Leveraging Variational Matrix Product States for the Analysis of Stochastic Dynamical Systems

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1. Abstract

The variational matrix product states (VMPS) technique is a key numerical tool in addressing many-body quantum system problems. Recent research has extended VMPS to classical stochastic dynamics. This study explores:
- Differences between applying VMPS to quantum models and classical stochastic models.
- Advantages of using VMPS in these contexts.

3. Algorithms

The VMPS algorithms for determining the target eigenstate and its eigenvalue quantum models of classical stochastic models involve:
- Constructing a matrix product operator $\hat{O}$ and an initial matrix product state $|\Psi_i\rangle$.
- Forming the effective operator by extracting an arbitrary matrix from $\langle\Psi_i|\hat{O}|\Psi_i\rangle$.
- Diagonalizing the effective operator by the appropriate algorithm and replacing the matrix by leading eigenvector.
- Move to the next lattice and return to the previous step.

Quantum Model

Employ the Lanczos algorithm for the Hermitian Hamiltonian.

Classical Stochastic Model

Employ the Arnoldi algorithm for the non-Hermitian infinitesimal Markov generator.

5. Efficiency

Monte Carlo Method

- The CPU time for obtaining the NESS of the SIS model on 50 sites is about 4.3×10^4 seconds. However, the deviations for the rare events are still large.

VMPS Algorithms

- The CPU time for obtaining the NESS of the SIS model on 50 sites is about 1.6×10^3 seconds for bond dimension= 1024, which is expensive in VMPS algorithms.

7.Conclusion

- **Exact NESS Distribution:** VMPS provides exact NESS distributions, making it highly effective for simulating rare events.
- **Efficiency:** VMPS is much more efficient than the Monte Carlo method when numerous samples are needed.
- **Convergence Near Criticality:** The algorithms' convergence rate decreases significantly near the critical point of the system, requiring more computational effort.

8. Further Questions

- Renormalization group insights can enhance SIS model understanding.
- How can VMPS be used for two-dimensional stochastic models?
- Can a time-dependent VMPS handle varying infection rates?
- Can VMPS be applied to stochastic systems without NESS, such as predator-prey models?

2. Models

Quantum Model: Transverse Ising Model

- The Hamiltonian operator of the transverse Ising model:

$$\hat{H} = -J\sum_i \sigma_i^x \otimes \sigma_{i+1}^x - h\sum_i \sigma_i^z$$

- The parameters are J and h that can describe the phase transitions of the system. The system is in the disordered phase when $J/h > 1$, in the ordered phase when $J/h < 1$, and at the critical point when $J/h = 1$.

Classical Stochastic Model: SIS Model

- The infinitesimal Markov generator of the SIS model:

$$\hat{W} = \beta\sum_{i=1}^{N-1}(\hat{n}_i\hat{w}_{i+1}^{01} + \hat{w}_i^{01}\hat{n}_{i+1}) + \gamma\sum_{i=1}^{N-1}\hat{w}_i^{10} + \hat{W}_{driv}(\alpha)$$

- The parameters are β and γ that can describe the phase transitions of the system. The system is in the endemic phase when $\beta/\gamma > 1.649$, in the absorbing phase when $\beta/\gamma < 1.649$, and at the critical point when $\beta/\gamma = 1.649$.

4. Results

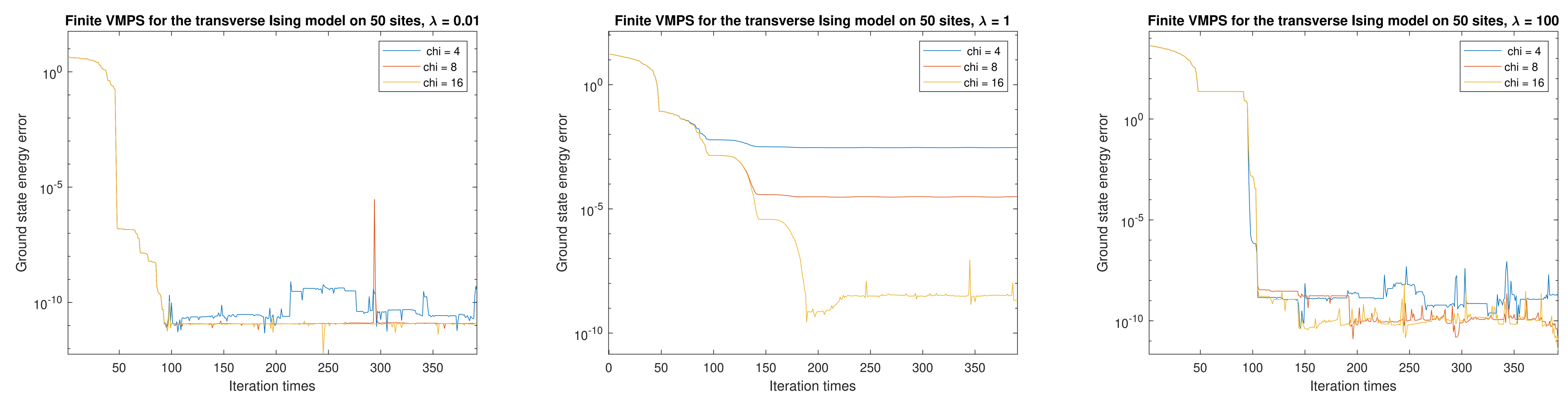

Figure: Numerical error of the ground state energy of the transverse Ising model employing VMPS.

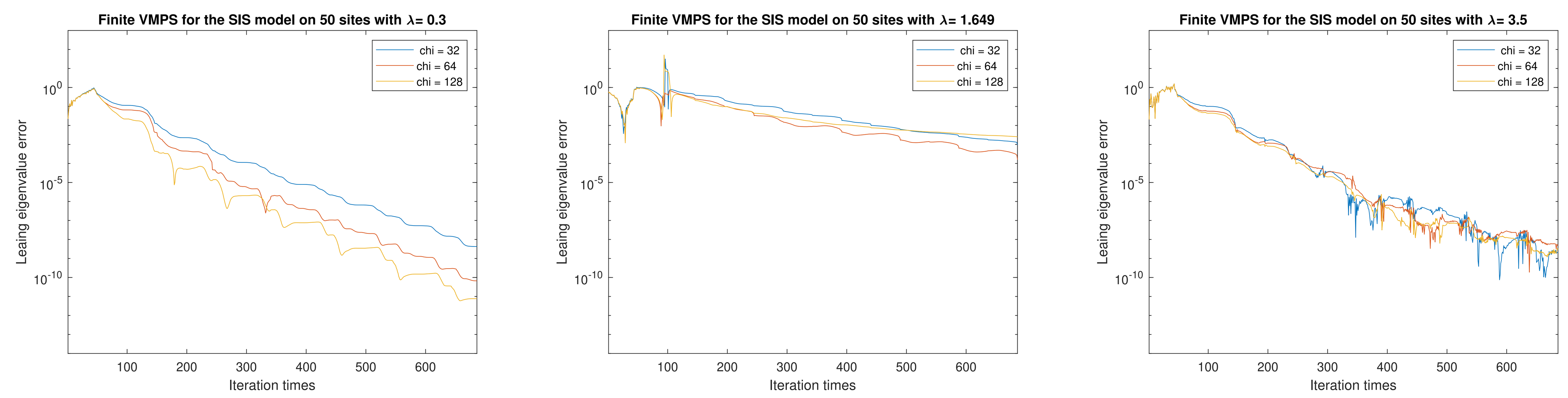

Figure: Numerical error of the leading eigenvalue of the SIS model employing VMPS.

6. The Scaled Cumulant Generating Function (SCGF)

- The Laplace transformation of the infinitesimal Markov generator is

$$\hat{W} = \beta\sum_{i=1}^{N-1}(\hat{n}_i w_{i+1}^{01} + \hat{w}_i^{01}\hat{n}_{i+1}) + \gamma\sum_{i=1}^{N-1} w_i^{10} + W_{driv}(\alpha, s).$$

The SCGF Θ is the leading eigenvalue of $W(s)$ and can be computed by the VMPS as well:

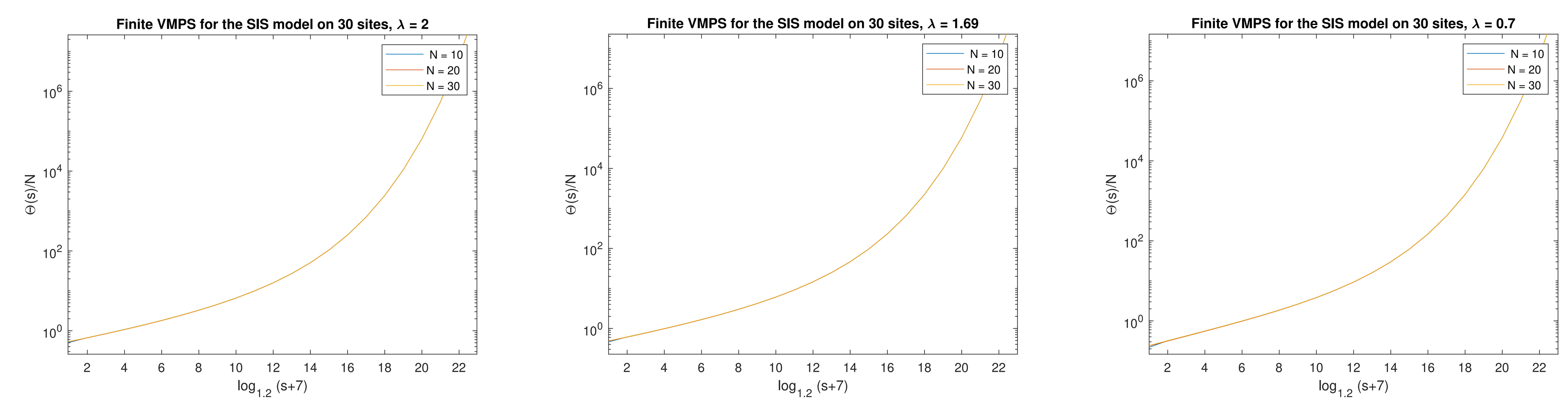

Figure: The SCGF density of the SIS system with different sizes and phases.

9. References

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