Lenstra Elliptic Curve Factorization

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MATH 317

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- ECM is third-fastest known factoring algorithm and the best algorithm for finding divisors not exceeding 50-60 digits.
- The largest factor found using ECM has 83 digits.

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- Observe f and g are inverses.

$$f(g(h)) = f(ah^{-1}) = (ah^{-1})^{-1}a = h$$

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- Therefore, $|G| = n|H|$.

A different perspective

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Lemma 2.2.5 Suppose that m, $n \in \mathbb{N}$ and $\gcd(m, n) = 1$. Then the map

$$\psi : (\mathbb{Z}/mn\mathbb{Z})^* \to (\mathbb{Z}/m\mathbb{Z})^* \times (\mathbb{Z}/n\mathbb{Z})^*$$

defined by

$$\psi(c) = (c \pmod{m}, c \pmod{n})$$

is a bijection.

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60	575	(1, 16)

- We compute $gcd(574, 1763) = 41$

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such that $4a^3 + 27 \in (\mathbb{Z}/N\mathbb{Z})^*$. This forces non singularity and ensures $P = (0, 1)$ is on the curve.

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- Definition 6.3.1 (Power Smooth). Let B be a positive integer. If n is a positive integer with prime factorization
$$n = \prod p_i^{e_i},$$
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- Example $30 = 2 \cdot 3 \cdot 5$ is B power smooth for $B \geq 5$, but $150 = 2 \cdot 3 \cdot 5^2$ is not 5-power smooth.

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- The idea of ECM is to replace modular exponentiation on $(\mathbb{Z}/N\mathbb{Z})^*$ by repeated addition of points on $E((\mathbb{Z}/N\mathbb{Z})^*)$
- Recall, by the Hasse-Weil bound we can reduce the size of our group by $2 \cdot \sqrt{p}$.

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- If Pollard $p-1$ fails, we have no choice but to increase B.
- However, ECM has a second option. We can choose another random elliptic curve.

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We can consider an analogous mapping

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- Note the quotations. There is a subtly in the difference between $E(\mathbb{Z}/N\mathbb{Z})$ and $\mathbb{Z}/N\mathbb{Z}$.

Implementation

- Generate a random elliptic curve $E \pmod{N}$ and let $P = (0, 1)$.
- Compute $m = lcm(1, 2, ..., B)$.
- Compute mP (don't be naive!).
- If the calculation fails, you have found a non-trivial factor of N.
- Otherwise, just generate a new Elliptic curve and try again.

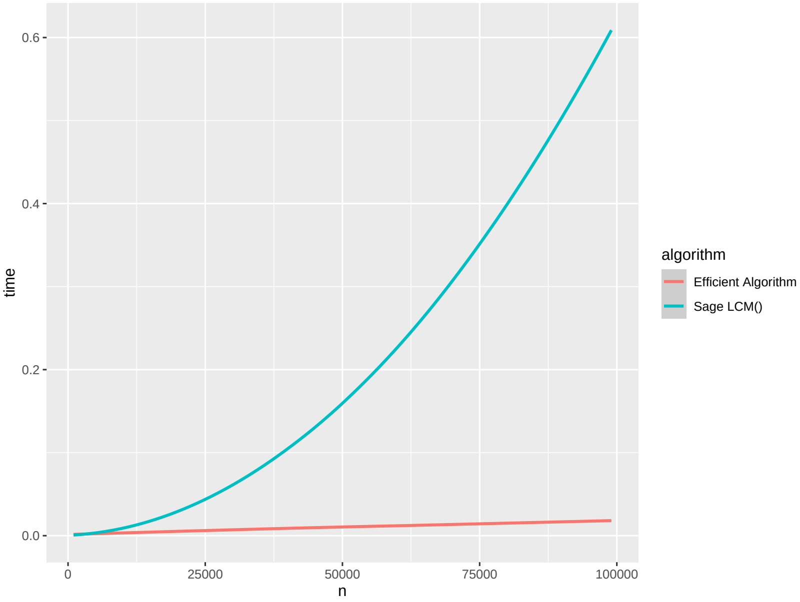
Computing lcm(1,2,...,B)

Recall,

$$lcm(1,2,...B) = \prod_{p \in P} p^r$$

where $r = \max\{r \in \mathbb{Z} \mid p^r \leq B\}$.

$$\begin{aligned} p^r &\leq B \\ r\log(p) &\leq \log(B) \\ r &\leq \log_p(B) \\ r &= \lfloor \log_p(B) \rfloor \end{aligned}$$


0.6
0.4
0.2
0.0
time
0
25000
50000
75000
100000
n
algorithm
Efficient Algorithm
Sage LCM()

Computing mP

$$mP = \overbrace{P + P + P \dots P}^{m \text{ times}}$$

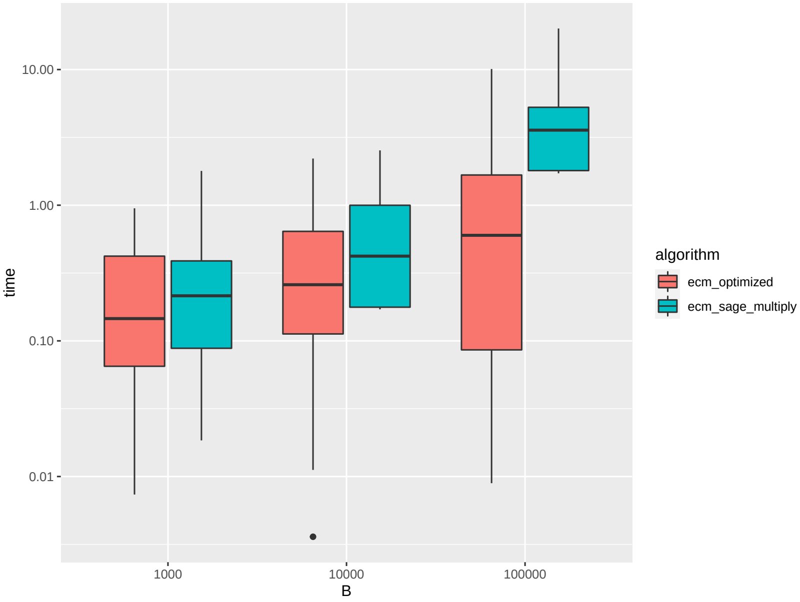
A very bad way to compute mP.

There are many algorithms for computing general elliptic curve point multiplication efficiently, and given the known make-up of m, we can save time by being thoughtful here.
Consider,

$$m_n = q_1^{r_1} \cdot q_2^{r_2} \dots q_n^{r_n}$$

then

$$m_n P = q_n^{r_n} \cdot m_{n-1} P$$


10.00
1.00
0.10
0.01
time
1000
10000
100000
B
algorithm
ecm_optimized
ecm_sage_multiply

Coded Example

```
def ecm(n, B=10^4, trials=100):
    R = Zmod(n)
    primes = list(prime_range(B+1))

    for _ in range(trials):
        while True:
            a = R.random_element()
            if gcd(4 * Integer(a)^3 + 27, n) == 1:
                break

        E = EllipticCurve([a, 1])
        P = E([0,1])

        try:
            for p in primes:
                P = P * p^floor(math.log(B,p))

        except ZeroDivisionError as e:
            return gcd(Integer(str(e).split()[2]), n)

    return -1
```

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- This implies computing mP given m is $O\left(\frac{B}{\log 2}\right)$.
- Therefore, computing mP should take roughly $O(B \log \log B)$

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- Recall Hasse-Weil bound lets us limit the group by $2 \cdot \sqrt{p}$
- $O\left(e^{\sqrt{2\log(p)\log(\log(p))}}\right)$

Demo