Title

Course

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DD MM YYYY

Contents


```
source("./additionalFiles/probAhelp.R")
source("./additionalFiles/probAdata.R")
figPath = "./Figures/"
```

2.

In this problem we want to implement the EM-algorithm. We have found the conditional expectation $Q(\lambda_0,\lambda_1) = Q(\lambda_0,\lambda_1|\lambda_0^{(t)},\lambda_1^{(t)})$. This corresponds to the E-step in the EM algorithm. In the M-step of the algorithm is to determine

$$(\lambda_0^{(t+1)},\lambda_1^{(t+1)}) = \text{argmax} \;\; Q(\lambda_0,\lambda_1).$$

This can be found be finding the partial derivates and $Q(\lambda_0,\lambda_1)$ and set them equal to zero.

$$\frac{\partial}{\partial\lambda_0}Q(\lambda_0,\lambda_1) = \frac{n}{\lambda_0} - \sum_{i=1}^{n}\left(u_i z_i + (1-u_i)\left(\frac{1}{\lambda_0^{(t)}} - \frac{z_i}{e^{\lambda_0^{(t)}z_i}-1}\right)\right) = 0$$

$$\frac{\partial}{\partial\lambda_1}Q(\lambda_0,\lambda_1) = \frac{n}{\lambda_1} - \sum_{i=1}^{n}\left((1-u_i)z_i + u_i\left(\frac{1}{\lambda_1^{(t)}} - \frac{z_i}{e^{\lambda_1^{(t)}z_i}-1}\right)\right) = 0$$

We solve these two equations for λ_0 and λ_1 respectively. This gives the M-step

$$\lambda_0^{(t+1)} = n/\sum_{i=1}^{n}\left(u_i z_i + (1-u_i)\left(\frac{1}{\lambda_0^{(t)}} - \frac{z_i}{e^{\lambda_0^{(t)}z_i}-1}\right)\right)$$
$$\lambda_1^{(t+1)} = n/\sum_{i=1}^{n}\left((1-u_i)z_i + u_i\left(\frac{1}{\lambda_1^{(t)}} - \frac{z_i}{e^{\lambda_1^{(t)}z_i}-1}\right)\right)$$

Let $\lambda^{(t)} = (\lambda_0^{(t)},\lambda_1^{(t)})$. We want to implement the EM-algorithm and we use the convergence criterion

$$d(x^{(t+1)}, x^t) = ||\lambda^{(t+1)} - \lambda^{(t)}||_2 < \epsilon.$$

The function below returns the conditional expectation, that is the E-step of the EM algorithm.

```
cond_expectation <- function(lambda0, lambda1, lambda0t, lambda1t, u, z) {
    n = length(u)
    exp = n * (log(lambda0) + log(lambda1)) - (lambda0 * sum(u * z + (1 - u) * (1/lambda0t -
        (z)/(exp(lambda0t * z) - 1)))) - (lambda1 * sum(u * z + (1 - u) * (1/lambda1t -
        (z)/(exp(lambda1t * z) - 1))))
    return(exp)
}
```

Under is a function that implement M-step.

```
M_step <- function(lambda0t, lambda1t, u, z) {
    n = length(u)
    lambda0next = n/(sum(u * z + (1 - u) * (1/lambda0t - (z)/(exp(lambda0t * z) -
        1))))
    lambda1next = n/(sum((1 - u) * z + u * (1/lambda1t - (z)/(exp(lambda1t * z) -
        1))))

    return(list(lambda0 = lambda0next, lambda1 = lambda1next))
}

lambda0 = 2
lambda1 = 3
list0 <- c()
list1 <- c()
for (i in 1:10) {
    lambda0 = M_step(lambda0, lambda1, u, z)$lambda0
    lambda1 = M_step(lambda0, lambda1, u, z)$lambda1
    list0 <- c(list0, lambda0)
    list1 <- c(list1, lambda1)
}
list0
```

```
##  [1] 0.01730962 0.01718016 0.01718015 0.01718015 0.01718015 0.01718015
##  [7] 0.01718015 0.01718015 0.01718015 0.01718015
```

```
EM_algorithm <- function(lambda, u, z, epsilon = 1e-04) {
    lambda0 = lambda[1]
    lambda1 = lambda[2]
    list0 <- c()
    list1 <- c()
    for (i in 1:30) {
        lambda0 = M_step(lambda0, lambda1, u, z)$lambda0
        lambda1 = M_step(lambda0, lambda1, u, z)$lambda1
        list0 <- c(list0, lambda0)
        list1 <- c(list1, lambda1)
    }
    return(list(lambdas0 = list0, lambdas1 = list1))
}
```

Under the EM algorithm is implemented

```
ME_algorithm <- function(lambda, u, z, epsilon = 1e-04) {
    i = 0
    lambda0 = lambda[0]
    lambda1 = lambda[1]
    lambda = c(lambda0, lambda1)
    lambdas0 = c()
    lambdas1 = c()
    norm = -Inf
    for (i in 1:50) {
        # M-step
        lambda0t = M_step(lambda0, lambda1, u, z)$lambda0
        lambda1t = M_step(lambda0, lambda1, u, z)$lambda1
        lambdas0 = c(lambdas0, lambda0t)
        lambdas1 = c(lambdas1, lambda1t)
        lambdat = c(lambda0t, lambda1t)
        # convergence

        norm = norm(lambdat - lambda, type = "2")
        lambda = lambdat
    }
    return(list(lambdas0 = lambdas0, lambdas1 = lambdas1))
}
```

```
lambdas <- EM_algorithm(c(2.5, 5), u, z)
lambdas$lambdas1
```

```
##  [1] 0.03852065 0.02731178 0.02728921 0.02728916 0.02728916 0.02728916
##  [7] 0.02728916 0.02728916 0.02728916 0.02728916 0.02728916 0.02728916
## [13] 0.02728916 0.02728916 0.02728916 0.02728916 0.02728916 0.02728916
## [19] 0.02728916 0.02728916 0.02728916 0.02728916 0.02728916 0.02728916
## [25] 0.02728916 0.02728916 0.02728916 0.02728916 0.02728916 0.02728916
```

4.

We want to find an analytical formula for $f_{Z_i,U_i}(z_i, u_i|\lambda_0, \lambda_1)$.

$$
\begin{aligned}
f_{Z_i,U_i}(z_i, u_i|\lambda_0, \lambda_1) &= P(\max(X_i, Y_i) = z, I(X_i \geq Y_i) = u_i|\lambda_0, \lambda_1) \\
&= u_i P(\max(X_i, Y_i) = z_i, X)
\end{aligned}
$$