Title

Course

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DD MM YYYY

Contents


```
source("./additionalFiles/probAhelp.R")
source("./additionalFiles/probAdata.R")
source("./additionalFiles/u.txt")
source("./additionalFiles/z.txt")
figPath = "./Figures/"
```

Problem C: The EM-algorithm and bootstrapping

Let $x_1, ...x_n$ and $y_1, ..., y_n$ be independet random variables, where

$$x_i \sim \text{Exp}(\lambda_0) \quad \text{and} \quad y_i \sim \text{Exp}(\lambda_1)$$

We observe

$$z_i = \max(x_i, y_i) \quad \text{for} \quad i = 1, ..., n$$

and

$$u_i = I(x_i \geq y_i) \quad \text{for} \quad i = 1, ..., n.$$

1.

The joint distribution of $(x_i, y_i), i = 1, ..n$ is given by

$$f(x, y|\lambda_0, \lambda_1) = \prod_{i=1}^{n} f_x(x_i|\lambda_0) \cdot f_y(y_i|\lambda_1)$$

$$= \prod_{i=1}^{n} \lambda_0 e^{-\lambda_0 x_i} \cdot \lambda_1 e^{-\lambda_1 y_i}.$$

This means that the log likelihood is given by

$$\ln f(x, y|\lambda_0, \lambda_1) = \sum_{i=1}^{n} \ln \lambda_0 + \ln \lambda_1 - \lambda_0 x_i - \lambda_1 y_i.$$

We want to find

$$E\left[\ln f(x, y|\lambda_0, \lambda_1)|z, u, \lambda_0^{(t)}, \lambda_1^{(t)}\right].$$

which is given by

$$\begin{aligned} Q(\lambda_0, \lambda_1|\lambda_0^{(t)}, \lambda_1^{(t)}) &= E\left[\sum_{i=1}^{n}(\ln \lambda_0 + \ln \lambda_1 - \lambda_0 x_i - \lambda_1 y_i)|z, u, \lambda_0^{(t)}, \lambda_1^{(t)}\right] \\ &= n(\ln \lambda_0 + \ln \lambda_1) - \lambda_0 \sum_{i=1}^{n} E \end{aligned}$$

2.

In this problem we want to implement the EM-algorithm. We have found the conditional expectation $Q(\lambda_0, \lambda_1) = Q(\lambda_0, \lambda_1|\lambda_0^{(t)}, \lambda_1^{(t)})$. This corresponds to the E-step in the EM algorithm. In the M-step of the algorithm is to determine

$$(\lambda_0^{(t+1)}, \lambda_1^{(t+1)}) = \text{argmax} \;\; Q(\lambda_0, \lambda_1).$$

This can be found be finding the partial derivates and $Q(\lambda_0, \lambda_1)$ and set them equal to zero.

$$\frac{\partial}{\partial \lambda_0} Q(\lambda_0, \lambda_1) = \frac{n}{\lambda_0} - \sum_{i=1}^{n}\left(u_i z_i + (1-u_i)\left(\frac{1}{\lambda_0^{(t)}} - \frac{z_i}{e^{\lambda_0^{(t)} z_i} - 1}\right)\right) = 0$$

$$\frac{\partial}{\partial \lambda_1} Q(\lambda_0, \lambda_1) = \frac{n}{\lambda_1} - \sum_{i=1}^{n}\left(u_i z_i + (1-u_i)\left(\frac{1}{\lambda_1^{(t)}} - \frac{z_i}{e^{\lambda_1^{(t)} z_i} - 1}\right)\right) = 0$$

We solve these two equations for λ_0 and λ_1 respectively. This gives the M-step

$$\lambda_0^{(t+1)} = n / \sum_{i=1}^{n}\left(u_i z_i + (1-u_i)\left(\frac{1}{\lambda_0^{(t)}} - \frac{z_i}{e^{\lambda_0^{(t)} z_i} - 1}\right)\right)$$

$$\lambda_1^{(t+1)} = n / \sum_{i=1}^{n}\left(u_i z_i + (1-u_i)\left(\frac{1}{\lambda_1^{(t)}} - \frac{z_i}{e^{\lambda_1^{(t)} z_i} - 1}\right)\right)$$

Let $\lambda^{(t)} = (\lambda_0^{(t)}, \lambda_1^{(t)})$. In the code below ,the EM-agorithm is implemented with the convergence criterion

$$d(x^{(t+1)}, x^t) = ||\lambda^{(t+1)} - \lambda^{(t)}||_2 < \epsilon$$