Algebraic Topology Aspects of Feynman Path Integral

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Abstract

This report delves into the intersection of topology, especially the homotopy theory, and the Feynman path integral in theoretical physics. Initially exploring foundational concepts in topology, such as topological spaces and quotient topology, the narrative progresses to algebraic topology and the fundamental group. This theoretical framework serves as a precursor to understanding homotopy theory, pivotal in algebraic topology and its application to the Feynman path integral. The article concludes by presenting critical theorems that bridge topology with the path integral framework, emphasizing their synergistic relationship in quantum mechanics.

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1 Introduction

Algebraic topology is a vibrant research field that employs algebraic structures to study the properties of topological spaces. One of the key algebraic structures within this field is homotopy, which draws immediate parallels to the Feynman path integral in physics. Homotopy theory offers a framework to perform path integrals in configuration spaces characterized by non-trivial topology. The algebraic tools provided by homotopy theory are crucial for analyzing path integrals in such complex configuration spaces. We explore the power of these theories and their practical application in the realm of quantum mechanics in this report.

2 Topology

Topology is a branch of mathematics concerned with the study of spaces and their properties under continuous transformations, without the need for precise measurements. It explores concepts such as continuity, connectedness, and compactness, focusing on the underlying structure rather than specific geometric details. By defining sets of points and the relationships between them, topology provides a framework for understanding spatial relationships across diverse fields, from pure mathematics to physics and engineering. With applications ranging from network analysis to the study of shapes and surfaces, topology offers a versatile toolkit for solving problems in various domains. The theoretical basis of topology is the definition of *topological space.*

2.1 Topological Spaces

Definition 1 ***Topological space***
Topological Space *is a space that the* **open sets** *(and therefore closed sets) are defined. Mathematically, topological space is a nonempty set X with a collection $\mathcal{O}$ of subsets of X satisfying the following conditions*

- $$\phi, X \in \mathcal{O}, \tag{2.1}$$
- $$\text{If } \{U_\alpha\}_{\alpha\in J} \subseteq \mathcal{O}, \text{ then } \bigcup_{\alpha\in J} U_\alpha \in \mathcal{O}, \tag{2.2}$$
- $$\text{If } U_1, \cdots, U_n \in \mathcal{O}, \text{ then } \bigcap_{i=1}^{n} U_i \in \mathcal{O}, \tag{2.3}$$

where J is an index set which may be infinite or even uncountable.

The sets in the collection $\mathcal{O}$ are called **open sets**. The set F is said to be **closed** if $X - F$ is open.

Remark 1 *A set may be open and closed at the same time in a topological space.*

Every metric space and manifold naturally falls within the realm of topological spaces. In fact, topological spaces serve as a generalization of metric spaces. Consequently, it becomes imperative to extend the definition of continuous functions, which is essential in topology, from metric spaces ($\epsilon - \delta$ definition) to accommodate this broader framework.

Definition 2 ***Continuous function***
Let X and Y be topological spaces and $f : X \to Y$ be a function. f is said to be ***continuous*** *if $f^{-1}(U)$ is open in X for all U open Y.*

Now we discuss the *quotient*, which is another essential definition in topology. We need to define the *equivalent relation* first.

Definition 3 ***Equivalent relation***
Let $\sim$ be a relation on a set A. $\sim$ is an equivalence relation if for all $a, b, c \in A$

$$\bullet \ a \sim a, \tag{2.4}$$

$$\bullet \ \text{If } a \sim b, \text{ then } b \sim a, \tag{2.5}$$

$$\bullet \ \text{If } a \sim b \text{ and } b \sim c, \text{ then } a \sim c. \tag{2.6}$$

Some simple examples of the equivalent relation are the equal relation and congruence modulo. We now can define the *equivalent class* with the definition of equivalent relation.

Definition 4 ***Equivalent class***
Let X be a set and $\sim$ be an equivalent relation in X. The ***equivalent class*** *of $x \in X$ denoted by $[x]$ is*

$$[x] = \{y \in X | y \sim x\}.$$

Now we discuss the *quotient*.

Definition 5 ***Quotient***
The ***quotient*** *of a set X and equivalent relation $\sim$ denoted by $X/\sim$ is*

$$X/\sim = \{[x] | x \in X\}.$$

The concept of equivalent class and quotient is *classification*, here is a simple example.

Example 2.1 ***Classification of shapes***
Let X be a set of all geometric shapes. The is an equivalent relation $\sim$ defined as $a \sim b$ if and only if a and b are the same shape. Then

$$X/\sim = \{\text{triangle}, \text{square}, \text{circle}, \cdots\}.$$

The definition of quotient can be any set with equivalent relations. If X is a topological space, we have the *quotient topology*.

Definition 6 ***Quotient topology***
Let X be a topological space and $\sim$ be an equivalent relation in X. Define the quotient map $\pi : X \to X/\sim$, $x \mapsto [x]$. Then $X/\sim$ is a topological space under the ***quotient topology*** *defined as*

$$U \subseteq X/\sim \text{ is open} \iff \pi^{-1}(U) \subseteq X \text{ is open}.$$

In topology, the quotient of an equivalent relation can be viewed as "sticking points together". A simple example is "making a circle".

Example 2.2 *Consider an interval $[0, 2\pi]$, which is a topological space, define an equivalent relation $0 \sim 2\pi$, then the quotient space $[0, 2\pi]/\sim$ can be viewed as a circle as shown in Figure 1.*

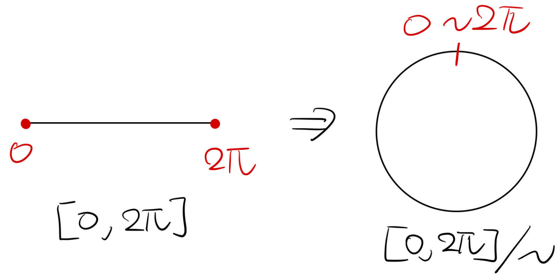

Figure 1: Making a circle is as quotient topology.

The concept of quotient In topology, we focus on the openness and closeness of the set. Alternatively, we can say we focus on the connection of the space. The spaces with the same connection are viewed as the same in topology. So many people say that "a cup is identical to a donut in topology". To justify if two topological spaces are identical or not, mathematicians came up with the concept of *topological invariants.* The most well-known one is *Euler characteristic.*

Example 2.3 ***Euler characteristic***
The Euler characteristic of a two-dimensional space is

$$\chi = F - V + E,$$

where F, V, E *is the number of faces, vertices, and edges.*

The abstract nature of topological spaces necessitates additional mathematical structures for in-depth study. One approach involves imbuing the topological invariants of topological spaces with algebraic structures, leading to a specialized research field known as *algebraic topology.*

2.2 Homotopy Theory in Algebraic Topology

In *algebraic topology Homotopy theory* is a branch of algebraic topology, that studies continuous mappings between topological spaces and explores when these mappings can be continuously deformed into one another. It provides insights into the shape of spaces, revealing connections and symmetries between them. Through concepts like homotopy equivalence and homotopy groups, it offers powerful tools for understanding the fundamental properties of spaces. Before discussing the homotopy theory, we need to discuss what is *path* in topology.

Definition 7 ***Path, loop, and product of paths***
Let X *be a topological space. A* ***path*** *in* X *is a continuous function* $f : [0, 1] \to X$*. If a path* f *satisfies* $f(0) = f(1) = x$*, then it is said to be a* ***loop*** *with the* ***bases pint*** x*. The* ***product*** *of two paths* f *and* g *satisfying* $f(1) = g(0)$ *is also a path and denoted by* $f \cdot g$ *defined as*

$$f \cdot g(t) = \begin{cases} f(2t), & \textit{if } 0 \le t \le \frac{1}{2}, \\ g(2t-1), & \textit{if } \frac{1}{2} \le t \le 1. \end{cases}$$

The meaning of the product of paths is connected, the visualization of the product of paths or loops is shown in Figure 2.

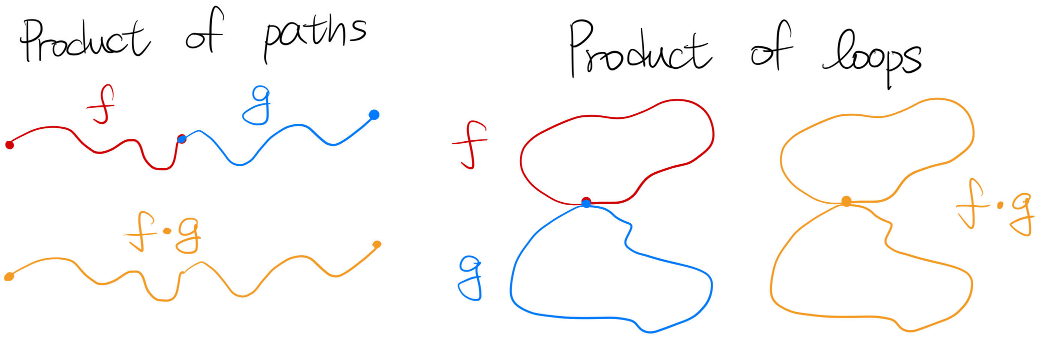

Figure 2: Product of paths and loops.

We can now define what is *homotopy* in algebraic topology.

Definition 8 ***Homotopy***
A ***homotopy*** *is a continuous function* $H : [0,1]^2 \to X$ *satisfying* $H(0,t) = a$ *and* $H(1,t) = b$ *for all* $t \in [0,1]$*. The paths* $f_0 = H(\cdot,0) : [0,1] \to X$ *and* $f_1 = H(\cdot,1) : [0,1] \to X$ *are said to be* ***homotopic*** *and denoted by* $f_0 \sim f_1$*.*

Intuitively, the second variable of the homotopy H can be viewed as the index of the function f_i, that is, $H(\cdot,i) = f_i$. Then we can imagine that two functions are *homotopic* means that two paths can transform to each other continuously as shown in Figure 3.

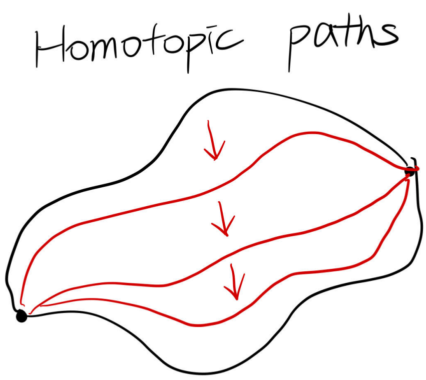

Figure 3: Homotopic paths.

The homotopic relation is an equivalent relation. Therefore, we have a natural equivalent class.

Definition 9 ***Homotopy class***
An equivalent class of a path f *under homotopy is said to be the* ***homotopy class*** *of* f*.*

With these definitions, we can start discussing *fundamental group*.

2.3 Fundamental Group

Now we can discuss the first *topological invariant* in this report, *fundamental group.*

Definition 10 ***Fundamental group***
Let X be a topological space. The ***fundamental group*** *is a group consisting of all homotopy classes of a loop $f : [0,1] \to X$ and is denoted by $\pi_1(X, x)$. That is,*

$$\pi_1(X,x) = \left\{ [f] \;\middle|\; f \textit{ is a loop with based point } x \textit{ in } X \right\}.$$

Remark 2 *One can check that fundamental groups are groups.*

We first look at the simplest nontrivial example, the fundamental group of a circle.

Example 2.4 *Fundemental group for one-dimensional circle*
By symmetry of the circle, the fundamental group is independent of the based point. The fundamental groups like that are denoted by

$$\pi_1(S^1, x) = \pi_1(S^1).$$

The fundamental group for one-dimensional is

$$\pi_1(S^1) \cong \mathbb{Z},$$

the strict proof of this example is a little long and complicated, but we can see the topological meaning of this result intuitively.
Consider a path

$$\omega : [0,1] \to S^1,\ t \mapsto (\cos(2\pi t), \sin(2\pi t))$$

is certainly a loop, and the corresponding homotopy class $[\omega]$ can generate the fundamental group, that is,

$$\pi_1(S^1) =< [\omega] >\cong< 1 >= \mathbb{Z}.$$

ω is a path that rotates clockwise on the circle for one time. Its inverse ω^{-1} is the path rotating in the opposite direction for one time. So we can see that the elements in $\pi_1(S^1)$ are the homotopy class of ω^n, that is, the paths rotating clockwise or anticlockwise for n times as shown in Figure 4.

In general, the fundamental group may be difficult to compute. The fundamental groups for some highly abstract spaces depend on those of basic topological spaces. We introduce an important theorem here.

Theorem 2.1 ***Decomposition of fundamental groups*** *Let X and Y be topological spaces and $x \in X,\ y \in Y$. Then*

$$\pi_1(X \times Y, (x,y)) = \pi_1(X,x) \times \pi_1(Y,y).$$

We can see another simple example is the fundamental group for ”donuts”.

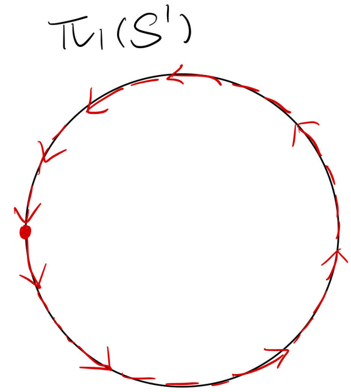

Figure 4: Visualization of $\pi_1(S^1)$.

Example 2.5 ***Fundamental group for two-dimensional ring surfaces***
A d-dimensional donuts-like ring surface is called a ***d-dimensional torus*** *and is defined by* $(S^1)^d$. *We often denote it by* T^d. *As shown in Figure 5, two homotopy classes can generate the fundamental group of this topological space and another one is the identity element. Mathematically, the fundamental group for the two-dimensional ring surface is isomorphic to the cartesian product of two fundamental groups of one-dimensional circle, that is,*

$$\pi_1(T^2) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}.$$

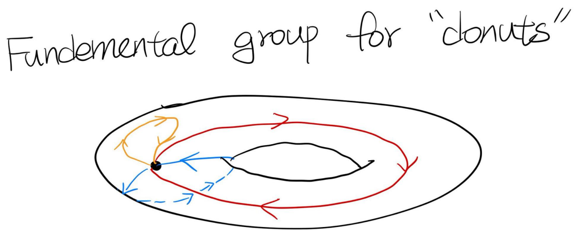

Figure 5: Fundamental group for "donuts".

2.4 Covering Space

The strict analysis and computation of the fundamental group require the concept of *covering space* and its *lifting properties*. These are what we will discuss in this section.

Definition 11 ***Covering space***
Let X *and* $\tilde{X}$ *be topological spaces and* $p : \tilde{X} \to X$ *be a continuous function.* $(\tilde{X}, p)$ *is called a* ***covering space*** *if for all* $x \in X$, *there exists an open neighborhood of* x *so that*

- $$p^{-1}(U) = \bigsqcup_{\lambda \in D} U_\lambda \subseteq \tilde{X}, \tag{2.7}$$
- $$p\big|_{U_i} \text{ is a homeomorphism for all } U_\lambda, \tag{2.8}$$

where D *is a discrete space and* U_λ *are called* ***sheets***.

Remark 3 *The covering spaces are NOT unique for a topological space.*

A visualized covering space is like Figure 6. It is a space "layer by layer", this is why u_λ are called sheets.

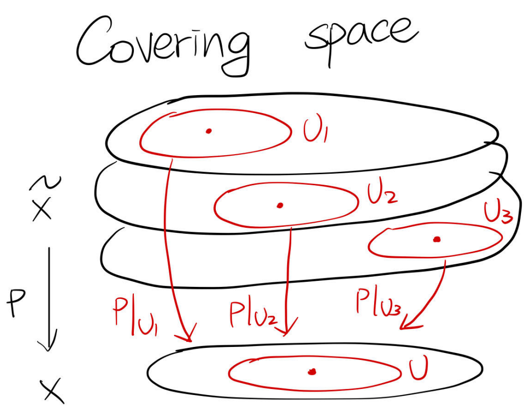

Figure 6: A visualized covering space, it is a space "layer by layer".

A critical property of the covering space is *lifting*, which makes the covering space significant.

Theorem 2.2 ***Lifting properties***
*Let X be a topological space, f be a path in X so that $f(0) = x$, and $(\tilde{X}, p)$ be the covering space for X. Then there exist paths $\tilde{f}_\lambda : [0,1] \to \tilde{X}$ for all $\tilde{x}_\lambda \in p^{-1}(x)$ so that $p\tilde{f}_\lambda = f$ and $\tilde{f}_\lambda(0) = \tilde{x}_\lambda$. Moreover, the homotopy also has the same property. These paths and homotopy are said to be **lifted**.*

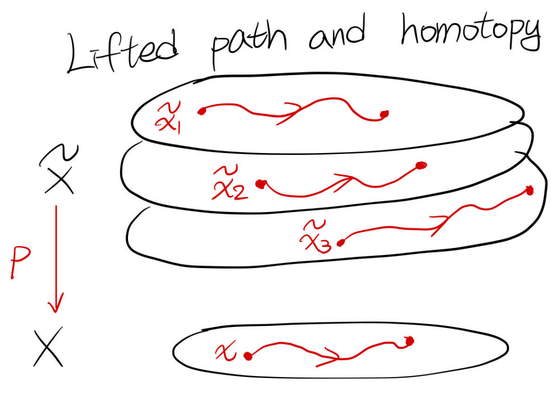

Figure 7: Lifted path and homotopy.

Some properties of the fundamental group may be difficult to view in the original topological space. Then we move the paths or loops in the original topological space to the covering space to do the analysis.

3 Feynman Path Integral

The Feynman path integral, introduced by Richard Feynman, revolutionized quantum mechanics by replacing the traditional wave function with a sum over all possible particle trajectories. This approach provides a unified framework for understanding both classical and quantum behavior, making it a powerful tool across various branches of theoretical physics.

The state of a particle is described by the probability amplitude of the jump from q_1 at time t_1 to q_2 at time t_2, denoted by $K[q_1, t_1; q_2, t_2]$, which is a complex number. The probability of the particle jumps from q_1 at time t_1 to q_2 at time t_1 is $\overline{K[q_1, t_1; q_2, t_2]}K[q_1, t_1; q_2, t_2]$, where $\overline{K[q_1, t_1; q_2, t_2]}$ is the complex conjugate of $K[q_1, t_1; q_2, t_2]$. The probability amplitude of the jumping is given by the "sum of amplitude of all paths", mathematically,

$$K[q_1, t_1; q_2, t_2] = \int \mathcal{D}\mathbf{m}\ e^{i\mathcal{S}[\mathbf{m};t]},$$

where $\int \mathcal{D}\mathbf{m}$ denotes the "functional integral" and can be imaged as $\sum_{\mathbf{m}\in\{all\ paths\}}$. Note that the functional integral may be difficult to compute or sometimes even does not exist. However, utilize the *saddle point approximation*. We can understand that the path satisfies $\delta\mathcal{S} = 0$ would give the path with the highest probability, this path is what we can observe under the classical limit, that is, in the macroscopic world. Therefore, this path is called *classical path*.

Remark 4 *Feynman path integral can explain the least action principle in Lagrangian mechanics.*

Feynman path integral is widely applied in quantum field theory and statistical mechanics. We discuss a simple example of how to compute the path integral.

Example 3.1 ***Free Particle***
Consider a free particle with unit mass in an independent one-dimensional real space $\mathbb{R}$. The Lagrangian for this free particle is

$$\mathcal{L} = \frac{1}{2}\dot{q}^2,$$

where q is the generalized position. What is the probability amplitude jump from q_i at time t_i to q_f at time t_f?
solution:
The action for this free particle is

$$\mathcal{S} = \int_{t_i}^{t_f} \frac{1}{2}\dot{q}^2 dt.$$

The probability amplitude is given by Feynman's time-slicing method

$$K[q_f, t_f; q_i, t_i] = \lim_{N\to\infty} \left(\frac{1}{2\pi i\xi}\right)^{N/2} \int_{-\infty}^{\infty} \prod_{j=1}^{N-1} dq_j \exp\left[i\frac{\xi}{2}\sum_{n=0}^{N-1}\left(\frac{q_{n+1}-q_n}{\xi}\right)^2\right],$$

where $\xi = (t_f - t_i)/N$. Since the integral parts are all Gaussian, the probability amplitude is

$$K[q_f, t_f; q_i, t_i] = \lim_{N\to\infty} \left(\frac{1}{2\pi i\xi}\right)^{N/2} \frac{1}{\sqrt{N}} (2\pi i\xi)^{(N-1)/2} \exp\left[i\frac{(q_f - q_i)^2}{2N\xi}\right]$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \left(\frac{1}{2\pi i N\xi}\right)^{1/2} \exp\left[i\frac{(q_f - q_i)^2}{2N\xi}\right] \\
&= \left(\frac{1}{2\pi i(t_f - t_i)}\right)^{1/2} \exp\left[\frac{i(q_f - q_i)^2}{2t_f - 2t_i}\right].
\end{aligned}
$$

4 Theorems for Path Integral

The concept of homotopy shares similarities with physicist Feynman's path integral framework. In this analogy, the trajectories followed in space align with the notion of homotopy classes. However, the inclusion of topology isn't universally necessary, particularly in uncomplicated spaces where trajectories are straightforward. The significance of topology in the path integral framework becomes evident when it is extended to varied configuration spaces, such as non-commutative and complex phase spaces, where the underlying structures may manifest non-trivial topological characteristics. Upon acknowledging the indispensable significance of topology within the framework of path integrals, we are prompted to introduce what stands as one of the paramount theorems in this domain. This theorem underscores the intrinsic relationship between topology and path integrals.

Theorem 4.1 ***Homotopy theorem for path integral***
Let the configuration space X be a topological space and $\mathcal{H}$ be the collection of all homotopy classes of the paths from q_1 to q_2.

$$K[q_1, t_1; q_2, t_2] = \sum_{\alpha\in\mathcal{H}} A_\alpha \int \mathcal{D}(\mathbf{m} \in \alpha) e^{i\mathcal{S}[\mathbf{m};t]}.$$

In particular, when the paths are loops, we can write

$$K[q, t_1; q, t_2] = \sum_{\alpha\in\pi_1(X,q)} A_\alpha \int \mathcal{D}(\mathbf{m} \in \alpha) e^{i\mathcal{S}[\mathbf{m};t]}.$$

The proof of this theorem is very complicated so we will skip it here. However, the philosophy of this theorem is not hard to understand. The essence of this theorem lies in the notion that each homotopy class is associated with an amplitude, denoted as $e^{i\mathcal{S}}$, which can be interpreted as a contribution. In simpler terms, the contribution corresponds to the significance of the class. Consequently, the total quantum amplitude is obtained by summing up the amplitudes of all homotopy classes, each weighted according to its respective significance.

Sometimes the topological space may be abstract, another significant theorem provides us with a method to move the path integral into the more concrete covering space.

Theorem 4.2 ***Lifting property of path integral***
Let X denote the configuration space and be a topological space. The path integral over the homotopy class α can be transferred to the covering space, yielding the relation

$$K_\alpha[q_f, t_f; q_i, t_i] = K^*[q^*_{f,\alpha}, t_f; q^*_i, t_i],$$

for some $q^*_i \in p^{-1}(q_i)$, $q^*_{f,\alpha} \in p^{-1}(q_f)$ corresponds to the lifted homotopy class of α with respect to q^*_i, and K^* denotes the path integral performed in the covering space.

Note that the lifted initial position q_i^* may be not unique; we can choose the most convenient one according to the problem.

Now, let's explore the critical aspect: how to derive the amplitude A_α as described in Theorem 4.1. This theorem offers a robust solution, enhancing the power of Theorem 4.1.

Theorem 4.3 ***Amplitude theorem for path integral***
Let X denote the configuration space and be a topological space. The probability amplitude can be written as

$$K[q_f, t_f; q_i, t_i] = \sum_{g \in \pi_1(X, q_f)} e^{-i\phi_g} K_\alpha[q_f, t_f; q_i, t_i],$$

where $\phi_g \in \mathbb{R}$ is a factor depends on $g \in \pi_1(X, q_f)$ and satisfies

$$\phi_{hg} = \phi_h + \phi_g \text{ (modulo } 2\pi\text{) for all } g, h \in \pi_1(X, q_f).$$

This theorem tells us that the amplitude in theorem 4.1 is nothing but a phase factor $e^{i\phi_g}$, where ϕ_g depends on $g \in \pi_1(X, q_f)$. The relation in theorem 4.3 states that the map $\pi_1(X, q_f) \to \mathbb{R}$, $g \mapsto \phi_g$ is a group homomorphism. That is, computing the amplitudes of homotopy classes is a *group representation* problem. Note that because $e^{i0} = e^{i2\pi}$, the values 1 and 2π are equivalent for the phase factor $e^{i\phi}$. Hence, in the context of the theorem, we consider the relation modulo 2π. With the homomorphism provided by the theorem, one can use the fundamental group to generate the phase factors and compute the probability amplitude.

With these theorems, we can do the path integral in the space with nontrivial topology. We will discuss some examples in the next section.

5 Application in Configuration Space with Finite Fundamental Groups

As theorem 4.3 asserts, computing the amplitudes of homotopy classes in configuration spaces is analogous to solving a group representation problem. Given the well-developed nature of representation theory for finite groups, we can utilize it to analyze both the topology of the configuration space and the amplitudes associated with each homotopy class.

5.1 Exchanging Positions of Particles

Consider two classical identical particles in a d-dimensional independent real space $\mathbb{R}^d$, where $d \geq 3$. In this section, we explore how their quantum state changes after exchanging positions.

First, we define the configuration space X. Since the space is independent, the absolute positions of the particles are meaningless. Hence, the state of these particles is described by the relative position $\mathbf{r} \in \mathbb{R}^d$. Since the particles are identical, we use the equivalence relation $\mathbf{r}_1 \sim \mathbf{r}_2$ if and only if $\mathbf{r}_1 = \pm\mathbf{r}_2$. Hence, the configuration space is given by

$$X = (\mathbb{R}^d - \{\mathbf{0}\}) / \sim .$$

The fundamental group of the configuration space is

$$\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z},$$

indicating that particles evolve as time and turn back to their original state can occur through paths in two homotopy classes, denoted α_1 and α_2. The physical interpretation of these classes is as follows:

Applying theorem 4.1, the probability amplitude can be expressed as

$$K[\mathbf{r}, t_f; \mathbf{r}, t_i] = A_{\text{direct}} \int \mathcal{D}(\mathbf{m} \in \alpha_1)\, e^{i\mathcal{S}(\mathbf{m};t)} + A_{\text{exchange}} \int \mathcal{D}(\mathbf{m} \in \alpha_2)\, e^{i\mathcal{S}(\mathbf{m};t)},$$

the two homotopy classes mean that two particles turn back to their original relative position and two particles exchange their relative position respectively. Shifting the path integral to the universal covering space $\mathbb{R}^d - \{\mathbf{0}\}$ with the covering projection $p : \mathbb{R}^d - \{\mathbf{0}\} \to (\mathbb{R}^d - \{\mathbf{0}\})/\sim$, $\mathbf{r} \mapsto [\mathbf{r}] = \mathbf{r}$. By the lifting properties, paths in the covering space start from points in $p^{-1}(\mathbf{r}) = \{\mathbf{r}, -\mathbf{r}\}$. Thus, by theorem 4.2, the probability amplitude can be expressed as

$$K[\mathbf{r}, t_f; \mathbf{r}, t_i] = A_{\text{direct}} K^*[\mathbf{r}, t_f; \mathbf{r}, t_i] + A_{\text{exchange}} K^*[-\mathbf{r}, t_f; \mathbf{r}, t_i],$$

where K^* denotes the path integral performed in the covering space and we choose the lifted $\mathbf{r} \in p^{-1}(\mathbf{r})$ be $\mathbf{r}$. According to theorem 4.3, the amplitudes in the two homotopy classes are $A_{\text{direct}} = 1$ and $A_{\text{exchange}} = e^{i\phi}$ for some $\phi \in \mathbb{R}$. Additionally, since ϕ need to satisfies $\phi + \phi = 0$, we know that $\phi \in \{0, \pi\}$. So the amplitude of particles exchanging positions is $e^{i\phi} \in \{-1, 1\}$. The exchange of particles results in two amplitudes: particles with amplitude 1 are called *bosons*, and those with amplitude -1 are called *fermions*.

5.2 Path Integral on 4-Manifolds with Finite Fundamental Groups

A 4-manifold is a mathematical space extending into four dimensions, offering a deeper exploration of geometry and topology beyond our familiar three-dimensional world. 4-manifolds are crucial in theoretical physics, influencing areas such as quantum field theory by offering a richer context for understanding fundamental interactions and complex phenomena. 4-manifolds are also significant in mathematics due to their interesting topological properties. Here is a well-known theorem.

Theorem 5.1 *For all* ***finite groups****, one can construct a smooth compact* 4*-manifold with it as its fundamental group.*

Constructing such a manifold is complex, so we will not delve into it here. However, using this theorem, we can explore certain 4-manifolds with finitely presented fundamental groups, revealing intriguing topology. By employing representation theory, we can determine the number of possible phase factor changes as the state evolves and returns to its initial state.

As an example, the group $\mathbb{Z}/4\mathbb{Z}$ is finite, by theorem 5.1, we have a corresponding 4-manifold M and can therefore consider the path integral on M. By theorem 4.3, the ordered collection of amplitudes of the homotopy classes (A_1, A_2, A_3, A_4) can be $(1, 1, 1, 1)$ or $(1, e^{i\pi/2}, e^{i\pi}, e^{3i\pi/2})$. If there is a system where this 4-manifold serves as the configuration space, two possibilities arise for the change in phase factor as the state evolves and returns to its original state at the end.

6 Application in Configuration Space with Infinite Fundamental Group

As we discussed in theorem 4.3, computing amplitudes is the same as a group representation problem. However, since the infinite group representation is still under development and requires more advanced mathematics, we will just discuss a simple example here.

6.1 Particle on the d-dimensional Torus

Consider a particle move on a d-dimensional torus T^d. Since the space that the particle is in is $T^d = (S^1)^d$ and the motion of a particle on the circle can be described by an angle $\phi \in [0, 2\pi]/\sim$, where $a \sim b$ if and only if $ab \in \{0, 2\pi\}$, the configuration space is

$$X = ([0, 2\pi]/\sim)^d.$$

This space is a topological space equivalent to T^d, so by theorem 2.1 the fundamental group of this space is

$$\pi_1(X) \cong \mathbb{Z}^d.$$

Since the fundamental group of X is nothing but the direct product of $\pi_1(S^1)$, we can consider the particle on S^1 first. By theorem 4.1 and theorem 4.3 the probability amplitude has the form

$$K[\phi_f, t_f; \phi_i, t_i] = \sum_{n\in\mathbb{Z}} e^{in\theta} K_n[\phi_f, t_f; \phi_i, t_i],$$

where $\theta \in \mathbb{R}$. Now we compute the probability amplitude for each homotopy class K_n. In the framework of complex mechanics, the action of the particle is

$$\mathcal{S} = \int_{t_i}^{t_f} \frac{1}{2}\dot{\phi}^2 dt.$$

The universal covering space is $\mathbb{R}$ equipped with the covering map $p : \mathbb{R} \to [0, 2\pi)$, $x \mapsto x - 2\pi\left[\frac{x}{2\pi}\right]_{\mathrm{G}}$, where $[\ ]_{\mathrm{G}}$ is the Gauss notation. By theorem 4.2, we can perform the path integral in the more concrete covering space $\mathbb{R}$. Recall what we have done in example 3.1, the only difference is an additional term 2π led to by the covering map. Therefore, the probability amplitude is

$$K[\phi_f, t_f; \phi_i, t_i] = \sum_{n\in\mathbb{Z}} e^{in\theta} \left(\frac{1}{2\pi i(t_f - t_i)}\right)^{1/2} \exp\left[\frac{i(\phi_f - \phi_i - 2\pi)^2}{2t_f - 2t_i}\right].$$

Now we can consider the particle on d-dimensional torus. We know that the fundamental group of d-dimensional torus is

$$\pi_1(T^d) = \pi_1(S^1)^d.$$

The same as what we discussed above, the probability amplitude of a particle on the d-dimensional torus is

$$K[\Phi_f, t_f; \Phi_i, t_i] = \sum_{n_1\in\mathbb{Z}} \cdots \sum_{n_d\in\mathbb{Z}} e^{i(n_1\theta_1 + \cdots + n_d\theta_d)} \left(\frac{1}{2\pi i(t_f - t_i)}\right)^{d/2} \prod_{j=1}^{d} \exp\left[\frac{i(\phi_{f,j} - \phi_{i,j} - 2\pi)^2}{2t_f - 2t_i}\right],$$

where $\Phi_i = (\phi_{f,1}, \cdots, \phi_{f,d})$, $\Phi_f = (\phi_{i,1}, \cdots, \phi_{i,d}) \in X$ and the collection of angles $\{\theta_j\}_{j=1}^d$ is linear independent over $\mathbb{Z}$.

7 Summary

Topology, an essential branch of mathematics, provides a solid framework for investigating path integrals in abstract spaces. Key concepts like homotopy, fundamental groups, and covering spaces imbue topological spaces with algebraic structures, enabling analysis through abstract algebraic theory. Theorems in homotopy theory further facilitate the application of path integral methods to multiply-connected spaces, while group representation theory supports their implementation.

In the domain of path integrals, a diverse range of topological structures awaits exploration and study. These encompass familiar Euclidean spaces as well as more intricate configurations, offering distinct insights into the dynamics of physical systems. For instance, the topological properties of configuration spaces and manifold structures with nontrivial fundamental groups can lead to unique phenomena.

By delving into the richness of topological concepts within quantum mechanics, researchers deepen their understanding of fundamental principles and uncover novel avenues for theoretical exploration and discovery. Particularly, advancements in representation theory for infinite groups contribute to a more comprehensive understanding of physics within the path integral framework.

A Appendix

You can see the slide version on my website.
Link: `https://kikiyenhaoyang.github.io/kikiyen/Web/ED.html`.

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