Introduction to Hamiltonian Dynamical Systems and The N-Body Problem

(by KR Meyer)

Exercise Solutions

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Problem 1.1

Solution.

(a) Let $\ell = mr \times R$ and compute it directly directly

$$\begin{aligned}\ell = mr \times R = m\begin{vmatrix} i & j & k \\ x & y & z \\ X & Y & Z \end{vmatrix} \\ &= m\left(yZi + zXj + xYk - yXk - xZj - zYi\right) \\ &= m(yZ - zY)i + m(zX - xZ)j + m(xY - yX)k \\ &= m(y\dot{z} - z\dot{y})i + m(z\dot{x} - x\dot{z})j + m(x\dot{y} - y\dot{x})k. \end{aligned} \tag{1}$$

(b) With loss of generaliti, take $\ell_x = m(y\dot{z} - z\dot{y})$ and $\ell_y = m(z\dot{x} - x\dot{z})$, which are respectively independent on $x, \dot{x}$ and $y, \dot{y}$, then

$$\begin{aligned}\{\ell_x, \ell_y\} &= \left(\frac{\partial \ell_x}{\partial x}\frac{\partial \ell_y}{\partial (m\dot{x})} + \frac{\partial \ell_x}{\partial y}\frac{\partial \ell_y}{\partial (m\dot{y})} + \frac{\partial \ell_x}{\partial z}\frac{\partial \ell_y}{\partial (m\dot{z})}\right) \\ &\quad - \left(\frac{\partial \ell_y}{\partial x}\frac{\partial \ell_x}{\partial (m\dot{x})} + \frac{\partial \ell_y}{\partial y}\frac{\partial \ell_x}{\partial (m\dot{y})} + \frac{\partial \ell_y}{\partial z}\frac{\partial \ell_x}{\partial (m\dot{z})}\right) \\ &= m\dot{y}(-mx) - m^2\dot{x}y = m^2(x\dot{y} - y\dot{x}). \end{aligned} \tag{2}$$

Thus, we can conclude that

$$\{\ell_x, \ell_y\} = m\ell_z. \tag{3}$$

Since $\{f, g\} = -\{g, f\}$, we also have

$$\{\ell_y, \ell_x\} = -m\ell_z. \tag{4}$$

(c) Since ℓ does not depend on time explicitly, we know that $\partial \ell_x/\partial t = \partial \ell_y/\partial t = \partial \ell_z/\partial t = 0$. The system admitting to two components of ℓ means

$$\frac{\mathrm{d}\ell_\alpha}{\mathrm{d}t} = \{\ell_\alpha, H\} = 0, \quad \frac{\mathrm{d}\ell_\beta}{\mathrm{d}t} = \{\ell_\beta, H\} = 0. \tag{5}$$

And use the Jacobi's identity

$$\{H, \{\ell_\alpha, \ell_\beta\}\} + \{\ell_\beta, \{H, \ell_\alpha\}\} + \{\ell_\alpha, \{\ell_\beta, H\}\} = 0 \tag{6}$$

we can write

$$\{H, \{\ell_\alpha, \ell_\beta\}\} = \{H, \pm m\ell_\gamma\} = \pm m\frac{\mathrm{d}\ell_\gamma}{\mathrm{d}t} = 0. \tag{7}$$

Thus, the third component ℓ_γ satisfies

$$\frac{\mathrm{d}\ell_\gamma}{\mathrm{d}t} = 0, \tag{8}$$

which means that the system admits the third component of the angular momentum.