Title

code - courseName

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dd-mm-yyyy

Contents

```r
source("chrisFunctions.r")
library(ggplot2)
```

GARCH

GARCH(p, q) is given by

$$r_t = \sigma_t \epsilon_t, \tag{1}$$

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j r_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2. \tag{2}$$

It can be shown that a GARCH(p, q) admits a non-Gaussian ARMA(p, q) for the squared process

$$r_t^2 \tag{3}$$

acf and pacf

For fun we consider the simple acf and pacf of the difference

$$\nabla x_t = x_t - x_{t-1}$$

```r
d = diff(df$Inflation)
par(mfrow = c(2, 2))
dacf = sampleAcf(d, 22)
dacf2 = acf(d)

dpacf = samplePacf(d, 22)
dpacf2 = pacf(d)
```

Now let's look at the return

$$\nabla r_t = \frac{x_t - x_{t-1}}{x_{t-1}}. \tag{4}$$

Problem: Some values are zero. Let's make sure that they are not. (Should not matter since we are looking at variance, right?)

```r
n = dim(df)[1]
shift = min(df$Inflation - 0.1)
inflationShifted = df$Inflation - shift
r = diff(inflationShifted)/inflationShifted[1:(n - 1)]
par(mfrow = c(1, 2))
dacf = sampleAcf(r, 22)

dpacf = samplePacf(r, 22)
```

Finally, let's consider the squared return, which is supposed to be ARMA(p,q)

```r
par(mfrow = c(1, 2))
dacf = sampleAcf(r^2, 22)

dpacf = samplePacf(r^2, 22)  # Verified with pacf fnc.
```

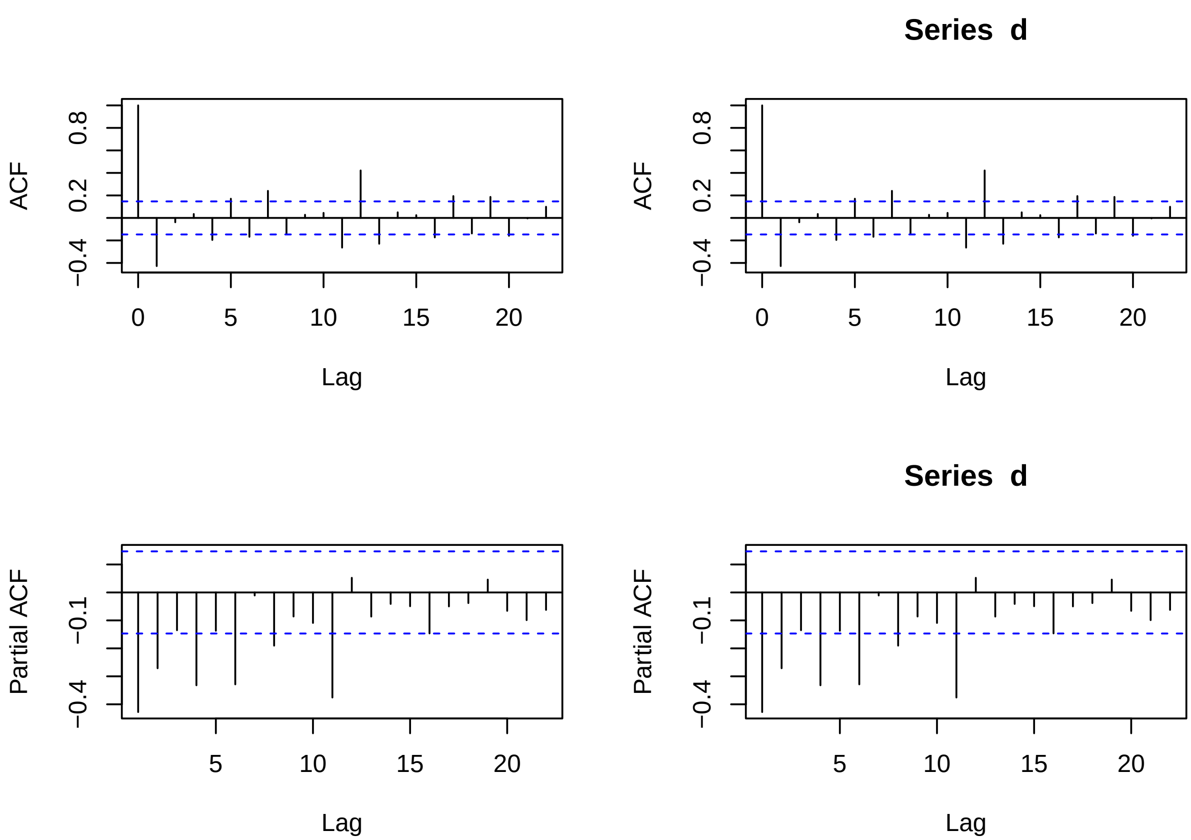

Figure 1: Manual to the left and built-in to the right

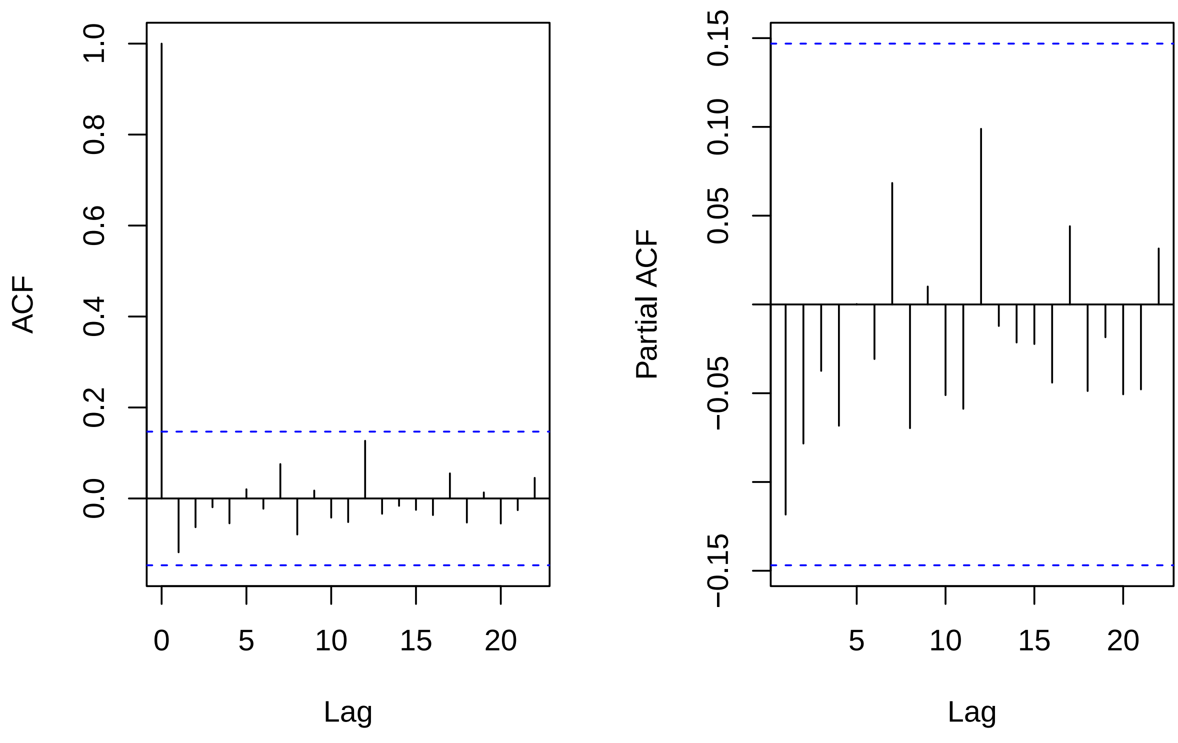

Figure 2: Return

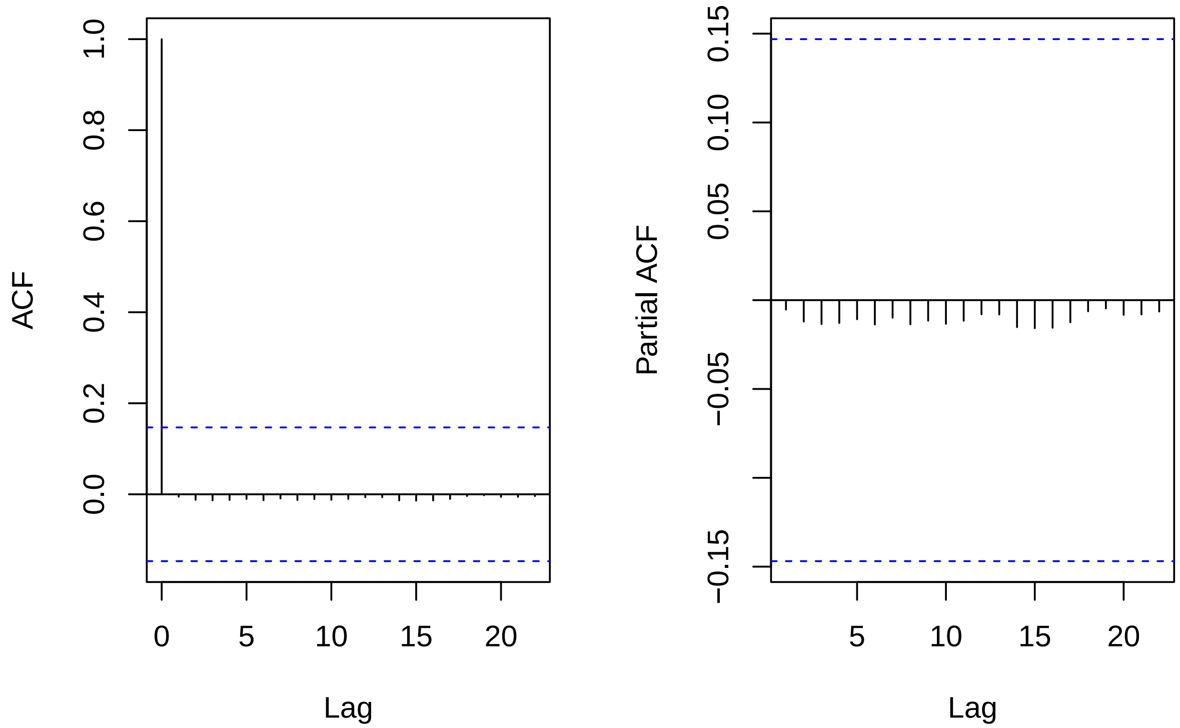

Figure 3: Squared return

Table 3.1. Behavior of the ACF and PACF for ARMA models

	AR(p)	MA(q)	ARMA(p,q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

(5)

GARCH(p,q) Implementation

Definitions for notational purposes:

- $\mathcal{R}_1 = (r_1, ..., r_{max(p,q)})$
- $\mathcal{R}_2 = (r_t, ..., r_{t+max(p,q)})$
- $\boldsymbol{\alpha} = (\alpha_0, ..., \alpha_p)$
- $\boldsymbol{\beta} = (\beta_1, ..., \beta_q)$
- $\boldsymbol{r}_{tp} = (1, r_{t-1}, ..., r_{t-p})^\top$
- $\sigma^2_{tq} = (\sigma^2_{t-1}, ..., \sigma^p_{t-q})^\top$

It can be shown that $r_t|\mathcal{R}_2 \sim \mathcal{N}(0, \sigma_t^2)$, $\mathcal{R}_2 = (r_t, ..., r_{t+max(p,q)})$. Because of the normality of the conditional return we can find MLE of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ parameters. That is, the likelihood to be maximized is given by

$$\begin{aligned} L(\boldsymbol{\alpha}, \boldsymbol{\beta}|\mathcal{R}_1) &= \prod_{t=\max(p,q)+1}^{n} f_{\boldsymbol{\alpha},\boldsymbol{\beta}}(r_t|\mathcal{R}_2) \\ &= \prod_{t=\max(p,q)+1}^{n} (2\pi\sigma_t^2)^{-1/2} \exp(-\frac{1}{2}\frac{r_t^2}{\sigma_t^2}), \end{aligned} \tag{6}$$

where $\mathcal{R}_1 = (r_1, ..., r_{max(p,q)})$ and σ_t^2 is given by equation (2). We define the criterion function l to be minimized as proportional to $-\ln L$, such that

$$\begin{aligned} l(\boldsymbol{\alpha}, \boldsymbol{\beta}|\mathcal{R}_1) &= \sum_{t=\max(p,q)+1}^{n} \ln(\sigma_t^2) + \frac{r_t^2}{\sigma_t^2} \\ &= \sum_{t=\max(p,q)+1}^{n} \ln(\alpha_0 + \sum_{j=1}^{p} \alpha_j r_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2) + \frac{r_t^2}{\alpha_0 + \sum_{j=1}^{p} \alpha_j r_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2} \\ &= \sum_{t=\max(p,q)+1}^{n} \ln(\boldsymbol{\alpha}\boldsymbol{r}_{tp} + \boldsymbol{\beta}\boldsymbol{\sigma}_{tq}^2) + \frac{r_t^2}{\boldsymbol{\alpha}\boldsymbol{r}_{tp} + \boldsymbol{\beta}\boldsymbol{\sigma}_{tq}^2}, \end{aligned} \tag{7}$$

where $\boldsymbol{r}_{tp} = (1, r_{t-1}, ..., r_{t-p})^\top$ and $\sigma^2_{tq} = (\sigma^2_{t-1}, ..., \sigma^p_{t-q})^\top$