Leveraging Variational Matrix Product States for the Analysis of Stochastic Dynamical Systems

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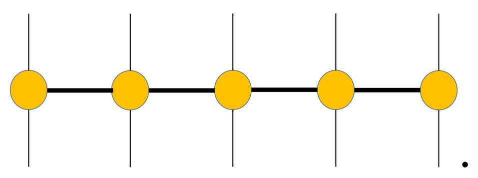
1. Abstract/Motivation

The Variational Matrix Product States (VMPS) technique is vital for solving large many-body quantum systems and has recently been extended to classical stochastic dynamics. We seek to apply VMPS to find the non-equilibrium steady state (NESS) of large lattice models. This study explores:

- Differences between applying VMPS to quantum models and classical stochastic models.
- The accuracy of utilizing VMPS to the classical stochastic models.
- Advantages of using VMPS in these contexts.

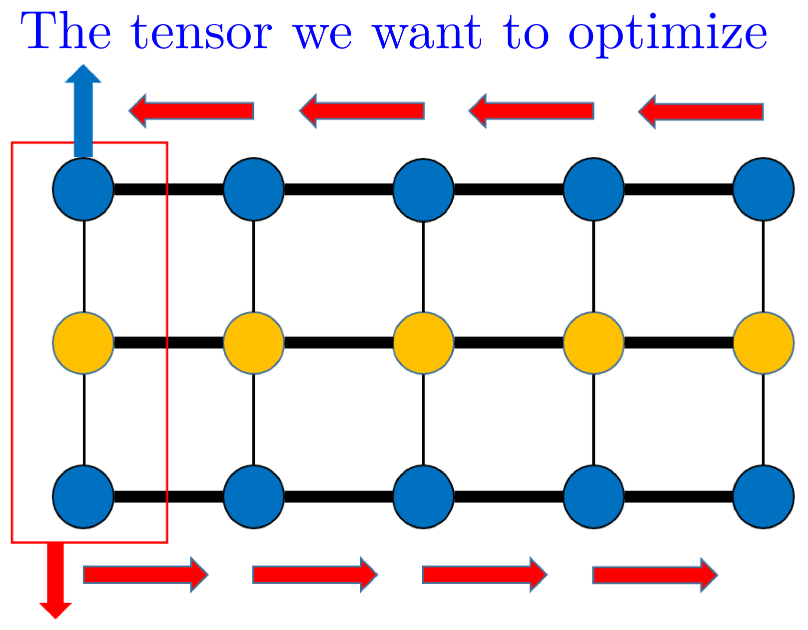
3. Algorithms

Treat the Hamiltonian or Markov generator as tensors as:



Our goal is to obtain the optimal blue tensor in the diagram, which is the wave function or the NESS for the models. Start with an initial guess, and iteratively refine it locally using diagonalization to minimize the tensor contraction of the blue and orange tensors in the diagram.

The tensor we want to optimize



Local diagonalization by the Lancsoz or Arnoldi algorithm

5. NESS

- Once the NESS is determined, it allows us to calculate the distributions of various physical quantities, including the gap length.

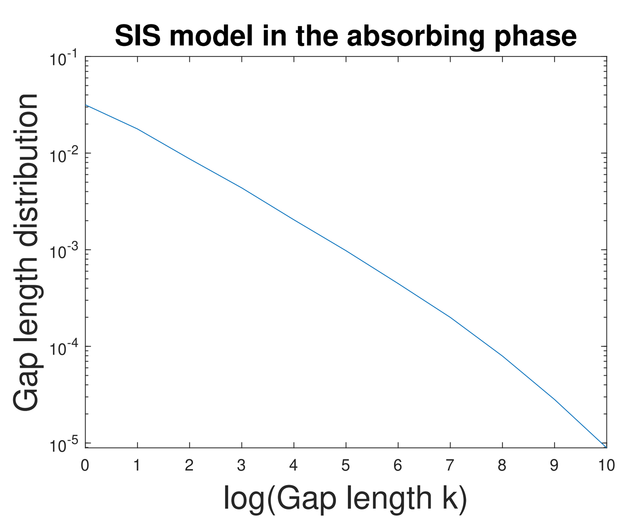


Figure 2: The gap length distribution in the NESS of the SIS model.

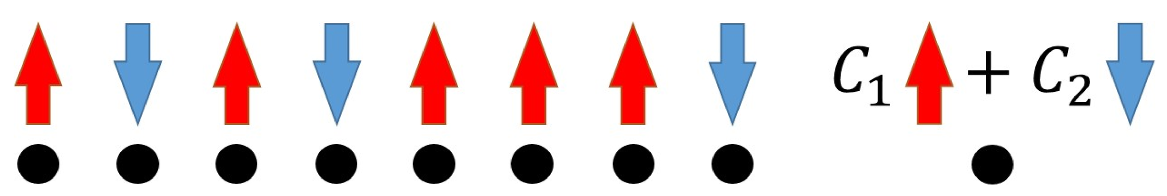
7. Conclusion

- **Target Eigenstate:** Changing the Hamiltonians of quantum models to the Markov generator enables the extraction of the target eigenstate for a lattice stochastic dynamical system.
- **NESS Distribution:** The VMPS method delivers precise NESS distributions, which proves highly effective for simulating rare events.
- **Convergence Near Criticality:** The rate of convergence diminishes notably as the system approaches its critical point.

2. Models

Quantum Model: Transverse Ising Model

The transverse Ising model describes spin interactions in a magnetic system under a transverse magnetic field, useful for studying quantum phase transitions.

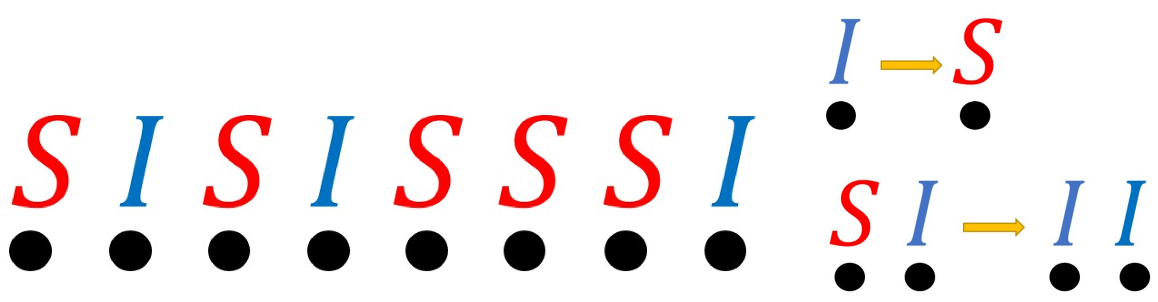


- The transverse Ising model is a quantum system that exhibits a phase transition with the order parameter J in the Hamiltonian $\hat{H}$, a complex matrix, due to quantum effects.

$J < J_c$: disordered phase
$J = J_c$: critical point
$J > J_c$: ordered phase.

Classical Stochastic Model: Susceptible-Infected-Susceptible Model (SIS Model)

The SIS model describes disease spread in epidemiology, where individuals alternate between susceptible and infected states through infection and recovery.



- The SIS model exhibits a phase transition with the order parameter λ in the Markov generator $\hat{W}$, a real matrix, as a result of its dynamics.

$\lambda < \lambda_c$: absorbing phase
$\lambda = \lambda_c$: critical point
$\lambda > \lambda_c$: active phase

4. Accuracy

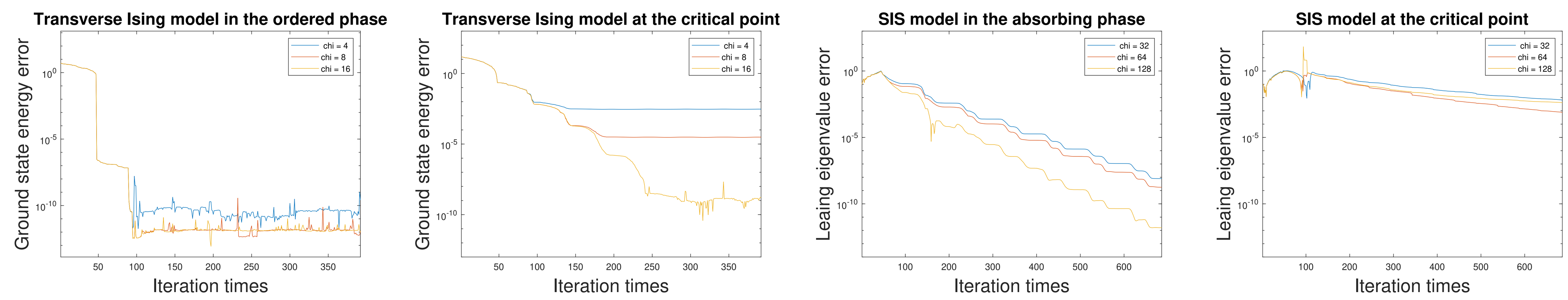


Figure 1: Numerical errors in the transverse Ising model and the SIS model when employing VMPS.

- VMPS yields highly accurate results for both models, showing its effectiveness for the SIS model.
- The rate of convergence and the computational resources required both increases significantly as the system approaches the critical point.
- The optimized eigenstate found corresponds to the NESS of the SIS model.

6. The Scaled Cumulant Generating Function

- Cumulants can detect rare events and phase transitions by analyzing the detailed characteristics of probability distributions, revealing how these distributions change with system parameters.
- The cumulants can be generated by the scaled cumulant generating function (SCGF) $\Theta(s)$.

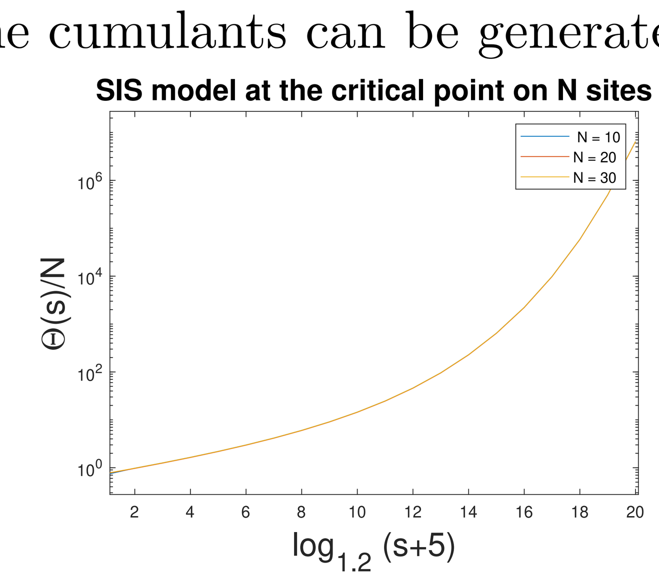


Figure 3: The rescaled SCGF $\Theta(s)/N$.

Computational Result

- Since the SCGF is the leading eigenvalue of $\hat{W}$'s Laplace transform, it can be computed using VMPS.
- The SCGF varies with the number of sites but becomes consistent after being rescaled.
- Consequently, all rescaled cumulants are identical for the NESS.

8. Further Questions

- How can VMPS be used for two-dimensional stochastic models?
- Can a time-dependent VMPS handle varying infection rates?
- Can VMPS be applied to stochastic systems with non-trivial NESS, such as predator-prey models?

References

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