Introduction to Hamiltonian Dynamical Systems and The N-Body Problem

(by KR Meyer)

Exercise Solutions

Author: Hao-Yang Yen (顏浩洋)
Department of Physics, National Tsing-Hua University
Email: kikiyen0715@gapp.nthu.edu.tw

Last modified: October 22, 2025

Problem 1.3

Solution.

(a) Compute the total differential directly

$$\begin{aligned}\frac{\mathrm{d}(2I)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\theta}^2 + (1-\cos\theta)\right) \\ &= \dot{\theta}\ddot{\theta} + \sin\theta\dot{\theta} = \dot{\theta}(\ddot{\theta} + \sin\theta) = 0, \qquad (1)\end{aligned}$$

which immediately implies that $2I$ is an integral.

(b) See Figure 1.

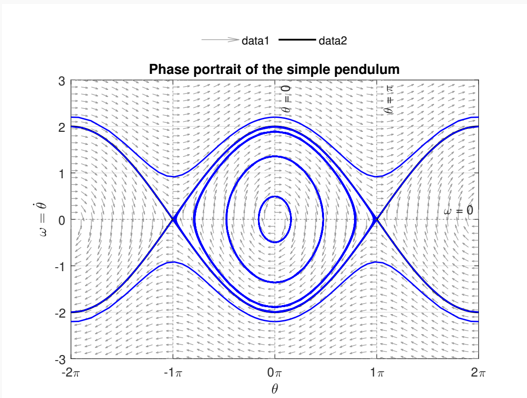

Figure 1: Phase portrait of the simple pendulum.

(c) Consider the substitution $y = \sin(\theta/2)$ and its derivative

$$\dot{y} = \frac{1}{2}\dot{\theta}\cos(\theta/2). \tag{2}$$

Express $2I$ in terms of $\dot{\theta}$ and $\sin(\theta/2)$, then we can get

$$2I = \frac{1}{2}\dot{\theta}^2 + (1 - \cos\theta) = \frac{1}{2}\dot{\theta}^2 + 2\sin^2(\theta/2), \tag{3}$$

where we want to express $\dot{\theta}/2$ in terms of y and $\dot{y}$. It is easy to see that

$$\frac{1}{2}\dot{\theta}^2 = 2\frac{\dot{y}^2}{1 - y^2}. \tag{4}$$

So the expression of $2I$ in terms of y and $\dot{y}$ is

$$2I = 2\frac{\dot{y}^2}{1 - y^2} + 2y^2. \tag{5}$$

So we obtain an ODE for y

$$\dot{y}^2 = (1 - y^2)(I - y^2). \tag{6}$$

Substitute $y = k\,\mathrm{sn}(t, k)$ into $\dot{y}^2$, we can obtain

$$\begin{aligned}\dot{y}^2 &= k^2\dot{\mathrm{sn}}^2(t, k) = I\,\dot{\mathrm{sn}}^2(t, k)\\ &= I\,(1 - \mathrm{sn}^2(t, k))(1 - k^2\,\mathrm{sn}^2(t, k))\\ &= (I - y^2)(1 - y^2),\end{aligned} \tag{7}$$

which means that $k\,\mathrm{sn}(t, k)$ indeed solves (6).