

Group Meeting #2

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Outline

① The SIS Model

- ▶ Introduction to the SIS model.
- ▶ Non-equilibrium phase transition.

② Theoretical Basis of the TN approach

- ▶ From Ising model to the SIS model.
- ▶ The tensor product formulation.
- ▶ The large deviation principle.



Introduction to the SIS Model

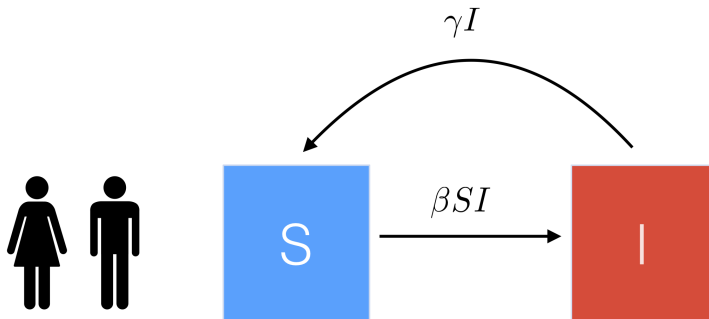


Figure: The SIS model.



Introduction to the SIS Model

- 1 The SIS model can be written as a differential equations system form.

$$\begin{cases} \frac{dS}{dt} = -\beta SI + \gamma I, \\ \frac{dI}{dt} = \beta SI - \gamma I. \end{cases}$$

- 2 There are several kinds of extensions to the SIS model, for example, the SIR model.
- 3 The SIS model is the simplest epidemic model that can describe the phase transitions.



Non-Equilibrium Phase Transition

- ① The SIS model has the differential equation form

$$\begin{cases} \frac{dS}{dt} = -\beta SI + \gamma I, \\ \frac{dI}{dt} = \beta SI - \gamma I. \end{cases}$$

- ② The model is characterized by two parameters β and γ .
- ③ The non-equilibrium phase transition in SIS models is described by the relation

$$\begin{cases} \text{Absorbing Phase: } \frac{\beta}{\gamma} \leq 1, \\ \text{Endemic Phase: } \frac{\beta}{\gamma} > 1. \end{cases}$$

- ④ Simulation:

<https://kikiyenhaoyang.github.io/kikiyen/Web/TM.html>



From Ising model to the SIS mode

- 1 The Ising model is the simplest statistical mechanics model

$$|\Psi\rangle = \Psi(s_1, \dots, s_N) \bigotimes_{i=1}^N |s_i\rangle.$$

- 2 Each lattice state $|s_i\rangle$ lives in the Hilbert space and therefore can be written as the linear combination of $|0\rangle$ and $|1\rangle$.
- 3 The basis for the Hilbert space is

$$\left\{ \bigotimes_{j=1}^N |i_j\rangle \right\}_{i_j \in \{0,1\}}.$$



From Ising Model to the SIS Model

- 1 The Hilbert space has dimension $2^N = \mathcal{O}(2^N)$, which grows rapidly.
- 2 With the TN, we can reduce it to $\mathcal{O}(N)$, which is much more efficient.
- 3 The SIS model is also a kind of binary model. Thus we can write it in tensor product formulation as well.



The Tensor Product Formulation

1 Measurement Postulate

Upon measurement, the outcome of an observable is one of its eigenvalues, and the probability of obtaining a particular outcome is given by the square of the absolute value of the projection of the state vector onto the corresponding eigenvector.



The Tensor Product Formulation

- ① In the lattice form SIS model. A lattice is in state $|0\rangle = |S\rangle$ or $|1\rangle = |I\rangle$ with probability $P_0(t)$ and $P_1(t)$ respectively. That is,

$$|P(t)\rangle = P_0(t) |0\rangle + P_1(t) |1\rangle ,$$

which is analog to

$$|\Psi\rangle = \Psi_0 |0\rangle + \Psi_1 |1\rangle .$$

- ② The state space of the SIS model is

$$\bigotimes_{i=1}^N \mathcal{H}_i, \text{ where } \mathcal{H} = \text{Span}\{|0\rangle, |1\rangle\}.$$



The Tensor Product Formulation

- 1 Since the SIS model is a kind of Markov chain, we can consider the infinitesimal Markov generator \hat{W} so that

$$\partial_t |P(t)\rangle = \hat{W} |P(t)\rangle ,$$

which is Schrödinger-equation-like.

- 2 **Time Evolution Postulate**

The time evolution of a quantum system is governed by the Schrödinger equation

$$-i\partial_t |\Psi\rangle = \hat{H} |\Psi\rangle ,$$

which describes how the state vector changes over time.



The Tensor Product Formulation

- ① The infinitesimal Markov generator \hat{W} for the SIS model is

$$\hat{W} = \beta \sum_{i=1}^{N-1} (\hat{n}_i \omega_{i+1}^{0 \rightarrow 1} + \hat{\omega}_i^{0 \rightarrow 1} n_{i+1}) + \gamma \sum_{i=1}^N \hat{\omega}_i^{1 \rightarrow 0} + \hat{W}_{driv}(\alpha),$$

where

$$\hat{n}_i = |1\rangle_i \langle 1|, \quad \hat{\omega}_i^{0 \rightarrow 1} = |1\rangle_i \langle 0| - |0\rangle_i \langle 0|, \quad \hat{\omega}_i^{0 \rightarrow 1} = |0\rangle_i \langle 1| - |1\rangle_i \langle 1|.$$

- ② The generator is *analog* to the quantum Hamiltonian.



The Tensor Product Formulation

① Quantum State Postulate

A quantum system is fully described by its state vector, usually denoted by $|\psi\rangle$ in Dirac notation. This state vector resides in a complex vector space known as a \mathbb{C} -Hilbert space.



The Tensor Product Formulation

Definition

Vector Space

Let F be a field. A **F -vector space** is set V equipped with two operators

$$+ : V \times V \rightarrow V,$$

$$\cdot : F \times V \rightarrow V$$

satisfies the following conditions

- $(V, +)$ is an abelian group,
- $a \cdot (b \cdot v) = (ab) \cdot v$ for all $a, b \in F$ and $v \in V$,
- $(a + b) \cdot v = a \cdot v + b \cdot v$ for all $a, b \in F$ and $v \in V$,
- $a(u + v) = au + av$ for all $a \in F$ and $u, v \in V$,
- $id_F \in F$ such that $id_F \cdot v = v$.

The Tensor Product Formulation

- 1 Though we analog the generator to the quantum Hamiltonian, there are many differences between the generator and the quantum Hamiltonian.
- 2 The generator is a **REAL** matrix and the the basis lives in a \mathbb{R} -Hilbert space.
- 3 We can not treat it as a quantum system.

