Sympcletic Geometry and Classical Mechanics

Hao-Yang Yen [1]

Interdisciplinary Program of Sciences, National Tsing Hua University, Hsinchu, 30013, Taiwan

[1] E-mail: `kikiyen0715@gapp.nthu.edu.tw`

Contents

When I was a student in classical mechanics class, I felt that it was a boring subject, the only mathematics used in the lectures is calculus and basic linear algebra. However, after I gained some knowledge in differential geometry, I felt that it was quite an intriguing subject. The geometric structure of classical mechanics is quite amazing. When I was the teaching assistant for statistical mechanics, I also introduced some geometric concepts to help students understand classical statistical mechanics.

This note offers a comprehensive introduction to Hamiltonian mechanics, intricately woven with the symplectic geometry that serves as its foundation.

1 Classical Mechanics

Just as we employ linear algebra to articulate the principles of quantum mechanics, the mathematical framework for classical mechanics is rooted in the theory of manifolds, with a special emphasis on smooth manifolds. In this discussion, we will explore Hamiltonian mechanics, which is the origin of symplectic geometry, as our starting point.

1.1 Lagranian Mechanics

Lagrangian mechanics is a reformulation of classical mechanics that provides a powerful framework for analyzing the motion of dynamical systems. Developed by Joseph-Louis Lagrange in the 18th century, this approach shifts the focus from forces to energy, emphasizing the principle of least action.

At the heart of Lagrangian mechanics is the Lagrangian function L, defined as the difference between the kinetic energy T and the potential energy V of a system:

$$L = T - V.$$

The central idea is that the actual path taken by a system between two points in configuration space is the one that minimizes the action, which is defined as the integral of the Lagrangian over time

$$S[q(t)] = \int L(q, \dot{q}, t)\ \mathrm{d}\,t,$$

where q represents the generalized coordinates and $\dot{q}$ the generalized velocities.

By applying the principle of least action, one derives the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0,$$

which govern the dynamics of the system. This formulation is particularly advantageous in dealing with complex systems and constraints, making it widely applicable in physics. Lagrangian mechanics serves as a foundational tool in fields ranging from classical mechanics and field theory.

The state space of Lagrangian systems is called *configuration spaces*. Suppose the generalized position space of a Hamiltonian system is a smooth manifold M, then the phase space is its tangent bundle TM.

Consider the Legendre transformation concerning the canonical momentum $p = \partial L/\partial q$ of the Lagrangian

$$\dot{q}_i p_i - L \ ,$$

we can obtain a new formulation of classical mechanics.

1.2 Hamiltonian Mechanics

Hamiltonian mechanics provides a framework for describing classical mechanical systems using pairs of conjugate variables: q_a representing generalized coordinates and p_a denoting generalized momenta. Here, the index a runs from 1 to n, where n signifies the degrees of freedom within the system. Central to this framework is the Hamiltonian function $H(q,p)$, which encapsulates the total energy of the system and is expressed through the equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

These equations, known as the Hamiltonian equations of motion, dictate the temporal evolution of the system's state.

The state spaces of the Hamiltonian systems are called *phase spaces*.

Since we know that the Hamiltonian and Lagrangian are the Legendre transformation of each other and we know that the configuration space of a Lagrangian system is a tangent bundle, we can describe the geometry of phase space directly.

Suppose the generalized position space of a Hamiltonian system is a $n-$dimensional smooth manifold M, then the phase space is its cotangent bundle T^*M.

The Legendre transformation transforms the following concepts

$$\begin{aligned} L &\longleftrightarrow H \\ \text{Tangent bundle } TM &\longleftrightarrow T^*M \text{ Cotangent bundle} \\ \text{Euler-Lagrange's equation} &\longleftrightarrow \text{Hamiltonian equations} \end{aligned}$$

In Hamiltonian mechanics, the trajectory of the system is determined by the principle of least action, which asserts that the true path taken by the system minimizes the action S. The action is formulated as follows

$$\begin{aligned} S[q(t), p(t)] &= \int \left(p_a \dot{q}_a - H(q,p)\right) \mathrm{d}\, t \\ &= \int p_\alpha dq_\alpha - H dt \ . \end{aligned}$$

To find the actual path, one must evaluate the variation of the action, leading to the condition $\delta S = 0$. This condition inherently results in the derivation of the Hamiltonian equations.

2 Symplectic Geometry for Classical Mechanics

2.1 Symplectic Form

The symplectic structure of phase space plays a pivotal role in Hamiltonian mechanics. On the cotangent bundle, there exists a standard symplectic form, which is the exterior derivative of a specific one-form. This one-form associates a vector in the tangent bundle of the cotangent bundle with an element of the cotangent bundle (a linear functional) by evaluating the vector against the differential of the projection from the cotangent bundle to the original manifold.

To demonstrate that this one-form indeed defines a symplectic form, we can utilize the local nature of symplectic forms. Since the cotangent bundle is locally trivial, we only need to verify this definition on $\mathbb{R}^n \times \mathbb{R}^n$. In this case, the one-form is defined as the sum of $y_i \, \mathrm{d}\, x_i$, and its differential is the sum of the standard symplectic form $\mathrm{d}\, y_i \wedge \mathrm{d}\, x_i$.

The symplectic 2-form ω, defined on the phase space M, is expressed as

$$\omega = \mathrm{d}\, p_i \wedge \mathrm{d}\, q^i \ ,$$

where the symbol $\wedge$ signifies the wedge product. This form is characterized by being non-degenerate and closed ($d\omega = 0$) clearly, thereby establishing M as a *symplectic manifold.*

Moreover, we can introduce the *symplectic potential*

$$\Theta = p_i \, \mathrm{d}\, q^i \ ,$$

which is a differential and satisfies $\mathrm{d}\, \Theta = \omega$. Therefore, the action can be rewritten as

$$S = \int (\Theta - H \, \mathrm{d}\, t) \ .$$

The symplectic form gives us its relation to the Poisson bracket

$$\{-,-\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathbb{R}, \ (f,g) \mapsto \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \ , \\ \text{where } \mathcal{F}(M) = \{f \mid f : M \to \mathbb{R}\} \ .$$

The Poisson brackets of phase space coordinates are

$$\{q_i, q_j\} = 0 \ , \quad \{p_i, p_j\} = 0 \ , \quad \{q_i, p_j\} = \delta_{ij} \ .$$

Therefore, the matrix representation of the Poisson bracket is

$$\Omega = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \ ,$$

which is known as *symplectic matrix.* Therefore, if we view the phase space as a vector space, then the Poisson bracket is a symplectic transformation, and $\{q_i\} \cup \{p_i\}$ is a Darboux basis. Alternatively, we can write

$$\{(q,p),(q,p)\} = \Omega \ .$$

We will see more about this in the next section.

2.2 Symplectomorphisms

The coordinates of a Hamiltonian system are not unique, consider a map $(q,p) \mapsto (Q,P)$, the new symplectic potential and action are

$$\omega' = \mathrm{d}\, P_i \wedge \mathrm{d}\, Q_i \ , \quad \Theta' = P_i \, \mathrm{d}\, Q_i \ , \quad S' = \int (\Theta' - H'(Q,P) \, \mathrm{d}\, t) \ .$$

Then the least action principle gives

$$0 = \delta S = \int (\Theta - H(q,p)\,\mathrm{d}\,t) \ ,$$
$$0 = \delta S' = \int (\Theta' - H'(Q,P)\,\mathrm{d}\,t) \ .$$

If the dynamics of the Hamiltonian system are identical under these two coordinates, they can only differ by one total differential

$$\mathrm{d}\,F = p_i\,\mathrm{d}\,q^i - P_i\,\mathrm{d}\,Q^i + (H' - H)\,\mathrm{d}\,t \ .$$

That is

$$\mathrm{d}\,\Theta = \mathrm{d}\,\Theta' + \mathrm{d}\,F \ ,$$

which gives the relation between the symplectic potential and symplectic form under two coordinates

$$\omega' = \mathrm{d}\,\Theta' = \mathrm{d}\,\Theta = \omega \ .$$

The maps preserving the symplectic form are called *canonical transformations*. The canonical transformations can be computed by the total differential $\mathrm{d}\,F$

$$p_i = \frac{\partial F}{\partial q_i} \ , \quad -P_i = \frac{\partial F}{\partial q_i} \ , \quad H' - H = \frac{\partial F}{\partial t} \ .$$

By the least action principle, the form of the Hamiltonian equations is preserved if and only if the symplectic form is preserved.

We can see that the most crucial structure in Hamiltonian mechanics is the symplectic form ω. When the symplectic form is preserved, the dynamics of the Hamiltonian systems are identical. Since the dynamics of two phase spaces are identical if they have the same symplectic form even if they are on different manifolds, we represent a phase space as a manifold M equipped with a symplectic form ω, this is called a *symplectic manifold* and denoted by (M, ω).

In differential geometry, the canonical transformations are a kind of *symplectomorphisms*. The symplectomorphisms preserve the symplectic from of the phase space and therefore preserve the dynamics of the Hamiltonian system as well. A symplectomorphism represents a diffeomorphism $\phi : (M, \omega_M) \to (N, \omega_N)$ that conserves the symplectic form

$$\phi^* \omega_N = \omega_M .$$

These transformations are essential in the exploration of canonical transformations and the invariance of physical laws across various coordinate systems.

2.3 Symplectic Group

Consider a transformation $(q,p) \mapsto (Q,P)$ (not necessarily canonical), we know that

$$(\dot{Q}, \dot{P})^T = \left[\frac{\partial(Q,P)}{\partial(q,p)}\right] (\dot{q}, \dot{p})^T \ .$$

The last term of this equation can be written as

$$(\dot{q}, \dot{p})^T = \Omega \frac{\partial H}{\partial (q,p)^T} = \Omega \left[\frac{\partial (Q,P)}{\partial (q,p)} \right]^T \frac{\partial H'}{\partial (Q,P)^T} \, .$$

The equation therefore can become

$$(\dot{Q}, \dot{P})^T = \left[\frac{\partial (Q,P)}{\partial (q,p)} \right] \Omega \left[\frac{\partial (Q,P)}{\partial (q,p)} \right]^T \frac{\partial H'}{\partial (Q,P)^T} \, .$$

When the condition

$$\left[\frac{\partial (Q,P)}{\partial (q,p)} \right] \Omega \left[\frac{\partial (Q,P)}{\partial (q,p)} \right]^T = \Omega$$

is satisfied the form of the Hamiltonian equations is preserved and therefore the symplectic form is preserved as well. Therefore, the condition of preserving the symplectic form is

$$\left[\frac{\partial (Q,P)}{\partial (q,p)} \right] \in \{ M \in GL(2n, \mathbb{R}) \mid M \Omega M = \Omega \} \, .$$

The set on the left-hand side is a matrix group and is called *symplectic group*, which is a Lie group. Hence, the canonical transformations can form a Lie group. Therefore, the phase space has a kind of special smooth symmetry. The study of this special symmetry is the origin of symplectic geometry. This symmetry implies several intriguing physics, one example is *Liouville's theorem*, which states that the density, that is, the symplectic form $\mathrm{d}\, q_i \wedge \mathrm{d}\, p^i$, of the phase space is conserved under canonical transformations.

3 Conclusion

In summary, this note provides a foundational exploration of classical mechanics and the underlying geometry, highlighting their interconnections within both the realms of physics and mathematics.

Mathematics	Physics
Cotangent Bundle T^*M	Configuration Space
Tangent Bundle TM	Phase Space
Darboux Basis	Position and Momentum
Symplectic Form	Dynamics
Symplectomorphisms	Canonical Transformations
Symplectic Group	Smooth Symmetry

The aim is to elevate the understanding of Hamiltonian systems through the lens of symplectic geometry, yielding deeper insights into classical mechanics as well as modern mathematical frameworks. This exploration sets the stage for further investigation into the rich tapestry of dynamical systems and their geometric interpretations.

4 References

[1] Lecture Note of Advanced Linear Algebra from prof. Meng-Kiat Chuah
[2] Goldstein, Herbert. Classical mechanics. Pearson Education India, 2011.
[3] Tu, Loring W. "Manifolds." An Introduction to Manifolds. New York, NY: Springer New York, 2011. 47-83.
[4] Arnol'd, Vladimir Igorevich. Mathematical methods of classical mechanics. Vol. 60. Springer Science & Business Media, 2013.