Introduction to Hamiltonian Dynamical Systems and The N-Body Problem

(by KR Meyer)

Exercise Solutions

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Problem 1.8

Solution.

(a) Since $x^T x = \|x\|^2 = 1$, we have $\mathrm{d}\|x\|/\mathrm{d}t = 0$. Then,

$$0 = \frac{\mathrm{d}(x^T x)}{\mathrm{d}t} = \dot{x}^T x + x^T \dot{x} = 2x^T \dot{x}. \tag{1}$$

Thus, we immediately have

$$x^T \dot{x} = 0. \tag{2}$$

$$0 = \frac{\mathrm{d}(x^T \dot{x})}{\mathrm{d}t} = \dot{x}^T \dot{x} + x^T \ddot{x} = \|\dot{x}\|^2 + \lambda x^T x = \|\dot{x}\|^2 + \lambda \|x\|^2 \tag{3}$$

(b) From the result of (a), since $\lambda = -\|\dot{x}\|^2$ and $\mathrm{d}\lambda/\mathrm{d}t = 0$, we can rewrite the equation of motion as

$$\ddot{x} = -\omega^2 x, \tag{4}$$

which is easy to solve

$$x = A\cos(\omega t) + B\sin(\omega t), \tag{5}$$

where $A, B \in \mathbb{R}^3$.

Now, consider the initial condition for the position. Let $x_0 \in \mathbb{R}^3$ be the initial position, then

$$x(0) = A = x_0. \tag{6}$$

We next consider the initial velocity $v_0 \in \mathbb{R}^3$, then again

$$\dot{x}(0) = -\omega A \sin(0) + \omega B \cos(0) = \omega B = v_0. \tag{7}$$

Thus, the solution is given by

$$x = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) \in S^2, \tag{8}$$

which is a perfect circle on S^2.

(c) Let $y = \dot{x}$, then if

$$\frac{\mathrm{d}\left(x^T \dot{x}\right)}{\mathrm{d}t} = 0 \tag{9}$$

(d) For $(x, y) \in T_1 S^2$, x, y, and $x \times y$ are unit vectors that are orthogonal to each other, that is, they form an orthonormal basis. Thus,

$$\det \begin{pmatrix} x & y & x \times y \end{pmatrix} = 1, \tag{10}$$

which implies that $\begin{pmatrix} x & y & x \times y \end{pmatrix} \in SO(3, \mathbb{R})$.

So we can consider two maps

$$\Psi : T_1 S^2 \to SO(3, \mathbb{R}), \quad (x, y) \mapsto \begin{pmatrix} x & y & x \times y \end{pmatrix}, \tag{11}$$

$$\Phi : SO(3, \mathbb{R}) \to T_1 S^2, \quad M \mapsto (Mi, Mj). \tag{12}$$

Since the cross product and linear transformation are smooth, Ψ and Φ are smooth as well.

In addition, for all $(x, y) \in T_1 S^2$

$$\begin{aligned} \Phi \circ \Psi(x, y) &= \Phi(\begin{pmatrix} x & y & x \times y \end{pmatrix}) \\ &= (\begin{pmatrix} x & y & x \times y \end{pmatrix} i, \begin{pmatrix} x & y & x \times y \end{pmatrix} j) = (x, y) \end{aligned} \tag{13}$$

and for all $M \in SO(3, \mathbb{R})$

$$\Psi \circ \Phi(M) = \Psi(Mi, Mj) = \begin{pmatrix} Mi & Mj & Mk \end{pmatrix} = M. \tag{14}$$

Therefore, $\Psi \circ \Phi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$, which means that $T_1 S^2 \cong SO(3, \mathbb{R})$.