Leveraging Variational Matrix Product States for the Analysis of Stochastic Dynamical Systems

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1. Abstract

The tensor network (TN) framework is a robust approach for solving many-body quantum system problems. The variational matrix product states (VMPS) technique provides an efficient numerical tool for these challenges. Recently, physicists have found that these methods can also be applied to classical systems, including stochastic dynamics problems. This research focuses on the difference between utilizing the VMPS to analyze the quantum models and the classical stochastic models and some advantages of the VMPS.

2. Models

Quantum Model: Transverse Ising Model

• The Hamiltonian operator of the transverse Ising model:

$$\hat{H} = -J\sum_i \sigma_i^x \otimes \sigma_{i+1}^x - h\sum_i \sigma_i^z$$

• The parameters are J and h that can describe the phase transitions of the system. The system is in the disordered phase when $J/h > 1$, in the ordered phase when $J/h < 1$, and at the critical point when $J/h = 1$.

Classical Stochastic Model: SIS Model

• The infinitesimal Markov generator of the SIS model:

$$\hat{W} = \beta\sum_{i=1}^{N-1}(\hat{n}_i\hat{w}_{i+1}^{01} + \hat{w}_i^{01}\hat{n}_{i+1}) + \gamma\sum_{i=1}^{N-1}\hat{w}_i^{10} + \hat{W}_{driv}(\alpha)$$

• The parameters are β and γ that can describe the phase transitions of the system. The system is in the endemic phase when $\beta/\gamma > 1.649$, in the absorbing phase when $\beta/\gamma < 1.649$, and at the critical point when $\beta/\gamma = 1.649$.

3. Algorithms

Algorithms to obtain the ground state energy and the ground state of the model:

Quantum Model

• Construct a matrix product operator $\hat{H}$ and an arbitrary initial matrix product state $|\Psi_i\rangle$.
• Construct the effective Hamiltonian by moving out an arbitrary matrix from $\langle\Psi_i|\hat{H}|\Psi_i\rangle$.
• Diagonalize the effective Hamiltonian by the Lancsoz algorithm and replace the leading eigenvector iteratively.

Classical Stochastic Model

Algorithm to obtain the leading eigenvalue and the non-equilibrium steady state of the model:
• Construct a matrix product operator $\hat{W}$ and an arbitrary initial matrix product state $|P_i\rangle$.
• Construct the effective Hamiltonian by moving out an arbitrary matrix from $\langle P_i|\hat{W}|P_i\rangle$.
• Diagonalize the effective Hamiltonian by the Arnoldi algorithm and replace the leading eigenvector iteratively.

4. Results

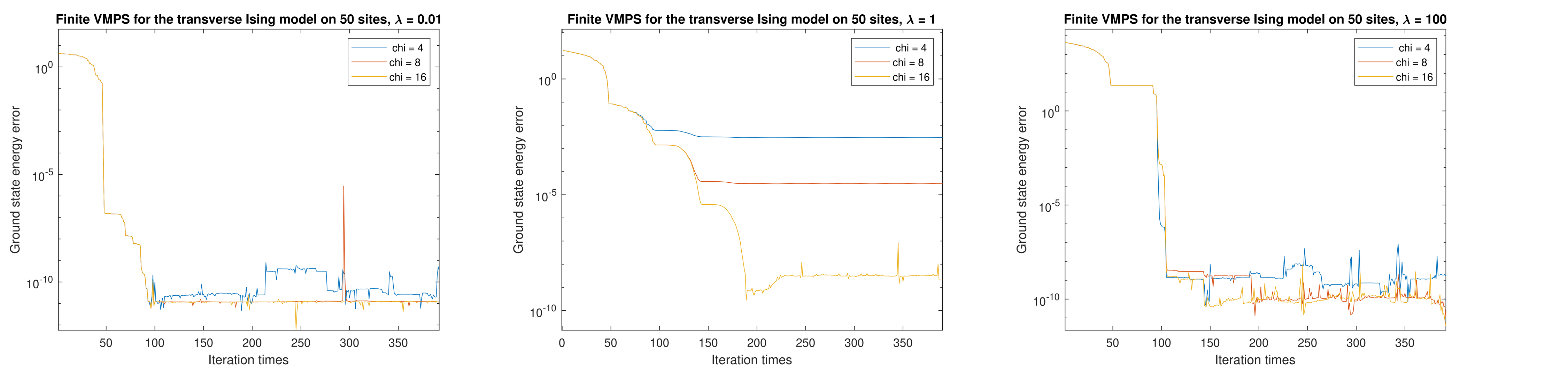

Figure: Numerical error of the ground state energy of the transverse Ising model employing VMPS.

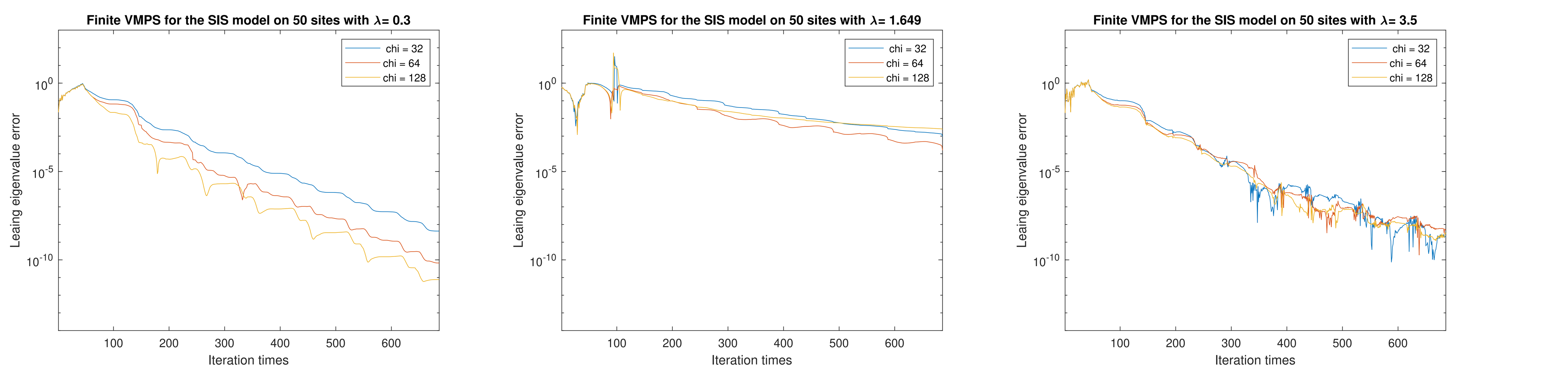

Figure: Numerical error of the leading eigenvalue of the SIS model employing VMPS.

5. The Scaled Cumulant Generating Function (SCGF)

• The Laplace transformation of the infinitesimal Markov generator is

$$\hat{W} = \beta\sum_{i=1}^{N-1}(\hat{n}_i w_{i+1}^{01} + \hat{w}_i^{01}\hat{n}_{i+1}) + \gamma\sum_{i=1}^{N-1} w_i^{10} + W_{driv}(\alpha, s).$$

The SCGF is the leading eigenvalue of $W(s)$ and can be computed by the VMPS as well:

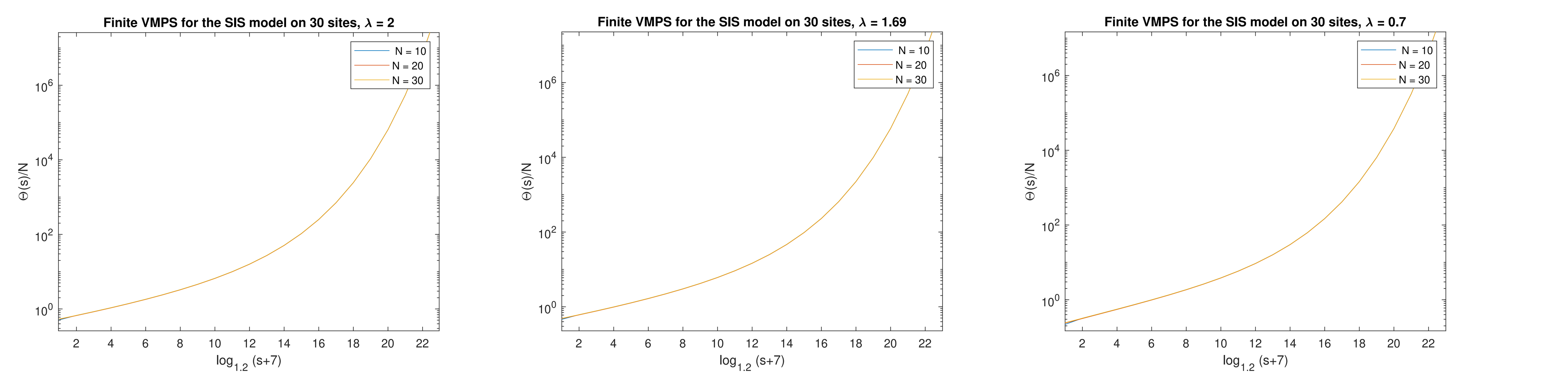

Figure: The SCGF density of the SIS system with different sizes and phases.

6.Conclusion

• **VMPS Accuracy:** The VMPS method is a precise tool for calculating stochastic systems' non-equilibrium steady state (NESS), using advanced diagonalization algorithms.
• **Convergence Near Criticality:** The algorithms' convergence rate decreases significantly near the critical point of the system, requiring more computational effort.
• **Exact NESS Distribution:** VMPS provides exact NESS distributions, making it highly effective for simulating rare events.

7. Further Questions

• While the VMPS is useful for the SIS model, the original framework may not fully clarify its application. Insights from the SIS model's renormalization group description can help improve understanding of the SIS model.
• The VMPS can be employed in the two-dimensional system now. How to leverage the VMPS to a two-dimensional stochastic model?
• The infection rate of the pathogen may change over time and this VMPS algorithm is therefore broken. Could we utilize the time-dependent VMPS to solve this problem?

8. References

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