $a = (a_1, a_2, \dots, a_n) \in A^n = \prod_{i=1}^{n} A_i$

⑥ Functions

$f: A\to B$ $\quad A \xrightarrow{f} B$

$(a, f(a))$: graph of $f: A\to B \in \Gamma_f \subset A\times B$

$\Gamma_f \subset X\times Y$

$= \{(x,y) \mid (x,y)\in X\times Y \text{ and } y = f(x)\}$

$f(X)$: image of X, $f^{-1}(Y)$: inverse image of Y

$f: A\to B$, A: domain, B: range

subjective onto : $B = f(A)$

injective one-one : $\forall b\in f(A)$, there is only one $a\in A$ s.t. $f(a) = b$

bijective one-one and onto

⑦ Relations

A relation R from the ele. of A to the ele. of B is a subset of $A\times B$

Reflexive : aRa $\forall a\in E$.

Symmetric : if aRb the bRa

Transitive : if aRb and bRc, then aRc

Equivalent Relation

Equivalent class of a : $\pi(a)$

the subset consisting of all x s.t. aRx

Equivalent classes :

either disjoint or identical

Quotient of E by Relation R :

E/R

Projection : $\pi : E \to E/R$ (onto)

⑧ Composition of Functions & Diagrams

Diagram
$$\begin{tikzcd} A \arrow[rr, "k"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

Commutative Diagram : $gf = k$

⑨ Inverse functions, Extensions, Restrictions

$f: A\to B$ inverse $g: B\to A$ inverse

$\Rightarrow$ Both one-one and onto

$\Leftrightarrow$ Necessary and Sufficient

Let $A\subset X$, $f: A\to Y$, $F: X\to Y$

if $\forall x\in A$ $f(x) = F(x)$

Author: kiking

$\to F$ an extension of f

$f = F|A$ = a restriction of F to A

Let $A \subset X$,

$i : A \to X$ where $i(x) = x$

$\Rightarrow$ inclusion mapping or function

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "i"'] & Y \\ & X \arrow[u, "F"'] \end{tikzcd}$$

$f = F|A = F_i$

(10) Arbitrary Products

Let $\{X_\alpha\}_{\alpha\in I}$: indexed family of sets

The Product of Sets $\{X_\alpha\}_{\alpha\in I}$, i.e.

$\prod_{\alpha\in I} X_\alpha$

consists of all functions x with domain the indexing set I s.t.

$\forall \alpha \in I,\ x(\alpha) \in X_\alpha$

Let $x \in \prod_{\alpha\in I} X_\alpha$.

$p_\alpha : \prod_{\alpha\in I} X_\alpha \to X_\alpha$ defined by $p_\alpha(x) = x(\alpha)$

$\Rightarrow$ α^{th} projection

Axiom of choices

if for each $\alpha \in I$ we can choose a point $x_\alpha \in X_\alpha$,

then we might construct $x \in \prod_{\alpha\in I} X_\alpha$ by setting $x(\alpha) = x_\alpha$.

$\Rightarrow$ Product of non-empty sets is non-empty

$x \in p_\alpha^{-1}(x_\alpha)$, where a point whose α^{th} coordinate is x_α, and other coordinates unrestricted.

Chapter 2 : Metric Spaces

(1) Intro

Continuity $\to$ w.r.t. subsets of neighbourhood of a point / open sets

(2) Metric Space

$d(a,b) = |a,b|$

Distance Function / Metric on X

① $d(x,y) \geq 0$

② $d(x,y) = 0$ iff $x = y$

③ $d(x,y) = d(y,x)$

④ $d(x,z) \leq d(x,y) + d(y,z)$

$\forall x,y,z \in X$

Metric Space : $(X,\ d : X\times X \to \mathbb{R})$

☆ (3) Continuity

☆ Original

Let $f : \mathbb{R} \to \mathbb{R}$. f conti. at point $a \in \mathbb{R}$,

if given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t.

$|f(x) - f(a)| < \varepsilon$

whenever

$|x - a| < \delta$

f is conti. if conti. at each point of $\mathbb{R}$.

Author: kiking

☆ Distance

Let (X,d) and (Y,d') be metric space, $a\in X$.
$f: X\to Y$ is conti. at $a\in X$
if given $\varepsilon>0$, $\exists\delta>0$ s.t.

$$d'(f(x), f(a))<\varepsilon$$

whenever $x\in X$ and

$$d(x,a)<\delta$$

④ Open Balls and Neighbourhoods

☆ Open Ball

$f:(X,d)\to(Y,d')$ is conti. at $a\in X$
iff given $\varepsilon>0$, $\exists\delta>0$ s.t.

$$f(B(a;\delta))\subset B(f(a);\varepsilon)$$

where $B(a;\delta)$ is the open ball at x of a radius δ.

$$B(a;\delta)\subset f^{-1}(B(f(a);\varepsilon))$$

A subset $N\subseteq X$ is a neighbourhood of a
if $\exists\delta>0$ s.t.

$$B(a;\delta)\subset N$$

(Metric Space definition)

☆ Neighborhood

Each Open Ball → Neighbourhood of each of their points.

$f:(X,d)\to(Y,d')$ is conti. at $a\in X$
iff for each neighbourhood M of $f(a)$,
$\exists$ neighbourhood N of a s.t.

$$f(N)\subset M$$

or equivalently

$$N\subseteq f^{-1}(M)$$

or when $f^{-1}(M)$ is also a neighbourhood of a

⇐ Proof

$$f(N)=f(B(a;\delta))\subset B(f(a);\varepsilon)\subset M$$

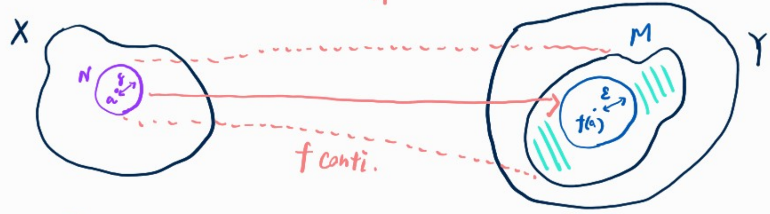
(f contin.)

⇒

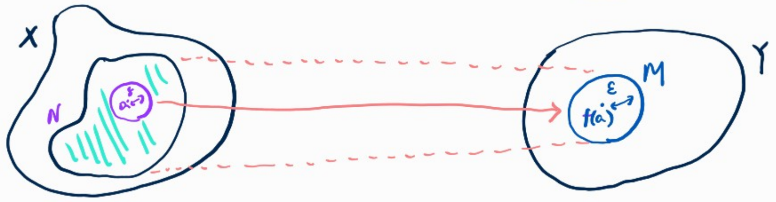
$$f(B(a;\delta))\subset f(N)\subset M=B(f(a);\varepsilon)$$

(neighborhood, neighborhood, neighborhood)

Proof ⇐



Proof ⇒



$\mathcal{B}_a$: A basis for the neighborhood system at a
if every neighborhood N of a
contains some element B of $\mathcal{B}_a$

e.g. if $a\in\mathbb{R}$, a basis = the collection of open intervals containing a

☆ No finite collection of subsets of $\mathbb{R}$

Author: kiking

is a basis for the system of neighborhoods at a

⑤ Limits

Let $a_1, a_2, \ldots,$ be a sequence of real numbers.
a is the limit of the sequence $a_1, a_2, \ldots$ if,
given $\varepsilon > 0$, $\exists$ pos N st.
whenever $n > N$, $|a - a_n| < \varepsilon$. $\Leftarrow$ $\lim_n a_n = a$

Let (X, d) be a metric space.
Let $a_1, a_2, \ldots$ be a sequence of points of X.
A point $a \in X$ is the limit of the sequence $a_1, a_2, \ldots$ if,
$\lim_n d(a, a_n) = 0$ $\Leftarrow$ $\lim_n a_n = a$

Corollary $\lim_n a_n = a$ iff $\forall$ neighbourhood V of a
$\exists N$ st. $a_n \in V$ whenever $n > N$

※ Limits

$f: (X, d) \to (Y, d')$ is conti. at $a \in X$
iff whenever $\lim_n a_n = a$ for a sequence $a_1, a_2, \ldots$ of points of X,
$\lim_n f(a_n) = f(a)$

Let $a \in X$ and $A \subseteq X$.
The distance between a and A $d(a, A)$
$=$ the greatest lower bound (g.l.b) of $d(a, x)$ $\forall x \in A$.

A bounded sequence of real numbers
has a convergent subsequence

⑥ Open Sets & Closed Sets

Def A subset O of a metric space
is open if O is a neighbourhood of each of its points.

Thm Open Set iff a union of open balls

※ Open Set

$f: (X, d) \to (Y, d')$ is continuous
iff $\forall$ open set O of Y,
the subset $f^{-1}(O)$ is an open subset of X

Def A subset F is closed
if $C(F)$ is open (the complement is open)

Author: kiking

Open and Close is NOT mutually exclusive,
e.g. ϕ and X is closed because
$\Rightarrow C(\phi) = X$, $C(X) = \phi$ are open

Def Limit Point

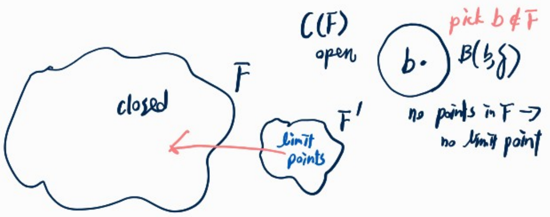
A point $b \in X$ is a limit point of a subset A if every neighborhood of b contains a point of A different from b

$\Rightarrow$ each $B(b, \frac{1}{n})$ contains a point a_n from A, and $\lim_n a_n = b$.

$\Rightarrow$ A limit point of a set is limit of a convergent seq. of points from A.

(Converse not true, e.g. when b is from $A \Rightarrow$ "isolated point")

Thm | A set $F \subset X$ is closed iff F contains all its limit point.



A set $F \subset X$ is closed iff each limit of sequence in F is also in F

✱ Closed Set

$f: (X,d) \to (Y,d')$ is continuous iff for each closed subset A of Y, the set $f^{-1}(A)$ is closed subset of X.

The union of closed sets need not to be a closed set,
E.g. $F_n = [\frac{1}{n}, 1]$ $\bigcup_{n=1}^{\infty} F_n = (0,1]$. 0 is a limit point but $\notin F$.

Closure $\bar{A} = A_{\text{limit points}} + A_{\text{isolated points}}$

$x \in \bar{A}$ iff $\exists$ a sequence in A converging to x

⑦ Subspaces & Equivalence of Metric Spaces

(Y, d') a subspace of (X,d)

if ① $Y \subset X$

Author: kiking

① $Y \subset X$

② $d' = d|_{Y \times Y}$.

$$\begin{tikzcd} Y \times Y \arrow[d, "i \times i"'] \arrow[r, "d'"] & \mathbb{R} \\ X \times X \arrow[ur, "d"'] & \end{tikzcd}$$

inclusion mapping

equals

diagram is commutative $\Rightarrow$ subspace

Metrically Equivalent / Isometric

defined by $f: A \to B$, $g: B \to A$: for $x, y \in A$, $d_B(f(x), f(y)) = d_A(x, y)$

for $u, v \in B$, $d_A(g(u), g(v)) = d_B(u, v)$

Then both f, g are continuous

Necessary & Sufficient Conditions:

① f is one-one

② f is onto

③ for each $x, y \in A$, $d_B(f(x), f(y)) = d_A(x, y)$

$$\begin{tikzcd} A \times A \arrow[d, "f \times f"'] \arrow[r, "d_A"] & \mathbb{R} \\ B \times B \arrow[ur, "d_B"'] & \end{tikzcd}$$

is commutative (f is distance-preserving)

Metrically Equivalent

$$\begin{tikzcd} A \arrow[rr, "i_A"] \arrow[dr, "f"'] & & A \\ & B \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, "i_B"] \arrow[dr, "g"'] & & B \\ & A \arrow[ur, "f"'] & \end{tikzcd}$$

Topologically Equivalent

(Chapter 3) The associated topological spaces are homomorphic.

$$\begin{tikzcd} A \times A \arrow[d, "f \times f"'] \arrow[r, "d_A"] & \mathbb{R} \\ B \times B \arrow[ur, "d_B"'] & \end{tikzcd} \qquad \begin{tikzcd} B \times B \arrow[d, "g \times g"'] \arrow[r, "d_B"] & \mathbb{R} \\ A \times A \arrow[ur, "d_A"'] & \end{tikzcd}$$

(f, g are inverse functions, continuous)

If two metric spaces are metrically equivalent, then they are topologically equivalent.

Converse is not true, e.g. a circle of radius 1 $\sim$ topologically equivalent to a circle of radius 2 but not metrically equivalent.

Corollary (X, d) (X, d') same underlying set

if $\exists K, K'$ pos s.t.

Author: kiking

$d'(x,y) \leq K \cdot d(x,y)$
$d(x,y) \leq K' \cdot d(x,y)$

then the identity mapping defines a topological equivalence between (X,d) and (X,d')

$d(x,y) = \max_i \{d_i(x_i, y_i)\}$

E.g. (R^n, d) and (R^n, d') where d = Max. distance func.

and d' = Euclidean distance func

are Topologically Equivalent since

$$d(x,y) \leq d'(x,y) \leq \sqrt{n}\, d(x,y)$$

Topological Equivalent

iff $\exists$ inverse functions

- establish a one-one correspondence between the open sets
- — // — the closed sets
- — // — the complete systems of neighborhood

of the two spaces.

Continuity equivalent statements

(X,d), (Y,d') metric spaces
$f: X \to Y$ $g: Y \to X$ = inverse functions

- f, g are continuous
- $O \subset X$ is open iff $f(O)$ is an open subset of Y
- $F \subset X$ is closed iff $f(F)$ is an closed subset of Y
- $\forall a \in X$, $N \subset X$,
  N is a neighborhood of a iff $f(N)$ is a neighborhood of $f(a)$

8) An Infinite Dimensional Euclidean Space

Hilbert Space

- Contains as subspaces Isometric Copies of the Euclidean spaces (R^n, d').
- A point $u \in H$ is a sequence $u_1, u_2, \ldots \in \mathbb{R}$ s.t. the series $\sum_{i=1}^{\infty} u_i^2$ is convergent
- Distance Metric

$$d(u,v) = \left[\sum_{i=1}^{\infty} (u_i - v_i)^2 \right]^{\frac{1}{2}}$$

☆ Need to prove that this series converges in H!!

Cauchy's Inequality / Schwarz's Lemma

Author: kiking

Lemma $\sum_i^n u_i v_i \le \left[\sum_i^n u_i^2\right]^{\frac{1}{2}} \left[\sum_i^n v_i^2\right]^{\frac{1}{2}}$

Collory Let $u, v \in H$. Let $U = \sum_i u_i^2$, $V = \sum_i v_i^2$

Then the series $\sum_i u_i v_i$ is absolutely convergent

and $\sum_i |u_i v_i| \le U^{\frac{1}{2}} V^{\frac{1}{2}}$

$= \sum_i |u_i||v_i| \le \left[\sum_i u_i^2\right]^{\frac{1}{2}} \left[\sum_i v_i^2\right]^{\frac{1}{2}} = U^{\frac{1}{2}} V^{\frac{1}{2}}$

Since partial sums of this series of positive terms are bounded and the series converges to a limit not greater than $U^{\frac{1}{2}} V^{\frac{1}{2}}$

i.e. $\left(\sum_i (u_i+v_i)^2\right)^{\frac{1}{2}} = \left(\sum_i |u_i^2 + 2u_i v_i + v_i^2|\right)^{\frac{1}{2}}$

$\le \left(\sum_i u_i^2 + 2\sum_i |u_i v_i| + \sum_i v_i^2\right)^{\frac{1}{2}}$

$\le \left(U + 2U^{\frac{1}{2}} V^{\frac{1}{2}} + V\right)^{\frac{1}{2}}$

$= \left(\left(U^{\frac{1}{2}} + V^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} = U^{\frac{1}{2}} + V^{\frac{1}{2}}$

Proof for $i = 1$ to n only!

Now: Subspace Property

Let E^n: collections of points $u = (u_1, u_2, \dots) \in H$ s.t. $u_j = 0 \ \forall j > n$.

To each point $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, associate $h(a) = (a_1, a_2, \dots, a_n, 0, 0, \dots) \in E^n$

$\Rightarrow$ h is a one-one mapping of $\mathbb{R}^n$ onto the subspace E^n of H.

$\Rightarrow$ Using $d'(a,b) = \left[\sum_i^n (a_i - b_i)^2\right]^{\frac{1}{2}}$ in $\mathbb{R}^n$

Isometry $\boxed{d'(a,b) = d(h(a), h(b))}$

$\Rightarrow$ Since E^n is a metric space (R^n, d') is a metric space

And h is an isometry of (R^n, d') with $(E^n, d|E^n)$.

(Exercises)

Define a vector space structure on Hilbert Space H.

$A(\vec{u}, \vec{v}) = \sum_{i=1}^{\infty} u_i v_i : V \times V \to \mathbb{R}$

$\vec{a}, \vec{b}, \vec{c}$: vectors $\in V$

α, β, γ : scalars $\in \mathbb{R}$

is a Positive Definite Bilinear form:

$$\left.\begin{array}{l} ① \ A(\alpha\vec{a} + \beta\vec{b}, \vec{c}) = \alpha A(\vec{a}, \vec{c}) + \beta A(\vec{b}, \vec{c}) \\ ② \ A(\vec{a}, \beta\vec{b} + \gamma\vec{c}) = \beta A(\vec{a}, \vec{c}) + \gamma A(\vec{b}, \vec{c}) \end{array}\right] \text{Bilinear}$$

$$③ \ A(\vec{x}, \vec{x}) > 0 \text{ unless } \vec{x} = \vec{0} \ \Big] \text{ Positive Definite}$$

$N(\vec{v}) = \left(A(\vec{v}, \vec{v})\right)^{\frac{1}{2}} : V \to \mathbb{R}$

Vector Space over $\mathbb{R}$: 7 Rules

a set V with 2 operations: Addition, Scalar Multiplication

$\forall u, v \in V$, $a \in \mathbb{R}$ $\xrightarrow{\text{def}}$ $u + v \in V$ $\quad a v \in V$

Author: kiking

is a Norm on V:

① $N(\vec{v}) \geq 0 \quad \forall \vec{v} \in \mathbb{R}$

② $N(\vec{v}) = 0$ iff $\vec{v} = \vec{0}$

③ $N(\vec{u}+\vec{v}) \leq N(\vec{u}) + N(\vec{v}) \quad \forall \vec{u}, \vec{v} \in V$

④ $N(\alpha \vec{v}) = |\alpha| N(\vec{v}) \quad \forall \alpha \in \mathbb{R}, \vec{v} \in V$

Set $d(\vec{u}, \vec{v}) = N(\vec{u} - \vec{v}) \quad \forall \vec{u}, \vec{v} \in V$

Then (V, d) is a metric space.

$$d(\vec{u}, \vec{v}) = N(\vec{u} - \vec{v}) = \left(A(\vec{u}-\vec{v}, \vec{u}-\vec{v})\right)^{\frac{1}{2}}$$
$$= \left(\sum_{i=1}^{\infty} (\vec{u}-\vec{v})(\vec{u}-\vec{v})\right)^{\frac{1}{2}}$$
$$= \left(\sum_{i=1}^{\infty} (\vec{u}-\vec{v})^2\right)^{\frac{1}{2}} = d$$

These are Continuous functions:

$a(\vec{u}, \vec{v}) = \vec{u} + \vec{v} : V \times V \to \mathbb{R}$

$b(\vec{v}) = -\vec{v} : V \to \mathbb{R}$

$c(\alpha, \vec{v}) = \alpha \vec{v} : \mathbb{R} \times V \to \mathbb{R}$

(1) V: a Commutative group under Addition, with identity element 0

(2) $a0 = 0, \quad \forall a \in \mathbb{R}$

(3) $0v = 0, \quad \forall v \in V$

(4) $1v = v, \quad \forall v \in V$

(5) $(\alpha+\beta)v = \alpha v + \beta v, \quad \forall \alpha, \beta \in \mathbb{R}, v \in V$

(6) $a(v+w) = av + aw, \quad a \in \mathbb{R}, \forall v, w \in V$

(7) $\alpha\beta v = \alpha(\beta v), \quad \forall \alpha, \beta \in \mathbb{R}, v \in V$

$(E^n, d|E^n)$
(V, d) $\Rightarrow$ Metric Spaces

Chapter 3: Topological Spaces

① Intro

Metric Space, discarding the distance function, retaining the open sets

$\Rightarrow$ Topological Space

All thms in Chapter 2 can be re-introduced via the characterization of the terms by means of open sets

Neighborhood Space

→ discard distance function

→ retain Neighborhood of points of the metric space

② Topological Space

X: non-empty set $\quad \mathcal{J}$: a collection of subsets of X

O_1: $X \in \mathcal{J}$

O_2: $\phi \in \mathcal{J}$

O_3: If $O_1, O_2, \ldots, O_n \in \mathcal{J}$, then $O_1 \cap O_2 \cap \cdots \cap O_n \in \mathcal{J}$ — finite intersection

O_4: If for each $\alpha \in I$, $O_\alpha \in \mathcal{J}$ then $\bigcup_{\alpha \in I} O_\alpha \in \mathcal{J}$ — arbitrary union

Open sets of Neighborhood Space $\Rightarrow$ Topological Space

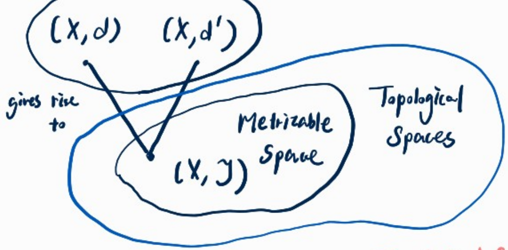
Then $(X, \mathcal{J})$ is a topological space. Members of $\mathcal{J}$ are called open sets. ※※※

If $\mathcal{J}$ is a collection of open sets of a metric space (X, d) then $(X, \mathcal{J})$ is a topological space associated with the metric space (X, d)

$\to (X, d)$ gives rise to $(X, \mathcal{J})$

Metric Space

Author: kiking



In passing from a metric space to a topological space

- Open sets are preserved
- Neighborhoods are preserved

In - Topological Space Definition || $N \subset X$ is a neighborhood of $a \in X$ if N contains an open set that contains a

Corollary || A subset $O \subset X$ is open iff O is a neighborhood of each of its points

Def || A subset $F \subset X$ is closed if the complement $C(F)$ is open

define a Neighborhood Space

Ex.

(1) Let $(X, \mathcal{J})$ be a metrizable topological space.

For each pair a, b of distinct point of X,

$\exists$ open set O_a and O_b s.t. $a \in O_a$ $b \in O_b$

and $O_a \cap O_b = \phi$.

$(\mathbb{Z}, \mathcal{J})$ where $\mathbb{Z}$ pos int, $\mathcal{J} = (\phi, \{1\}, \{1,2\}, \{1,2,3\}, \dots)$ is not a metrizable topological space

(2) $(\mathbb{R}^n, d)$ and $(\mathbb{R}^n, d')$ where

$$d = \max_i |x_i - y_i|$$

$$d' = \left(\sum_i (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

gives rise to the same topological space

Consider $(\mathbb{R}^n, \mathcal{J})$ where $\mathcal{J}$ is the collection of opensets of $\mathbb{R}^n$

3 Neighborhood and Neighborhood Space

Sharable Neighborhood properties between Metric Space & Topological Space:

Let $\mathcal{N}_x$: the collection of all neighborhoods of x

"complete system of neighborhood at point x"

Topological Space: no more "open ball"

N_1: For each $x \in X$, $\mathcal{N}_x \neq \phi$

N_2: For each $x \in X$ and $N \in \mathcal{N}_x$, $x \in N$

N_3: For each $x \in X$ and $N \in \mathcal{N}_x$, if $N' \supset X$ then $N' \in \mathcal{N}_x$

N_4: For each $x \in X$ and $N, M \in \mathcal{N}_x$, $N \cap M \in \mathcal{N}_x$

N_5: For each $x \in X$ and $N \in \mathcal{N}_x$, $\exists$ $O \in \mathcal{N}_x$ s.t. $O \subset N$ and $O \in \mathcal{N}_y$ for each $y \in O$ (O is a neighborhood of each of its point)

Not - sharable properties: Hausdorff axiom

In Metric Space

Author: kiking

In Metric Space,

given two distinct points $x, y \in (X,d)$

$\exists$ neighborhood N, M of x and y respectively

s.t. $N \cap M = \phi$

In Topological Space,

Not true : e.g.

$Y = \{a,b\}$ $a \neq b$ $\mathcal{J} = \{\phi, \{a\}, Y\}$

$(Y, \mathcal{J})$ is a topological space

$\Rightarrow$ the only neighborhood M of $b = Y$

$\Rightarrow$ $\forall$ neighborhood N of $a = \{\{a\}, Y\}$

$N \cap M = \{a\} \neq \phi$

Hausdorff axiom :

a topological space satisfying the hausdorff axiom is a Hausdorff space // "separated space"

Many authors requires a topological space to be a Hausdorff space!

Let X be a set. If a collection $\mathcal{N}_x$ of subsets of X (neighborhoods of x) satisfies $N_1 \sim N_5$, this object is a Neighborhood space.

Neighborhood Space

Def A subset O is open if it's a neighborhood of each of its points

Lemma ϕ, whole space, finite intersection of open sets, arbitrary union of open sets $\Rightarrow$ all open

Start with a Topological Space

- define Neighborhoods
- Underly set and the complete systems of neighborhood of the points of the set yield a Neighborhood Space

$(X, \mathcal{N}) = a(X, \mathcal{J})$ Neighborhood Space, Topological Space

Start with a Neighborhood Space

- define Open Sets
- Yields a Topological Space

$(Y, \mathcal{J}') = a'(Y, \mathcal{N}')$ Topological Space, Neighborhood Space

For each Topological Space $(X, \mathcal{J})$, $(X, \mathcal{J}) = a'(a(X, \mathcal{J}))$

Neighborhood Space $(X, \mathcal{N})$, $(X, \mathcal{N}) = a(a'(X, \mathcal{N}))$

$\Rightarrow$ 1-1 correspondence

Proof: Let O be an open set $\in \mathcal{J}$. (in Topological Space)

$\leftarrow \rightarrow O$ is a neighborhood of each of its points

Def : if N is a neighborhood of a point, N contains an open set that contains the point

Since O is open, O itself is the open set that contains each of its point

$\Rightarrow O$ is a neighborhood of each of its points

$\leftarrow \rightarrow O$ is an open subset of the Neighborhood Space

$\leftarrow \rightarrow O$ in the Neighborhood Space

Author: kiking

Let ...

Example

Let $\mathbb{Z}$: set of positive ints. For each $n \in \mathbb{Z}$,
Let $O_n = \{n, n+1, n+2, \dots\}$.
Let $\mathcal{J} = \{\phi, O_1, O_2, \dots, O_n\}$
Then $(\mathbb{Z}, \mathcal{J})$ is a Topological Space.

Let $\mathcal{U}$ be a Neighborhood of n
If for each integer $m \geq n$, $m \in \mathcal{U}$
Then $(\mathbb{Z}$, collections of $\mathcal{U})$ is a Neighborhood Space

Satisfy O_1, O_2, O_3, O_4 $\longleftrightarrow$ Satisfy $N_1 \sim N_5$

Ex. Base Neighborhood Space

A collection $\mathcal{B}_x$ of subsets of X satisfy

- BN_1: For each $x \in X$, $\mathcal{B}_x \neq \phi$
- BN_2: For each $x \in X$ and $\mathcal{U} \in \mathcal{B}_x$, $x \in \mathcal{U}$
- BN_3: For each $x \in X$ and $\mathcal{U}, V \in \mathcal{B}_x$, $\mathcal{U} \cap V$ contains an element $W \in \mathcal{B}_x$
- BN_4: For each $x \in X$ and $\mathcal{U} \in \mathcal{B}_x$, $\exists\, O \subset \mathcal{U}$ s.t. $x \in O$ and for each $y \in O$, O contains an element $V_y \in \mathcal{B}_y$

Where $\mathcal{B}_x$ is a base for the neighborhoods at x,
i.e. for each neighborhood N of x,
$\exists$ neighborhood $\mathcal{U} \in \mathcal{B}_x$ s.t. $\mathcal{U} \subset N$.

⊙ Neighborhoods of a base neighborhood space yields a Neighborhood Space
⊙ Base Neighborhood Space $\Leftrightarrow$ Neighborhood Space $\Rightarrow$ Many to One
(since many possible bases for the neighborhood at a point x)

4 Closure, Interior, Boundary

Closure A point x is in the closure of A, if all points in A is arbitrary close to x

In Metric Space,
$d(x, A) = 0$
or each neighborhood N of x contains a point of A

Def. In Topological Space,
For each neighborhood N of x,
$N \cap A \neq \phi$

Thm $\Leftrightarrow$ Def

Closure of a subset $A \Rightarrow \overline{A}$

Lemma. A closed set F contains $A \Rightarrow \overline{A} \subset F$

Lemma. If a point $x \notin \overline{A}$, then $x \notin F$ for some closed set F

Thm. $\overline{A} = \bigcap_{\alpha \in I} F_\alpha$ the family of all closed sets containing A.
(Characteristics. Smallest Closed set containing A)

Characteristics A is closed iff $A = \overline{A}$.

Collection of closures of subset A in a topological space
$\to$ yields a Closure Space $\leftrightarrow$ 1:1 correspondence to a topological space

Author: kiking

CL_1 : $\overline{\phi} = \phi$

CL_2 : $\overline{X} = X$

CL_3 : For each $A \subset X$, $A \subset \overline{A}$

CL_4 : For each pair $A, B \subset X$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$

CL_5 : For each $A \subset X$, $\overline{\overline{A}} = \overline{A}$

Interior

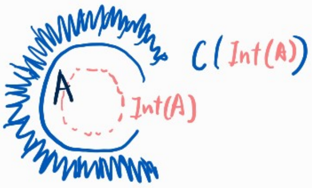
A point x is in the interior of A if A is a neighborhood of x.

Interior of $A \to Int(A)$

Lemma. An open set O contained in $A \to O \subset Int(A)$

|| Thm : $Int(A) = \bigcup_{\alpha \in I} O_\alpha$ the family of all open sets contained in A

(Characteristics : $Int(A)$ is the largest open set contained in A)



$C(Int(A)) = \overline{C(A)}$

$Int(A) = C(\overline{C(A)})$

$C(\overline{A}) = Int(C(A))$

Boundary

A point x is in the boundary of A if $x \in \overline{A}$ and $x \in \overline{C(A)}$

closure of A — closure of the complement of A

(x is arbitrary close to both A and $C(A)$)

Boundary of $A \to Bdry(A) = \overline{A} \cap \overline{C(A)}$

(closed since an intersection of two closed sets)

Ex.

A subset $A \subset X$ is dense if $\overline{A} = X$.

⊖ If for each open set O we have $A \cap O \neq \phi$, then A is dense

⊖ Rational Density Theorem : For $\mathbb{R}$ between any two real numbers, 正常人
$\exists$ rational number in between 痴线佬

Proof: Archimedean Principle : For $\mathbb{R}$, if $c, d > 0$, $\exists N > 0$ s.t. $Nc > d$.

⊖ → can prove that rational numbers are dense

⑤ Functions, Continuity, Homeomorphism

Continuous : $f : (X, \mathcal{J}) \to (Y, \mathcal{J}')$ is continuous at a point $a \in X$

Author: kiking

if for each neighborhood N of $f(a)$, $f^{-1}(N)$ is a neighborhood of a.

f is continuous if f is continuous at each point of X.

Thm: ⊖ Continuous iff for each open subset O of Y,
$f^{-1}(O)$ is an open subset of X.

⊖ Continuous iff for each closed subset F of Y,
$f^{-1}(X)$ is a closed subset of X.

!! Important: iff inverse image is open

$\neq$ open mapping = the image of each open set is open

A function can be open mapping but not continuous,

e.g. $f: \mathbb{R} \to Y = \begin{cases} f(x) = a & \forall x \geq 0 \\ f(x) = b & \forall x < 0 \end{cases}$

(Let $Y = \{a, b\}$ and all subsets open)

Then for each open subset U of $\mathbb{R}$, $f(U)$ is open
but for e.g. open subset $\{a\}$ of Y, $f^{-1}(\{a\}) = [0, \infty]$ is closed
$\Rightarrow$ not continuous.

⊖ Continuous iff for each subset A of X,
$f(\bar{A}) \subset \overline{f(A)}$

Homomorphism: Topological spaces are homomorphic
if $\exists$ inverse functions $f: X \to Y$, $g: Y \to X$ > f, g define a homormorphism between $(X, \mathcal{J})$ and $(Y, \mathcal{J}')$
s.t. f, g are continuous

Topologically Equivalent: Let (X, d) and (Y, d') be metric spaces.
The two are topologically equivalent iff the two associated topological spaces $(X, \mathcal{J})$ and $(Y, \mathcal{J}')$ are homeomorphic.

Necessary & Sufficient Conditions: A function $f: (X, \mathcal{J}) \to (Y, \mathcal{J}')$ s.t.

1. f is one-one
2. f is onto
3. A subset O in X is open iff $f(O)$ is open.

(Ex. $\forall$ point $x \in X$, $\forall$ subset N of X,
N is a neighborhood of x iff $f(N)$ is a neighborhood of $f(x)$)

Ex. $f: (X, 2^X) \to (Y, \mathcal{J}')$ f is always continuous,
$f: (X, \mathcal{J}) \to (Y, \{\phi, Y\})$

Author: kiking

... where $\mathcal{J}$, $\mathcal{J}$ are any topology on X / Y.

⑥ Subspaces

If Y is a subspace of X,
each open subset O' of Y is the restriction to Y
of an open subset O of X.

Def. Y is a subspace of X if
① $Y \subset X$
② The open subsets O' of Y: $O' = O \cap Y$
for some open sets O of X.
Note: The relatively open subsets of Y
are in general not open in X.

Number of subspaces in a topological space X
= Number of subsets in X.
Proof: ① Define collection $\mathcal{J}'$ to be subsets O' of Y of the form $O' = O \cap Y$ (from X)
② $(Y, \mathcal{J}')$ is a topological space → members are open
→ $(Y, \mathcal{J}')$ is a subspace
The topology $\mathcal{J}'$ of Y is induced by the topology $\mathcal{J}$ of X

(Relative) Neighborhoods on Y → $\mathcal{J}'$: relative topology of Y

Thm N' a relative neighborhood of a in Y
iff $N' = N \cap Y$
where N a neighborhood of a in X

Thm A subset F' is relatively closed in Y
iff $F' = F \cap Y$
for some closed subset F of X

Example
Let $a < b < c < d$.
Let $Y = [a,b] \cup (c,d)$ be considered a subspace of $\mathbb{R}$.
Then the subset $[a,b]$ is relatively open & relatively closed.
Why? $[a,b] = \underbrace{[a,b]}_{\in \mathbb{R}} \cap Y$ ← relatively closed

$[a,b] = \underbrace{(a-\varepsilon, a+\varepsilon)}_{\in \mathbb{R}} \cap Y$ where $\varepsilon < c-b$.
↖ relatively open
(c,d) = Complement of $[a,b]$ ← also relatively open & close

Def Let $\mathcal{J}_1, \mathcal{J}_2$ be two topologies on set Y.
$\mathcal{J}_1$ is weaker than $\mathcal{J}_2$ if $\mathcal{J}_1 \subset \mathcal{J}_2$

If Y is a subspace of $(X, \mathcal{J})$
Then the relative topology $\mathcal{J}'$ on Y is the weakest topology
s.t. the inclusion mapping $i: (Y, \mathcal{J}') \to (X, \mathcal{J})$

is continuous

Ex. A subspace of a Hausdorff space is a Hausdorff space.
A subspace of a metrizable space is a metrizable space.

7 Products

$X = \prod_{i=1}^{n} X_i$ The unions of the products of open sets constitute a topology

Def: The topological space $(X, \mathcal{J})$
where $\mathcal{J}$ is the collection of subsets of X
that are union of sets of the form
$O_1 \times O_2 \times \cdots \times O_n$
and each O_i an open subset of X_i,
is called the product of the topological spaces $(X_i, \mathcal{J}_i)$

$\{O_\alpha\}_{\alpha \in I}$ a collection of open sets in X
is a basis for the open sets of X,
if each open set is a union of members of $\{O_\alpha\}_{\alpha \in I}$

A subset N is a neighborhood of a point $a = (a_1, a_2, \ldots, a_n) \in N$
iff N contains a subset of the form $N_1 \times N_2 \times \cdots \times N_n$
where N_i is a neighborhood of a_i

A collection $\mathcal{N}_a$ of neighborhoods of a
is a basis for the neighborhoods of a
if each neighborhood N of a contains a member of $\mathcal{N}_a$

$\{(X_\alpha, \mathcal{J}_\alpha)\}_{\alpha \in A}$: indexed family of topological spaces.
Topological product of this family =
$X = \prod_{\alpha \in A} X_\alpha$ with $\mathcal{J}$ = all unions of sets of the form
$p_{\alpha_1}^{-1}(O_{\alpha_1}) \cap \cdots \cap p_{\alpha_k}^{-1}(O_{\alpha_k})$, where $O_{\alpha_i} \in \mathcal{J}_{\alpha_i}$, $i = 1, \ldots, k$

Ex. Cantor set D
consists of all real numbers $a \in [0,1]$
which can be represented as triadic decimals
$a = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ s.t. $a_n \in \{0, 2\}$ $\forall n$

Ex. Family of open intervals with rational end points
is a basis for the topology of the real line.

8 Identification Topologies

Let $p: E \to B$ continuous function
mapping topological space E onto
topological space B

Author: kiking

p: identification if
for each subset U of B,
$f^{-1}(U)$ is open implies U is open

Let $p: X \to Y$ continuous function
from a topological space X onto
a set Y

The identification topology on Y determined by p
consists of those sets U s.t.
$p^{-1}(U)$ is open in X. and then p is an identification

$$\begin{tikzcd} & X \arrow[dl, "\pi_f"'] \arrow[dr, "f"] & \\ X/\sim_f \arrow[rr, "f^*"] \arrow[drr, "f^*"'] & & (Y, f^*(\mathcal{J})) \\ & & (Y, S) \end{tikzcd}$$

$f^*(\mathcal{J})$: topology of $\mathcal{J}$ in X carried over to Y
→ smallest / weakest topology s.t. f is continuous

i: inclusion mapping continuous

$\sim_f$: equivalent relation
s.t. $x \sim_f x'$
if $f(x) = f(x')$

S: any topology on Y

Examples!!!

R: Real line S: Circle H: helix in $\mathbb{R}^3$

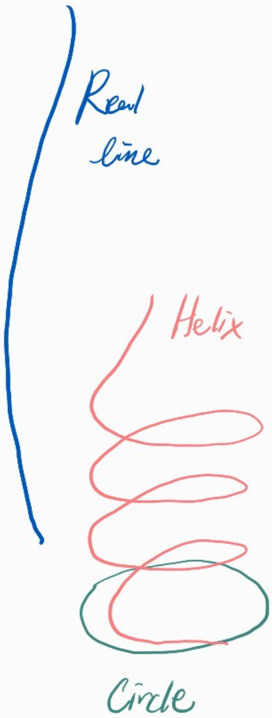
$p(t): R \to S = (\cos 2\pi t, \sin 2\pi t)$
a continuous mapping of the real line onto the circle

p is an identification mapping
i.e. if $U \subset S$ is s.t. $p^{-1}(U)$ is open then U is open

$g(t): R \to H = (\cos 2\pi t, \sin 2\pi t, t)$
a homomorphism of the real line onto the helix

Let S = sets of points (x, y, z) in $\mathbb{R}^3$
defined by $x^2 + y^2 = 1$, $z = 0$

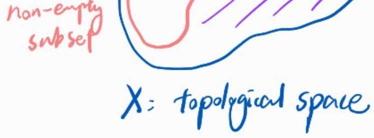
Then the projection of H onto S by
$(\cos 2\pi t, \sin 2\pi t, t) \to (\cos 2\pi t, \sin 2\pi t, 0)$
is also an identification mapping



Shrinking a subset to a point

Let X/A:
$\{X - A\} \cup \{a^*\}$

Author: kiking

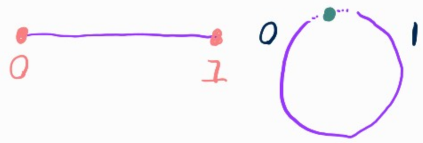


Define $f: X \to X/A$

$= \begin{cases} x & \forall x \in X - A \\ a^* & \forall x \in A \end{cases}$

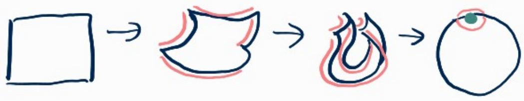
The space (X/A, identification topology by f) is obtained by shrinking A to a point.

Let $\dot{I} = \{0, 1\}$ be the boundary of the unit interval $[0,1]$.
Then $I/I^{\circ} = \{ (0,1) \} \cup \{\text{a point}\}$ is homomorphic to a circle.



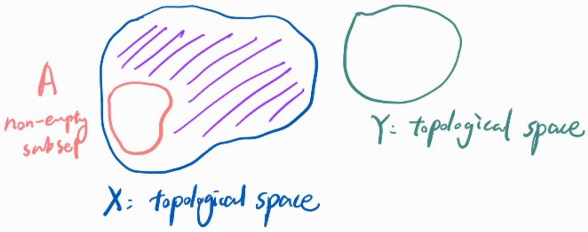
If the boundary of a square is shrink to a point, the resulting space is homeomorphic to the surface of a globe or a 2-sphere

gathering the string on the bdy of elastic sheet to a point



Generalize: attaching a space X to a space Y.

Let X, Y topological spaces, A non-empty closed subset of X
X, Y disjoint, $f: A \to Y$ given



Let $X \cup_f Y = (X - A) \cup Y$
and identification topology by $\varphi = \begin{cases} x & \forall x \in X - A \\ f(x) & \forall x \in A \end{cases}$

$\to$ A space attaching X to Y

Let I^2: unit square in $\mathbb{R}^2$
A: union of its two vertical edges
$\{(x,y) \in \mathbb{R} \mid x = 0, 0 \le y \le 1 \text{ or } x = 1, 0 \le y \le 1\}$
Y: $[0,1]$ the unit interval
Define $f: A \to Y$ by $f(x,y) = y$
Then $I^2 \cup_f Y$ is the cylinder formed by identifying the two vertical edges of I^2.

Author: kiking

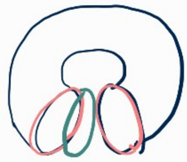
Ex.

Let C a circular cylinder, S_1, S_2 boundary circles

Let $f: S_1 \cup S_2 \to S$

which is an identification mapping that map S_1, S_2 homomorphically into a third circle S

Then $C \cup_f S$ is a torus



Define a relation in $\mathbb{R}^2$ by

$(x,y) \sim (x',y')$ if $x-x'$, $y-y'$ are integers.

$\sim$ is an equivalent relation.

Let T be the collection of equivalence sets

and $\varphi: \mathbb{R}^2 \to T$ the mapping for each point $\to$ its equivalence set

Give T the identification topology defined by φ.

Then T is homeomorphic to a torus.

Let a circular cylinder $S \times [0,1]$

A: a subset given by $S \times [1]$

Then $S \times [0,1] / A$ is a cone over S,

homomorphic to

homomorphic to a disc $D = \{(x,y) \mid (x,y) \in \mathbb{R}, x^2+y^2 = 1\}$

⑨ Categories and Functors

	Category C	collections of objects A	for each ordered pair (X,Y) of objects a set $H(X,Y)$ the maps of X into Y +	A rule of composition associative and identity exists
Chp 1	Category C_S	Class of all sets A_S	Sets of all functions from X to Y for $X, Y \in A_S$	e.g. $f \in H(X,Y)$ $g \in H(Y,Z)$ $h \in H(Z,W)$; $gf \in H(X,Z)$ $h(gf) = (hg)f$; $\exists\, 1_X \in H(X,X)$ s.t. $\forall g \in H(X,Y)$ $g 1_X = g$, $h 1_X = h$ $h \in H(W,X)$
Chp 2	Category C_M	All metric spaces	Continuous functions	Passage from a metric space (X,d) to its associated topological space $(X,\mathcal{T})$ is a functor from C_M to C_T.
Chp 3	Category C_T	All topological spaces	Continuous mappings	
Eg.	Category C_G	All groups	Homomorphisms	

Group G: a set G together with a function which associates to each ordered pair g_1, g_2 of elements of G an element $g_1 g_2 \in G$ s.t.

(1) $g_1(g_2 g_3) = (g_1 g_2) g_3 \quad \forall g_1, g_2, g_3 \in G$

Author: kiking

(2) $\exists\, e \in G$ identity s.t. $\forall g \in G$, $eg = ge = g$

(3) $\forall g \in G$ $\exists\, g^{-1} \in G$ inverse of g s.t. $gg^{-1} = g^{-1}g = e$

Homomorphism: f from a Group G to a Group K is a function $f: G \to K$ s.t. $f(e) = e'$ if e and e' are identities of G and K resp. and $\forall g, g' \in G$, $f(gg') = f(g)f(g')$

Def Let C and C' be categories with objects A and A' respectively.

A (covariant) functor $F: C \to C'$ is a pair of functions F_1 and F_2 s.t.

$F_1 : A \to A'$

$F_2 : H(X,Y) \to H'(F_1(X), F_1(Y))$ for each ordered pair X, Y of objects of A

s.t.

$F_2(1_X) = 1_{F_1(X)}$

$F_2(gf) = F_2(g)F_2(f)$ $\quad \forall f \in H(X,Y),\ g \in H(Y,Z)$

Denote $f \in H(X,Y)$ by $X \xrightarrow{f} Y$.

If $F: C \to C'$ is a functor, then $F_1(X) \xrightarrow{F_2(f)} F_1(Y)$.

F_2 preserves identities, and

if
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$
commutative then so is
$$\begin{tikzcd} F_1(X) \arrow[r, "F_2(f)"] \arrow[dr, "F_2(h)"'] & F_1(Y) \arrow[d, "F_2(g)"] \\ & F_1(Z) \end{tikzcd}$$

F carries commutative diagrams into commutative diagrams.

Examples of functors

① Passage from a metric space (X,d) to its associated topological space $(X, \mathcal{T})$:
a functor from C_M to C_T

② A functor from C_T to itself:

Let Z: fixed topological space.

To each topological space $X \in C_T$

- associate the topological space $F_1(X) = X \times Z$

To each continuous function $f \in H(X,Y)$

- associate the function $F_2(f)$ defined by
$(F_2(f))(x,z) = (f(x), z)$ $\quad \forall (x,z) \in F_1(X)$

Then $F_2(f): F_1(X) \to F_1(Y)$ is continuous and $F = (F_1, F_2)$ is a functor.

Chapter 4: Connectedness

① Introduction

- A subspace of a topological space is "connected" if it is all "of one piece", i.e.
impossible to decompose the subspace into two disjoint non-empty open sets

Fact:

Author: kiking

Fact: On real lines,

single points & intervals are non-empty connected subsets

Fact: The continuous image of a connected set is necessary a connected set

→ Intermediate Value Theorem

A continuous function $f: [a,b] \to \mathbb{R}$ must assume all values between $f(a)$ and $f(b)$

⊖ Second type of connectedness: "path-connectedness"

Each pair of points may be "connected" by a path/arc

Stronger condition than connectedness

⊖ Third type of connectedness: "Simple Connectedness"

A topological space is simply connected

if there are no holes in it, to prevent the continuous shrinking of each closed arc to a point

The degree to which a topo space failed to be simply connected may be measured by an algebraic topological invariant: the fundamental group of the space.

② Connectedness

X is Connected: only X and ϕ are simultaneously open and closed

disconnected: iff $\exists$ non-empty open (closed) subsets P, Q

s.t. $P \cup Q = X$, $P \cap Q = \phi$ ← P, Q are both open and closed, but not X nor ϕ

E.g. $A = [0,1] \cup (2,3)$ is disconnected

Since $[0,1]$ is closed in $\mathbb{R}$, but relatively open in A

Since $[0,1] = (-\frac{1}{2}, \frac{3}{2}) \cap A$

and $[0,1] \neq \phi$ $[0,1] \neq A$.

→ Lemma: iff $\exists$ open (closed) subsets P, Q of X

s.t. $A \subset P \cup Q$, $P \cap Q \subset C(A)$

and $P \cap A \neq \phi$, $Q \cap A \neq \phi$

Thm Let X, Y be topological spaces and let $f: X \to Y$ be continuous

If A is a connected subset of X, then $f(A)$ is a connected subset of Y

Connectedness is preserved under continuous mapping

Lemma Let $f: X \to Y$ be a continuous mapping of X onto Y

let X be connected; then Y is connected

Lemma Let X, Y be homeomorphic topological spaces

then X is connected iff Y is connected

Topological property:

if each topological space homeomorphic to the given space must also possesses this property.

⇒ Connectedness is a topological property

Lemma Let $Y = \{0, 1\}$.

A topological space X is connected iff

the only continuous mappings $f: X \to Y$ are the constant mappings

Proof Let $f: X \to Y$ be a continuous non-constant mapping. → Prove X is disconnected

Then $P = f^{-1}(\{0\})$ and $Q = f^{-1}(\{1\})$ are both non-empty.

Thus $P \neq \phi$ and $P \neq X$.

$\{0\}$ $\{1\}$ open subsets of Y, f continuous → P, Q open subsets of X

but $P = C(Q) \Rightarrow$ P and Q are open and close simultaneously

$\Rightarrow X$ is disconnected

Conversely, suppose X is disconnected → Prove $\exists$ non-constant continuous mapping

Then $\exists$ non-empty open subsets P, Q s.t.

$P \cap Q \neq \phi$ and $P \cup Q = X$.

Define $f: X \to Y = \begin{cases} 0 & \text{if } x \in P \\ 1 & \text{if } x \in Q \end{cases}$

f is continuous since they are from open subsets

... four open subsets,

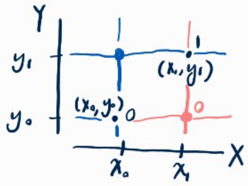
$\phi, \{0\}, \{1\}, Y$ of Y and

$f^{-1}(\phi)=\phi$, $f^{-1}(\{0\})=P$, $f^{-1}(\{1\})=Q, X$

So that the inverse image of an open set is open.

Thm X, Y connected topological space. Then $X \times Y$ is connected.

Proof: Show that the only continuous mapping $f: X \times Y \to \{0,1\}$ are constant mappings.



Suppose $\exists f$ non-constant continuous mapping.

Then $\exists (x_0,y_0)\ (x_1,y_1) \in X \times Y$ s.t.

$f(x_0,y_0)=0 \quad f(x_1,y_1)=1$.

Suppose $f(x_1,y_0)=0$

Define an "imbedding" $i_{x_1}: Y \to X \times Y$ by $i_{x_1}(y)=(x_1,y)$

i_{x_1} is continuous $\to f i_{x_1}: Y \to \{0,1\}$ is continuous

But $(f i_{x_1})(y_0) = f(x_1,y_0)=0 \quad (f i_{x_1})(y_1)=f(x_1,y_1)=1$

$\Rightarrow \exists$ non-constant mapping of Y into $\{0,1\}$

Contradicting the connectedness of Y!

$f: X \times Y \to \{0,1\}$ must remain constant on each of the connected subsets $\{x_0\} \times Y$ and $X \times \{y_1\}$.

Corollary: if $X_1, X_2, \dots, X_n$ are connected topological spaces, then $\prod_{i=1}^{n} X$ is a connected topological space

Lemma: Let $\{X_\alpha\}_{\alpha \in I}$ be an indexed family of topological spaces each of which is connected.

Let x, x' be two points of $X = \prod_{\alpha \in I} X_\alpha$ s.t. $x(\alpha)=x'(\alpha)$ except on a finite set of indices $I' \subset I$ and let $f: X \to \{0,1\}$ be continuous.

Then $f(x)=f(x')$. (Alternating a finite set of coordinates cannot change the value of a continuous function $f: X \to \{0,1\}$)

Thm: $X = \prod_{\alpha \in I} X_\alpha$ is connected if each X_α is connected.

③ Connectedness on the Real Line

Def A subset A of the real line is an interval if A contains at least 2 distinct points, and if given points $a, b \in A$ with $a<b$, then for each point x s.t. $a<x<b$, it follows that $x \in A$.

Thm Apart from empty set and single points, the only connected subsets of real line are intervals (iff)

Ex A homeomorphism $f: [a,b] \to [a,b]$ carries end-points into end-points.

Monotone Increasing: Let A, B subsets of $\mathbb{R}$.

$f: A \to B$ is monotone increasing if $x, y \in A$ and $x<y$ imply $f(x)<f(y)$

$f: A \to B$ monotone increasing $\to f: A \to B$ is one-one.

$f: [a,b] \to [f(a), f(b)]$ monotone increasing and continuous. $\to f$ is a homeomorphism.

④ Some applications of Connectedness

Intermediate-Value-Theorem

Let $f: [a,b] \to \mathbb{R}$ be continuous. Let $f(a) \neq f(b)$.

Then for each number V between $f(a)$ and $f(b)$

$\exists$ a point $v \in [a,b]$ s.t. $f(v)=V$.

Author: kiking

Proof $[a,b]$ is connected hence $f([a,b])$ is connected and thus an interval.

Now $f(a), f(b) \in f([a,b])$. Thus if V is between $f(a)$ and $f(b)$, $V \in f([a,b])$ since $f([a,b])$ is an interval.

Thus, $\exists\, v \in [a,b]$ s.t. $f(v) = V$.

Corollary

Let $f: [a,b] \to \mathbb{R}$ be continuous. If $f(a) f(b) < 0$, then $\exists\, x \in [a,b]$ s.t. $f(x) = 0$

Fixed-Point Theorem

Special Case of

Let $f: [0,1] \to [0,1]$ be continuous.

Then $\exists\, z \in [0,1]$ s.t. $f(z) = z$.

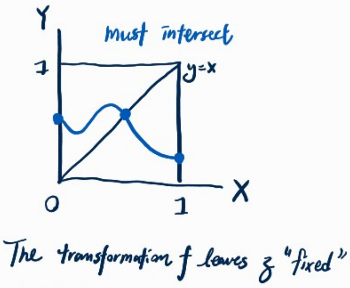
Proof. When $f(0) = 0$ or $f(1) = 1$, it is true.

Let $g: [0,1] \to \mathbb{R}$ defined by

$$g(x) = x - f(x)$$

g is continuous and $g(0) = -f(0) < 0$, $g(1) = 1 - f(1) > 0$

By corollary, $\exists\, z \in [0,1]$ s.t. $g(z) = 0$, i.e. $f(z) = z$.



The transformation f leaves z "fixed"

Thm Let X, Y be homeomorphic topological spaces.

Then each continuous function $h: X \to X$ possesses a fixed point iff each continuous function $k: Y \to Y$ possesses a fixed point.

Since any two closed intervals $[a,b]$, $[c,d]$ are homeomorphic, Fixed-point Theorem holds for any closed intervals too.

Brouwer Fixed-Point Theorem

Let $f: I^n \to I^n$ be continuous.

Then $\exists$ a point $z \in I^n$ s.t. $f(z) = z$.

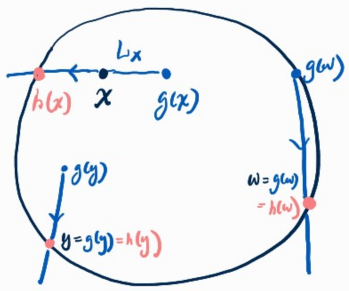
When $n=2$,

Suppose $\exists\, g$ s.t. $g(x) \neq x\ \forall x$.

Construct a new transformation $h(x)$ that carries each point of the disc into a bdry point, and leave each bdry point fixed.

h is not continuous, since one can't retract the head of a drum onto the rim — must introduce discontinuity.

Hence contradiction.



Borsuk-Ulam Theorem

Let $f: S^n \to S^n$ be continuous.

Then $\exists$ a pair of antipodal points $z, -z \in S^n$ s.t. $f(z) = f(-z)$

When $n=1$ (S^1 is a circle)

Let $F: S^1 \to \mathbb{R}$ be $F(x) = f(x) - f(-x)$

Then $F(z) = 0$ for some $z \in S^1$.

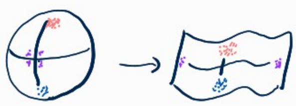
Proof:

Value of F decided by the diameter of S^1, and the extremety x or $-x$.

After rotating π radian, F value changes it sign (pos → neg or neg → pos)

Since F is continuous, $\exists\, z \in S^1$ s.t. $F(z) = 0$

When $n=2$ (S^2 is a sphere)



"Is it possible to draw a map of the surface of the earth on a flat sheet of paper s.t. distinct points on the surface of the earth correspond to distinct points on the map, and nearby points on the surface of the earth correspond to nearby points on the map?" → No

Otherwise the existence of the correspondence imply the existence of a continuous function $f: S^2 \to \mathbb{R}^2$ that was one-one, but this possibility was ruled out by Borsuk-Ulam Theorem

(5) Compact

Author: kiking

Components and Local Connectedness

Component of a:

Thm For each point $a \in X$ topological space,
$\exists$ non-empty subset $Cmp(a)$ where $Cmp(a)$ is connected,
and if D is any connected subset of X containing a,
then $D \subset Cmp(a)$

Corollary In a topological space X, let $b \in Cmp(a)$.
Then $Cmp(b) = Cmp(a)$.

Corollary Define $a \sim b$ if $b \in Cmp(a)$. Then $\sim$ is an equivalent relation.

Component of a topological space X: if it is a component of some point $a \in X$.
They constitute a partition of X into maximal connected subsets.

Def Let X be a set, $\{P_\alpha\}_{\alpha \in I}$ an indexed family of non-empty subsets of X.
$\{P_\alpha\}_{\alpha \in I}$ is called a partition of X if

(i) $X = \bigcup_{\alpha \in I} P_\alpha$;

(ii) If $\alpha, \beta \in I$, $\alpha \neq \beta$, then $P_\alpha \cap P_\beta = \phi$.

Thm Let A be a connected subset of a topological space X.
Let $A \subset B \subset \bar{A}$. Then B is also connected.

Corollary The closure of a connected set is also connected.

Corollary In a topological space, each component is a closed set.

Proof. Let $A = Cmp(a)$. $a \in \bar{A}$ so $\bar{A} \subset Cmp(a) = A$.
So $\bar{A} = A$ and A is closed.

Each component is closed, but might not be open.

Example:
X: $\{0\}$ and all numbers of the form $\frac{1}{n}$, n pos int on $\mathbb{R}$.
The only connected set containing 0 is $\{0\}$, thus $Cmp(0) = \{0\}$.
$\{0\}$ is not a neighborhood of $0 \to \{0\}$ is not open.

$\to$ A sufficient condition for the components in a space to be open is that the space is "locally connected"

Def A topological space X is locally connected at a point $a \in X$ if each neighborhood N of a contains a connected neighborhood U of a.
A topological space X is locally connected if it is locally connected at each of its points. (Eg. Euclidean space R^n and the standard n-cube I^n)

Collary Let X be a locally connected topological space and let Q be a component. Then Q is open.

Proof. Let $a \in Q$. Since X is locally connected,
$\exists$ a connected neighborhood N of a.
But $Q = Cmp(a)$, By Thm, $N \subset Q$.
Hence Q is a neighborhood of a and also that of each of its points.
Hence Q is open.

Lemma

A topological space is locally connected at a point $a \in X$
iff $\exists$ a basis for the neighborhoods at a

Author: kiking

Composed of connected subsets of X.

Ex.

→ A non-empty connected subset of a topological space that is both open and closed is a component.

→ Let X be a topological space that has a finite number of components. Then each component of X is both open and closed.

→ Local Connectedness is a topological property.

⑥ Path-connected Topological Spaces

Def Let X be a topological space.
A continuous function $f:[0,1]\to X$ is a path in X.
The path f connects or joins the point $f(0)$ to point $f(1)$.
$f(0)$: initial point of the path f ; $f(1)$: terminal point of the path f.

if initial = terminal : the path f is a closed path or a loop

If f is a path in X,
$f([0,1])$ is a curve in X.

Def A topological space X is path-connected,
if $\forall$ pair of points $u,v\in X$, $\exists$ a path f connecting u to v.

A non-empty subset A of X is path-connected
if the topological space A in the relative topology is path-connected.

Eg $\mathbb{R}$ is path-connected.
Let $f:[0,1]\to\mathbb{R}$ be defined as $f(t)=a+(b-a)t$ $\forall a,b\in\mathbb{R}$.
Then f connects a and b by $f(0)=a$, $f(1)=b$.
Euclidean spaces $\mathbb{R}^n$, standard spheres S^n : path-connected.

Thm Let Y be a topological space.
If $\exists$ a path-connected topological space X
and a continuous mapping $g:X\to Y$ that is onto,
then Y is path-connected.

Path-connectedness is a topological property.

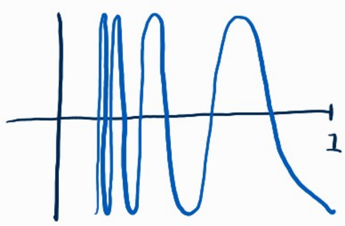
Thm If X is path-connected, then X is connected.

Proof Suppose X is disconnected.
Then $\exists$ a proper subset P of X that is open and closed.
Since P is proper, pick $a\in P$ and $b\in C(P)$.
Since X is path-connected, let $f:[0,1]\to X$ be a path from a to b.
Now consider $f^{-1}(P)$. Since f is continuous, $f^{-1}(P)$ is open and closed.
$f^{-1}(P)$ is also a proper subset of $[0,1]$ since $0\in f^{-1}(P)$ but $1\notin f^{-1}(P)$.
But then that leads to $[0,1]$ not a connected set, contradiction.

The converse is false!

Eg. A topological space that is connected but not path-connected:

All points on a plane s.t. $\begin{cases} x=0, & -1\le y\le 1 \\ 0<x\le 1, & y=\cos\frac{\pi}{x} \end{cases}$



As x approaches 0,
the oscillation of the graph $y=\cos\frac{\pi}{x}$
becomes more rapid.

Proof : Z is connected :

Author: kiking

Let Z_1 be the set of points $(0,y)$, $-1 \le y \le 1$

Z_2 the complement sets (x,y), $0 < x \le 1$ and $y = \cos\frac{\pi}{x}$.

Z_2 is connected because $F(t) = (t, \cos\frac{\pi}{t})$ is a continuous mapping of the connected interval $(0,1]$ onto Z_2.

Now $Z_1 \subset \overline{Z_2}$, so $\overline{Z_2}$ is the entire space $Z \to Z$ is connected.

Z is not path-connected:

Suppose Z is path-connected.

$\exists$ path $F: [0,1] \to Z$ with $F(0) = (0,1) \in Z_1$ and $F(1) = (1,-1) \in Z_2$

Write $F(t) = (F_1(t), F_2(t))$. Then F_1, F_2 are continuous.

Let $U = F_1^{-1}(\{0\})$. U is a closed bounded subset of $\mathbb{R}$ so it contains a least upper bound t^*. Since $F_1(1) \neq 0$, $t^* < 1$.

Show: since $\cos\frac{\pi}{x}$ oscillate for values of x close to 0,

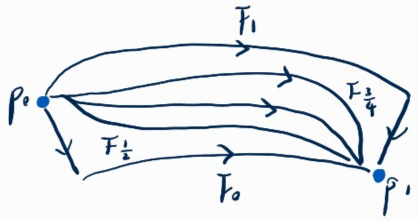
F_2 cannot be continuous at t^*.

Ex $\mathbb{R}^n$, I^n (the unit cube), and $S^n (n>0)$ are path-connected.

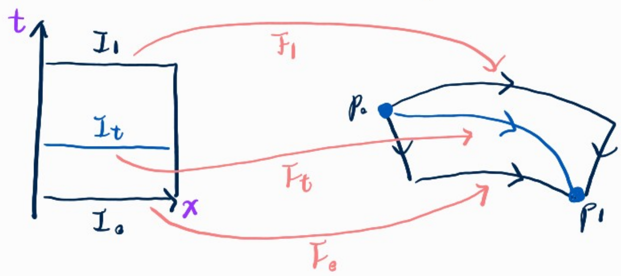
7 Homotopic Paths and the Fundamental Group

Annulus: collections of points on and between two concentric circles

it's path connected.



the concept of regarding two paths as "close"



Def Let F_0, F_1 be two paths in a topological space X with the same initial point $p_0 = F_0(0) = F_1(0)$ and the same terminal point $p_1 = F_0(1) = F_1(1)$

F_0 is homotopic to F_1 if $\exists$ continuous function $H: I^2 \to X$ s.t.

$$\left.\begin{array}{l} H(0,t) = p_0 \\ H(1,t) = p_1 \end{array}\right\} 0 \le t \le 1 \quad \text{Endpoints are preserved}$$

$$\left.\begin{array}{l} H(x,0) = F_0(x) \\ H(x,1) = F_1(x) \end{array}\right\} 0 \le x \le 1 \quad \text{At end states, matches the functions } F_0, F_1$$

H is a homotopy connecting F_0 to F_1

F_0 is deformable into the F_1 with fixed end points.

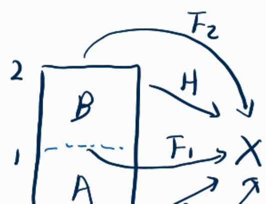
Thm If F is a path homotopic to a path G, $F \simeq G$.

(i) $F_0 \simeq F_0$

(ii) if $F_0 \simeq F_1$ then $F_1 \simeq F_0$

(iii) if $F_0 \simeq F_1$ and $F_1 \simeq F_2$, then $F_0 \simeq F_2$.

$\simeq$ is an equivalence relation.



Author: kiking

The equivalence class of a path F is $[\![F]\!]$.

Lemma Let A, B be closed subsets of a topological space Z.
Let $g: A \to X$ and $h: B \to X$ be continuous functions with the property that

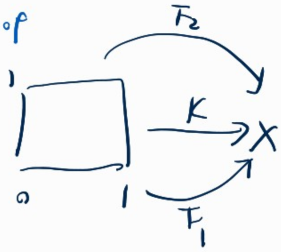
for $z \in A \cap B$, $g(z) = h(z)$.

Then the function $k: A \cup B \to X$ defined by

$$k(z) = \begin{cases} g(z), & z \in A \\ h(z), & z \in B \end{cases}$$

is a continuous extension of g and h.

use this for proof



Def An equivalent set of homotopic paths is called homotopy class of paths.

At a point z in a topological space Z, the collection of homotopy classes of closed paths at z is denoted by $\Pi(Z, z)$.

The Fundamental Group

$[\![e_z]\!] \in \Pi(Z, z)$

the constant path defined by $e_z(t) = z$, $0 \le t \le 1$.

☆ $\Pi(Z, z)$ may be converted into a group with $[\![e_z]\!]$ as its identity.

Def Let $F, G: I \to Z$ be closed paths at $z \in Z$.

Define $F \cdot G: I \to Z$ by

$$(F \cdot G)(t) = F(2t) \qquad 0 \le t \le \tfrac{1}{2}$$
$$(F \cdot G)(t) = G(2t-1), \qquad \tfrac{1}{2} \le t \le 1$$

Since $F(1) = G(0) = z$ (when $t = \frac{1}{2}$), by the Lemma, $F \cdot G$ is a closed path at z (a continuous extension).

$F \cdot G$ is the product or concatenation of F and G, or F followed by G.

Still in the set — **Lemma** In $\Pi(Z, z)$, let $[\![F]\!] = [\![F']\!]$, $[\![G]\!] = [\![G']\!]$,
then $[\![F \cdot G]\!] = [\![F' \cdot G']\!]$. (let $[\![F]\!] \cdot [\![G]\!] = [\![F \cdot G]\!]$ in $\Pi(Z, z)$)

Exists Identity — **Lemma** $[\![F]\!] \cdot [\![e_z]\!] = [\![e_z]\!] \cdot [\![F]\!] = [\![F]\!] \ \forall [\![F]\!] \in \Pi(Z, z)$

Def Let $F: I \to Z$ be a path.

Define $F^{-1}: I \to Z$ by $F^{-1}(t) = F(1-t)$

Exist Inverse — **Lemma** For each $[\![F]\!] \in \Pi(Z, z)$, $[\![F]\!] \cdot [\![F^{-1}]\!] = [\![F^{-1}]\!] \cdot [\![F]\!] = [\![e_z]\!]$

Associative — **Lemma** $([\![F]\!] \cdot [\![G]\!]) \cdot [\![K]\!] = [\![F]\!] \cdot ([\![G]\!] \cdot [\![K]\!])$

$$\forall [\![F]\!], [\![G]\!], [\![K]\!] \in \Pi(Z, z)$$

Ex

Two groups G and G' are isometric if there are homomorphisms

$$h: G \to G' \text{ and } h': G' \to G$$

s.t. $h'h$ is the identity mapping of G
hh' is the identity mapping of G'

Homomorphism f from a Group G to a Group K is a function $f: G \to K$ s.t. $f(e) = e'$ if e and e' are identities of G and K resp. and $\forall g, g' \in G$, $f(gg') = f(g)f(g')$

If $f: X \to Y$ is a homomorphism of the topological space X with the space Y

Topological spaces are homomorphic

if $\exists$ inverse functions $f: X \to Y$ > f, g define a homomorphism

Author: kiking

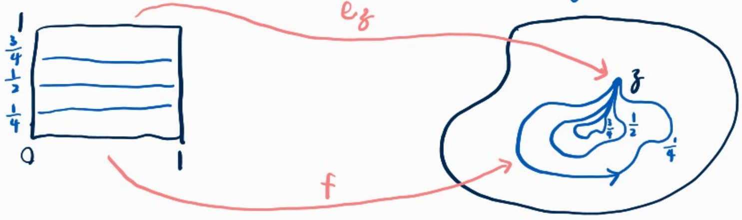
$g: Y \to X$

s.t. $f(x) = y$ then $\Pi(X,x)$ is isomorphic to $\Pi(Y,y)$ | s.t. f, g are continuous

homotopic class = a group ... homotopic class = a group

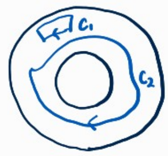
8) Simple Connectedness

Def A topological space Z is simply connected if at each point $z \in Z$ $\exists$ only one homotopy class of closed paths, i.e. the fundamental group $\Pi(Z, z)$ consists precisely $[\![e_z]\!]$.



The deformation corresponds to the fact that the curve traced out by f does not enclose any holes in the space Z.

An Annulus is not simply connected, e.g.
C_1 is homotopic to a constant path,
C_2 is not homotopic to a constant path.



Thm Let Z be a path-connected topological space and let $z \in Z$.
Z is simply connected iff $\exists$ exactly one homotopy class of closed paths at z.

Def. A homomorphism $a: G \to G'$ of a group G into a group G' which has an inverse is called an isomorphism.
G and G' are said to be isometric.

Conclusion In a path-connected space, the fundamental group at any two points are isometric.

Ex A product of simply connected spaces is simply connected.
For each positive integer n, $\mathbb{R}^n$ and I^n are simply connected.

Chapter 5 : Compactness

1) Intro.

Compactness (like connectedness and arcwise connectedness) is a global property that depends on the nature of the entire space.

Advantage in compact spaces: one might study the whole space by studying a finite number of open subsets.

2) Compact topological spaces

Author: kiking

Def Let X be a set, B be a subset of X, $\{A_\alpha\}_{\alpha\in I}$ indexed family of subsets of X.
The collection $\{A_\alpha\}_{\alpha\in I}$ is a covering of B, or covers B if $B \subset \bigcup_{\alpha\in I}\{A_\alpha\}$.

If the indexing set I is finite, $\{A_\alpha\}_{\alpha\in I}$ is a finite covering of B

Def Let X be a set, $\{A_\alpha\}_{\alpha\in I}$ and $\{B_\beta\}_{\beta\in J}$ be two coverings of a subset C of X. If for each $\alpha\in I$, $A_\alpha = B_\beta$ for some $\beta\in J$, then $\{A_\alpha\}_{\alpha\in I}$ is a subcovering of the covering $\{B_\beta\}_{\beta\in J}$.

Def Let X be a topological space, B a subset of X.
A covering $\{A_\alpha\}_{\alpha\in I}$ of B is an open covering of B if for each $\alpha\in I$, A_α is an open subset of X.

Def A topological space X is said to be compact if for each open covering $\{U_\alpha\}_{\alpha\in I}$ of X, $\exists$ a finite subcovering $\{U_\beta\}_{\beta\in J}$.

Def A subset C of a topological space X is compact, if C is a compact topological space in the relative topology.

Thm A subset C of a topological space X is compact iff for each open covering $\{U_\alpha\}_{\alpha\in I}$ of C, U_α open in C, $\exists$ a finite subcovering $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ of C.

Thm A topological space X is compact iff whenever for each $x\in X$ a neighborhood N_x of X is given, $\exists$ a finite number of points $x_1, x_2, \dots, x_n$ of X s.t. $X = \bigcup_{i=1}^{n} N_x$.

Thm A topological space X is compact iff whenever a family $\{F_\alpha\}_{\alpha\in I}$ of closed sets is s.t. $\bigcap_{\alpha\in I} F_\alpha = \phi$, then $\exists$ a finite subset of indices $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ s.t. $\bigcap_{i=0}^{n} F_{\alpha_i} = \phi$

Thm Let $f: X\to Y$ be continuous, A be a compact subset of X. Then $f(X)$ is a compact subset of Y.

Corollary. Let the topological spaces X and Y be homeomorphic. Then X is compact iff Y is compact.

Author: kiking

※ Not every subset of a compact space is compact.

E.g. The closed interval $[0,1]$ is compact,
but the open interval $(0,1)$ is not compact,
for this open covering $\{U_n = (\frac{1}{n}, 1-\frac{1}{n})\}\ \forall n = 3,4,5,\dots$
we have $\frac{1}{k} \notin \bigcup_{n=3}^{k} U_n$ for $k > 3$. $\frac{1}{4} \notin (\frac{1}{4}, \frac{3}{4}) \cup (\frac{1}{3}, \frac{2}{3})$

thus this union of every finite subcollections must fail to contain some point of $(0,1)$
i.e. no finite subcovering.

Thm Let X be compact. Then each closed subset of X is compact.

Thm Let X be a Hausdoff space.
If a subset F of X is compact,
then F is closed.

Hausdroff axiom:
given two distinct points $x, y \in (X,d)$
$\exists$ neighborhood N, M of x and y respectively
s.t. $N \cap M = \phi$

Proof show $C(F)$ is open by showing
for each point $z \in C(F)$ $\exists$ a neighborhood of z contained in $C(F)$,
equivalently, $N_z \cap F = \phi$.

Corollary Let X be a compact Hausdoff space.
Then a subset F of X is compact iff it is closed.

Thm Let $f: X \to Y$ be a one-one continuous mapping of
the compact space X onto a Hausdorff space Y.
Then f is a homomorphism.

Ex. Real line R is not compact.

③ Compact subsets of the real line

Def A subset A of R^n is bounded if $\exists$ a real number K
s.t. for each $(x_1, x_2, \dots, x_n)$ of A, $|x_i| \leq K$ for $1 \leq i \leq n$.

E.g. every closed interval $[a,b]$ is bounded for $[a,b] \subset [-K, K]$
where $k = \max\{|a|, |b|\}$.

Lemma If A is a compact subset of R, then A is closed and bounded.

Lemma The closed interval $[0,1]$ is compact.

Corollary Each closed interval $[a,b]$ is compact.
Since $[a,b]$ is homomorphic to $[0,1]$, and compactness is a topological property.

"$\Rightarrow$"

Thm A subset A of the real line is compact
iff A is closed and bounded.

Author: kiking

"⇐" if A is closed and bounded, $A \subset [-K, K]$ for $K > 0$, but $[-K, K]$ is a compact space.

Ex. The unit-cube I^n is a compact subspace of R^n.

④ Products of Compact Spaces.

Fundamental: the product of two compact spaces is itself compact.

Thm Let X and Y be compact topological spaces; then $X \times Y$ is compact.

Corollary Let $X_1, X_2, \dots, X_n$ be compact topological spaces; then $\prod_{i=1}^{n} X_i$ is also compact.

Thm A subset A of R^n is compact iff A is closed and bounded.

Ex A topological space X is locally connected if each point $x \in X$ has at least one compact neighborhood.
The real line R, and R^n are locally compact.

⑤ Compact Metric Spaces

Def Let X be a topological space and A a subset of X.
A point $a \in X$ is called an accumulation point of A if each neighborhood of a contains infinitely many distinct points of A.

Imp to specify which topological space X

e.g. in real line $\mathbb{R}$, the subset $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots\}$ has the accumulation point 0,
but in the topological space $(0, +\infty)$,
the same set A has no accumulation point.

Lemma Let X be a Hausdorff space (and thus Metric space) and A a subset of X.
A point $a \in X$ is an accumulation point of A iff a is a limit point of A.

Limit point:
a is a limit point of a subset A if every neighborhood of a contains a point of A different from a.

Thm Let X be a compact space; then every infinite subset K of X has at least one limit point in X

Proof Assume no limit point; then $\{\dots\dots\}$ K is finite, contradiction.

→ Let X be a compact Hausdorff (esp. Metric) space, then every infinite subset A of X has at least one point of accumulation in X.

Author: kiking

Main Result

Thm Let (X,d) be a metric space. Bolzano-Weierstrass property of a topological space: Each infinite subset of X has at least one accumulation point, iff it is compact.

Proof

Lemma Let (X,d) be a metric space having the Bolzano-Weierstrass property. Then, for each positive integer n, $\exists$ a finite set of points $x_1^n, x_2^n, \dots, x_p^n$ of X s.t. the collection of open balls $B(x_1^n, \frac{1}{n}), B(x_2^n, \frac{1}{n}), \dots, B(x_p^n, \frac{1}{n})$ covers X.

Lemma Let (X,d) be a metric space having the Bolzano-Weierstrass property. Then for each open covering $\{O_\alpha\}_{\alpha \in I}$ of X, there is a positive number ε s.t. each open ball $B(x;\varepsilon)$ is contained in an element O_β of this covering.

Corollary Let (X,d) be a metric space s.t. having the Bolzano-Weierstrass property. Then each open covering $\{O_\alpha\}_{\alpha \in I}$ of X has a Lebesgue number ε_L, i.e. the least upper bound of ε.

Let $\{O_\alpha\}_{\alpha \in I}$ be an open covering and ε_L be the Lebesgue number. Choose n s.t. $\frac{1}{n} < \varepsilon_L$. By Lemma $\exists$ a finite set $\{x_1, x_2, \dots, x_p\}$ of points of X s.t. the open balls $B(x_1;\frac{1}{n}), B(x_2;\frac{1}{n}), \dots, B(x_p;\frac{1}{n})$ cover X. Furthermore, by lemma for each $i = 1, 2, \dots, p$, $\exists\, \beta_i \in I$ s.t. $B(x_i;\frac{1}{n}) \subset O_{\beta_i}$. Hence $O_{\beta_1}, O_{\beta_2}, \dots, O_{\beta_p}$ is a finite subcovering of $\{O_\alpha\}_{\alpha \in I}$. #

Corollary Let X be a subspace of $\mathbb{R}^n$. The followings are equivalent:

(1) X is compact

(2) X is closed and bounded

(3) X has the Bolzano-Weierstrass property.

Consequences of

Def Let $f(X,d) \to (Y,d')$ be a function from a metric space (X,d) to (Y,d'). f is uniformly continuous if, for each $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. whenever $d(x,y) < \delta$, then $d'(f(x), f(y)) < \varepsilon$.

dependent on ε only but not x, i.e. uniformly throughout X to yield $d'(f(x), f(y)) < \varepsilon$.

Compared to Continuity: if f is continuous then for each $x \in X$ and each $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $d(x,a) < \delta$ implies $d'(g(x), g(a)) < \varepsilon$

dependent on x and ε

Corollary If f is uniformly continuous, then f is continuous.

Thm Let $f: (X,d) \to (Y,d')$ ...

Author: kiking

Let $f:(X,d) \to (Y,d')$ be a continuous function from a compact metric space X to a metric space Y. Then f is uniformly continuous.

Ex

Def In a metric space X, a sequence $a_1, a_2, \dots$ of points of X is a Cauchy sequence if for each $\varepsilon>0$, $\exists N$ st. $d(a_n, a_m) < \varepsilon$ whenever $n, m > N$.

Def A metric space is complete if every Cauchy sequence in X converges to a point in X.

☆ A compact metric space is complete.

- R^n is complete (since every Cauchy sequence lies in a bounded closed subset of R^n ...)
- A compact metric space is bounded with respect to d, i.e. $\exists K>0$ s.t. $d(x,y) \le K$ $\forall x,y \in R$

6 Compactness and the Bolzano-Weierstrass property

(1) First property : Compactness ☆ stronger

(2) Second property : Bolzano-Weierstrass property ☆ weaker

Hausdorff space : compact $\to$ Bolzano-Weierstrass

Metric space : Compact $\leftrightarrow$ Bolzano-Weierstrass

For some spaces : compact $\not\leftarrow$ Bolzano-Weierstrass

Thm Let E be a subspace of a topological space X with the Bolzano-Weierstrass property. Then every countable open covering of E has a finite subcovering.

Lindelöf's Thm Let X be a topological space that has a countable basis for the open sets. Then each open covering has a countable subcovering.

Corollary. Let X be a topological space that has a countable basis for the open sets. Then X is compact iff X has the Bolzano-Weierstrass property.

With countable basis : Completely separable / satisfy the second axiom of countability.

Ex

☆ $\mathbb{R}$ satisfies the second axiom of countability :

the collection of open intervals $B(p;q)$, $p,q>0$ rational (countable) are a basis for the open sets of $\mathbb{R}$

(by rational density theorem: $\exists$ rational number between any two real numbers)

implies

Def First axiom of countability :

If at each point $x \in X$, $\exists$ a countable basis for the complete system of neighborhoods at x.

7 Surfaces by Identification

Author: kiking

$p: [0,1] \to S^1$ (circle) by $p(t) = (\cos 2\pi t, \sin 2\pi t)$

p is a continuous function defined on a compact space onto a Hausdorff space.

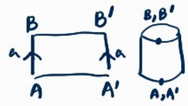
Lemma Let $f: X \to Y$ be a continuous mapping of a compact space X onto a Hausdorff space Y. Then a subset B of Y is closed iff $f^{-1}(B)$ is a closed subset of X.

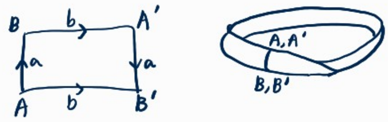
Corollary Let $f: X \to Y$ be a continuous mapping of a compact space X onto a Hausdorff space Y. Then Y has the identification topology determined by f.

Relation $x \sim_f x'$ if $f(x) = f(x')$

Corollary The mapping $f^*: X/\sim_f \to Y$ induced by a continuous function $f: X \to Y$ of a compact space onto a Hausdorff space is a homeomorphism.

Identifying:

- Cylinder 

- Möbius strip 

- Torus 

- Klein Bottle
- Projective Plane

} identify pairs of edges of a polygon with $2n$ sides: "2-manifolds"

Each 2-manifold is homeomorphic to a 2-manifold whose surface symbol is:

orientable {
(1) $abb^{-1}a^{-1}$: homeomorphic to a sphere
(2) $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_p b_p a_p^{-1} b_p^{-1}$, $p \geq 1$: a torus / a sphere with p handles
}

(3) $abab$: a projective plane

(4) $a_1 a_1 \cdots a_q a_q$, $q > 1$: attaching q Möbius strip to a sphere with q circular regions removed.

Author: kiking