

Chapter 1. General

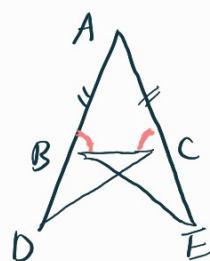
1800s: wave equation \rightarrow 1864: Maxwell \rightarrow 1888: Hertz confirmed \rightarrow 1896: Marconi
radio transmission

Pure Mathematicians \rightarrow Theoretician \rightarrow Experimental Scientist \rightarrow Practical man

150 yrs

Power of mathematics: Intuition and Rigour

Chapter 2. Motion



Proving Isosceles Triangles' base angles are equal

- ① Euclid's proof: congruent triangles
- ② Motion: flipping over \rightarrow Zeno's paradox (movement)
- ③ Rigid motion via Transformations

Chapter 3. Arithmetics

$b \equiv c \pmod{a}$ b and c are congruent to the modulus a
 arithmetics mod n : addition \checkmark subtraction \checkmark multiplication \checkmark
 division: prime modulus \checkmark \times composite modulus

Fermat's number: $2^{2^n} + 1 \rightarrow$ prime?

✓: 3, 5, 17, 257, 65537

✗: $2^{3^a} + 1 \equiv 0 \pmod{641}$

Fermat's Theorem: $x^{p-1} \equiv 1 \pmod{p}$

$$3^{5-1} = 9 \cdot 9 = 81 \equiv 1 \pmod{5}$$

$$\begin{aligned} & (1 \cdot x)(2 \cdot x) \dots ((p-1) \cdot x) \\ &= x^{p-1} (1 \cdot 2 \dots (p-1)) \\ &= 1 \cdot 2 \dots (p-1) \cancel{(mod p)} \quad \left. \begin{array}{l} \text{prime multiplication table:} \\ \text{all nums appear once} \end{array} \right\} \end{aligned}$$

Wilson's Theorem: $1 \cdot 2 \dots (p-1) \equiv -1 \pmod{p}$

(test for prime)

$= \frac{1 \cdot (2 \cdot 4) \cdot (3 \cdot 5) \cdot 6}{1 \cdot 1 \cdot 1 \cdot (-1)} \equiv -1 \quad \leftarrow \text{pair off with reciprocal!}$

Chapter 4. Set

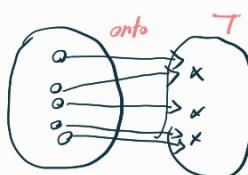
Euclidean Geometry as a part of Set Theory

e.g. a point lies on a line, geometrically
 if it's a set-theoretic member of a line

Chapter 5. Function

$f: D \rightarrow T$ Domain Target

$f(\text{domain})$: range of f
 equal T : onto T
 f is a surjection



Surjection + injection = bijection = inverse exists

onto one another

Chapter 6

Abstract Algebra

Abstract from Arithmetics <

~~A~~ | Rings: 6 rules

- (1) Associative Law of Addition $(ab)c = a(b+c)$
- (2) Commutative Law of Addition $a+b = b+a$
- (3) Existence of Zero $a+0 = a = 0+a$
- (4) Existence of Additive Inverse $a+(-a) = 0 = (-a)+a$
- (5) Associative Law of Multiplication $(ab)c = a(bc)$

Commutative Ring with Unity

~~A~~ | Multiplication

- (6) Commutative Law of Multiplication $ab = ba$
- (7) Existence of Unity $1a = a = a1$

~~A~~ | (8) Distribution Laws

$$\begin{cases} a(b+c) = ab+ac \\ (ab)c = a(bc) \end{cases}$$

~~A~~ | Fields: 10 Rules

- (9) Existence of Multiplicative Inverse $aa^{-1} = 1 = a^{-1}a$
- (10) $0 \neq 1$ for non-zero element $0 \neq 1$

Examples of Fields / Rings

|| \rightarrow The Set of constructible lengths by Platonic restriction
ie. Ruler, compasses

$$r+s, r-s, rs, \frac{r}{s}, p+q\sqrt{r}$$

$\Rightarrow \sqrt[3]{2}$ is not constructible



Field

|| \rightarrow Congruence classes ($+$, \cdot are well-defined)
 \Rightarrow (1)-(8) are satisfied \rightarrow Commutative Ring with Unity.
 \Rightarrow prime modulus \rightarrow Field.

|| \rightarrow Polynomial Rings : $R[x]$
 Coefficients are real numbers

Ring

|| \rightarrow Complex numbers $ai+b$

\Rightarrow Congruence classes of Polynomials
 to the modulus x^2+1

Field

$$x^2 \equiv -1 \pmod{x^2+1}$$

$$\text{all poly } \equiv ax+bx \pmod{x^2+1}, a, b \in \mathbb{R}$$

$$\Rightarrow (ax+b)(-ax+b) \equiv a^2 + b^2 \pmod{x^2+1}$$

Inverse can be defined!

Game of Solitaire



Prove finite final positions
for end of game
left with one peg

Init: all holes except
centre contains a peg

Rule: jump a peg over another
horizontally / vertically adjacent peg (not diagonally)
into an empty hole
 \Rightarrow remove the jumped-over peg
(board size doesn't matter)

Win: Remove all pegs but one (+ end in centre)

Explain from
the endings!

Define: ① Integer coordinates on the board

② Situation = a set of pegs on the board

Solve: ③ $A(S) = \sum p^{k+l}$ for adding p^{k+l} $\forall (k, l)$ pegs on the board
 $B(S) = \sum p^{k-l}$ for adding p^{k-l} —
where t, x of p is defined by a field with 4 elements

+	a	p	q
o	o	p	q
l	l	o	p
p	p	g	o
q	q	p	l
f	f	p	o

x	a	p	q
o	o	o	o
l	l	o	p
p	p	g	o
q	q	p	l
f	f	g	o

Then each legal move from S to $T \Rightarrow A(S) = A(T)$

e.g. $p^{k+l} \Rightarrow$ a move to the left: $p^{k-2+l} - (p^{k+l} + p^{k+l+2}) = 0$

similarly, $B(S) = B(T)$ for any legal move

Note: $(A(S), B(S))$ has 16 pairs (4 elements in a field)

\Rightarrow separate the board positions into 16 sets,
s.t. a series of moves does not change which set the positions
belong to

Init: $A(S) = B(S) = I$

Final: One single peg on board $\Rightarrow (p^{k+l}, p^{k-l}) = (1, 1)$

$\Rightarrow k+l, k-l$ multiples of 3

$\Rightarrow (0, 0)$ is one of the valid soln.

Chapter 7.

(Symmetry) Group

Group = 5 rules

① A set G

② An operation " $*$ " $\forall x, y \in G$
s.t. $x * y \in G$

③ Associative: $x * (y * z) = (x * y) * z$

Define an operation " $*$ "

Author: Kiking * : $G \times G \rightarrow G$

④ Existence of Identity $I \in G$ s.t.

$$I * x = x = x * I$$

⑤ Existence of inverse $x' \forall x \in G$ s.t.

$$x * x' = I = x' * x$$

Example : $(\mathbb{Z}, +)$, Identity: 0

$(\mathbb{R}, +)$, Identity: 0

$(\mathbb{Q}/\{0\}, \times)$, Identity: 1

$(\text{Symmetries of } S \in \mathbb{R}^2, \times)$, Identity: I

Symmetry Group

Symmetry: a rigid motion $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

s.t. \forall point $x \in S, f(x) \in S$

(a rigid motion of the plane which leaves S in the same place)

rule ①: symmetries is a set

②: symmetry is closed under multiplication

③: true for function

④: Identity symmetry is defined

⑤: Inverse of symmetry is defined

Non-Example : $(z \in \mathbb{Z} \text{ s.t. } z \in (-10, 10), +)$ violate (2)

$(\mathbb{Z}_{\geq 1}, +)$ violate (4)

$(\mathbb{Z}, \rightarrow)$ violate (3)

(\mathbb{Q}, \times) violate (5) since $0 \times 0 \neq 1$

but 1 is the only possible identity

Group & Subgroups: Subgroup: subset $H \subseteq G$ s.t. H forms a group

Order of a Group: # elements in a finite Group

Lagrange's Theorem: Order of subgroups must divide order of Group

Numerical
relation of
Groups

Proof: Consider a subgroup $J \subseteq K$

if multiply J with each element in K,

will result in some distinct cosets s.t.

→ one of them is J → no distinct cosets share common elements,

→ every elements of K lie in some coset → each coset has equal size

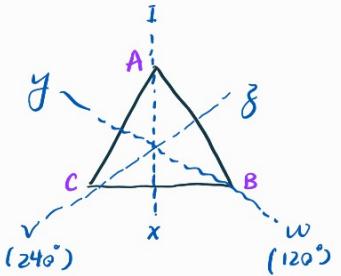
Sylow's Theorem: If h divides the order of a Group G,
and h is a power of prime \Rightarrow
then G has a subgroup of order h

Properties

Isomorphic groups

Given two groups G and H ,they are isomorphic if \exists a bijection $f: G \rightarrow H$

s.t. $f(\alpha * \beta) = f(\alpha) * f(\beta)$

Example

✓ Symmetries of a Triangle
is isomorphic to

✓ Permutations of a 3-element Set

Same:

Order of 6

Difference

- ① 6 subgroups vs 4 subgroups
- ③ Addition mod 6 satisfies Commutative Law
(Not for symmetry groups)

x	I	w	v	<u>x</u>	y	z
I	I	w	v	x	y	z
w	w	v	I	<u>z</u>	x	y
v	v	I	w	y	z	x
x	x	y	z	I	w	v
y	y	z	x	v	I	w
z	z	x	y	w	v	I

Multiplication Table

x	*	w
I	\downarrow	\downarrow
A	A	B
B	C	C
C	C	A

Symmetries

x	I	w	v	x	y	z
A	A	B	C	A	C	B
B	B	C	A	C	B	A
C	C	A	B	B	A	C

Permutations

Non-Examples

✗ (Integers mod 6, +)

17 wallpaper groups (from Arabian works)

- || → 17 plane symmetry groups p¹, p², pm, pg, cm, pmm, p⁴g, p⁴m, p⁴g, cmm, pgg, p³g, p³m, p⁶, p⁶m
- tilings with translational symmetries in 2 independent directions
- frieze patterns = in 1 direction
- 230 symmetry groups in 3D

Axiomatics

→ deduct properties → ...

Define terms → List laws → axioms
e.g. Ring, Field, Group assumptions + can be divorced from reality
axiomatic systems

"Are the group axioms true?" is an non-sense question

Euclid Axioms:

(6) Given any line, and any point not on the line,
 \exists exactly one line \parallel to the first line and passing the point

\Rightarrow Not provable from other axioms only means that

\Rightarrow "Not true" for Earth/Round surfaces \rightarrow Earth doesn't fit in Euclidean Geometry

\Rightarrow maybe that's why Euclid states this axiom explicitly

An axiomatic theory must be consistent (David Hilbert)

If we can prove two contradictory theorems,
 then we can prove anything.

Prove by Contradiction

Prove anything p is True:

① Assume p is false

② Observe a and b are contradictory, deduced from the axioms.

③ So p must not be false

$\rightarrow p$ is True, QED.

Example = Field

Law(9): if change to Existence of Multiplicative Inverse
 & elements

\Rightarrow inconsistent,

$$\text{since } (0 \cdot 0) \cdot 0^{-1} = 0 \cdot 0^{-1} = 1 \quad \begin{matrix} 0 \neq 1 \\ \text{but} \\ \text{Associative Law} \end{matrix}$$

$$0 \cdot (0 \cdot 0^{-1}) = 0 \cdot 1 = 0 \quad \text{Law}(5)$$

Consistency, Completeness, Independence (David Hilbert)

Have enough axioms to prove the truth/falsity of any conceivable statements of the system

Cannot add extra axioms anymore: either deductible or pointless

No axioms can be deducted from others

Model = example of an axiom system

e.g. Coordinate Geometry: algebraic model of the axioms of Euclidean Geometry

To prove Independence:

Construct a model that satisfy some but not all axioms of the axiom system

→ If exists, that means the failed axioms are not deducible from other axioms → independent, QED.

Euclidean Geometry

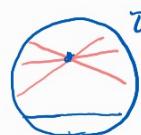
\exists exactly one line

parallel to the given line and pass through the given point

Hyperbolic Geometry

\exists infinitely many lines

parallel (i.e. not meeting) the given line

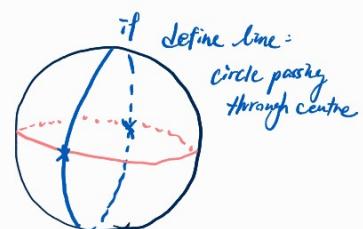


if define line, point
lie inside T

Elliptic Geometry

\exists no line parallel to the given line

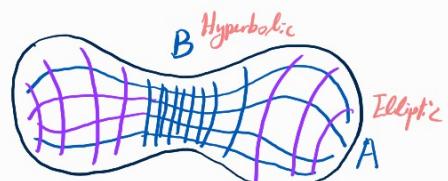
(i.e. two lines always meet-up)



if define line =
circle passing
through centre

Riemann: Curved Geometry

Near A: Ellipse Near B: Hyperbola \Rightarrow



3D: Curved Space \leftarrow Einstein: Space-Time

curvature \leftrightarrow gravitational/attraction of matters

Elliptic Time? Get back before set out!

e.g. Radio stars at diametrically opposite points in sky

\Rightarrow could be just one star seen from two opposite directions

Chapter 9

Canting
Finite & Infinite

Infinite
cardinals

Cardinals (Transfinite numbers)

Galileo: \exists bijection between a set and a smaller subset of itself

e.g. $f(n) = n^2 \rightarrow$ bijection of \mathbb{N}

\Rightarrow Same Cardinal of \mathbb{N}

\aleph_0 : alpha-zero = Cardinal of \mathbb{N}

Examples of \aleph_0 : odd / even numbers (both infinite)

\aleph_0 $\aleph_0 + \aleph_0 = \aleph_0$ (e.g. $\{\text{odd}\} \cup \{\text{even}\} = \mathbb{N}$)

Q: Rational Number is also \aleph_0 ($\frac{p}{q}$: n th distinct rational number)
 $\dots \frac{1}{2}, \frac{1}{3}, \dots, \frac{2}{2}, \frac{3}{3}, \dots$)

larger cardinals

\mathbb{R} : Real Number has a larger Cardinal!

Assume \exists bijection0 $\rightarrow A$. a_1, a_2, a_3, \dots 1 $\rightarrow B$. b_1, b_2, b_3, \dots

can always construct
a number not in the
R.H.S!
Contradiction.

Set of all subsets of any A → always have larger cardinal than A → Proof: Assume such bijection $h(x)$ exists.

Injection: easy to see

$\begin{cases} \text{if } t \in T \\ \quad \rightarrow \text{violate } T \\ \text{if } t \notin T \\ \quad \rightarrow \text{should be in } T \end{cases}$

$\boxed{\{x \mid x \notin h(x)\}}$

Assume Surjective.

these elements are $h(t)$ for some t Contradicts. Such bijection h doesn't exist. → larger cardinals

\mathbb{R} : real numbers

→ algebraic numbers

satisfy polynomial equations

$$a_n x^{n-1} + a_{n-1} x^{n-2} + \dots + a_0 = 0, a_i \in \mathbb{Q}_{\text{rational numbers}}$$

longer cardinal → transcendental numbers
 e.g. π, e (not satisfy above polynomial equations)

Proof of Existence (Cantor's Theorem)

① Algebraic numbers are countable

→ Has Cardinal \aleph_0 m^d distinct algebraic numbers→ observe for each polynomial of degree d ,
only a finite number of polynomials has height h

Finite height \geq height 2 height 3
 Count: $p_1(x) p_2(x) p_3(x) p_4(x) \dots$

Count: β_1, β_2, \dots Roots Roots RootsFinite Roots of $p_1(x)=0$ By definition, each root is an algebraic number

along the sequence

② Real numbers are uncountable

⇒ Transcendental numbers exist

⇒ They have larger cardinal than \aleph_0 since Cardinal of $\mathbb{R} - \aleph_0 > \aleph_0$

Topology

Topological spaces



Two topological spaces are topologically equivalent if

$\exists f: A \rightarrow B$ st. (i) f is a bijection

(ii) f is continuous

(iii) f^{-1} is continuous

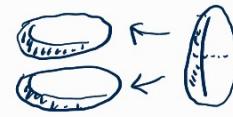
} pass to one another
in a continuous way,
and also come back
in a continuous way.



Example of unidirectional continuous



(points originally close to each other)
remain so



(points around dividing line → further apart)

Möbius Strip, Klein Bottle

Non-orientable ("one-sidedness") Topological Property



Cannot embed in
3D without crossing

Hairy Ball Theorem

No perfectly smooth system of hairs exists



Earth Wind System
→ Cyclone must exist

But if a planet is a Torus,
smooth wind system exists!



Chapter 11

Network

Euler's Formula $V - E + F = 1$

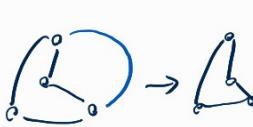
On Sphere,

$$V - E + F = 2$$

(Considering the region
outside the graph)

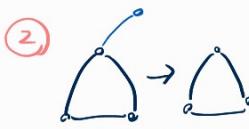
Proof: Collapse

①



$$E - 1, F - 1$$

②



$$V - 1, E - 1$$

Both Collapses doesn't alter $V - E + F$

\Rightarrow Sequences of Collapses lead to a single vertex

$$\Rightarrow V = 1 \neq (V = F = 1 \text{ on sphere})$$

Non-Planar Graph

$N > N_{\text{plane}}$



Any non-planar network must contain within it one of these two

$$\text{Proof: } V = 6, E = 9$$

$$\text{Assume } F = 24 \text{ (+2 for outside face)}$$

Notice all closed loop must have either 4 or 6 edges.

$$5 \text{ faces of } \geq 4 \text{ edges} \Rightarrow 20 \text{ double-counted edges}$$

$\Rightarrow \geq 10 E$ Contradict.

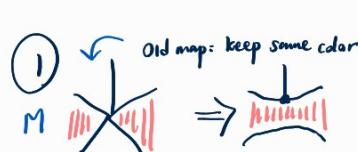
5 vs 4 coloring Theorem

Proof of 5-color Theorem (A.B. Kempe, 1879)

For any given map,

reduce its faces such that if it is 5-colorable, M'
then the original map is 5-colorable, M

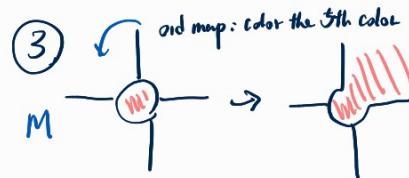
Reduce faces until the final map contains ≤ 5 faces
 \Rightarrow Original map is 5-colorable



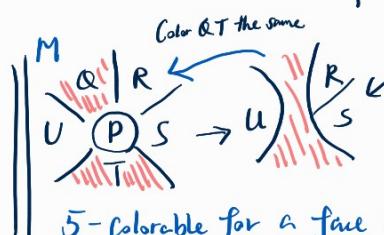
Reduce vertices with ≥ 3 faces



Remove 3-sided faces



① \rightarrow Each V connects 3 edges
Each E lies on 2 faces
 $3V = 2E = aF$, $a = \frac{\text{avg. # sides of faces}}{\text{# faces}}$
 $\Rightarrow a = 6 - \frac{12}{F} \leq 6$
 \Rightarrow Every face has ≥ 5 sides
 \Rightarrow At least one face has 5 sides



Original map is 5-colorable
5-colorable for a face with exactly 5 faces \uparrow with face reduction

4-Color Theorem

\rightarrow 1972, Appel-Haken Proof

Define Minimal Criminal: the map having least regions that is NOT 4-colorable

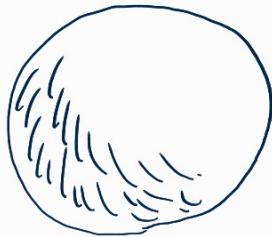
Define Unavoidable Configuration: every map must contain this configuration

Define Reducible Configuration: Configuration not appear in minimal criminal

Proof: Find an Unavoidable set of reducible configurations
 $\Rightarrow \geq 1800$ configurations! Computational intensive

Chapter 12

Topological Invariants



Difficult to prove topologically inequivalent spaces

Hole is not in the torus, it's in the surrounding space

→ One way: find topological property differences

! every closed curve on sphere divide it in two
but \exists closed curve on torus that don't divide



Surface on a Topological Space

(i) Triangulable



✓ Sphere, Torus,

(ii) Connected (in one piece)

Klein Bottle, Projective Plane

(iii) Without edges

✗ Möbius Strip

Euler Characteristics of a Surface

$V - E + F$ invariant of any map on the surface

→ invariant for topologically equal space

→ Topological Invariants

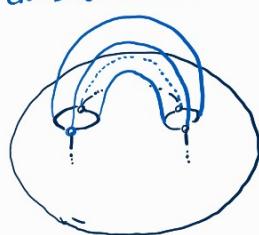
2 topological invariants

<u>S</u>	<u>X(S)</u>	<u>Orientable?</u>
sphere	2	yes
torus	0	yes
double torus	-2	yes
projective plane	1	no
Klein Bottle	0	no

Standard Surface

Std Orientable Surface

Genus n : Glue n handles



$$\text{Sphere: } X(S) = 2$$

$$\text{Torus: } V: \text{equal}$$

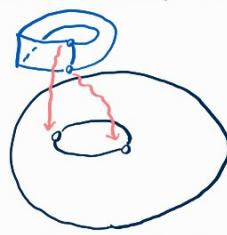
$$E: +2$$

$$F: -2+2 \rightarrow \text{equal}$$

$$\Rightarrow X(S) = 0 \\ = 2 - 2n$$

Std Non-orientable Surface

Genus n : Glue n Möbius strip



$$\text{Sphere: } X(S) = 2$$

$$V: \text{equal}$$

$$E: +1$$

$$F: -1+1 \rightarrow \text{equal}$$

$$\Rightarrow X(S) = 1 \\ = 2 - n$$

Enter Uniqueness! Proving
show that all standard surfaces are topologically distinct.

Any Surface is topologically equivalent to a Standard Surface.

Proof:

$$\begin{aligned} & \text{maximal dual tree} \quad \text{Original Network + Edges not met: } M \\ & X(S) = X(M) + X(C) \\ & \frac{1}{1} \leq 1 \leftarrow \\ & \leq 2 \end{aligned}$$

$$\begin{aligned} & X(S) = X(M) + X(C) \leftarrow \\ & \text{if } X(S) \neq 2, X(C) \text{ is not a tree} \\ & M, C \text{ disjoint} \rightarrow \text{closed curve } C \\ & \rightarrow \text{divides } S \\ & \rightarrow M \text{ must cut through } C \\ & \rightarrow \text{contradiction} \\ & \text{So } X(S) = 2, X(C) \text{ a tree} \\ & M, C \text{ a flatten disc, meet along edges} \\ & \Rightarrow \text{a sphere} \end{aligned}$$

On any surface, apply a "surgery": draw a curve on it that don't divide the surface into two. Remove narrow strip around the curve.

If the strip is a cylinder, put back two discs to fill the hole, keeping the arrow on the disc to remind the original direction, $+2 X(S)$

If the strip is a mobius strip, put back only one disc, $+1 X(S)$

Assertion: Any surface has $X(S) \leq 2$.

Then the sequence of surgeries must stop. But it only stops when we no longer find the closed curve.

Assertion: A surface that is disconnected by any closed curve or if is topologically equivalent to a sphere.

So Surgery Stop \Rightarrow We obtain a sphere.

\rightarrow A continuous transformation to a sphere.

\rightarrow Now sewing back: either sewing back a handle or a Mobius Strip. Also a continuous transformation to a standard surface.

\therefore Any surface is topologically equivalent to a standard surface.

Color-Mapping of Any Surface

With $X(S) = n$:

$$\frac{1}{2} (7 + \sqrt{49 - 24n})$$

colors suffices.

Not exact for Klein Bottle
 \rightarrow need only 6
 formula gives 7

Chapter 13

Algebraic Topology

Associate a group with a space, s.t.

Topologically equivalent spaces \Rightarrow invariants have Isomorphic groups

* Fundamental group of space S : (Poincaré)

$\pi_1(S)$

= (Set of Homotopy classes of Loops,

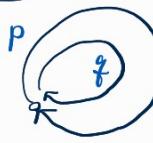
Operation * of Composition)



path composition, $a \circ b = a \cdot b$

At time t , point at $p(t)$. From A to B to C $\rightarrow p^* f$
 where p defined at interval $t_1 \leq t \leq t_2$, f $t_2 \leq t \leq t_3$,
 and $p^* f \rightarrow t_1 \leq t \leq t_3 - t_2$

Group Rule : Multiplicative Inverse I



$p^* q \neq$ trivial loop \xrightarrow{I} time pass
 (but still a loop \rightarrow closed)



$$\text{But } [p]^* [q] = [p^* q] \rightarrow$$

$\therefore [p]^{-1}$ is well-defined

Run faster \rightarrow the trivial loop

If two spaces S and T

are topologically equivalent

\Rightarrow the fundamental groups $\pi(S)$ and $\pi(T)$
 are isomorphic groups

① S, T topologically equiv. $\rightarrow f: S \rightarrow T$ continuous, f^{-1} cont.

② f defines a function F s.t.

$$[f(p)] = F([p])$$

Homotopy Class of a loop on T \rightarrow Homotopy Class of all loops on S , get transformed by F



defined by f

③ $F\left(\frac{\text{loop}}{[p]} * \frac{\text{loop}}{[q]}\right) = F\left(\frac{\text{loop}}{[p]}\right) * F\left(\frac{\text{loop}}{[q]}\right)$ \leftarrow Bijection exists
 $\Rightarrow [p], [q]$ are Isomorphic

Order of the group $\pi(S)$: (# elements in a finite group)

\rightarrow When S is a disc, \mathbb{R} , \mathbb{R}^2 , ball...

only have trivial element I . (order = 1)

All loops can be deformed to a trivial loop (point)

\rightarrow When S is a Circle, order = order of \mathbb{Z}

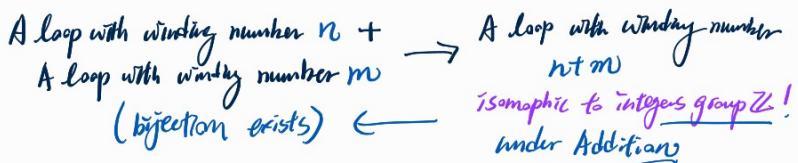
Winding Number of the loop (directional)

Each cannot be deformed to each other



Consider putting the loop to a spring

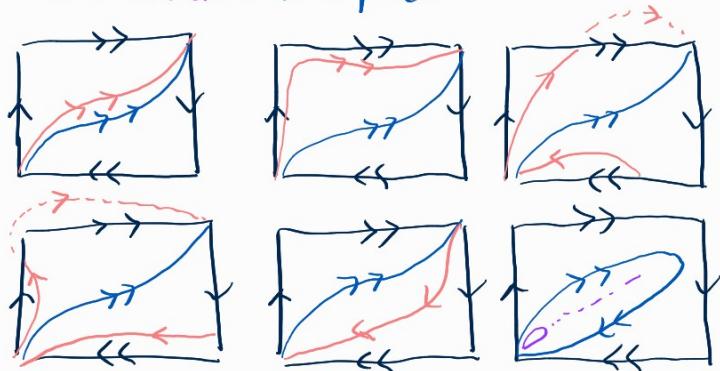
Each spring distinguished by the same endpoint



→ When S is a Projective Plane, order = 2



Alone cannot shrink to a point, but going around it twice can shrink to a point.



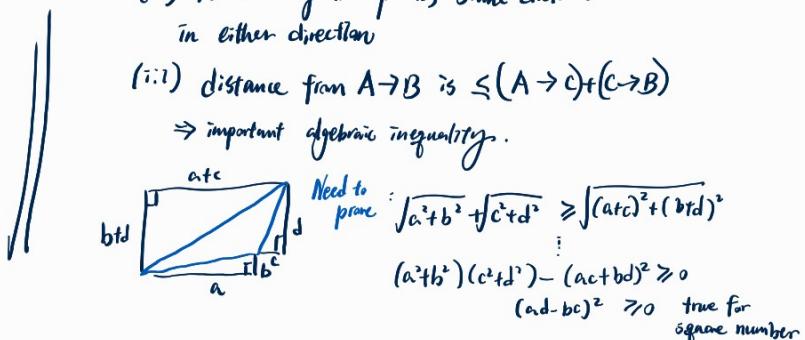
Going around once get twisted,
going twice back to normal

Chapter 14 Hyperspace

Distance: (i) between any two points is positive

(ii) between any two points, same distance in either direction

(iii) distance from $A \rightarrow B$ is $\leq (A \rightarrow C) + (C \rightarrow B)$
 ⇒ important algebraic inequality.



2-space: Polygon

3-space: Polyhedra:

5 Regular



4-space: Polytope.

Solid: the "faces", which are 3-d regular polyhedra
 (in regular polyhedra, the faces are 2-d regular polygons)

6 Regular

Simplex: 5 tetrahedron ↙ solids count

Hypercube: 8 cube Tesseract

16-cell: 16 tetrahedron

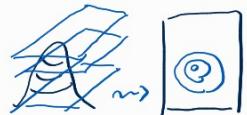
24-cell = 24 octahedron
 120-cell = 120 dodecahedron
 600-cell = 600 tetrahedron

5-space, 6-space, ...

all has 3 regular polytopes only.

Projections

2-space creatures understand 3-space:
 or series of cross-sections



3-space creatures understand 4-space:

cross-sections as 3-d objects, starting to see full

E.g. Hypercube project to 3-space: always a Cube

Stacking Sections: Appear, Growing, Shrinking, Disappear

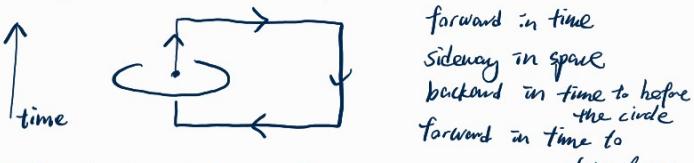
Hypersphere: .. 0 0 0 ..

Hypercube: Appear, Stay as a cube for a while, Disappear

More thought Exercises:



Untying a knot in 4-space



Linking of two circles in 3-space

\Rightarrow In 4-d, link a sphere with a circle

\Rightarrow or knot a sphere (same way as knot a circle in 3d)

Swinging Pendulum:

p : position, q : velocity



\rightarrow q

\rightarrow p

\rightarrow p

\rightarrow p

diagram: phase diagram

(p, q) - plane: phase space

2-d since the state is determined by 2 numbers (position coordinate, velocity)

Any dynamical system has

phase space, one dim for each position var.
 one dim for each velocity var.

\Rightarrow Sun, Earth, Moon + Gravitational Attraction

\Rightarrow 18-dim for the phase space

State of the system at any time \rightarrow

a point in this 18-dim phase space.

As System develop:

a point describe a path

diff initial points \rightarrow a family of paths

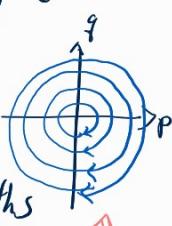
Fluid = flow along the paths

Newton's Law of Conservation of Energy

this imaginary fluid behaves exactly like a genuine fluid //

\Rightarrow General theory of Fluid Dynamics

\rightarrow applied to the general theory of Dynamical Systems

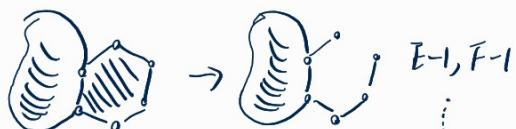


N-dim of Euler's Formula: (Euler-Poincaré formula)

$$F_0 - F_1 + F_2 - \dots + F_n = 1$$

Proof: similar to 2-d,

by observing that the eq. doesn't change when "collapsing"



Algebraic Topology generalized to higher dim

[Path from a line segment] [Joining end to end]
 \downarrow [n-dim Hypercube] [joining face to face]

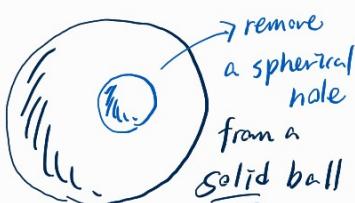
A group: hypercube whose boundary is squashed up to a single point



Elements: Homotopy classes of n-dim "path"s

n-th homotopy group $\pi_n(S)$ of the space S

Fundamental group: $\pi_1(S)$, 1st of a whole series of algebraic invariants



remove a spherical hole from a solid ball

$\pi_1(S)$ cannot detect it: any path can still shrink to a point

$\pi_2(S)$ can detect it: if put a square surrounding it

Higher Homotopy group can detect the differences

whose boundary is squashed to a point (paper being curved the hole) then this square cannot be shrunk to a point in S

Poincaré conjecture:

If S has the same sequence of H_n s as an n -sphere, then S is an n -sphere up to topological equivalence

- When $n=2 \rightarrow$ proven true in Ch.12 Topo. Invariant Assertion 2

• When $n \geq 5 \rightarrow$ true

• When $n=4 \rightarrow$ true, proven in 1982
Michael Freedman, Fields Medal 1986

• When $n=3 \rightarrow$ true, proven in 2003

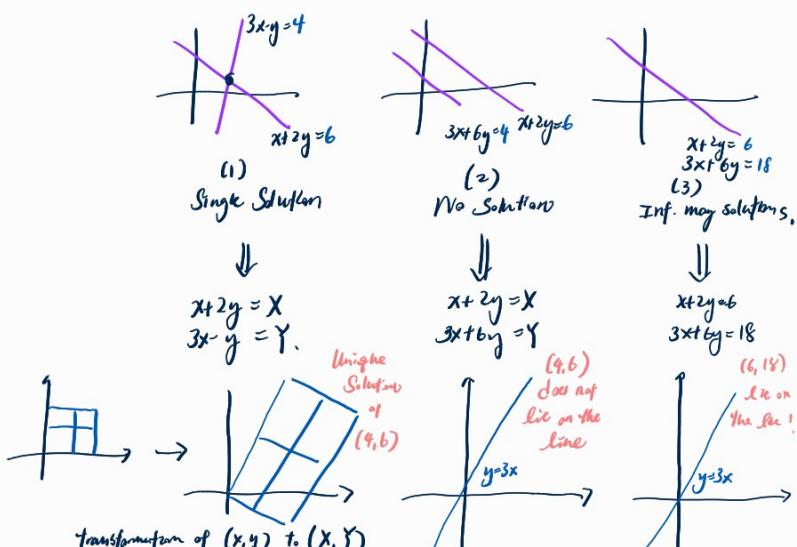
Grigori Perelman
offered but declined Fields Medal 2006

Chapter 15 Linear Algebra

A proper understanding of linear algebra requires a synthesis of 3 points of views:

Textbooks tend to focus on only one aspect

* (i) the underlying geometrical motivation



Solutions depend on
geometric properties of
linear transformation

$$\begin{aligned} T(x,y) &= (x+2y, 3x-y) \\ &= \begin{pmatrix} x+2y \\ 3x-y \end{pmatrix} \end{aligned}$$

Range of $T = \text{plane}$
Solution Space = point

$$\begin{aligned} S(x,y) &= \begin{pmatrix} x+2y \\ 3x+6y \end{pmatrix} \\ &= \begin{pmatrix} x+2y \\ 3x+6y \end{pmatrix} \end{aligned}$$

Range of $S = \text{line}$
Solution Space = line
(may or not exist)

3 Eq. in 3 unknowns:

Range	Solution Space
\mathbb{R}^3 Solid	point \mathbb{R}^0
\mathbb{R}^2 Plane	Line \mathbb{R}^1
\mathbb{R}^1 Line	Plane \mathbb{R}^2
\mathbb{R}^0 Point	Solid. \mathbb{R}^3

Trivial Transformation

$$\begin{aligned} Z(x,y) &= \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} \end{aligned}$$

Range of $Z = \text{point}$
Solution Space = plane

(2 Eq. in 2 unknowns)

$$\begin{aligned} &\Rightarrow \text{Dimension of the Range} + \\ &\quad \text{Dimension of the Solution Space} \\ &= n \text{ (in } \mathbb{R}^n) \end{aligned}$$

(ii) the matrix-theoretic technique

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

several transformation:

If $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$

i.e. $UT(x,y) = U(x,y) = (\bar{x}, \bar{y})$

Then $\begin{pmatrix} Aa+Bc & Ab+Bd \\ Ca+Dc & Cb+Dd \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$

↳ Suggests us to define the product of Matrices:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} Aa+Bc & Ab+Bd \\ Ca+Dc & Cb+Dd \end{pmatrix}$$

With this we could compute a series of linear transformation.

E.g. Rotate by θ then by ϕ

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos \theta \cos \phi - \sin \theta \sin \phi & -\cos \theta \sin \phi - \sin \theta \cos \phi \\ \sin \theta \cos \phi + \cos \theta \sin \phi & -\sin \theta \sin \phi + \cos \theta \cos \phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix}$$

Trigonometry "addl formula"

$$\cos(\phi+\theta) = \cos \phi \cos \theta - \sin \phi \sin \theta$$

$$\sin(\phi+\theta) = \sin \phi \cos \theta + \cos \phi \sin \theta$$

(iii) Abstract formulation in Algebra

(attempts to avoid using coordinates in theory)

Characterization of Linear Transformation:

functions $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ s.t. $\forall p, q, r, s, a,$

looks like Isomorphism in Group Theory under addition $T((p,q)+(r,s)) = T(p,q)+T(r,s)$ Addition

under multiplication $T(a(p,q)) = aT(p,q)$ Multiplication

Vector Space over \mathbb{R} : 7 Rules

a set V with 2 operations: Addition, Scalar Multiplication
 $\forall u, v \in V, a \in \mathbb{R} \rightarrow$ denote $u+v \in V \quad aV \in V$

(1) V : a Commutative group under Addition, with identity element 0

(2) $a0 = 0, \forall a \in \mathbb{R}$

(3) $0v = v, \forall v \in V$

(4) $1v = v, \forall v \in V$

- (5) $(\alpha + \beta)v = \alpha v + \beta v, \forall \alpha, \beta \in \mathbb{R}, v \in V$
- (6) $\alpha(v+w) = \alpha v + \alpha w, \alpha \in \mathbb{R}, \forall v, w \in V$
- (7) $\alpha\beta v = \alpha(\beta v), \forall \alpha, \beta \in \mathbb{R}, v \in V$

Standard Vector Spaces = $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \dots$

Others: Polynomial Ring $\mathbb{R}[x]$, $\mathbb{R}[x, y]$, $\mathbb{R}[x, y, z], \dots$

\Rightarrow arise in the solution of differential equations,
certain parts of group theory,
modern formulation of the calculus

Linear Transformation Definition:

A function $T: V \rightarrow W$ where V, W arbitrary Vector Spaces,
s.t. $T(u+v) = T(u) + T(v)$
 $T(\alpha u) = \alpha T(u)$
 $\forall u, v \in V, \alpha \in \mathbb{R}$

✓ One can prove all desired theorems about linear transformation

✓ No particular coordinates are involved \rightarrow proof are very clean and direct

✗ To perform calculations in particular cases \rightarrow use matrix methods

Chapter 16 Real Analysis

3 Cornerstones of modern maths:

Algebra, Topology, Analysis

(Mathematical Logic:
hold the bricks
together)

Analysis : Study of infinite processes

- infinite series
- limits
- differentiation
- integration

Example

Infinite Addition :

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \dots$$

Then

$$S = (1-1) + (1-1) + \dots = 0$$

$$S = 1 - (1-1) - (1-1) - \dots = 1$$

without a meaning of
infinite addition,

$$\begin{aligned} 1 - S &= 1 - (1 - 1 + 1 - 1 + \dots) \\ &= 1 - 1 + 1 - 1 + \dots \\ &= S \Rightarrow S = \frac{1}{2} \end{aligned}$$

all are correct!

Consider the limit of the subsequent sums,

Series: $a_1 + a_2 + a_3 + \dots$

$$\text{Apprx. Sum} = b_1 = a_1$$

$$b_2 = a_1 + a_2$$

$$b_3 = a_1 + a_2 + a_3$$

\vdots

\rightarrow A sequence b_n tends to the limit L if the difference $b_n - L$

can be as small as we please by taking n sufficiently large.

$\rightarrow S$ does not have a limit,

but e.g. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$, $b_n = 2 - \frac{1}{2^{n-1}}$ has a limit 2

* A sequence tend to a limit is convergent.

* An infinite series $a_1 + a_2 + a_3 + \dots$ is the limit L of the sequence b_n of approx. sums, provided that limit exists.

However, from sum to law of algebra \rightarrow not always possible.

Example, Convergent series

$$K = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \dots$$

has limit $\log_e 2$.

$$\text{But } 2K = 2 - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \frac{2}{5} - \frac{2}{6} \dots$$

$$= (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \dots$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$= K!$$

Problem: We have to guess the limit L to prove convergent.

Consider the error terms: if $\exists k > 0$ st.

$$-k \leq a_{n+1} + \dots + a_{n+m} \leq k$$

for every m , the the error terms is small

no guessing of L ,

\Rightarrow no infinite sum.

\rightarrow can expect the series to converge.

The completeness axiom (of real numbers):

[Rational numbers have gaps (irrational numbers in between, but not real numbers)]

Real numbers = limit of a given sequence of which the error terms became arbitrary small.
(infinite series of rational numbers)

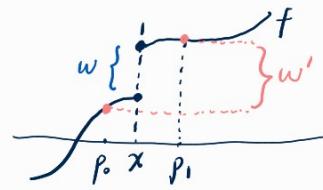
Continuity:

"no jumps"

f is cont. at a point x

if by choosing p_0 and p_1 sufficiently close to x ,
we can make $f(p_0)$ and $f(p_1)$ as close together as we please.

f is continuous if it is cont. at all point x .



e.g. $f(x) = x^2$

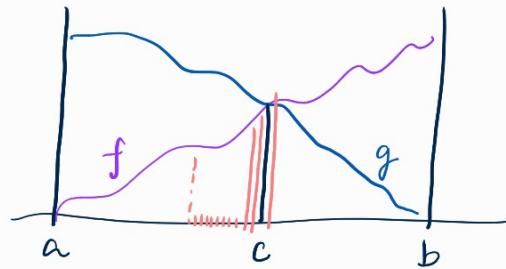
Why continuous at 0:

by choosing $-k < p_0 < 0$ and $0 < p_1 < k$,
we have $-2k^2 < p_0^2 - p_1^2 < 2k^2$

By choosing k small enough, we have $2k^2$ as small as we please

(no function can be dist. cont. at all irrational points
but continuous at rational points)

Theorem



if we have two continuous function f and g on the real line, s.t.

$$f(a) < g(a) \quad f(b) > g(b)$$

then \exists some points c s.t.

$$f(c) = g(c)$$

Proof:

divide the interval into 10 parts.

For the 1st point where f is larger than g ,
divide it up to 10 parts again.

The end point of the intervals form a sequence
 p_1, p_2, p_3, \dots

Complete Axiom \Rightarrow the square converge to a real number.

Definition of continuity $\Rightarrow f(p) = g(p)$

Chapter 17. The theory of Probability

- Combinatorial Probability

- Independence
- Paradoxical Dice
- Binomial Bias
- Random Walks

Chapter 18 Computers & their Uses

- Binary Notation
- A Ball-bearing Computer
- The Structure of Computers
- Writing a Program
- The Use of Computers

Chapter 19 Application of Modern Mathematics

- How to Maximize Profits (Linear Programming)
- The Eightfold Way (Atomic Theory
→ Representation of Groups)
- Catastrophe Theory (Discontinuous Theory
e.g. Zeeman Catastrophe Machine)

Chapter 20 Foundations

Frege's Naïve Set Theory

- Russell's Paradox (e.g. Village Barber)
 $\{P \text{ not-}P\}$
- $C = \{x \mid x \text{ has property } P\}$
 C is the class of all sets x which have property P
 If x has property P, does not deduce $x \in C$

unless x is a set

- Those classes which are not sets are called proper classes.
- Hilbert Programme (whether every problem can be solved)
 - vs
 - Gödel Theorem
 - (1) if axiomatic set theory is consistent,
 \exists theorems which can neither be proved or disproved,
 - (2) there is no constructive procedure which will prove axiomatic set theory to be consistent
- Undecidability
 - From Gödel Theorem (1), in ordinary arithmetic there exist statements P st. neither P nor $\neg P$ can be proved.
 \Rightarrow Undecidable statements
 - E.g.1. Whether Diophantine equation has a solution \Rightarrow undecidable
 (Matijasevič's proof) \Rightarrow "a formula of prime"
 - E.g.2 Continuum Hypothesis
 - Chapter 9 \rightarrow is the cardinal c of the real numbers the next cardinal after \aleph_0 ?
 - A: Yes & No.
 - \rightarrow Independent of other axioms of set theory.
 - i.e. Add an axiom saying its true, will not make set theory inconsistent.

Appendix

- The Four-Color Theorem
- Polynomials & Primes
- Chaos (Dynamical systems)

e.g. $x_{t+1} = kx_t(1-x_t)$ (Rabbit Birth)

$$0 \leq k \leq 4, 0 \leq x_t \leq 1$$

deterministic but random)

- fixed point $f(x)=x$
- periodical point $f^{(n)}(x)=x$
- orbit: $x, f(x), f^{(2)}(x), \dots, f^{(n)}(x), \dots$
- attractor, repeller

General → Motion (Geometry) → Arithmetics (Modular)

Chp. 4-5 → Set → Function

Chp. 6-7 → Abstract Algebra → (Symmetry) Group

Chp. 8-9 → Axiomatics → Counting (Cardinal)

Chp. 10-12 → Topology → Networks (Graph) → Topological Invariants

Chp. 13-14 → Algebraic Topology → Hyperspace

Chp. 15-16 → Linear Algebra → Real Analysis

Chp. 17-19 → Probability → Computer → Modern Applications

Chp. 20 → Foundations (Naive Set Theory, Gödel Theorem)

→ Appendix

Author: kiking