Introduction to Hamiltonian Dynamical Systems and The N-Body Problem

(by KR Meyer)

Exercise Solutions

Author: Hao-Yang Yen (顏浩洋)
Department of Physics, National Tsing-Hua University
Email: `kikiyen0715@gapp.nthu.edu.tw`

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Problem 1.9

Solution.

The Lagrangian defined

$$L(x, \dot{x}) = \frac{1}{2} g_{ij}(x)\, \dot{x}^i \dot{x}^j \tag{1}$$

is actually the kinetic-energy Lagrangian of a Riemannian metric $g_{ij}(x)$, where $g_{ij}(x)$ is positive definite.
The Lagrange equations read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^s}\right) - \frac{\partial L}{\partial x^s} = 0 \quad s = 1, 2. \tag{2}$$

We compute the two terms separately.
For the first term, since L is quadratic in the velocities and $g_{ij} = g_{ji}$, we have

$$\frac{\partial L}{\partial \dot{x}^s} = \frac{1}{2}\left(g_{sj}\dot{x}^j + g_{is}\dot{x}^i\right) = g_{sj}\dot{x}^j. \tag{3}$$

Differentiating in time gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^s}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(g_{sj}\dot{x}^j\right) = \partial_i g_{sj}\, \dot{x}^i \dot{x}^j + g_{sj}\, \ddot{x}^j, \tag{4}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and we used the chain rule.
The second term is much simplier

$$\frac{\partial L}{\partial x^s} = \frac{1}{2}\, \partial_s g_{ij}\, \dot{x}^i \dot{x}^j. \tag{5}$$

Combine them together to get the full Lagrange equations

$$g_{sj}\, \ddot{x}^j + \partial_i g_{sj}\, \dot{x}^i \dot{x}^j - \frac{1}{2}\, \partial_s g_{ij}\, \dot{x}^i \dot{x}^j = 0. \tag{6}$$

Symmetrize the middle term in the indices (i, j)

$$\partial_i g_{sj}\, \dot{x}^i \dot{x}^j = \frac{1}{2}\big(\partial_i g_{sj} + \partial_j g_{si}\big)\dot{x}^i \dot{x}^j. \tag{7}$$

Hence,

$$g_{sk}\, \ddot{x}^k + \frac{1}{2}\Big(\partial_i g_{sj} + \partial_j g_{si} - \partial_s g_{ij}\Big)\dot{x}^i \dot{x}^j = 0. \tag{8}$$

Multiplying by the inverse metric g^{ks} (so that $g^{ks} g_{sk} = \delta^k_k$) yields

$$\ddot{x}^k + \Gamma^k{}_{ij}(x)\, \dot{x}^i \dot{x}^j = 0, \tag{9}$$

where

$$\Gamma^k{}_{ij} = \frac{1}{2}\, g^{ks}\Big(\partial_i g_{sj} + \partial_j g_{si} - \partial_s g_{ij}\Big). \tag{10}$$

Remarks.

- The indices are kept consistent: the free index in the final equation is k, while i, j, s are dummy (summed) indices.
- The symmetry $g_{ij} = g_{ji}$ is part of the definition of a (Riemannian) metric and is used to simplify $\partial L / \partial \dot{x}^s$ and to justify symmetrizing in (i, j).
- The combination $\frac{1}{2}\big(\partial_i g_{sj} + \partial_j g_{si} - \partial_s g_{ij}\big)$ is precisely the Levi–Civita Christoffel symbol with one index lowered; raising the index with g^{ks} gives $\Gamma^k{}_{ij}$.
- The resulting ODE is the geodesic equation: its solutions are curves that locally extremize arc length with respect to g.