Author: kiking

Chapter ① Preliminaries

1.1 Sets and Functions

1.2 Mathematical Induction

1.3 Finite and Infinite Sets

Cantor's Theorem If A is any set, then there is no surjection of A onto the set $P(A)$ of all subsets of A.

Chapter ② The Real Numbers

2.1 The Algebraic and Order Properties of $\mathbb{R}$

Trichotomy Property of $\mathbb{R}$

If $a \in \mathbb{R}$, then exactly one of the followings hold:

$a \in \mathbb{R}$, $a = 0$, $-a \in \mathbb{P}$

Arithmetic-Geometry Mean Inequality

$$(a_1 a_2 \cdots a_n)^{\frac{1}{2}} \leq \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

Bernoulli's Inequality

If $x > -1$, then

$$(1+x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}$$

2.2 Absolute Value and the Real Line

Triangle Inequality

If $a, b \in \mathbb{R}$, then $|a+b| \leq |a| + |b|$

2.3 The Completeness property of $\mathbb{R}$

Completeness Property of $\mathbb{R}$ (also: Supremum Property)

Every non-empty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$ (least upper bound)

Archimidean Property of $\mathbb{R}$

If $x \in \mathbb{R}$, then $\exists\, n_x \in \mathbb{N}$ s.t. $x \leq n_x$.

Corollary (1) If $S := \{\frac{1}{n}, n \in \mathbb{N}\}$ then $\inf S = 0$

(2) If $t > 0$, $\exists\, n_t \in \mathbb{N}$ s.t. $0 < \frac{1}{n_t} < t$

(3) If $y > 0$, $\exists\, n_y \in \mathbb{N}$ s.t. $n_y - 1 \leq y \leq n_y$

The Density Theorem

If $x, y \in \mathbb{R}$ and $x < y$, $\exists$ rational number $r \in \mathbb{Q}$ s.t. $x < r < y$.

2.4 Intervals

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$\mathbb{R}$ is uncountable.

Chapter ③ Sequence and Series

3.1 Sequence and their limits

Def A sequence $X = (x_n)$ in $\mathbb{R}$ converges to $x \in \mathbb{R}$, or x is a limit of (x_n), if $\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \geq K(\varepsilon)$, the term x_n satisfy $|x_n - x| < \varepsilon$.

3.2 Limit Theorems

Thm A convergent sequence of real numbers is bounded.

3.3 Monotone Sequences

Monotone Convergence Theorem

A monotone sequence of real numbers is convergent iff it is bounded.

e.g. $e_n := (1+\frac{1}{n})^n \ \forall n \in \mathbb{N}$ is bounded and increasing; it is convergent and converge to the Euler number $e \approx 2.78\ldots$

3.4 Subsequences and the Bolzano-Weierstrass Theorem

Monotone Subsequence Theorem

If $X = (x_n)$ is a sequence of real numbers, then $\exists$ a subsequence of X that is monotone.

Bolzano-Weierstrass Theorem

A bounded sequence of real numbers has a convergent subsequence.

3.5 The Cauchy Criterion

Def A sequence $X = (x_n)$ of real numbers is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists H(\varepsilon) \in \mathbb{N}$ s.t. $\forall n, m \geq H(\varepsilon)$, x_n, x_m satisfy $|x_n - x_m| < \varepsilon$

Cauchy Convergence Criterion

A sequence of real number is convergent iff it is a Cauchy sequence.

3.6 Properly Divergent Sequences

Def $\lim(x_n) = +\infty$ ($= -\infty$) if $\forall \alpha \in \mathbb{R}$ ($\forall \beta \in \mathbb{R}$), (properly divergent)
$\exists K(\alpha) \in \mathbb{N}$ s.t. if $n \geq K(\alpha)$ ($\geq K(\beta)$), $x_n > \alpha$ ($< \beta$).

3.7 Introduction to Infinite Series

Def If $X := (x_n)$ is a sequence in $\mathbb{R}$, Σx_n is the (infinite) series generated by X, which is a sequence $S := (s_k)$ where

$s_1 := x_1$
$s_2 := x_1 + x_2$
$\vdots$

s_k: partial sum of the series

Geometric Series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots$$ generated by the sequence $X = (r^n)_{n=0}^{\infty}$

diverges if $|r| \geq 1$

Harmonic Series

$\sum_{n=0}^{\infty} \frac{1}{n}$ diverges

n-th term test

If the series Σx_n converges, then the limit of the sequence (x_n) is 0.

Cauchy Criterion for Series

Σx_n converges iff $\forall \varepsilon > 0$, $\exists M(\varepsilon) \in \mathbb{N}$ s.t. if $n > m \geq M(\varepsilon)$, ...

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$\sum x_n$ converges iff $\forall \varepsilon > 0$, $\exists M(\varepsilon) \in \mathbb{N}$ s.t. if $m > n \geq M(\varepsilon)$, then

$$|S_m - S_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon$$

Chapter 4: Limits

4.1 Limits of Functions

Def Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point of A if $\forall \delta > 0$, $\exists x \in A$, $x \neq c$ s.t. $|x - c| < \delta$.

Def Let $A \subseteq \mathbb{R}$, c be a cluster point of A.
For a function $f: A \to \mathbb{R}$, a real number L is a limit of f at c,
if $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ s.t. if $x \in A$ and $0 < |x - c| < \delta(\varepsilon)$, then $|f(x) - L| < \varepsilon$.
Also says that f converges to L at c.

4.2 Limit Theorems

4.3 Some Extensions of the Limit Concept

One-sided Limits: if $0 < x - c < \delta(\varepsilon)$
$0 < c - x < \delta(\varepsilon)$

Chapter 5: Continuous Functions

5.1 Continuous Functions

f is continuous at c if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Sequential Criterion for Continuity

A function $f: A \to \mathbb{R}$ is continuous at a point $c \in A$
iff $\forall$ sequence (x_n) in A that converges to c,
the sequence $f((x_n))$ converges to $f(c)$.

Discontinuity: if $\exists (x_n)$ s.t. $\lim_n (x_n) = c$
but $\lim_n f((x_n)) \neq f(c)$

Dirichlet's function

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

discontinuous

Let $\lim_n (x_n) = c$ rational,
but (x_n) a sequence of irrational numbers. Density Thm asserts this must exist.
Then $f((x_n)) = 0 \neq f(c) = 1$

Thomae's function

$h: \{x \in \mathbb{R} : x > 0\} \to \mathbb{R}$

$$= \begin{cases} 0 & \text{for irrational number } x > 0 \\ \frac{1}{n} & \text{for rational number } \frac{m}{n},\ m, n \text{ no common factor except } 1 \\ 1 & \text{for } x = 0 \end{cases}$$

h is continuous at every irrational number in A
discontinuous at every rational number in A

Let (x_n) a seq of rational numbers converge to an irrational number b. $f((x_n)) = \frac{1}{n_k} \to 0$ (Proof by Archimedean Property)
Let (x_n) a seq. of irrational numbers converge to a rational number c. Then $f((x_n)) = 0 \neq f(c)$

5.2 Combinations of Continuous Functions

5.3 Continuous Functions on Intervals

Bolzano's Intermediate Value Theorem

Let I be an interval and let $f: I \to \mathbb{R}$ be continuous on I.
If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$,
then $\exists\, c \in I$ between a and b s.t. $f(c) = k$.

5.4 Uniform Continuity

Uniform Continuity Theorem

Let I be a closed bounded interval and let $f: I \to \mathbb{R}$ be continuous on I.
Then f is uniformly continuous on I.

Def Let $A \subseteq \mathbb{R}$ ...

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Def Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. f is a Lipschitz function if

$$\frac{|f(x)-f(u)|}{|x-u|} \le K \quad \text{for some constant } K>0$$

all slopes on f are bounded

Thm If $f: A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.

Note: $\sqrt{x}$ is uniformly continuous, but not a Lipschitz function on $[0,2]$.

Approximation

↪ $\exists$ step function, piece-wise linear functions, polynomial functions s.t.

$\forall \varepsilon>0$, $|f(x)-S_\varepsilon(x)|<\varepsilon \quad \forall x \in I$. Weierstrass Approximation Theorem

5.5 Continuity and Gauges

5.6 Monotone and Inverse Functions

Thm Let $I \subset \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be monotone on I.

Then the set of points $D \subseteq I$ at which f is discontinuous is a countable set.

Continuous Inverse Theorem

Let $I \subset \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be strictly monotone and continuous on I.

Then the function g inverse to f is strictly monotone and continuous on $J := f(I)$

Chapter 6 : Differentiation

6.1 Derivative

Def Let $I \subset \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$, $c \in I$.

A real number L is the derivative of f at c

if $\forall \varepsilon>0$, $\exists \delta(\varepsilon)>0$ s.t. if $x \in I$ satisfies $0<|x-c|<\delta(\varepsilon)$,

then
$$\left|\frac{f(x)-f(c)}{x-c} - L\right| < \varepsilon \qquad f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

f is differentiable at c and $f'(c) = L$. given by the limit

Thm If f is differentiable at c, f is continuous at c.

Converse not true: e.g. $f(x)=|x|$, f is continuous at 0 but not differentiable at 0.

since $\lim_{x\to 0^+} \frac{|x|}{x} = 1$, $\lim_{x\to 0^-} \frac{|x|}{x} = -1$

Weierstrass Function

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \cos(3^n x)$$

continuous at every point, but derivative does not exist anywhere.

Chain Rule

Let I, J be intervals in $\mathbb{R}$, $g: I \to \mathbb{R}$ and $f: J \to \mathbb{R}$ s.t. $f(J) \subseteq I$. let $c \in J$.

If f is differentiable at c and g is differentiable at $f(c)$,

then $f \circ g$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

6.2 The Mean Value Theorem

Interior Extremum Theorem

Let c be an interior point of the interval I

at which $f: I \to \mathbb{R}$ has a relative extremum.

If the derivatives of f at c exists, then $f'(c)=0$.

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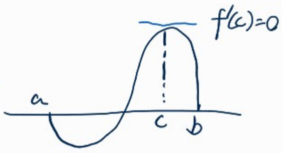
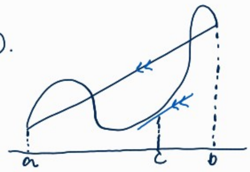
e.g. $f(x)=|x|$, $x=0$ minimum but $f'(0)$ not exists.

Rolle's Theorem



Suppose f continuous on $I := [a,b]$ (closed), and $f'(x)$ exists $\forall x \in (a,b)$ (open), and that $f(a)=f(b)=0$. Then $\exists\, c \in (a,b)$ s.t. $f'(c)=0$.

Mean Value Theorem



Suppose f continuous on $I := [a,b]$, and $f'(x)$ exists $\forall x \in (a,b)$. Then $\exists\, c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Darboux's Theorem

If f is differentiable on $I=[a,b]$, and if k between $f'(a)$ and $f'(b)$, then $\exists\, c \in (a,b)$ s.t. $f'(c)=k$. f' has the Intermediate Value Property.

6.3 L'Hospital's Rules

Cauchy Mean Value Theorem

Let f, g continuous on $[a,b]$ and differentiable on (a,b). Assume $g'(x) \neq 0$ $\forall x \in (a,b)$. Then $\exists\, c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

L'Hospital's Rule, I & II

Let $-\infty \le a < b < \infty$, f, g differentiable on (a,b) s.t. $g'(x) \neq 0$ $\forall x \in (a,b)$. Suppose that

I $\quad \displaystyle\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x)$ $\quad\Big|\quad$ $\displaystyle\lim_{x\to a^+} g(x) = \pm\infty$ $\quad$ II

(a) If $\displaystyle\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\displaystyle\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$ $\quad\Big|\quad$ (a) — " —

(b) If $\displaystyle\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, then $\displaystyle\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$ $\quad\Big|\quad$ (b) — " —

6.4 Taylor's Theorem

Taylor's Theorem

Let $n \in \mathbb{N}$, let $I := [a,b]$, $f: I \to \mathbb{R}$ s.t. $f, f', f'', \dots, f^{(n)}$ are continuous on I and that $f^{(n+1)}$ exists on (a,b).

If $x_0 \in I$, then $\forall x \in I$, $\exists$ a point c between x and x_0 s.t.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1} \\ &= P_n(x) + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Lagrange form / derivative form of the remainder}} \end{aligned}$$

Convex Functions

Let $I \subset \mathbb{R}$ be an interval. $f: I \to \mathbb{R}$ is convex on I if $\forall t$ satisfying $0 \le t \le 1$ and any points $x_1, x_2 \in I$, we have

$$f((1-t)x_1 + x_2) \le (1-t)f(x_1) + t f(x_2)$$

Let $f: I \to \mathbb{R}$ have a second derivative. Then f convex iff $f''(x) \ge 0$ $\forall x \in I$.

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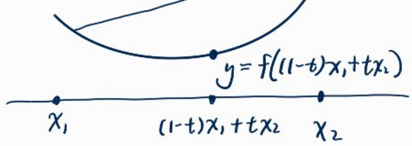


Newton's Method

Let $I := [a,b]$ and $f: I \to \mathbb{R}$ be twice differentiable on I.
Suppose that $f(a)f(b)<0$ and there are constants m, M s.t.

$$|f'(x)| \geq m > 0 \text{ and } |f''(x)| \leq M \quad \forall x \in I$$

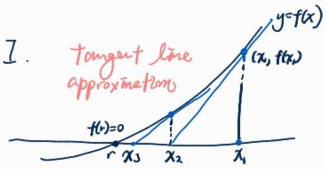
Let $K = \frac{M}{2m}$. Then $\exists$ a subinterval I^* containing a zero r of f s.t.

$\forall x_1 \in I^*$ the sequence (x_n) defined by

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)} \quad \forall n \in \mathbb{N}$$

belongs to I^* and (x_n) converges to r.

Moreover, $|x_{n+1} - r| \leq K|x_n - r|^2 \quad \forall n \in \mathbb{N}$.

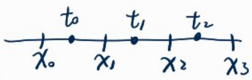


Approximate x_n s.t.
$f(x_n) \approx 0$
$(x_n) \to r$

Chapter 7 : Riemann Integral

7.1 Riemann Integral

Def $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ tagged partition



$\|P\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}$ norm of partition

Riemann Sum

$$S(f;\dot{P}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

Def $f: [a,b] \to \mathbb{R}$ is Riemann Integrable on $[a,b]$
if $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ s.t.
if $\dot{P}$ is any tagged partition of $[a,b]$ with $\|\dot{P}\| < \delta(\varepsilon)$, then

$$|S(f;\dot{P}) - L| < \varepsilon$$

The set of all Riemann Integrable functions: $R[a,b]$.
L is uniquely determined and $L = \int_a^b f$ or $\int_a^b f(x)\,dx$

Thm if $g \in R[a,b]$ and $f(x) = g(x)$ except for a finite number of points in $[a,b]$,
then $f \in R[a,b]$ and $\int_a^b f = \int_a^b g$.

Boundedness Theorem

If $f \in R[a,b]$, then f is bounded on $[a,b]$.

Thomae's function

$h: \{x \in \mathbb{R} : x > 0\} \to \mathbb{R}$

$$= \begin{cases} 0 & \text{for irrational number } x > 0 \\ \frac{1}{n} & \text{for rational number } \frac{m}{n}, \ m, n \text{ no common factor except } 1 \\ 1 & \text{for } x = 0 \end{cases}$$

h is continuous at every irrational number in A
discontinuous at every rational number in A, not monotone
h is Riemann Integrable with integral $= 0$

7.2 Riemann Integrable Functions

Cauchy Criterion

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$f: [a,b] \to \mathbb{R}$ belongs to $\mathcal{R}[a,b]$

iff $\forall \varepsilon > 0$, $\exists \eta_\varepsilon > 0$ s.t. if $\dot{P}$ and $\dot{Q}$ are any tagged partitions of $[a,b]$ with $\|\dot{P}\| < \eta_\varepsilon$ and $\|\dot{Q}\| < \eta_\varepsilon$, then

$$|S(f;\dot{P}) - S(f;\dot{Q})| < \varepsilon$$

Squeeze Theorem

Find partitions to prove Riemann Integrable is hard! Squeeze is easier

$f: [a,b] \to \mathbb{R}$ belongs to $\mathcal{R}[a,b]$

iff $\forall \varepsilon > 0$, $\exists$ functions α_ε and $\omega_\varepsilon \in \mathcal{R}[a,b]$ with

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \quad \forall x \in [a,b]$$

and s.t. $$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$$

Thm If f is $\begin{cases} \text{a step function} \\ \text{a continuous function} \\ \text{a monotone function} \end{cases}$, then $f \in \mathcal{R}[a,b]$

7.3 The Fundamental Theorem

The Fundamental Theorem of Calculus (First Form)

Suppose $\exists$ a finite set $E \subset [a,b]$ and functions $f, F: [a,b] \to \mathbb{R}$ s.t.

(a) F is continuous on $[a,b]$ ✓ if F is differentiable at every point

(b) $F'(x) = f(x)$ $\forall x \in [a,b] \setminus E$ ✓ even if f is not defined at some point

(c) $f \in \mathcal{R}[a,b]$! even if F is differentiable at every point, may fail

Then $$\int_a^b f = F(b) - F(a)$$

e.g. $K(x) = \begin{cases} x^2 \cos\left(\frac{1}{x^2}\right) & \text{for } x \in (0,1] \\ 0 & \text{for } x = 0 \end{cases}$

$K'(x) = \begin{cases} 2x \cos\left(\frac{1}{x^2}\right) + \left(\frac{2}{x}\right) \sin\left(\frac{1}{x^2}\right) & \text{for } x \in (0,1] \\ 0 & \text{for } x = 0 \end{cases}$

So K is continuous and differentiable on $[a,b]$ but K' is unbounded, so $K' \notin \mathcal{R}[0,1]$ and the theorem does not apply.

The Fundamental Theorem of Calculus (Second Form)

Let $f \in \mathcal{R}[a,b]$, f continuous at $c \in [a,b]$.

Then the indefinite integral

$$F(z) = \int_a^z f \quad \text{for } z \in [a,b]$$

is differentiable at c and $F'(c) = f(c)$.

if f is continuous on $[a,b]$, then F is the anti-derivative of f.

Def $Z \subset \mathbb{R}$ is a null set if $\forall \varepsilon > 0$, $\exists$ a countable collection $\{(a_k, b_k)\}_{k=1}^\infty$ of open intervals s.t.

$$Z \subseteq \bigcup_{k=1}^\infty (a_k, b_k) \quad \text{and} \quad \sum_{k=1}^\infty (b_k - a_k) < \varepsilon$$

Def $Q(x)$ holds almost everywhere on I, if $\exists$ a null set $Z \subset I$ s.t. $Q(x)$ holds $\forall x \in I \setminus Z$

"$Q(x)$ for a.e. $x \in I$"

E.g. $\mathbb{Q}$, of rational numbers on $[0,1]$ is a null set.

Lebesgue's Integrability Criterion

A bounded function $f: [a,b] \to \mathbb{R}$ is Riemann Integrable iff f is continuous almost everywhere on $[a,b]$.

Dirichlet's function

$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ discontinuous

E.g. Dirichlet's function is discontinuous on every point on $[0,1]$, so it is NOT Riemann Integrable.

E.g. Thomae's function is discontinuous on rational points i.e. null set, so it is Riemann Integrable.

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7.4 The Darboux Integral

Def Lower integral of f on I:

$$L(f) := \sup \{ L(f;P) : P \in \mathcal{P}(I) \}$$

$L(f;P)$: lower sum $= \sum_{k=1}^{m} \left\{ \inf_{x \in [x_{k-1}, x_k]} \{f(x)\} \right\} \cdot (x_k - x_{k-1})$

Upper integral of f of I:

$$U(f) := \inf \{ U(f;P) : P \in \mathcal{P}(I) \}$$

$U(f;P)$: upper sum $= \sum_{k=1}^{m} \left\{ \sup_{x \in [x_{k-1}, x_k]} \{f(x)\} \right\} \cdot (x_k - x_{k-1})$

Darboux Integral

Let $I := [a,b]$ and $f: I \to \mathbb{R}$ be bounded.

Then f is Darboux integrable on I iff $L(f) = U(f)$, and the Darboux Integral is $L(f) = U(f)$.

Equivalence Theorem

A function f on $I = [a,b]$ is Darboux integrable iff it is Riemann integrable.

7.5 Approximate Integration

Equal Partitions

$$T_n(f) := \frac{1}{2}(L_n(f) + R_n(f))$$
$$= h_n \left(\frac{1}{2} f(a) + \sum_{k=1}^{n-1} f(a + k h_n) + \frac{1}{2} f(b) \right)$$

P_n: the partition $a < a + h_n < a + 2h_n < \cdots < a + n h_n = b$

Let f monotone on $[a,b]$. Then

$$\left| \int_a^b f - T_n(f) \right| \leq |f(b) - f(a)| \cdot \frac{(b-a)}{2n}$$

Trapezoid's Rule
Midpoint Rule
> by piecewise linear functions

Simpson's Rule by parabolic arcs

Chapter 8 Sequences of Functions

8.1 Pointwise and Uniform Convergence

Pointwise Convergence

A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges to a function $f: A_0 \to \mathbb{R}$, iff $\forall \varepsilon > 0$ and each $x \in A_0$, $\exists K(\varepsilon, x) \in \mathbb{N}$ s.t. if $N > K(\varepsilon, x)$, then

$$|f_n(x) - f(x)| < \varepsilon.$$

f: limit on A_0 of the sequence (f_n) $\quad f = \lim (f_n)$ on A_0

(f_n): convergent on A_0 / converges pointwise on A_0 $\quad f_n \to f$ on A_0

E.g. Let $f_n := \frac{x}{n}$. Then $\lim (f_n) = x \lim \frac{1}{n} = x \cdot 0 = 0$.

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Uniform Convergence

A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges uniformly on $A_0 \subseteq A$ to a function $f: A_0 \to \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists K(\varepsilon) \in \mathbb{N}$ s.t. if $n \geq K(\varepsilon)$, then

$$|f_n(x) - f(x)| < \varepsilon.$$

(f_n): uniformly convergent on A_0 $\quad f_n \rightrightarrows f$ on A_0

E.g. $f_n := \frac{x}{n}$, $f = 0$. Let $x_k = n_k = 1$, then $|f_{n_k}(x_k) - f(x_k)| = 1 > \varepsilon$.
$\to$ not uniformly convergent.

Def If $A \subseteq \mathbb{R}$ and $\varphi: A \to \mathbb{R}$ is a function, then φ is bounded on A if the set $\varphi(A)$ is a bounded subset of $\mathbb{R}$. If φ is bounded, define the uniform norm of φ on A as

$$\|\varphi\|_A := \sup\{|\varphi(x)| : x \in A\}$$

Lemma A sequence (f_n) of bounded functions converges uniformly on A to f iff $\|f_n - f\|_A \to 0$.

E.g. $f_n := \frac{x}{n}$ cannot be applied since f_n is unbounded.

Cauchy Criterion for Uniform Convergence

Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Then it converges uniformly on A to a bounded function f iff $\forall \varepsilon > 0$, $\exists H(\varepsilon) \in \mathbb{N}$ s.t. $\forall n, m \geq H(\varepsilon)$,

$$\|f_m - f_n\|_A \leq \varepsilon$$

8.2 Interchange of Limits

Thm (f_n) sequence of continuous functions on $A \subseteq \mathbb{R}$.
$(f_n) \rightrightarrows f$ on A. Then f is continuous on A.

Thm (f_n) sequence of functions on J bounded interval on $\mathbb{R}$.
$\exists x_0 \in J$ s.t. $f_n(x_0)$ converges

(f_n') sequence of derivatives exists on J.
$(f_n') \rightrightarrows g$ on J

Note: (f_n) differentiable
$(f_n) \rightrightarrows f \not\Rightarrow f$ differentiable.
E.g. Weierstrass function
If considering (f_n) = partial sums $\to$ differentiable
but $f \to$ not differentiable.

Then $(f_n) \rightrightarrows$ a function f on J that has a derivative at every point in J, and $f' = g$.

Thm (f_n) sequence of functions $\in \mathcal{R}[a,b]$,
$(f_n) \rightrightarrows f$ on $[a,b]$. Then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

Bounded Convergence Theorem

(f_n) sequence of functions $\in \mathcal{R}[a,b]$.
$(f_n) \to$ a function $f \in \mathcal{R}[a,b]$. converges pointwise
$\exists B > 0$ s.t. $|f_n(x)| \leq B$ $\forall x \in [a,b]$. Then $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.
bounded

Dini's Theorem

Author: kiking

(f_n) monotone sequence of continuous functions on $I := [a,b]$ closed & bounded

$(f_n) \to$ a continuous function f

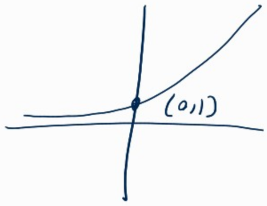
Then the convergence of the sequence is uniform.

8.3 The Exponential and Logarithmic Functions.

Exponential Function

The unique function $E: \mathbb{R} \to \mathbb{R}$ s.t.

$$E'(x) = E(x) \quad \forall x \in \mathbb{R}$$
$$E(0) = 1$$



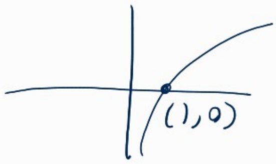
$e := E(1)$ is the Euler number. $\lim_{x \to -\infty} E(x) = 0$

$\lim_{x \to \infty} E(x) = \infty$

Inverse function: Logarithm

$\lim_{x \to -\infty} L(x) = -\infty$

$\lim_{x \to \infty} L(x) = \infty$



Power Function

If $\alpha \in \mathbb{R}$ and $x > 0$, then

$$x^\alpha := e^{\alpha \ln x} = E(\alpha L(x))$$

extend from rational power to arbitrary real number

8.4 Trigonometric Functions

Thm $\exists$ functions $C: \mathbb{R} \to \mathbb{R}$ s.t.

(i) $C''(x) = -C(x)$ and

$S''(x) = -S(x) \quad \forall x \in \mathbb{R}$

(ii) $C(0) = 1 \qquad S(0) = 0$

$C'(0) = 0 \qquad S'(0) = 1$

Proof Define (C_n) (S_n)

(i) $C_1(x) = 1$

$S_1(x) = x$

(ii) $S_n(x) = \int_0^x C_n(t)\,dx$

(iii) $C_{n+1}(x) = 1 - \int_0^x S_n(t)\,dx \quad \forall n \in \mathbb{N},\ x \in \mathbb{R}$

$\Rightarrow \quad S_n'(x) = C_n(x)$

$C_{n+1}'(x) = -S_n(x)$

$\Rightarrow \quad C_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$

$S_{n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

By Induction

Define $S(x) := \lim S_n(x)$

$C(x) := \lim C_n(x)$

Let $\cos x := C(x)$

$\sin x := S(x)$

Then $C'(x) = -S(x)$, $S'(x) = C(x) \quad \forall x \in \mathbb{R}$

$(C(x))^2 + (S(x))^2 = 1 \quad \forall x \in \mathbb{R}$

Thm If $f: \mathbb{R} \to \mathbb{R}$ s.t. $f''(x) = -f(x)$,

then $\exists\, \alpha, \beta \in \mathbb{R}$ s.t.

$$f(x) = \alpha C(x) + \beta S(x)$$

Thm C is even: $C(-x) = C(x)$

S is odd: $S(-x) = -S(x)$

Addition Formulas

$$C(x+y) = C(x)C(y) - S(x)S(y)$$
$$S(x+y) = S(x)C(y) + C(x)S(y)$$

Chapter 9 Infinite Series

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9.1 Absolute Convergence

Def Let $X := (x_n)$ be a sequence in $\mathbb{R}$.
$\sum x_n$ is absolutely convergent if $\sum |x_n|$ is convergent in $\mathbb{R}$.
A series is conditionally convergent if it is convergent but not absolutely convergent.

E.g. Alternative Harmonic Series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent
since Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent because of the subtraction

Thm If a series in $\mathbb{R}$ is absolutely convergent, it is convergent.

Thm If a series $\sum x_n$ is convergent, then any series obtained from grouping it also converges to the same value.

Rearrangement Theorem

If a series $\sum x_n$ is absolutely convergent, then any rearrangement $\sum y_k$ of $\sum x_k$ also converges to the same value.

9.2 Tests for Absolute Convergence

Limit Comparison Test II

Suppose (x_n) (y_n) non zero real sequence and the limit exists in $\mathbb{R}$.

$$r := \lim \frac{|x_n|}{|y_n|}$$

(a) If $r \neq 0$, then $\sum x_n$ is abs. convergent iff $\sum y_n$ is abs. convergent

(b) If $r = 0$, if $\sum y_n$ is abs. convergent, then $\sum x_n$ is abs. convergent

Root Test

(a) If $\exists\, r \in \mathbb{R}$ with $r < 1$ and $K \in \mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} < r \quad \forall n \geq K$$

then $\sum x_n$ is abs. convergent.

(b) If $\exists\, K \in \mathbb{N}$ s.t.

$$|x_n|^{\frac{1}{n}} \geq 1 \quad \forall n \geq K$$

then $\sum x_n$ is divergent.

Suppose the limit

$$r := \lim |x_n|^{\frac{1}{n}}$$

exists. Then

$$r = \begin{cases} < 1, & \sum x_n \text{ abs. convergent} \\ > 1 & \sum x_n \text{ divergent} \end{cases}$$

Ratio Test

Suppose $X := (x_n)$ a sequence of non-zero real numbers.

(a) If $\exists\, r \in \mathbb{R}$ with $0 < r < 1$ and $K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \leq r \quad \forall n \geq K$$

then $\sum x_n$ is abs. convergent.

(b) If $\exists\, K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \geq 1 \quad \forall n \geq K$$

then $\sum x_n$ is divergent.

Let (x_n) non-zero sequence in $\mathbb{R}$.
Suppose the limit

$$r := \lim \left|\frac{x_{n+1}}{x_n}\right|$$

exists in $\mathbb{R}$. Then

$$r = \begin{cases} < 1 & \sum x_n \text{ abs. convergent} \\ > 1 & \sum x_n \text{ convergent} \end{cases}$$

Integral Test

Let f be a positive, decreasing function on $\{t : t \geq 1\}$.
Then $\sum_{k=1}^{\infty} f(k)$ converges iff the Improper integral exists.

$$\int_1^{\infty} f(t)\,dt = \lim_{b \to \infty} \int_1^b f(t)\,dt$$

Raabe's Test

Let $X := (x_n)$ be a sequence of non-zero real numbers.

(a) If $\exists\, a > 1$ and $K \in \mathbb{N}$ s.t.

$$\left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n} \quad \text{for } n \geq K$$

then $\sum x_n$ is abs. convergent.

(b) If $\exists\, a \leq 1$ and $K \in \mathbb{N}$ st

Let

$$a := \lim \left(n \cdot \left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right)\right)$$

whenever the limit exists. Then

$$a = \begin{cases} > 1 & \sum x_n \text{ abs. convergent} \end{cases}$$

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$\left|\frac{x_{n+1}}{x_n}\right| \geqslant 1-\frac{a}{n}$ for $n \geqslant K$,

then Σx_n is not abs. convergent.

Σx_n not abs. convergent.

Euler's Constant

$$C_n := 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} - \ln n$$

is convergent. $\approx 0.5772\dots$ Unsolved: rational or irrational

9.3 Tests for Non-Absolute Convergence

Alternating Series Test

Dirichlet's Test

Abel's Test

9.4 Series of Functions

Def If (f_n) a sequence of functions defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$, the sequence of partial sums (S_n) of the infinite series Σf_n is defined for x in D by

$$S_1(x) := f_1(x)$$
$$S_2(x) := f_2(x)$$
$$\vdots$$
$$S_{n+1}(x) := S_n(x) + f_{n+1}(x)$$

If the sequence (S_n) converges to a function f on D, the infinite series of function Σf_n converges to f on D. Σf_n denote the series / the limit function when it exists.

Def If $\Sigma |f_n(x)|$ converges $\forall x \in D \Rightarrow \Sigma f_n$ is abs. convergent on D.

If (S_n) of partial sums is uniformly convergent on D to f, Σf_n is uniformly convergent on D.

Thm If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ $\forall n \in \mathbb{N}$ and $\Sigma f_n \rightrightarrows f$ on D, then f is continuous.

Thm Suppose the real-valued $f_n \in \mathcal{R}[a,b]$.

$\Sigma f_n \rightrightarrows f$ on $J=[a,b]$, then $f \in \mathcal{R}[a,b]$ and $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$

Thm (f_n) sequence of functions on J bounded interval on $\mathbb{R}$.

$\exists\, x_0 \in J$ s.t. $\Sigma f_n(x_0)$ converges

$\Sigma(f_n')$ sequence of derivatives exists on J.

$\Sigma(f_n')$ converges uniformly on J

Then $\Sigma(f_n) \rightrightarrows$ a function f on J that has a derivative at every point in J, and $f' = \Sigma(f_n')$

Cauchy Criterion

Weierstrass M-Test

> Tests for Uniform Convergence of Series

Let (M_n) be a sequence of positive real numbers s.t. $|f_n(x)| \le M_n$ for $x \in D$, $n \in \mathbb{N}$. If the series ΣM_n is convergent, then Σf_n is uniformly convergent on D

Power Series

Σf_n is a power series around $x=c$ if f_n has the form

$$f_n(x) = a_n(x-c)^n$$

for $a_n, c \in \mathbb{R}$, $n \in \mathbb{N}$

for simplicity, $c=0 \Rightarrow \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \dots + a_n x^n$

Radius of Convergence

Let $\Sigma a_n x^n$ be power series. If the sequence $(|a_n|^{\frac{1}{n}})$ is bounded, set

$$\rho := \lim\sup (|a_n|^{\frac{1}{n}})$$

Then the radius of convergence of $\Sigma a_n x^n$ is

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$$R := \begin{cases} 0 & \text{if } \rho = +\infty \\ \frac{1}{\rho} & \text{if } 0<\rho<+\infty \\ +\infty & \text{if } \rho = 0 \end{cases}$$

The interval of convergence is $(-R, R)$

Cauchy - Hadamard Theorem

If R is the radius of convergence of the power series $\sum a_n x^n$,
then if $|x| < R \rightarrow$ the series is abs. convergent.
$|x| > R \rightarrow$ the series is divergent.

Chapter 10 The Generalized Riemann Integral

10.1 Definations and Main Properties

Gauge on $[a,b]$: a strictly positive function $\delta: [a,b] \rightarrow (0,\infty)$

δ-fine tagged partition $\dot{P}$: $t_i \in I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$ for $i=1,\dots,n$

Generalized Riemann Integral

A function $f: [a,b] \rightarrow \mathbb{R}$ is generalized Riemann Integrable on $[a,b]$
if $\exists\, L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$, $\exists$ a gauge δ_ε on $[a,b]$ s.t.
if $\dot{P}$ is any δ_ε-fine partition of $[a,b]$, then

$$|S(f;\dot{P}) - L| < \varepsilon$$

Collection of generalized Riemann integrable functions: R^*

※ Gauges more useful than norm $\|\dot{P}\|$:
More dedicated control: length of the largest subinterval
use of small subintervals
large subintervals
when function vary rapidly / remain constant

E.g. Dirichlet's function $\notin R[0,1]$ but $\in R^*[0,1]$, with integral $=0$. Set $\delta_\varepsilon = \begin{cases} \frac{\varepsilon}{2^{k+2}} & \text{when } x \text{ rational} \\ 1 & \text{when } x \text{ irrational} \end{cases}$

~ Cauchy Criterion
~ Squeeze Theorem
> Similar to that of Riemann Integral

The Fundamental Theorem of Calculus (stronger)

The Fundamental Theorem of Calculus (First Form)

Suppose $\exists$ a ~~finite~~ countable set $E \subset [a,b]$ and functions $f, F: [a,b] \rightarrow \mathbb{R}$ s.t.

(a) F is continuous on $[a,b]$ ✓ if F is differentiable at every point

(b) $F'(x) = f(x)$ $\forall x \in [a,b] \setminus E$ ✓ even if f is not defined at some point

~~Then (c)~~ Then $f \in R^*[a,b]$

~~Then~~ and $\int_a^b f = F(b) - F(a)$

The Fundamental Theorem of Calculus (Second Form)

Let $f \in R^*[a,b]$, ~~f continuous at $c \in [a,b]$.~~

Then the indefinite integral

$$F(z) = \int_a^z f \quad \text{for } z \in [a,b]$$

~~if f is continuous on $[a,b]$,~~
~~then F is the anti-derivative of f.~~

(a) F is continuous on $[a,b]$

(b) $\exists$ a null set Z s.t. if $x \in [a,b] \setminus Z$, then
F is differentiable at x and $F'(x) = f(x)$.

(c) If f is continuous at $c \in [a,b]$, then $F'(c) = f(c)$

The Multiplication Theorem

If $f, g \in R[a,b]$, then $f \cdot g \in R[a,b]$.

If $f \in R^*[a,b]$ and g a monotone function on $[a,b]$, then $f \cdot g \in R^*[a,b]$

10.2 Improper and Lebesgue Integral

Hake's Theorem

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If $f:[a,b]\to\mathbb{R}$, then $f\in\mathcal{R}^*[a,b]$ iff $\forall\, \gamma\in[a,b)$, the restriction of f to $[a,\gamma]\in\mathcal{R}^*[a,b]$ and

$$\lim_{\gamma\to b^-}\int_a^{\gamma} f = A\in\mathbb{R}$$

In this case $\int_a^b f = A$

* The generalized Riemann Integral does not need to be extended by taking such limit

Fact: Generalized Riemann Integrable functions are not "absolute integrals";
$\exists$ generalized Riemann integrable functions whose absolute values is not generalized Riemann integrable.

Lebesgue Integrable Functions

$f\in\mathcal{R}^*[a,b]$ is Lebesgue integrable on $[a,b]$ if $|f|\in\mathcal{R}^*[a,b]$, denoted by $\mathcal{L}[a,b]$

E.g. Step functions
Polynomials
Bounded Measurable Functions

Def If $f\in\mathcal{L}[a,b]$, define seminorm $= \|f\| := \int_a^b |f|$.
If $f,g\in\mathcal{L}[a,b]$, define $dist(f,g) = \|f-g\| := \int_a^b |f-g|$

Completeness Theorem

A sequence (f_n) of functions on $\mathcal{L}[a,b]$ converges to a function $f\in\mathcal{L}[a,b]$ iff $\forall\,\varepsilon>0$, $\exists\, H(\varepsilon)$ s.t. if $m,n\geq H(\varepsilon)$, then

$$\|f_m-f_n\| = dist(f_m,f_n) < \varepsilon.$$

10.3 Infinite Intervals

Generalized Riemann Intergrable
Lebesgue Integrable
Hake's Theorem
Fundamental Theorem

} generalize to unbounded closed intervals $[a,\infty)$, $(-\infty,b]$ $(-\infty,\infty)$

10.4 Convergence Theorems

Thm (f_n) sequence of functions $\in\mathcal{R}^*[a,b]$,
$(f_n)\rightrightarrows f$ on $[a,b]$. Then $f\in\mathcal{R}^*[a,b]$ and $\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$.

Def (f_k) sequence in $\mathcal{R}^*(I)$ is equi-integrable if $\forall\,\varepsilon>0$, $\exists$ gauge δ_ε on I s.t. if $\dot{\mathcal{P}}$ is any δ_ε-fine partition of I and $k\in\mathbb{N}$, then

$$\left|S(f_k;\dot{\mathcal{P}}) - \int_I f_k\right| < \varepsilon$$

Equi-integrability Theorem

If $(f_k)\in\mathcal{R}^*(I)$ is equi-integrable on I and if $f(x) = \lim f_k(x)$ $\forall x\in I$, then $f\in\mathcal{R}^*(I)$ and $\int_I f = \lim_{k\to\infty}\int_I f_k$

Monotone Convergence Theorem

Let $(f_k)\in\mathcal{R}^*(I)$ be monotone and $f(x) = \lim f_k(x)$ almost everywhere on I.
Then $f\in\mathcal{R}^*(I)$ iff the sequence of integrals $\left(\int_I f_k\right)$ is bounded in $\mathbb{R}$, in which case

$$\int_I f = \lim_{k\to\infty}\int_I f_k.$$

(Lebesgue) Dominated Convergence Theorem

Let (f_n) a sequence in $\mathcal{R}^*(I)$ and let $f(x) = \lim f_k(x)$ almost everywhere on I.
If $\exists$ functions α,ω in $\mathcal{R}^*(I)$ s.t.

$$\alpha(x)\leq f_k(x)\leq \omega(x) \quad \text{for almost every } x\in I,$$

then $f\in\mathcal{R}^*(I)$ and $\int_I f = \lim_{k\to\infty}\int_I f_k$.

Moreover, if $\alpha,\omega\in\mathcal{L}(I)$, then f_k and $f\in\mathcal{L}(I)$ and

$$\|f_k - f\| = \int_I |f_k - f| \to 0$$

Thm $f:[a,b]\to\mathbb{R}$ is (Lebesgue) measurable if $\exists$ a sequence (s_k) of step functions on $[a,b]$ s.t.

Thm: holds for (s_k) sequence of continuous functions

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$f(x) = \lim_{k\to\infty} S_k(x)$ for almost every $x \in [a,b]$

The collection of all measurable functions: $\mathcal{M}[a,b]$.

E.g. $\left\{\begin{array}{l}\text{Step functions} \\ \text{Monotone functions} \\ \text{Continuous functions}\end{array}\right.$ are measurable functions

E.g. $\left\{\begin{array}{l}\text{Dirichlet function} \\ \text{Thomae's function}\end{array}\right.$ are measurable functions

Since $\mathbb{Q} \cap [0,1]$ is a null set, take $(S_k) = 0$-function. Then $S_k(x) \to f(x)$ for $x \in [0,1] \setminus \mathbb{Q}$

Measurability Theorem

If $f \in \mathcal{R}^*(a,b)$, then $f \in \mathcal{M}[a,b]$

Converse not true: measurable does not imply integrable

Integrability Theorem

Let $f \in \mathcal{M}[a,b]$. Then $f \in \mathcal{R}^*[a,b]$ iff $\exists\, \alpha, \omega \in \mathcal{R}^*[a,b]$ s.t. $\alpha(x) \le f(x) \le \beta(x)$ for almost every $x \in [a,b]$.

Moreover if either $\alpha, \omega \in \mathcal{L}[a,b]$, then $f \in \mathcal{L}[a,b]$.

Chapter 11 A Glimpse into Topology

11.1 Open and Closed Sets in $\mathbb{R}$

Cantor Set

The Cantor set $\mathbb{F}$ is the intersection of the sets F_n, $n \in \mathbb{N}$, obtained by successive removal of open middle thirds, starting with $[0,1]$

(i) Total length of the removed interval is 1

$$L = \frac{1}{3} + \frac{2}{3^2} + \dots + \frac{2^n}{3^{n+1}} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\cdot\underbrace{\frac{1}{1-\frac{2}{3}}}_{\text{formula for geometric series}} = 1$$

$\Rightarrow \mathbb{F}$ has length 0

(ii) $\mathbb{F}$ contains no non-empty open intervals as a subset

(iii) $\mathbb{F}$ has infinitely (even uncountably) many points.

infinitely: $\mathbb{F}$ contains all endpoints of intervals in the form $\frac{2k}{3^n}$ for $k \in \mathbb{N}$

uncountably: More points than $\frac{2k}{3^n}$; prove by contradiction (otherwise $[0,1]$ is countable which is false)

11.2 Compact Sets

every open cover of K has a finite subcover

Heine-Borel Theorem

A subset K of $\mathbb{R}$ is compact iff it is closed and bounded.

Bolzano-Weierstrass: every bounded sequence has a converging subsequence.

Thm A subset K of $\mathbb{R}$ is compact iff every sequence in K has a subsequence that converges to a point in K.

11.3 Continuous Functions

11.4 Metric Spaces

Thm A metric space (S,d) is complete if each Cauchy sequence converges to a point of S

E.g. $\mathbb{Q}$ is not complete, e.g. (x_n) rational converges to $\sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$

C with metric $d_\infty(f,g) = \sup\{|f(x)-g(x)| : x \in [0,1]\}$ is complete

C with metric $d_1(f,g) = \int_0^1 |f-g|$ is not complete

Semimetric space

A space with a semimetric d where $d(x,y)=0$ if $x=y$

$\Rightarrow$ might not converge to a unique limit but $d(x,y)=0 \not\Rightarrow x=y$

identification $\to$ equivalent classes.

Appendix E

A continuous but nowhere differentiable function

Let $f_0 : \mathbb{R} \to \mathbb{R} = \operatorname{dist}(x, \mathbb{Z})$
$= \inf\{|x-k|, k \in \mathbb{Z}\}$

A "sawtooth" function with slope ± 1 on $\left[\frac{k}{2}, \frac{(k+1)}{2}\right]$

$f_m := \left(\frac{1}{4^m}\right) f_0(4^m x)$ with $0 \le f_m(x) \le \frac{1}{2\cdot 4^m}$ for $m > 0$ Also "sawtooth"

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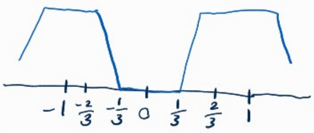
(Let $g := \sum g_n(x)$. Weierstrass-M implies g is continuous)

A space-filling curve

Let $\varphi: \mathbb{R} \to \mathbb{R}$ be the continuous, even function with period 2

$$\varphi(t) := \begin{cases} 0 & \text{for } 0 \le t \le \frac{1}{3} \\ 3t-1 & \text{for } \frac{1}{3} \le t \le \frac{2}{3} \\ 1 & \text{for } \frac{2}{3} \le t \le 1 \end{cases}$$



For $t \in [0,1]$, define

$$f(t) := \sum_{k=0}^{\infty} \frac{\varphi(3^{2k} t)}{2^{k+1}} \qquad g(t) := \sum_{k=0}^{\infty} \frac{\varphi(3^{2k+1} t)}{2^{k+1}}$$

Since $0 \le \varphi(t) \le 1$, Weierstrass M test implies f, g are continuous on $[0,1]$

An arbitrary point (x_0, y_0) in $[0,1] \times [0,1]$ is the image under (f, g) of some point $t_0 \in [0,1]$

Let x_0, y_0 have a binary (base 2) representation:

$$x_0 = \frac{a_0}{2} + \frac{a_2}{2^2} + \frac{a_4}{2^3} + \cdots \qquad y_0 = \frac{a_1}{2} + \frac{a_3}{2^2} + \frac{a_5}{2^3} + \cdots$$

Let t_0 have ternary (base 3) representation:

$$t_0 = \sum_{k=0}^{\infty} \frac{2a_k}{3^{k+1}} = \frac{2a_0}{3} + \frac{2a_1}{3^2} + \frac{2a_2}{3^3} + \cdots$$

Note if $a_0 = \begin{cases} 0, \Rightarrow 0 \le t_0 \le \frac{1}{3} \Rightarrow \varphi(t_0) = 0 \\ 1, \Rightarrow \frac{2}{3} \le t_0 \le 1 \Rightarrow \varphi(t_0) = 1 \end{cases} \Rightarrow \varphi(a_0) = a_0$

Similarly $\forall n \in \mathbb{N}\ \exists m_n \in \mathbb{N}$ s.t.

$$3^n t_0 = 2m_n + \frac{2a_n}{3} + \frac{2a_{n+1}}{3^2} + \cdots$$

and $\varphi(3^n t_0) = a_n$ since φ has period 2

$$f(t_0) = \sum \frac{\varphi(3^{2k} t_0)}{2^{k+1}} = \sum \frac{a_{2k}}{2^{k+1}} = x_0$$

$$g(t_0) = \sum \frac{\varphi(3^{2k+1} t_0)}{2^{k+1}} = \sum \frac{a_{2k+1}}{2^{k+1}} = y_0$$

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