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Chapter 1 Pure Mathematics

Introduction

Euclidean geometry as pure mathematics

"Euclidean geometry": Greece, ~ 600 and 300 B.C.
codified by Euclid in *The Elements*
→ definitions
→ axioms (assumptions)
→ single deductive chain of 465 theorems

Self-sufficency is the hallmark of pure mathematics

Games

Games:			e.g. Euclidean Geometry
①	Objects to play with	Chessboard & Chessmen	→ A plane, some points & lines
②	An opening arrangement	Initial game setup	→ List of Axioms
③	Rules	How the pieces move	→ Rules of formal logic
④	A goal	"Checkmate"	→ Prove profound & interesting theorems

☆ Games similar to Pure Mathematics:

the objects with which a game is played have no meaning outside the context of a game
the essence is its abstract structure; the objects is a visual aid or a fairy tale
☆ Euclidean never define plane, lines, points!

Why study pure mathematics?

① Pure math is applicable — freedom of interpretation
→ applied mathemetics

② Pure math is a culture clue — logic → "correct" from Western rationality
→ study of deduction

pure math → the only completely deductive study

③ Pure meth is fun the world's best game

Chapter 2 Graphs

Sets

Paradox

The Pythagorean Paradox (~600 B.C): $\sqrt{2}$ → intuition: rational
→ logical: must not be rational ✓

"Rigor": theorems discovered by intuition
demonstrated by logic

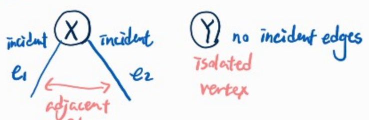
Russell's Paradox (1902): ordinary set: A = {1,2,3,4}
"~~extraordinary~~" set: A = {1, 2, 3, 4, A}
Not a "set" anymore, call it a Class

"theory of types": set: exclude collections that are elements of themselves

Graphs

Graph = {Vertex set, Edge set}
↓ finite non-empty
↓ empty / two-element subset of the vertex set

If {X, Y} is an edge: {X,Y} is incident to each of X and Y ⟷ vice versa



Graph Diagrams

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Graph Diagrams

edges

Multi-graph : allow "skeins" (several edges joining the same pair of vertices)

Pseudograph : Multi-graph + allow "loops" (vertices joining themselves)

Digraph : edges have directions

Common Graphs

Path graph : except this

Cyclic graph on v vertices : C_v — vertex set $\{1,2,\dots,v\}$, edge set $\{\{1,2\},\{2,3\},\dots,\{v-1,v\},\{v,1\}\}$

C_3 C_4

$C_3 = K_3$

Null grah on v vertices : N_v — vertex set $\{1,2,\dots,v\}$, edge set $\{\phi\}$

N_1 N_2

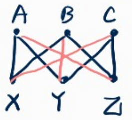
$K_1 = N_1$

Complete graph on v vertices : K_v — vertex set $\{1,2,\dots,v\}$, all possible edges

K_4 K_2

total edges : $\frac{1}{2}(v-1)(v)$

Utility graph: UG



exactly this graph

Complement of G : $\overline{G}$ — same vertex set as G, edge set : all 2-element subsets of the vertex set not already in edge set of G

Subgraph: vertex set, edge set are subsets of the original graph

if erase vertex, must erase all incident edges

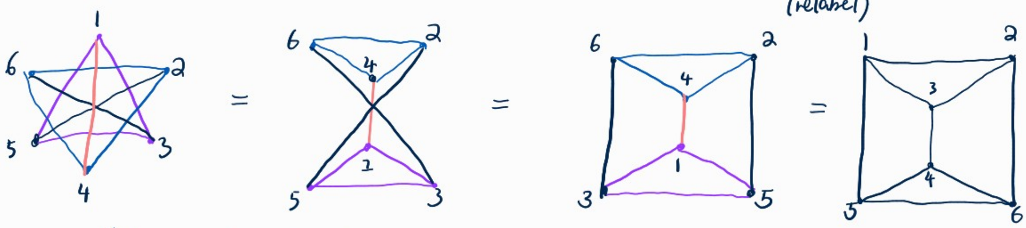
Isomorphism

Two graphs are isomorphic if $\exists$ a 1:1 correspondance between their vertex sets s.t. adjacency of vertices is preserved.

Recognizing Isomorphic Graphs

Method of Steel balls and Rubber bands

(relabel)



✓ same number of vertices

✓ same number of edges

✓ same distribution of degrees

✓ same number of components

degree of a vertex = num of incident edges

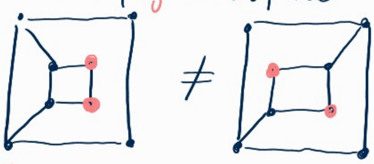
✓ same distribution of subgraphs (i.e. isomorphic subgraphs)

Isomorphic $\rightarrow$ four conditions ✓

$\cancel{\leftarrow}$

Vertices of degree 2 are separated

e.g.



Equal ≠ Isomorphic (equal : vertex set, edge set exactly the same inc. label)

"A is B" commonly mean "A is isomorphic to B"

Polya Enumeration Theorem

Compute the number of graphs given the vertex count

Chapter 3 Planar Graphs

Introduction

A graph is planar if it is isomorphic to a graph drawn in plane "Pl... "

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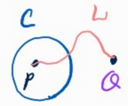
without edge-crossings. Otherwise a graph is non-planar. Planarity

UG, K_5, and the Jordan Curve Theorem

Jordan Curve Theorem no points repeated except start point / same endpoints

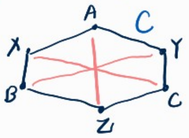


If C is a continuous simple closed curve in a plane, then C divides the rest of the plane into two regions having C as the common boundary.
If a point P in one of these regions is joined to a point Q in the other region by a continuous curve L in the plane, then L intersects C.

Corollary
If two points of C is joined by a continuous curve L in the plane having no other points common with C, then except for its endpoints, L is completely contained in one of the regions determined by C.

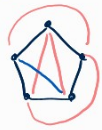
Thm UG is non-planar

Proof:



{A, Z} {B, Y} {C, Z}
two of them must be in one region divided by C (pigeonhole principle)

Thm K_5 is non-planar (with the smallest number of vertices)



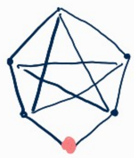
Thm Subgraph of a planar graph is planar. erasing vertices / edges cannot create edge-crossings.

Thm Supergraph of a non-planar graph is non-planar. suppose planar; contradiction

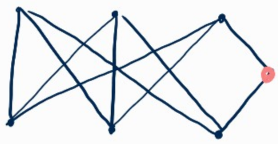
Subgraph: selective erasing; Supergraph: selective augmenting

Are there more non-planar graphs?

Examples of non-planar graphs (infinite)
that are not supergraph of UG or K_5



(1) prove not supergraph
(2) prove non-planar: assume planar. remove vertex and joining the edges become K_5/UG $\to$ non-planar contradiction.



Expansions

Def If some new vertices of degree 2 are added to some edges of a graph, the resulting graph is an expansion of the original graph.

Expansions $\neq$ Supergraphs
both are augmentations of a graph, but by different procedures.
"some new vertices of degree 2 are added to some edges"

the only allowed operation for Expansions

the only forbidden operation for Supergraphs

e.g. C_4 expansion of K_3 expansion of

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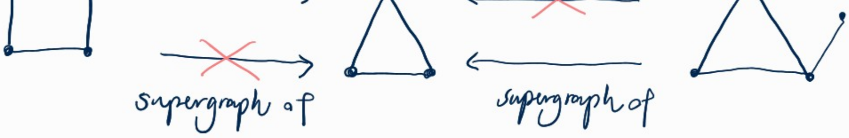
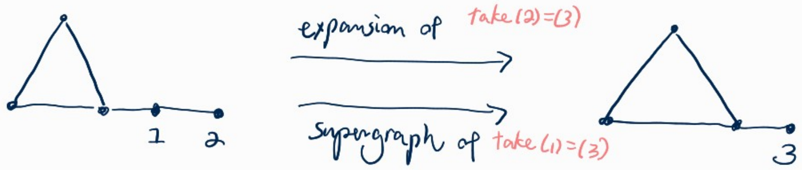


e.g.



Thm Expansion of UG or K_5 is non-planar.

Kuratowski's Theorem

Every non-planar graph is a supergraph of an expansion of UG or K_5.

Corollary The set of all planar graph is equal to the set of all graphs that are not supergraphs of expansions of UG and K_5.

Determining whether a graph is planar or non-planar

non-planar: $\exists$ a subgraph that is isomorphic to a UG or K_5 or the expansion of UG or K_5

Chapter 4 Euler's Formula

Introduction

Leonhard Euler "the Seven Bridges of Königsberg" (1736)
→ earliest known work of graph theory

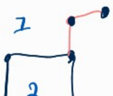
Def A walk in a graph:
a sequence $A_1 A_2 A_3 \dots A_n$ of (not necessarily distinct) vertices, A_1 joined by an edge to A_2, ...
The walk $A_1 A_2 A_3 \dots A_n$ join A_1 and A_n.

Def A graph is connected if every pair of vertices is joined by a walk (otherwise: disconnected)
e.g. every cyclic / complete graph is connected
e.g. every null graph is disconnected except for N_1.

Def A planar graph (drawn in plane without edge-crossing) cuts the plane into regions called faces of the graph. ← only planar graph has faces ← only define with cross-free drawings
(generalization of Jordan Curve Theorem)
(number of faces is independent of the drawing)

Def A graph is polygonal if it is
* planar
* connected
* every edge borders on two different faces

e.g.



not polygonal

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 since two edges only border on face #1

 not polygonal since disconnected

non-planar → not polygonal since no faces

Euler's Formula

For planar connected graph, $v+f-e=2$.

Mathematical Induction

Proof: ① Euler's formula on polygonal graphs
② Euler's formula on planar connected graphs that are not polygonal.

Some consequences of Euler's formula

Thm If G is planar and connected with $v \geq 3$,
then $\underbrace{\frac{3}{2}f \leq e}\ \underbrace{\leq 3v-6}$

Proof

Case ①: G has a face bounded by < 3 edges.
→ G must be P_3 •—•—• (path graph) $f=1$, $e=2$, $v=3$

Case ②: Every face of G is bounded by ≥ 3 edges.
$\rightarrow \sum_i 3 \leq \sum_i$ num. of edges bounding face $i \leq 2e$ equal if G is polygonal
$\rightarrow 3f \leq 2e \rightarrow \underbrace{\frac{3}{2}f \leq e}$
$\rightarrow v+f-e=2 \Rightarrow 2 \leq v+\frac{3}{2}e-e \Rightarrow \underbrace{e \leq 3v-6}$

Thm If G is planar and connected with $v \geq 3$
and G is not a supergraph of K_3.
then $2f \leq e \leq 2v-4$

Thm If G is planar (and connected)
then G has a vertex of degree ≤ 5

Algebraic Topology

Euclidean geometry: connected with the "metric" properties of figures
Topology: properties of figures preserved by "continuous deformations"
e.g. Jordan Curve Theorem on "continuous simple closed curve"
Henri Poincaré: "qualitative" subject
Algebra: studies sets where "operations" are defined.
high school algebra: on Real numbers of ⊕ ⊖ ⊗ ⨸
Algebraic Topology: algebraic methods applied to topological problems

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convert → take a T problem, convert to A problem

solve → try to solve the A problem

reconvert → reconvert the A solution into T terms

result: a solution to the T problem

e.g. analytical geometry

means of convert: associate every geometry point the "coordinates"

Straight lines → equations → system of equations, ...

Euler's Formula

↳ means of convert: graph theory (topology) ↓ high school algebra (algebra)

K_5 is non-planar ↓ Proof: K_5 connected, but $e \nleq 3v-6$, contradiction

"disadvantage": less conductive to understanding

Exercises

Def A component of a graph is a connected subgraph that is not contained in a larger connected subgraph.

Generalization of Euler's Formula

If G is planar, $v+f-e=1+p$ where p is the # of components

When G is connected, $p=1 \Rightarrow$ reduces to $v+f-e=2$

Def The connectivity of a graph c is the smallest number of vertices whose removal (+ incident edges) results in either K_1 or a disconnected graph.

Def A bridge in a graph is an edge whose removal increases # of components.

Thm If the connectivity is ≥ 6, the graph is non-planar.

Chapter 5 Platonic Graphs

Introduction

1. historical: "Platonic solids" (after Plato)
2. heuristic: spectacular warning if over-indulge the natural tendency
3. pedagogical: power of Euler's formula

Def A graph is regular if all vertices have the same degree

"regular of degree d"

e.g. C_v (cyclic): regular of degree 2

K_v (complete): regular of degree $v-1$

N_v (null): regular of degree 0

UG: regular of degree 3

Def A graph is plantonic if it is

* polygonal
  * planar — isomorphic to a graph without edge-crossing
  * connected
  * every edge borders on two different faces
* regular

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all faces bounded by same number of edges

e.g. C_v (cyclic) : plantonic

K_v (complete) : only K_1, K_3, K_4 plantonic

N_v (null) : only N_1

UG : not plantonic, since not planar

Thm Other than ① K_1, ② Cyclic graphs, there are only 5 plantonic graphs.

Proof ... algebra with Euler's formula ...

$(n-2)(d-2) < 4$ where n: # edges bounding a face of a polygonal graph

$\Rightarrow$ only five solutions! d: the degree of each vertex of G

... more algebra on v, e, f ...

tetrahedron	cube (hexahedron)	dodecahedron	octahedron	icosahedron
$n=3$, $d=3$	$n=4$ $d=3$	$n=5$, $d=3$	$n=3$, $d=4$	$n=3$, $d=5$
$f=4$	$f=6$	$f=12$	$f=8$	$f=20$

History

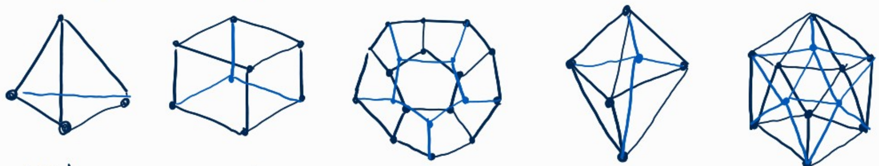
Terminology : "regular polygon" (not graph) = edges same length, angle same size — infinitely many

"regular polyhedron" = congruent regular polygons for faces, corner angles same size

Only 5 regular polyhedron

every plantonic graph 1:1 corresponds to a regular polygon (cyclic graph) or regular polyhedron (5 plantonic graph) except K_1 (trivial)

isomorphic (3-d drawings)



regular tetrahedron	regular cube (hexahedron)	regular dodecahedron	regular octahedron	regular icosahedron
$n=3$, $d=3$	$n=4$ $d=3$	$n=5$, $d=3$	$n=3$, $d=4$	$n=3$, $d=5$
$f=4$	$f=6$	$f=12$	$f=8$	$f=20$

Pythagoreans : universe (dodeca) + {fire, earth, air, water}

Kelper : five regular polyhedra $\longleftrightarrow$ six known planets

Exercises

Def A dual graph of a planar graph is formed by:

① taking a cross-free drawing of the planar graph

② place a dot inside each face

③ joining two dots whenever the borders of two faces have one or more edge in common

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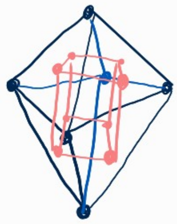
→ Only a planar graph have a dual

→ A dual is always planar and connected

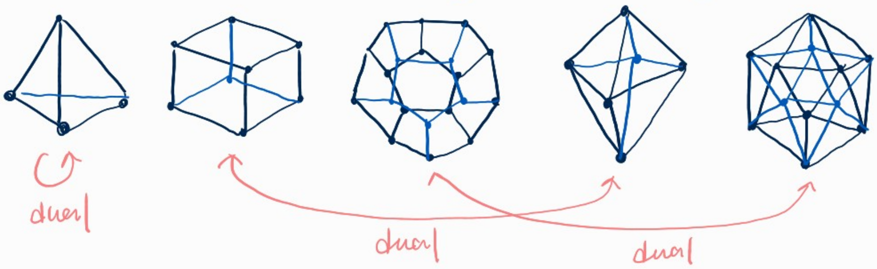
→ A dual is not always unique (from different cross-free drawings)

☆ Plantonic graph : dual is unique



K_1 : dual is itself

C_v = dual is always K_2



Chapter 6 Coloring

Chromatic number

Def. A graph is colored if adjacent vertices have been assigned with different colors.

Def Chromatic number X : the smallest number of colors which a graph can be colored.

Coloring planar graphs

The Four Color Conjecture (Theorem) : Every planar graph has $X \leq 4$.

The Five Color Theorem ← easier to prove

"every planar graph has at least one vertex with degree less than or equal to 5"

the version for 4 is false e.g. Isocahedron ($d=5$)

Coloring maps

Map coloring problem : find the smallest number m s.t. the faces of every planar graph can be colored with $\leq m$ colors s.t. faces sharing a harder have different colors.