Symmetry, Integrability, and Calculus of Variations

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Contents

Preface

This note serves as supplementary material for the 2025 Theoretical Mechanics course, in which I am a teaching assistant. In modern physics and mathematics, symmetry and integrability provide an elegant and powerful language for describing physical systems, appearing in classical mechanics, quantum mechanics, and mathematical physics. At the same time, the calculus of variations is a central mathematical tool in classical mechanics. From a more modern perspective, it is deeply connected to symmetry and integrability. Unfortunately, most elementary physics textbooks do not present this connection in a formal or rigorous way.

Alongside teaching, I am conducting research in mathematical physics, where I frequently encounter differential geometry and Lie groups. Through this work, I have come to appreciate how powerful symmetries and integrable structures are in the theoretical description of physical systems.

In this note, I present a more rigorous viewpoint on the calculus of variations, symmetry, and integrability, emphasizing concepts that are most useful in physics while avoiding excessively difficult or abstract aspects of differential geometry and Lie group theory (but possibly added in a future version). Beyond supporting my teaching, preparing these materials also gives me an opportunity to reflect on how to distil abstract mathematics into tools that are physically meaningful and practically useful. My goal is to offer students a rigorous and contemporary perspective that helps them solve physical problems more effectively and logically, and, I hope, inspires some of them to pursue mathematical research in the future.

Chapter 1

Calculus of Variation

1.1 Functionals

The goal calculus of variations is to find an extreme value of a **functional**. Informally, the functionals are functions that map functions to a real number, which are important in many different places in physics.

Example 1.1: Linear density

Let ρ be the density of anything like mass or charge of a line, then its center is defined as:

$$x_c = \frac{\int_a^b x\,\rho(x)\,\mathrm{d}x}{\int_a^b \rho(x)\,\mathrm{d}x}.$$

We can view $x_c \in \mathbb{R}$as a function of the density $\rho(x)$, i.e., we write it $x_c[\rho(x)]$. In this sense, $x_c[\rho(x)]$ is a functional: it takes a function $\rho(x)$ as input and returns a real number.

Euler's method Euler considered the functionals as so-called "infinitely many variable functions", that is, consider a functional $J[f(x)]$, the variables of these "infinitely many variable functions" are $f_1, f_2, \cdots, f_n$, where $n \to \infty$.

For example, for the most classic functional

$$J[f(x)] = \int_a^b f(x)\,\mathrm{d}x,$$

one can descretize the interval $[a, b]$ into $\bigcup_{i=1}^{n-1}[x_i, x_{i+1}]$. From calculus, one can know that the integral is precisely the summation under the limit $n \to \infty$,

$$\sum_{i=1}^{n-1} f_i \cdot (x_{i+1} - x_i).$$

This is the basic spirit of the classical calculus of variations, which is useful in some classical problems.

This method, however, is certainly not rigorous from the modern viewpoint, since the functionals can take more complicated and abstract forms that may not be approximated by the finite-difference method. For example, the functional

$$J[f(x)] = \begin{cases} 1, & f \text{ is integrable,} \\ 0, & f \text{ is not integrable,} \end{cases}$$

certainly can not be approximated by the finite-difference method. This is why we need to study the calculus of variations from a more modern viewpoint.

This point of view suggests that, just as for ordinary functions of finitely many variables, we may try to find extrema of a functional $J[f]$ by studying how J changes when we vary each "component" f_i independently. In the continuum limit, this will become the idea of a *variation* of f in the "direction" of some test function h.

1.2 Euler–Lagrange equation

In the modern viewpoint, we characterize extrema of a functional J by looking at its *first variation*. Given a function f and a "direction" h, we consider a family

$f_\varepsilon = f + \varepsilon h$ and define

$$\delta J[f;h] := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J[f+\varepsilon h]\Big|_{\varepsilon=0}.$$

For many functionals that appear in physics, especially of the form

$$J[f] = \int_a^b L(x, f(x), f'(x))\,\mathrm{d}x,$$

one can show that the first variation can be written as

$$\delta J[f;h] = \int_a^b F(x)\,h(x)\,\mathrm{d}x$$

for some continuous function F depending on f and its derivatives.

If f is an extremum of J under variations that vanish at the endpoints, we expect that $\delta J[f;h] = 0$ for all such test functions h. Intuitively, this should force $F(x) = 0$ for all x. The following fundamental lemma makes this precise.

Lemma 1.1: Fundamental lemma of the calculus of variations

Let $f \in C(a,b)$. Suppose that

$$\int_a^b f(x)\,h(x)\,\mathrm{d}x = 0$$

for all $h \in C_c^\infty(a,b)$. Then $f(x) \equiv 0$ on (a,b).

Proof. Assume that there exists $x_0 \in (a,b)$ such that $f(x_0) \neq 0$. By continuity, there exists an interval $(c,d) \subset (a,b)$ containing x_0 such that either $f(x) > 0$ for all $x \in (c,d)$, or $f(x) < 0$ for all $x \in (c,d)$.
Choose $h \in C_c^\infty(a,b)$ with $h(x) > 0$ on (c,d) and $h(x) = 0$ outside (c,d), for instance the bump function

$$h(x) = \begin{cases} \exp\Big(-\dfrac{1}{(x-c)(d-x)}\Big), & x \in (c,d), \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int_a^b f(x)h(x)\,\mathrm{d}x = \int_c^d f(x)h(x)\,\mathrm{d}x$$

is strictly positive in the first case and strictly negative in the second case, so it cannot be 0. This contradicts the assumption that $\int_a^b f(x)h(x)\,\mathrm{d}x = 0$ for all $h \in C_c^\infty(a,b)$.
Therefore $f(x) \equiv 0$ on (a,b).

□

Example 1.2: Thomson's theorem (equipotential conductors)

The electrostatic energy of a system of conductors can be written as a functional of the surface charge density σ:

$$W[\sigma] = \frac{1}{2}\sum_i \oint_{S_i} \phi\,\sigma\,\mathrm{d}a,$$

where S_i are the conductor surfaces and ϕ is the electrostatic potential generated by σ.
Suppose that the total charge on each surface S_i is fixed and equal to Q_i:

$$\oint_{S_i} \sigma\,\mathrm{d}a = Q_i.$$

We use the variational method with Lagrange multipliers λ_i to show that, at the energy minimum, each conductor surface is an equipotential.
Define the constrained functional

$$\mathcal{L}[\sigma] = \frac{1}{2}\sum_i \oint_{S_i} \phi[\sigma]\,\sigma\,\mathrm{d}a - \sum_i \lambda_i \left(\oint_{S_i} \sigma\,\mathrm{d}a - Q_i\right).$$

Here $\phi[\sigma]$ denotes the potential generated by σ:

$$\phi[\sigma](\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\sum_i \oint_{S_i} \frac{\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}a'.$$

Take an arbitrary variation $\delta\sigma$ on the surfaces (compatible with the fixed boundary of the region) and consider

$$\sigma_\varepsilon = \sigma + \varepsilon\,\delta\sigma.$$

The variation of $\mathcal{L}$ is

$$\delta\mathcal{L} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathcal{L}[\sigma_\varepsilon] \tag{1.1}$$

$$= \frac{1}{2}\sum_i \oint_{S_i} \big(\phi\,\delta\sigma + \sigma\,\delta\phi\big)\,\mathrm{d}a - \sum_i \lambda_i \oint_{S_i} \delta\sigma\,\mathrm{d}a, \tag{1.2}$$

where

$$\delta\phi(\mathbf{x}) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \phi[\sigma_\varepsilon](\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\sum_i \oint_{S_i} \frac{\delta\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}a'.$$

To express $\sigma\,\delta\phi$ in terms of $\delta\sigma$, we use the symmetry of the Coulomb kernel:

$$\sum_i \oint_{S_i} \sigma(\mathbf{x})\,\delta\phi(\mathbf{x})\,\mathrm{d}a = \frac{1}{4\pi\varepsilon_0}\sum_{i,j} \oint_{S_i}\oint_{S_j} \frac{\sigma(\mathbf{x})\,\delta\sigma(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}a\,\mathrm{d}a' \tag{1.3}$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i,j} \oint_{S_j}\oint_{S_i} \frac{\sigma(\mathbf{x}')\,\delta\sigma(\mathbf{x})}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}a'\,\mathrm{d}a \tag{1.4}$$

$$= \sum_i \oint_{S_i} \phi(\mathbf{x})\,\delta\sigma(\mathbf{x})\,\mathrm{d}a. \tag{1.5}$$

Therefore,

$$\delta\mathcal{L} = \sum_i \oint_{S_i} \big(\phi - \lambda_i\big)\,\delta\sigma\,\mathrm{d}a.$$

At an extremum we must have $\delta\mathcal{L} = 0$ for all admissible variations $\delta\sigma$. By the fundamental lemma of the calculus of variations (applied on each surface S_i), this is only possible if

$$\phi(\mathbf{x}) - \lambda_i = 0 \quad \text{for all } \mathbf{x} \in S_i,$$

that is,

$$\phi(\mathbf{x}) = \lambda_i \quad \text{on } S_i.$$

Hence the potential is constant on each conductor surface: each S_i is an equipotential surface.

We now apply the fundamental lemma to the first variation of a standard class of functionals arising in mechanics. Consider

$$J[q] = \int_a^b L(x, q(x), q'(x))\,\mathrm{d}x,$$

where L is sufficiently smooth and the admissible functions q satisfy fixed boundary conditions $q(a) = q_a$, $q(b) = q_b$. The condition that q makes J stationary under all variations vanishing at the endpoints will lead to a differential equation for q, the *Euler–Lagrange equation*.

Theorem 1.1: Euler–Lagrange equation

Let $L\colon [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function and consider the functional

$$J[q] = \int_a^b L\big(x, q(x), q'(x)\big)\,\mathrm{d}x,$$

defined on C^2 functions q satisfying fixed boundary conditions

$$q(a) = q_a, \qquad q(b) = q_b.$$

If q makes J stationary under all variations $\eta \in C_c^\infty(a,b)$ (i.e. $\eta(a) = \eta(b) = 0$) in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J[q + \varepsilon\eta]\bigg|_{\varepsilon=0} = 0 \quad \text{for all } \eta,$$

then q satisfies the Euler–Lagrange equation

$$\frac{\partial L}{\partial q}\big(x, q(x), q'(x)\big) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial q'}\big(x, q(x), q'(x)\big)\right) = 0, \qquad x \in (a,b).$$

Proof. Take an arbitrary $\eta \in C_c^\infty(a,b)$ with $\eta(a) = \eta(b) = 0$ and consider $q_\varepsilon(x) = q(x) + \varepsilon\eta(x)$. Then

$$J[q_\varepsilon] = \int_a^b L\big(x, q(x) + \varepsilon\eta(x), q'(x) + \varepsilon\eta'(x)\big)\,\mathrm{d}x.$$

Differentiating with respect to ε at $\varepsilon = 0$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J[q_\varepsilon]\Big|_{\varepsilon=0} = \int_a^b \left(\frac{\partial L}{\partial q}(x, q, q')\, \eta(x) + \frac{\partial L}{\partial q'}(x, q, q')\, \eta'(x) \right) \mathrm{d}x.$$

Integrating the second term by parts and using $\eta(a) = \eta(b) = 0$ yields

$$\int_a^b \frac{\partial L}{\partial q'}\, \eta' \, \mathrm{d}x = \left[\frac{\partial L}{\partial q'}\, \eta \right]_a^b - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial q'} \right) \eta \, \mathrm{d}x = - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial q'} \right) \eta \, \mathrm{d}x.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J[q_\varepsilon]\Big|_{\varepsilon=0} = \int_a^b \left(\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial q'} \right) \right) \eta(x) \, \mathrm{d}x.$$

By assumption this derivative is zero for all $\eta \in C_c^\infty(a, b)$, so by the fundamental lemma of the calculus of variations we conclude that

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial q'} \right) \equiv 0 \quad \text{on } (a, b).$$

This is precisely the Euler–Lagrange equation. □

Remark 1.1: Non-uniqueness of the Lagrangian

The Lagrangian is *not* unique at the level of the equations of motion. In particular, if two Lagrangians differ by a total derivative,

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{\mathrm{d}}{\mathrm{d}t} F(q, t),$$

then L and L' give identical Euler–Lagrange equations. The proof of this statement is quite straightforward and therefore left as an exercise.

Example 1.3: Shortest path between two points

Consider curves $y = y(x)$ joining two fixed points (x_a, y_a) and (x_b, y_b) with $x_a < x_b$. The length of such a curve is

$$J[y] = \int_{x_a}^{x_b} \sqrt{1 + (y'(x))^2}\, \mathrm{d}x.$$

We want to find y that makes $J[y]$ stationary under variations with fixed endpoints $y(x_a) = y_a$, $y(x_b) = y_b$.
Here

$$L(x, y, y') = \sqrt{1 + (y')^2},$$

so

$$\frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}.$$

The Euler–Lagrange equation gives

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial y'}\right) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = 0.$$

Thus

$$\frac{y'}{\sqrt{1 + (y')^2}} = C$$

for some constant C. Solving for y' we obtain that y' is constant, hence y is an affine function,

$$y(x) = mx + b,$$

i.e. the extremals are straight lines between the two points.

Example 1.4: Fermat's principle and Snell's law

Fermat's principle states that the path taken by a light ray between two points is such that the travel time is stationary (usually minimal) compared to nearby paths.

Consider a two-dimensional situation with coordinates (x, y), where the refractive index depends only on the height, $n = n(y)$ (a stratified medium). We look for a curve $y = y(x)$ connecting two fixed points (x_a, y_a) and (x_b, y_b) with $x_a < x_b$. The travel time along a path $y(x)$ is

$$T[y] = \frac{1}{c} \int_{x_a}^{x_b} n\big(y(x)\big) \sqrt{1 + \big(y'(x)\big)^2}\, \mathrm{d}x,$$

where c is the speed of light in vacuum.
This is a functional of the form

$$T[y] = \int_{x_a}^{x_b} L\big(x, y(x), y'(x)\big)\, \mathrm{d}x, \qquad L(x, y, y') = \frac{1}{c}\, n(y) \sqrt{1 + (y')^2}.$$

We assume that y is C^2 and that the endpoints are fixed: $y(x_a) = y_a$, $y(x_b) = y_b$.
The Euler–Lagrange equation reads

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial L}{\partial y'} \right) = 0.$$

Here

$$\frac{\partial L}{\partial y} = \frac{1}{c}\, n'(y) \sqrt{1 + (y')^2}, \qquad \frac{\partial L}{\partial y'} = \frac{1}{c}\, n(y)\, \frac{y'}{\sqrt{1 + (y')^2}}.$$

Since L has no explicit x-dependence, we can use the Beltrami identity

$$L - y' \frac{\partial L}{\partial y'} = \text{constant}.$$

A short computation gives

$$L - y' \frac{\partial L}{\partial y'} = \frac{1}{c}\, n(y) \sqrt{1 + (y')^2} - \frac{1}{c}\, n(y)\, \frac{(y')^2}{\sqrt{1 + (y')^2}} = \frac{1}{c}\, \frac{n(y)}{\sqrt{1 + (y')^2}},$$

so we obtain

$$\frac{n(y)}{\sqrt{1 + (y')^2}} = \text{constant}.$$

Geometrically, if θ denotes the angle between the ray and the normal (vertical) direction, then

$$\sin\theta = \frac{1}{\sqrt{1+(y')^2}},$$

and the above relation can be written as

$$n(y)\,\sin\theta = \text{constant along the ray.}$$

In the special case where $n(y)$ is piecewise constant and changes abruptly across a horizontal interface (say from n_1 to n_2), this condition reduces to

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

which is precisely Snell's law of refraction. Thus Fermat's principle is a variational principle whose Euler–Lagrange equation encodes the familiar law of refraction in geometrical optics.

Example 1.5: Electrostatics as a variational principle

Consider a static charge distribution $\rho(\mathbf{x})$ in a region $\Omega \subset \mathbb{R}^3$. The electrostatic potential $\phi(\mathbf{x})$ determines the electric field $\mathbf{E} = -\nabla\phi$ and, in vacuum, the electrostatic energy can be written as

$$U[\phi] = \int_\Omega \left[\frac{\varepsilon_0}{2}\,|\nabla\phi(\mathbf{x})|^2 - \rho(\mathbf{x})\,\phi(\mathbf{x})\right] \mathrm{d}^3\mathbf{x},$$

where ε_0 is the permittivity of free space. We assume that the potential is fixed on the boundary $\partial\Omega$, i.e. $\phi|_{\partial\Omega}$ is given.
We regard $U[\phi]$ as a functional of ϕ and look for stationary points under variations that vanish on the boundary. Take

$$\phi_\varepsilon(\mathbf{x}) = \phi(\mathbf{x}) + \varepsilon\,\eta(\mathbf{x}),$$

where $\eta(\mathbf{x})$ is a smooth function with compact support in Ω or $\eta|_{\partial\Omega} = 0$. Then

$$U[\phi_\varepsilon] = \int_\Omega \left[\frac{\varepsilon_0}{2}\,|\nabla(\phi + \varepsilon\eta)|^2 - \rho(\phi + \varepsilon\eta)\right] \mathrm{d}^3\mathbf{x}.$$

Differentiating with respect to ε at $\varepsilon = 0$ gives the first variation

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon} U[\phi_\varepsilon]\right|_{\varepsilon=0} = \int_\Omega \left[\varepsilon_0\,\nabla\phi\cdot\nabla\eta - \rho\,\eta\right] \mathrm{d}^3\mathbf{x}.$$

We now integrate the first term by parts (using the divergence theorem):

$$\int_\Omega \nabla\phi\cdot\nabla\eta\,\mathrm{d}^3\mathbf{x} = \int_\Omega \nabla\cdot(\eta\nabla\phi)\,\mathrm{d}^3\mathbf{x} - \int_\Omega \eta\,\nabla^2\phi\,\mathrm{d}^3\mathbf{x}.$$

The first integral on the right-hand side becomes a surface term

$$\int_{\partial\Omega} \eta\,\frac{\partial\phi}{\partial n}\,\mathrm{d}S,$$

which vanishes because $\eta = 0$ on $\partial\Omega$. Hence

$$\int_\Omega \nabla\phi\cdot\nabla\eta\,\mathrm{d}^3\mathbf{x} = -\int_\Omega \eta\,\nabla^2\phi\,\mathrm{d}^3\mathbf{x}.$$

Substituting this into the expression for the first variation, we obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon} U[\phi_\varepsilon]\right|_{\varepsilon=0} = \int_\Omega \left[-\varepsilon_0\,\nabla^2\phi(\mathbf{x}) - \rho(\mathbf{x})\right] \eta(\mathbf{x})\,\mathrm{d}^3\mathbf{x}.$$

If ϕ makes U stationary under all such variations, then this derivative must vanish for all test functions η with $\eta|_{\partial\Omega} = 0$. By the (multidimensional version of the) fundamental lemma of the calculus of variations, it follows that

$$-\varepsilon_0\,\nabla^2\phi(\mathbf{x}) - \rho(\mathbf{x}) = 0 \quad \text{for all } \mathbf{x}\in\Omega,$$

or equivalently,

$$\nabla^2\phi(\mathbf{x}) = -\frac{\rho(\mathbf{x})}{\varepsilon_0},$$

which is precisely Poisson's equation for electrostatics in vacuum.

Thus, the familiar field equation of electrostatics can be obtained from a variational principle: the physical potential makes the electrostatic energy functional stationary (in fact, minimal) under variations with fixed boundary values.

1.2.1 Extrema of functionals and variations

In ordinary calculus, a function $g\colon \mathbb{R} \to \mathbb{R}$ has a (local) minimum at x_0 if $g(x) \geq g(x_0)$ for all x sufficiently close to x_0. For functionals we use a completely analogous idea.

We first fix a class of *admissible functions* $\mathcal{A}$, for example

$$\mathcal{A} = \{\, f \in C^1([a,b]) : f(a) = A,\ \ f(b) = B \,\}.$$

Definition 1.1: Extrema of a functional

Let $J\colon \mathcal{A} \to \mathbb{R}$ be a functional and $f_0 \in \mathcal{A}$.

- We say that f_0 is a **(local) minimum** of J if there exists $\delta > 0$ such that for all $f \in \mathcal{A}$ with $\|f - f_0\| < \delta$ we have
$$J[f] \geq J[f_0].$$
- We say that f_0 is a **(local) maximum** of J if there exists $\delta > 0$ such that for all $f \in \mathcal{A}$ with $\|f - f_0\| < \delta$ we have
$$J[f] \leq J[f_0].$$

Here $\|\cdot\|$ can be taken, for example, as the supremum norm on $[a,b]$; for our purposes, the precise choice will not be essential.

In practice, instead of comparing $J[f]$ with all nearby functions, we consider *variations* of the form

$$f_\varepsilon(x) = f_0(x) + \varepsilon\, h(x),$$

where h is an admissible "direction" (often $h(a) = h(b) = 0$ so that f_ε has the same boundary values as f_0) and ε is a small real parameter.

Definition 1.2: Stationary point and first variation

Let $f_0 \in \mathcal{A}$. If for every admissible h the function

$$\varepsilon \mapsto J[f_0 + \varepsilon h]$$

has derivative at $\varepsilon = 0$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J[f_0 + \varepsilon h]\Big|_{\varepsilon=0} = 0,$$

then we say that f_0 is a **stationary point** (or **extremal**) of J. We denote

$$\delta J[f_0; h] := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J[f_0 + \varepsilon h]\Big|_{\varepsilon=0}$$

and call $\delta J[f_0; h]$ the *first variation* of J at f_0 in the direction h.

As in elementary calculus, a (local) minimum or maximum of J must be a stationary point. Therefore our first task in the calculus of variations is to find stationary points by solving the condition

$$\delta J[f_0; h] = 0 \quad \text{for all admissible } h.$$

In the next subsection we will see that, for a large class of functionals, this condition can be written as

$$\int_a^b F(x)\, h(x)\, \mathrm{d}x = 0 \quad \text{for all } h,$$

and the fundamental lemma of the calculus of variations will then imply that $F(x) \equiv 0$.

1.2.2 Systems with many degrees of freedom

So far we have considered a single function $q(x)$. In many physical systems we have several degrees of freedom, which we collect into a vector

$$\mathbf{q}(x) = (q_1(x), \ldots, q_N(x)).$$

A typical functional is

$$J[\mathbf{q}] = \int_a^b L\big(x, \mathbf{q}(x), \mathbf{q}'(x)\big)\,\mathrm{d}x,$$

where $L\colon [a,b] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is C^2.

The variation is now taken with respect to each component q_α independently:

$$q_\alpha(x) \mapsto q_\alpha(x) + \varepsilon\,\eta_\alpha(x), \qquad \eta_\alpha(a) = \eta_\alpha(b) = 0, \quad \alpha = 1, \dots, N.$$

A calculation completely analogous to the one-dimensional case gives

$$\delta J = \int_a^b \sum_{\alpha=1}^{N} \left(\frac{\partial L}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial q'_\alpha} \right) \eta_\alpha(x)\,\mathrm{d}x.$$

Requiring $\delta J = 0$ for all variations η_α and using the fundamental lemma componentwise, we obtain the system of Euler–Lagrange equations

$$\frac{\partial L}{\partial q_\alpha} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial q'_\alpha} = 0, \qquad \alpha = 1, \dots, N.$$

1.2.3 Field-theoretic Euler–Lagrange equations

In field theory the basic objects are fields $\phi_a(x)$ depending on several independent variables $x = (x^1, \dots, x^d)$ (for instance $x = (t, \mathbf{x})$ in relativistic notation). A typical action functional has the form

$$S[\{\phi_a\}] = \int_\Omega \mathcal{L}\big(x, \phi_a(x), \partial_\mu \phi_a(x)\big)\,\mathrm{d}^d x,$$

where Ω is a region in $\mathbb{R}^d$, $\mu = 1, \dots, d$, and $\mathcal{L}$ is called the *Lagrangian density*.

We consider variations

$$\phi_a(x) \mapsto \phi_a(x) + \varepsilon\,\eta_a(x),$$

with η_a smooth and vanishing on the boundary of Ω (or with compact support in Ω). A straightforward generalization of our previous calculation gives

$$\delta S = \int_\Omega \sum_a \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \eta_a(x)\, \mathrm{d}^d x.$$

If S is stationary under all such variations, then by the (multidimensional) fundamental lemma we obtain the field Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = 0, \qquad \text{for each field index } a.$$

Many of our previous examples can be reinterpreted in this framework. For instance, the electrostatic example corresponds to a scalar field $\phi(\mathbf{x})$ in three dimensions with Lagrangian density

$$\mathcal{L}(\mathbf{x}, \phi, \nabla\phi) = \frac{\varepsilon_0}{2} |\nabla\phi|^2 - \rho(\mathbf{x})\, \phi,$$

whose Euler–Lagrange equation is precisely Poisson's equation $\nabla^2 \phi = -\rho/\varepsilon_0$.

1.2.4 From variational principles to symmetries

In this chapter we have developed the calculus of variations and derived the Euler–Lagrange equations for a wide class of functionals, together with several examples from mechanics, optics, and electromagnetism.

In the next chapter we will specialize to the action functional

$$S[q] = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, \mathrm{d}t$$

for mechanical systems and study the role of symmetries of the Lagrangian. Noether's theorem will then provide a deep link between continuous symmetries of the action and conservation laws, such as energy, momentum, and angular momentum.

Chapter 2

Symmetry

2.1 Symmetries and Groups

Consider a square in the plane. If you rotate it by 90°, the square looks the same. If you do the same with a generic rectangle, rotating by 90° gives a different rectangle (the width and height are exchanged). We may say that the square and the rectangle have different "symmetries".

This idea is intuitive, but to make it quantitative, we need a mathematical language. For example, do a rectangle and a rhombus have the same symmetry? To answer such questions systematically we introduce the notion of a *group*, which abstracts the algebraic structure behind symmetries.

Definition 2.1: Group

A **group** is a set G equipped with a binary operation $\star$ such that:

- **Associativity:** $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$.
- **Identity:** There exists $e \in G$ such that $e \star a = a \star e = a$ for all $a \in G$.
- **Inverses:** For each $a \in G$ there exists $a^{-1} \in G$ such that $a \star a^{-1} =$

$$a^{-1} \star a = e.$$

If, in addition, $\star$ is commutative, the group is called an **abelian group**.

Example 2.1: Integers under addition

The set of integers $\mathbb{Z}$ with addition $+$ is a group:

- Associative: $(a + b) + c = a + (b + c)$.
- Identity: 0.
- Inverse: for each a, the inverse is $-a$.
- Commutative: $a + b = b + a$.

This is the simplest example of an **abelian group**.

Example 2.2: Rotations in the plane

Consider rotations of the plane $\mathbb{R}^2$ about the origin by an angle θ.

- Composition of rotations is associative.
- The identity rotation corresponds to $\theta = 0$.
- Each rotation by θ has an inverse rotation by $-\theta$.

These rotations form a group usually denoted $SO(2)$. It is the symmetry group of a circle and will be important when we discuss angular momentum.

Example 2.3: Reflection symmetries of a square

The set of all rotations and reflections that map a square onto itself forms a group, denoted D_4 (the dihedral group of order 8). This is the **symmetry group** of the square.

In physics we often have a configuration space Q (for example $Q = \mathbb{R}^n$ for a particle moving in $\mathbb{R}^n$), and a group G whose elements act on Q as transformations

$$g \colon Q \to Q, \qquad q \mapsto g \cdot q.$$

Such a group describes the symmetries of the system. We now make this precise for Lagrangian systems.

Example 2.4: Galilean symmetry in Newtonian mechanics

In Newtonian mechanics, space and time are described by coordinates $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3$. The basic symmetries of free space and time are:

- **Time translations:** $t' = t + a$, $\mathbf{x}' = \mathbf{x}$.
- **Spatial translations:** $t' = t$, $\mathbf{x}' = \mathbf{x} + \mathbf{b}$.
- **Rotations:** $t' = t$, $\mathbf{x}' = R\,\mathbf{x}$, where $R \in SO(3)$.
- **Galilean boosts:** $t' = t$, $\mathbf{x}' = \mathbf{x} + \mathbf{v}\,t$, where $\mathbf{v}$ is a constant velocity.

More generally, a *Galilean transformation* has the form

$$t' = t + a, \qquad \mathbf{x}' = R\,\mathbf{x} + \mathbf{v}\,t + \mathbf{b},$$

with parameters $a \in \mathbb{R}$ (time shift), $\mathbf{b} \in \mathbb{R}^3$ (space shift), $\mathbf{v} \in \mathbb{R}^3$ (boost velocity) and $R \in SO(3)$ (rotation).
The set of all such transformations forms a group under composition:

- The composition of two Galilean transformations is again of the same form.

- There is an identity transformation $(a, \mathbf{b}, \mathbf{v}, R) = (0, \mathbf{0}, \mathbf{0}, I)$.
- Each transformation has an inverse (with parameters $-a$, $-R^{-1}\mathbf{b}$, $-R^{-1}\mathbf{v}$, R^{-1}).

This group is called the **Galilean group**. It is the symmetry group of Newtonian space-time and leaves the free-particle Lagrangian

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{m}{2} |\dot{\mathbf{x}}|^2$$

invariant. In the following, Noether's theorem will relate these symmetries to the conservation of energy (time translations), linear momentum (space translations), angular momentum (rotations), and a quantity associated with Galilean boosts.

2.1.1 Linear symmetry groups as matrix groups

Many continuous symmetry groups in physics can be realized as groups of matrices. For example:

- $SO(3)$: 3×3 real matrices R with $R^T R = I$ and $\det R = 1$. They preserve the Euclidean length $|\mathbf{x}|^2$ in $\mathbb{R}^3$.
- $O(N)$: $N \times N$ real matrices R with $R^T R = I$. They preserve the quadratic form $\mathbf{x} \cdot \mathbf{x}$ in $\mathbb{R}^N$.
- $U(N)$: $N \times N$ complex matrices U with $U^\dagger U = I$. They preserve the Hermitian inner product on $\mathbb{C}^N$.
- $SU(N)$: unitary matrices with $\det U = 1$.

Such groups act linearly on vectors $q \in \mathbb{R}^N$ (or $\mathbb{C}^N$) via $q \mapsto Rq$ or $q \mapsto Uq$, and frequently appear as symmetry groups of Lagrangians. We will see examples of internal $O(N)$ and $U(1)$ symmetries below.

Remark 2.1: T

e set of all infinitesimal generators of a matrix group (heuristically, derivatives of one-parameter subgroups at $s = 0$) forms a vector space of matrices closed under the commutator $[A, B] = AB - BA$. This is called a *Lie algebra*. For example, the Lie algebra of $SO(3)$ consists of all real 3×3 antisymmetric matrices and is closely related to the algebra of angular momentum operators in quantum mechanics.

2.1.2 Internal $O(N)$ symmetry

Consider N real scalar fields $\phi_a(x)$, $a = 1, \ldots, N$, with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \sum_{a=1}^{N} \partial_\mu \phi_a \, \partial^\mu \phi_a - V\left(\sum_{a=1}^{N} \phi_a^2 \right),$$

where V is some function of the invariant combination $\phi_1^2 + \cdots + \phi_N^2$.

This Lagrangian is invariant under $O(N)$ rotations in the internal space:

$$\phi_a(x) \mapsto \phi'_a(x) = R_{ab} \, \phi_b(x), \qquad R \in O(N).$$

Indeed, the kinetic term is a sum of squares of derivatives, and the potential depends only on $\sum_a \phi_a^2$, both of which are invariant under orthogonal transformations.

For a one-parameter subgroup $R(s) = \exp(sA)$ with A antisymmetric, the infinitesimal transformation is

$$\delta\phi_a = \left. \frac{\partial}{\partial s} \right|_{s=0} \phi'_a = A_{ab} \, \phi_b,$$

with $A_{ab} = -A_{ba}$. By Noether's theorem, the associated conserved current is

$$j^\mu = \sum_{a=1}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \, \delta\phi_a = \sum_{a,b} \partial^\mu \phi_a \, A_{ab} \, \phi_b.$$

For each independent antisymmetric matrix A we obtain a conserved current. Equivalently, one can define

$$J^\mu_{ab} = \phi_a \, \partial^\mu \phi_b - \phi_b \, \partial^\mu \phi_a,$$

which satisfies

$$\partial_\mu J^\mu_{ab} = 0 \qquad (\text{for all } a < b).$$

These J^μ_{ab} generate the $O(N)$ internal symmetry.

2.1.3 Discrete symmetries

Discrete symmetries are generated by transformations of finite order. Important examples in physics include:

- **Parity** P: space inversion $\mathbf{x} \mapsto -\mathbf{x}$.
- **Time reversal** T: $t \mapsto -t$ (together with appropriate transformation of velocities or momenta).
- **Charge conjugation** C: exchange of particles and antiparticles in relativistic field theories.

These symmetries do not form one-parameter continuous groups, so they are not directly covered by Noether's theorem. Nevertheless, they play an essential role in:

- classifying states (parity of bound states in quantum mechanics),
- selection rules for transitions,

- and in the study of fundamental interactions (violation of P, CP, etc.).

In this chapter, we focus on continuous symmetries and Noether's theorem. A more complete discussion of P, T, C and their combinations belongs to a course in quantum mechanics or quantum field theory.

2.2 Noether's theorem

In mechanics, or even general relativity and quantum field theory, the physical systems could be described by the Lagrangian. Therefore, the symmetry of the physical systems is defined by the interaction between the group and the Lagrangian.

Definition 2.2: Symmetry of the Lagrangian

Let $L(q, \dot{q}, t)$ be a Lagrangian with $q \in Q \subseteq \mathbb{R}^n$. A group G acting on Q is called a **symmetry group** of L if for every $g \in G$ and every trajectory $q(t)$ we have

$$L\big(g \cdot q(t), \frac{\mathrm{d}}{\mathrm{d}t}(g \cdot q(t)), t\big) = L\big(q(t), \dot{q}(t), t\big),$$

or more generally, if L changes only by a total time derivative

$$L\big(g \cdot q(t), \frac{\mathrm{d}}{\mathrm{d}t}(g \cdot q(t)), t\big) = L\big(q(t), \dot{q}(t), t\big) + \frac{\mathrm{d}}{\mathrm{d}t} F_g\big(q(t), t\big)$$

for some function F_g.
In this case, we say that the transformation $q \mapsto g \cdot q$ is a *symmetry* of the Lagrangian (or of the action).

We now focus on a continuous one-parameter subgroup of symmetries.

Definition 2.3: One-parameter symmetry

A **one-parameter symmetry** of L is a family of transformations $\{h_s\}_{s \in \mathbb{R}}$ acting on Q such that

- h_0 is the identity on Q;
- $h_{s+t} = h_s \circ h_t$ for all s, t;
- for each fixed q, the map $s \mapsto h_s(q)$ is C^1;
- the family $\{h_s\}$ is a symmetry of L in the sense of the previous definition.

The *infinitesimal generator* (or *Noether vector field*) of the symmetry is

$$\xi(q,t) := \left.\frac{\partial}{\partial s}\right|_{s=0} h_s(q,t).$$

Theorem 2.1: Noether's theorem for Lagrangian systems

Let $L(q, \dot{q}, t)$ be a Lagrangian and $\{h_s\}$ a one-parameter symmetry of L such that

$$L\big(h_s(q), \tfrac{\mathrm{d}}{\mathrm{d}t} h_s(q), t\big) = L(q, \dot{q}, t)$$

for all s (no explicit boundary term, for simplicity). Let

$$\xi(q,t) = \left.\frac{\partial}{\partial s}\right|_{s=0} h_s(q,t)$$

be the corresponding infinitesimal generator.
Then for any solution $q(t)$ of the Euler–Lagrange equations, the quantity

$$Q(t) = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}, t) \cdot \xi(q,t) = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}, t)\, \xi_i(q,t)$$

is conserved:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 0.$$

Proof. Let $q(t)$ be a solution of the Euler–Lagrange equations. By assumption the Lagrangian is invariant under the symmetry:

$$L\big(h_s(q), \tfrac{\mathrm{d}}{\mathrm{d}t} h_s(q), t\big) = L(q, \dot{q}, t)$$

for all s, so in particular

$$0 = \frac{\partial}{\partial s}\bigg|_{s=0} L\big(h_s(q), \tfrac{\mathrm{d}}{\mathrm{d}t}h_s(q), t\big).$$

Using the chain rule, we obtain

$$0 = \frac{\partial L}{\partial q}(q,\dot q,t)\cdot\xi(q,t) + \frac{\partial L}{\partial \dot q}(q,\dot q,t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\xi(q,t),$$

where $\xi(q,t) = \partial_s h_s(q,t)|_{s=0}$ and the dot denotes the standard Euclidean scalar product. On the other hand, along a solution $q(t)$ the Euler–Lagrange equations give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}.$$

Therefore

$$\begin{aligned} 0 &= \frac{\partial L}{\partial q}\cdot\xi + \frac{\partial L}{\partial \dot q}\cdot\dot\xi \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q}\right)\cdot\xi + \frac{\partial L}{\partial \dot q}\cdot\dot\xi \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q}\cdot\xi\right). \end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(t) = 0, \qquad Q(t) = \frac{\partial L}{\partial \dot q}\cdot\xi(q,t),$$

so Q is conserved along the motion. □

Remark 2.2: Intuitive picture

Intuitively speaking, Noether's theorem states that a smooth symmetry (described by the one-parameter family $\{h_s\}$) singles out a direction in configuration space along which the physical situation does not change. The corresponding conserved quantity Q measures how the motion projects onto

this "redundant" direction, and effectively removes one independent degree of freedom from the dynamics.
A simple intuitive analogy is a round table with several identical cups on it. If you rotate the entire table by any angle, the relative arrangement of the cups is unchanged; only the numbering of the seats has shifted. The angle by which you rotate the table is a "symmetric" direction: changing it does not change the physical configuration.
If the table can rotate freely without friction, once it starts spinning it keeps the same angular velocity. This constant angular velocity (times the moment of inertia) is the conserved angular momentum. Here the rotational symmetry allows us to treat the overall angle of the table as a redundant coordinate, while the conserved angular momentum L is the Noether charge associated with this symmetry.

Remark 2.3: Scope of Noether's theorem used here

Strictly speaking, what we have stated and proved is a coordinate version of *Noether's first theorem*. We work with Lagrangians $L(q, \dot{q}, t)$ on configuration spaces $Q \subseteq \mathbb{R}^N$ (or a finite collection of fields $\phi_a(x)$), and with one-parameter families of smooth transformations $q \mapsto h_s(q, t)$ or $\phi_a \mapsto \phi_a + s\,\delta\phi_a$.
The full mathematical formulation of Noether's first theorem treats the configuration space as a differentiable manifold, the symmetry group as a Lie group acting smoothly on this manifold, and the Lagrangian as a function (or density) on suitable jet bundles. In that setting, both the statement and the proof use more advanced tools from differential geometry and Lie group theory. However, this "coordinate" form of Noether's theorem already covers essentially all examples encountered in elementary mechanics and many basic field-theory models, and is sufficient for these notes. For a fully geometric treatment, the reader may consult, for example, [2] or the original paper of Emmy Noether

(English version) [7] for a more old-fashioned method.

From the point of view of Lagrangian mechanics, symmetries fall roughly into three classes:

- **Space-time symmetries**: transformations acting on $(t, \mathbf{x})$, such as time translations, spatial translations, rotations, and Galilean (or Lorentz) boosts. These lead to conservation of energy, momentum, angular momentum, etc.
- **Internal symmetries**: transformations acting on "internal" degrees of freedom but not on space-time itself, such as phase rotations of a complex scalar field or rotations between several components of a multiplet of fields. These often lead to conserved charges (isospin, particle number, etc.).
- **Discrete symmetries**: symmetries generated by transformations of order two or a finite group, such as parity (space inversion $\mathbf{x} \mapsto -\mathbf{x}$), time reversal, and charge conjugation. These do not give a conserved current in the same way as continuous symmetries, but play an important role in selection rules and classification of states.

The reason why Noether's theorem is so powerful and beautiful is that it connects "symmetry" and conserved quantities in a more mathematically rigorous sense. We will apply Noether's theorem to see what the conserved quantities are in the next section.

2.2.1 Spatial translation symmetry and linear momentum

Consider a single particle of mass m moving in $\mathbb{R}^3$ with coordinates $q = (x, y, z)$ and Lagrangian

$$L(q, \dot{q}) = \frac{m}{2}|\dot{q}|^2 - V(q),$$

where $|\dot{q}|^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$.

Suppose the potential is invariant under translations in some fixed direction $\mathbf{a} \in \mathbb{R}^3$, i.e.

$$V(q + s\,\mathbf{a}) = V(q) \quad \text{for all } s \in \mathbb{R},\ q \in \mathbb{R}^3.$$

Then the one-parameter family of transformations

$$h_s(q) = q + s\,\mathbf{a}$$

is a symmetry of L, because $|\dot{q}|$ is unchanged and V is invariant.

The infinitesimal generator is

$$\xi(q) = \left.\frac{\partial}{\partial s}\right|_{s=0} h_s(q) = \mathbf{a}.$$

By Noether's theorem, the conserved quantity is

$$Q = \frac{\partial L}{\partial \dot{q}} \cdot \xi = (m\dot{q}) \cdot \mathbf{a} = \mathbf{p} \cdot \mathbf{a},$$

where $\mathbf{p} = m\dot{q}$ is the linear momentum.

Since $\mathbf{a}$ can be any vector along the symmetry direction, this implies that the corresponding component of the momentum is conserved. In particular, if V does not depend on x, then p_x is conserved; if V does not depend on y, then p_y is conserved, and so on.

2.2.2 Rotational symmetry and angular momentum

Again consider a particle with Lagrangian

$$L(q, \dot{q}) = \frac{m}{2}|\dot{q}|^2 - V(q), \qquad q \in \mathbb{R}^3.$$

Assume that the potential depends only on the distance from the origin,

$$V(q) = V(|q|),$$

so the system is invariant under spatial rotations.

Let $\mathbf{n}$ be a unit vector giving the axis of rotation. For each s we let h_s be the rotation of $\mathbb{R}^3$ by an angle s about the axis $\mathbf{n}$. This defines a one-parameter symmetry of L.

For small s, the rotation can be written as

$$h_s(q) = q + s\,(\boldsymbol{\omega} \times q) + O(s^2),$$

where $\boldsymbol{\omega} = \mathbf{n}$ is a constant vector along the axis. Hence the infinitesimal generator is

$$\xi(q) = \left.\frac{\partial}{\partial s}\right|_{s=0} h_s(q) = \boldsymbol{\omega} \times q.$$

Noether's conserved quantity is

$$\begin{aligned} Q &= \frac{\partial L}{\partial \dot{q}} \cdot \xi = (m\dot{q}) \cdot (\boldsymbol{\omega} \times q) \\ &= \boldsymbol{\omega} \cdot (q \times m\dot{q}) \\ &= \boldsymbol{\omega} \cdot \mathbf{L}, \end{aligned}$$

where $\mathbf{L} = q \times \mathbf{p}$ is the angular momentum.

Since $\boldsymbol{\omega}$ is an arbitrary constant vector along the symmetry axis, this implies that the component of $\mathbf{L}$ along that axis is conserved. If the system is invariant under all rotations (central potential), then the full angular momentum vector $\mathbf{L}$ is conserved.

2.2.3 Time-translation symmetry and energy conservation

Consider a Lagrangian $L(q, \dot{q}, t)$. If L does not depend explicitly on time, i.e.

$$\frac{\partial L}{\partial t} = 0,$$

we say that the system is invariant under time translations $t \mapsto t + s$. Noether's theorem then implies the conservation of *energy*.

Instead of giving the most general version of Noether's theorem including time transformations, we can give a direct derivation. Define the quantity

$$E(q, \dot{q}) = \sum_i \frac{\partial L}{\partial \dot{q}_i}\, \dot{q}_i - L = \mathbf{p} \cdot \dot{q} - L.$$

Along a solution of the Euler–Lagrange equations we have

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \sum_i \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\, \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\, \ddot{q}_i \right) - \frac{\mathrm{d}L}{\mathrm{d}t} \\
&= \sum_i \frac{\partial L}{\partial q_i}\, \dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\, \ddot{q}_i - \left(\sum_i \frac{\partial L}{\partial q_i}\, \dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\, \ddot{q}_i + \frac{\partial L}{\partial t} \right) \\
&= -\frac{\partial L}{\partial t}.
\end{aligned}$$

Thus, if L has no explicit time dependence,

$$\frac{\partial L}{\partial t} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}E}{\mathrm{d}t} = 0.$$

Therefore, E is conserved along the motion. This is the familiar energy conservation law arising from invariance under time translations.

2.2.4 Global gauge symmetry and charge conservation

We now consider a simple field-theoretic example. Let $\phi(x)$ be a complex scalar field on spacetime, with Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(|\phi|^2),$$

where V depends only on $|\phi|^2 = \phi^* \phi$. This system is invariant under *global phase rotations*

$$\phi(x) \mapsto \phi'(x) = e^{\mathrm{i}s}\, \phi(x), \qquad \phi^*(x) \mapsto \phi'^*(x) = e^{-\mathrm{i}s}\, \phi^*(x),$$

where $s \in \mathbb{R}$ is a constant parameter. This is often called a global $U(1)$ gauge symmetry.

The infinitesimal form of the transformation is obtained by expanding to first order in s:

$$\delta\phi = \frac{\partial}{\partial s}\bigg|_{s=0} \phi' = \mathrm{i}\,\phi, \qquad \delta\phi^* = \frac{\partial}{\partial s}\bigg|_{s=0} \phi'^* = -\mathrm{i}\,\phi^*.$$

The field-theoretic version of Noether's theorem states that the conserved current associated with this symmetry is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\,\delta\phi^*,$$

and the corresponding conservation law is

$$\partial_\mu j^\mu = 0.$$

We compute

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi^*, \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi.$$

Therefore

$$\begin{aligned} j^\mu &= \partial^\mu \phi^* \,(\mathrm{i}\phi) + \partial^\mu \phi \,(-\mathrm{i}\phi^*) \\ &= \mathrm{i}\left(\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*\right). \end{aligned}$$

Noether's theorem then guarantees that

$$\partial_\mu j^\mu = 0.$$

The conserved quantity (often interpreted as a "charge") is obtained by integrating the time component j^0 over space:

$$Q = \int_{\mathbb{R}^3} j^0(t, \mathbf{x})\, \mathrm{d}^3\mathbf{x}, \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} = 0.$$

Thus global $U(1)$ gauge symmetry leads to a conserved charge. In quantum theory this is typically associated with particle number or electric charge.

2.2.5 Beyond mechanics

In this section, we briefly illustrate Noether's theorem in several non-mechanical contexts, emphasizing the common pattern: a variational problem, a continuous symmetry, and a corresponding conserved quantity or first integral.

Example 2.5: Geodesics on a surface of revolution and Clairaut's relation

Consider a surface of revolution obtained by rotating a plane curve $(f(r), 0, g(r))$ about the z-axis. A point on the surface can be described by coordinates (r, φ):

$$\mathbf{x}(r, \varphi) = \big(f(r)\cos\varphi,\ \ f(r)\sin\varphi,\ \ g(r)\big).$$

A curve on the surface can then be written as $t \mapsto (r(t), \varphi(t))$. The arc length functional of such a curve is

$$S[r, \varphi] = \int_{t_1}^{t_2} L\big(r, \varphi, \dot{r}, \dot{\varphi}\big)\, \mathrm{d}t,$$

with

$$L(r, \varphi, \dot{r}, \dot{\varphi}) = \sqrt{E(r)\,\dot{r}^2 + G(r)\,\dot{\varphi}^2},$$

where

$$E(r) = f'(r)^2 + g'(r)^2, \qquad G(r) = f(r)^2.$$

Geodesics on the surface are the extremals of S.
The key observation is that the surface is generated by rotation about the z-axis, so the Lagrangian does not depend explicitly on φ:

$$\frac{\partial L}{\partial \varphi} = 0.$$

Thus φ is a cyclic coordinate. By Noether's theorem (or directly by the Euler–Lagrange equation), the conjugate momentum

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = \frac{G(r)\,\dot{\varphi}}{\sqrt{E(r)\,\dot{r}^2 + G(r)\,\dot{\varphi}^2}}$$

is conserved along geodesics:

$$p_\varphi = \text{constant}.$$

To interpret this geometrically, it is convenient to reparametrize the curve by arc length s, so that $L \equiv 1$ and

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 E(r) + \left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 G(r) = 1.$$

In this parametrization,

$$p_\varphi = G(r)\,\frac{\mathrm{d}\varphi}{\mathrm{d}s} = f(r)^2\,\frac{\mathrm{d}\varphi}{\mathrm{d}s}.$$

Let $\alpha(s)$ be the angle between the geodesic and a meridian (curve of constant φ). One can show that

$$\sin\alpha(s) = f(r)\,\frac{\mathrm{d}\varphi}{\mathrm{d}s},$$

so the conservation of p_φ is equivalent to

$$f(r)\,\sin\alpha = \text{constant along the geodesic.}$$

This is Clairaut's relation for geodesics on a surface of revolution. Here, the rotational symmetry in φ leads, via Noether's theorem, to a conserved quantity which has a clear geometric meaning and simplifies the study of geodesics.

Example 2.6: Dirichlet energy and translation symmetry

Let $\Omega \subset \mathbb{R}^n$ be a domain and consider the Dirichlet energy functional

$$J[u] = \int_\Omega \frac{1}{2}|\nabla u(\mathbf{x})|^2\,\mathrm{d}^n\mathbf{x},$$

defined for sufficiently smooth functions u with fixed boundary values on $\partial\Omega$. The Euler–Lagrange equation for J is Laplace's equation $\Delta u = 0$, so

minimizers are harmonic functions.
If Ω and the boundary conditions are invariant under translations in some direction $\mathbf{a}$, then the Lagrangian density

$$\mathcal{L}(\mathbf{x}, u, \nabla u) = \frac{1}{2}|\nabla u|^2$$

is invariant under

$$\mathbf{x} \mapsto \mathbf{x} + s\mathbf{a}, \qquad u(\mathbf{x}) \mapsto u(\mathbf{x} + s\mathbf{a}).$$

This is a space-translation symmetry of a purely mathematical variational problem. The field-theoretic Noether theorem then yields a divergence-free quantity, which can be interpreted as a conserved "energy flux" associated with harmonic functions in the direction $\mathbf{a}$. This is an example of Noether's theorem in the context of potential theory and partial differential equations, without any explicit mechanical interpretation.

Example 2.7: Finite-dimensional analogue: radial symmetry in minimization

Consider the finite-dimensional function

$$f(x, y) = g(x^2 + y^2),$$

where $g\colon [0, \infty) \to \mathbb{R}$ is a smooth function. The problem of finding minima of f is invariant under rotations in the plane:

$$(x, y) \mapsto R_\theta(x, y), \qquad R_\theta \in SO(2),$$

because $x^2 + y^2$ is unchanged.
As a consequence, the value of f depends only on the radius $r = \sqrt{x^2 + y^2}$ and not on the angle. The angular variable is a "redundant" degree of freedom, eliminated by the rotational symmetry; the minimization reduces effectively

to a one-dimensional problem for r. Noether's theorem can be viewed as the infinite-dimensional, dynamical analogue of this idea: a continuous symmetry selects a direction in configuration space along which the physics is unchanged, and the associated conserved quantity plays the role of a constant label for motion along that redundant direction.

Summary. In classical mechanics, the connections between "symmetries" and "conserved quantities" may be seen obviously. However, in modern physics or mathematical physics, such as general relativity or topological quantum field theory, the symmetries and conserved quantities can be incredibly abstract and non-physical. As a consequence, it may be difficult to relate the symmetries and conserved quantities only through imagination. Fortunately, Noether's theorem, by providing us with a systematic and mathematically rigorous approach to finding the conserved quantities utilizing symmetries, elegantly solves this challenge.

Chapter 3

Integrability

For a long time, "symmetry" has been viewed as playing the most central role in physics and mathematics, particularly in dynamical systems and differential geometry. However, a more novel concept called *integrability* replaces it, giving a more general framework for describing and finding conserved quantities.

An N-second-order equations of motion describe a degree-of-freedom mechanical system, or equivalently, by $2N$ first-order equations for positions and velocities. Without any conserved quantities, a generic trajectory can wander through a large portion of this $2N$-dimensional space.

Consider the conserved quantities, each conserved quantity $F(q, \dot{q}, t)$ is an *integral of motion*:

$$\frac{\mathrm{d}}{\mathrm{d}t} F(q(t), \dot{q}(t), t) = 0.$$

Geometrically, this means that the trajectory is confined to a level set $F =$ constant, a $(2N-1)$-dimensional hypersurface. Each additional independent integral of motion further reduces the dimensionality of the region in which the trajectory can move.

Noether's theorem provides a systematic way to generate such integrals of motion: every continuous symmetry of the Lagrangian yields a conserved quantity.

A simple illustration is the motion in a central potential $V(r)$:

- Rotational symmetry implies conservation of the angular momentum vector $\mathbf{L}$, which forces the motion to lie in a fixed plane through the origin. This reduces the problem from three spatial dimensions to two.
- In suitable polar coordinates (r, φ) on that plane, the angle φ does not appear explicitly in the Lagrangian and is therefore a cyclic coordinate. Its conjugate momentum $p_\varphi = L_z$ is conserved, reducing the dynamics to an effective one-dimensional problem for $r(t)$ with an effective potential.

Here we see explicitly how symmetries and their Noether charges "eat up" degrees of freedom and turn a seemingly complicated three-dimensional problem into a one-dimensional integrable one.

In more complicated systems, having many independent symmetries (and hence many conserved quantities) is often a signal of integrability, whereas systems with very few symmetries tend to exhibit irregular or even chaotic behaviours. In this sense, symmetry provides the "rigid structure" that makes integrable dynamics possible.

Remark 3.1

The word "integrable" here is completely unrelated to the word "integrable" in calculus. The reason why mathematicians use this word here is that the conserved quantities are also called "integral of motion".

3.1 Internal $O(N)$ symmetry revisited

Consider N real scalar fields $\phi_a(x)$, $a = 1, \ldots, N$, with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \sum_{a=1}^{N} \partial_\mu \phi_a \, \partial^\mu \phi_a - V\left(\sum_{a=1}^{N} \phi_a^2 \right),$$

where V is some function of the invariant combination $\phi_1^2 + \cdots + \phi_N^2$.

This Lagrangian is invariant under $O(N)$ rotations in the internal space:

$$\phi_a(x) \mapsto \phi'_a(x) = R_{ab}\,\phi_b(x), \qquad R \in O(N).$$

Indeed, the kinetic term is a sum of squares of derivatives, and the potential depends only on $\sum_a \phi_a^2$, both of which are invariant under orthogonal transformations.

For a one-parameter subgroup $R(s) = \exp(sA)$ with A antisymmetric, the infinitesimal transformation is

$$\delta\phi_a = \left.\frac{\partial}{\partial s}\right|_{s=0} \phi'_a = A_{ab}\,\phi_b,$$

with $A_{ab} = -A_{ba}$. By Noether's theorem, the associated conserved current is

$$j^\mu = \sum_{a=1}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\,\delta\phi_a = \sum_{a,b} \partial^\mu \phi_a\, A_{ab}\,\phi_b.$$

For each independent antisymmetric matrix A we obtain a conserved current. Equivalently, one can define

$$J^\mu_{ab} = \phi_a\,\partial^\mu \phi_b - \phi_b\,\partial^\mu \phi_a,$$

which satisfies

$$\partial_\mu J^\mu_{ab} = 0 \qquad (\text{for all } a < b).$$

These J^μ_{ab} generate the $O(N)$ internal symmetry.

Example 3.1: Central potential and reduction to an effective one-dimensional problem

Consider a particle of mass m moving in a central potential $V(r)$, where $r = |\mathbf{x}|$. In spherical coordinates (r, θ, φ) the Lagrangian is

$$L = \frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2 + r^2 \sin^2\theta\,\dot\varphi^2\right) - V(r).$$

- **Rotational symmetry and planar motion.** The potential depends

only on r, so the system is invariant under spatial rotations. As a consequence, the angular momentum vector $\mathbf{L}$ is conserved. This implies that the motion takes place in a fixed plane through the origin. By choosing coordinates so that this plane is the equatorial plane, we can set $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$. The dynamics effectively reduces from 3D to 2D with coordinates (r, φ).

- **Cyclic coordinate φ.** In the reduced Lagrangian

$$L_{\text{red}} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r),$$

the angle φ does not appear explicitly:

$$\frac{\partial L_{\text{red}}}{\partial \varphi} = 0.$$

Thus φ is a cyclic coordinate and its conjugate momentum

$$p_\varphi = \frac{\partial L_{\text{red}}}{\partial \dot{\varphi}} = mr^2\dot{\varphi}$$

is conserved. This is just the magnitude of the angular momentum L_z.

Using the conservation of p_φ we can express $\dot{\varphi}$ in terms of r:

$$\dot{\varphi} = \frac{p_\varphi}{mr^2}.$$

Substituting this back into the energy

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r)$$

gives an effective one-dimensional problem for $r(t)$:

$$E = \frac{m}{2}\dot{r}^2 + V_{\text{eff}}(r), \qquad V_{\text{eff}}(r) = V(r) + \frac{p_\varphi^2}{2mr^2}.$$

Thus, by using rotational symmetry (Noether: angular momentum conservation) and the cyclic coordinate φ, the original three-dimensional problem reduces to the motion of a fictitious particle in one dimension with effective potential $V_{\text{eff}}(r)$. This is a concrete example of how symmetries and conserved quantities "eliminate" degrees of freedom and make the system integrable by quadratures.

In one dimension the conservation of energy always allows us to solve the motion by quadratures, i.e. by integrating an explicit first-order equation for $r(t)$. In higher dimensions, having sufficiently many conserved quantities can make a system *integrable* in a similar sense.

3.2 Integrals of motion and integrability

In the previous section we have seen that continuous symmetries of the Lagrangian lead to conserved quantities via Noether's theorem. These conserved quantities are special cases of a more general notion.

Definition 3.1: Integral of motion

Let $L(q, \dot{q}, t)$ be a Lagrangian with generalized coordinates $q = (q_1, \dots, q_N)$, and let $q(t)$ be a solution of the Euler–Lagrange equations. A function

$$F\colon \{(q, \dot{q}, t)\} \to \mathbb{R}$$

is called an **integral of motion** (or **constant of motion**) if along every solution $q(t)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} F\big(q(t), \dot{q}(t), t\big) = 0.$$

Equivalently, F is constant in time for all motions of the system.

Energy, linear momentum, and angular momentum in classical mechanics are all examples of integrals of motion. Noether's theorem tells us that each continuous symmetry of the Lagrangian produces such an F.

Remark 3.2

Heuristically, each independent integral of motion reduces the "effective dimension" of the space of possible motions by one. This makes it easier to solve the equations of motion, and in favorable cases the system becomes *integrable* in the sense that the trajectories can be found by explicit integration (quadrature).

3.2.1 One-dimensional systems

The simplest situation is that of a single degree of freedom $q(t)$ with a Lagrangian $L(q, \dot{q})$ that does not depend explicitly on time.

Theorem 3.1: Energy and one-dimensional integrability

Let $L(q, \dot{q})$ be a Lagrangian with no explicit time dependence, $\partial L/\partial t = 0$. Define the *energy*

$$E(q, \dot{q}) = \dot{q}\,\frac{\partial L}{\partial \dot{q}} - L(q, \dot{q}).$$

Then along any solution of the Euler–Lagrange equation E is an integral of motion. Moreover, in one dimension the equation of motion can be reduced to a first-order equation and solved by quadrature using E.

Proof. The conservation of E has already been derived: one differentiates E with respect to t and uses the Euler–Lagrange equation to show that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\,\frac{\partial L}{\partial t} = 0.$$

In one dimension, the relation $E(q, \dot{q}) = \text{constant}$ can often be solved for $\dot{q}$ as

a function of q and E:

$$\dot{q} = f(q; E).$$

This turns the second-order equation into a first-order separable equation

$$\frac{\mathrm{d}q}{\mathrm{d}t} = f(q; E) \quad \Longrightarrow \quad \int_{q_0}^{q(t)} \frac{\mathrm{d}q'}{f(q'; E)} = t - t_0.$$

Thus the motion can be obtained (at least implicitly) by a single integration. In this sense a one-dimensional autonomous Lagrangian system is always integrable by quadrature. □

Example 3.2: Simple harmonic oscillator

Consider the harmonic oscillator

$$L(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2,$$

with no explicit time dependence. The energy is

$$E = \frac{m}{2}\dot{q}^2 + \frac{k}{2}q^2.$$

Solving for $\dot{q}$ gives

$$\dot{q} = \pm\sqrt{\frac{2}{m}\Big(E - \frac{k}{2}q^2\Big)}.$$

Separating variables,

$$\frac{\mathrm{d}q}{\sqrt{E - \frac{k}{2}q^2}} = \sqrt{\frac{2}{m}}\,\mathrm{d}t.$$

Integrating both sides yields (after some algebra) the familiar sinusoidal solution

$$q(t) = A\cos(\omega t + \phi), \qquad \omega = \sqrt{\frac{k}{m}},$$

with amplitude A and phase ϕ determined by E and the initial condition. Here the conservation of E has reduced the problem to a single quadrature.

3.2.2 Cyclic coordinates and reduction

For systems with several degrees of freedom, integrability in a strict sense is more subtle. However, even without a full theory we can see how individual integrals of motion simplify the dynamics by *reducing* the number of effective degrees of freedom.

Definition 3.2: Cyclic coordinate

Let $L(q, \dot{q}, t)$ be a Lagrangian with $q = (q_1, \dots, q_N)$. We say that q_k is a **cyclic** (or **ignorable**) coordinate if

$$\frac{\partial L}{\partial q_k} = 0$$

for all $(q, \dot{q}, t)$ in the domain considered.

If q_k is cyclic, the corresponding Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0.$$

Thus the generalized momentum

$$p_k := \frac{\partial L}{\partial \dot{q}_k}$$

is an integral of motion. This is precisely the situation described by Noether's theorem when the symmetry is a translation $q_k \mapsto q_k + s$.

In practice one can treat p_k as a constant parameter, solve for $\dot{q}_k$ in terms of p_k and the remaining variables, and substitute back into the energy or into the remaining equations. In this way a cyclic coordinate can be eliminated from the dynamics, reducing the problem by one degree of freedom. This is a concrete manifestation of the intuitive idea that "one continuous symmetry eliminates one degree of freedom".

3.2.3 Angular momentum and invariant surfaces of motion

So far we have mostly worked in configuration space and used symmetries to reduce the number of coordinates (for example, central potentials lead to planar motion and an effective one-dimensional radial equation). It is often useful to visualize the effect of conserved quantities directly in *phase space*.

For a single particle in three dimensions, the phase space has coordinates $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3$ and is therefore 6-dimensional. A Hamiltonian of the form

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2m}|\mathbf{p}|^2 + V(|\mathbf{x}|)$$

describes motion in a central potential $V(r)$ with $r = |\mathbf{x}|$. The equations of motion can be written as Hamilton's equations, but for our purposes it is enough to know that the energy

$$E = H(\mathbf{x}, \mathbf{p})$$

and the angular momentum vector

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}$$

are integrals of motion.

Fixing the energy to a value E_0 restricts the motion to the *energy surface*

$$\Sigma_{E_0} = \{\, (\mathbf{x}, \mathbf{p}) : H(\mathbf{x}, \mathbf{p}) = E_0 \,\} \subset \mathbb{R}^6,$$

which is a 5-dimensional hypersurface in phase space. Trajectories starting on Σ_{E_0} can never leave it.

Because the potential is central, the three components of $\mathbf{L}$ are also constants of motion. In particular, the magnitude

$$L^2 = \mathbf{L} \cdot \mathbf{L} = |\mathbf{x} \times \mathbf{p}|^2$$

is conserved. If we fix both E and L^2 to some values E_0 and ℓ^2, the motion is confined to the intersection

$$\Sigma_{E_0,\ell^2} = \{\, (\mathbf{x},\mathbf{p}) : H(\mathbf{x},\mathbf{p}) = E_0,\ |\mathbf{x}\times\mathbf{p}|^2 = \ell^2 \,\},$$

which has (generically) dimension $6-2=4$.

We may also fix one component of the angular momentum, say L_z, to a value m_ℓ. Then the allowed phase-space points satisfy three independent constraints

$$H(\mathbf{x},\mathbf{p}) = E_0, \qquad |\mathbf{x}\times\mathbf{p}|^2 = \ell^2, \qquad L_z(\mathbf{x},\mathbf{p}) = m_\ell.$$

Their intersection

$$\Sigma_{E_0,\ell^2,m_\ell} = \{\, (\mathbf{x},\mathbf{p}) : H = E_0,\ L^2 = \ell^2,\ L_z = m_\ell \,\}$$

has (again, generically) dimension $6-3=3$. Every trajectory with these fixed values of (E, L^2, L_z) is entirely contained in this 3-dimensional subset of phase space, and different triples (E_0, ℓ^2, m_ℓ) correspond to disjoint invariant subsets.

Geometrically, one can think of the full phase space as being partitioned into many such invariant "slices" labelled by the constants of motion. The dynamics does not mix different slices: once the initial condition is chosen, the system evolves forever within the corresponding slice.

For a central potential, we have already seen that:

- conservation of $\mathbf{L}$ confines the motion to a plane in configuration space;
- choosing polar coordinates in that plane, the angle φ is cyclic and the conjugate momentum $p_\varphi = L_z$ is constant;
- the remaining radial motion $r(t)$ is described by a one-dimensional effective potential $V_{\text{eff}}(r)$ and can be solved by quadrature.

In phase space, this reduction corresponds to:

- restricting from the full 6-dimensional space to the energy shell (5D),
- then to the subset with fixed L^2 (4D),
- then to the subset with fixed L_z (3D),
- and finally, within this 3D invariant set, the actual trajectory is a 1D curve.

Thus, the conserved quantities (E, L^2, L_z) carve phase space into lower-dimensional invariant regions on which the motion is confined. This is a concrete example of how angular momentum conservation provides a strong form of "partial integrability": although the system starts out with three spatial degrees of freedom, the presence of several integrals of motion reduces the effective complexity of the trajectories to that of a one-dimensional problem.

Summary. A continuous symmetry produces a conserved quantity (Noether charge). Each conserved quantity restricts the motion to a lower-dimensional region of the state space. If we have "enough" independent conserved quantities, the motion is so constrained that it can be solved by quadratures: the system is (in this naive sense) integrable.

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