Lenstra Elliptic Curve Factorization

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MATH 317

2021

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- The largest factor found using ECM has 83 digits.

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Lemma 2.2.5 Suppose that $m,\ n \in \mathbb{N}$ and $\gcd(a, n) = 1$. Then the map

$$\psi : (\mathbb{Z}/mn\mathbb{Z})^* \to (\mathbb{Z}/m\mathbb{Z})^* \times (\mathbb{Z}/n\mathbb{Z})^*$$

defined by

$$\psi(c) = (c \pmod{m}, c \pmod{n})$$

is a bijection.

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6	570	(37, 11)
60	575	(1, 16)

- We compute $gcd(574, 1763) = 41$

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such that $4a^3 + 27 \in (\mathbb{Z}/N\mathbb{Z})^*$. This forces non singularity and ensures $P = (0, 1)$ is on the curve.

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- Definition 6.3.1 (Power Smooth). Let B be a positive integer. If n is a positive integer with prime factorization

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- Example $30 = 2 \cdot 3 \cdot 5$ is B power smooth for $B \geq 5$, but $150 = 2 \cdot 3 \cdot 5^2$ is not 5-power smooth.

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- The idea of ECM is to replace modular exponentiation on $(\mathbb{Z}/N\mathbb{Z})^*$ by repeated addition of points on $E((\mathbb{Z}/N\mathbb{Z})^*)$
- Recall, by the Hasse-Weil bound we can reduce the size of our group by $2 \cdot \sqrt{p}$.

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3. Try to compute mP. If at some point we cannot compute a sum of points, then some denominator g is not coprime to N, then $gcd(g, N)$ is a nontrivial divisor of N.

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- If Pollard $p - 1$ fails, we have no choice but to increase B.
- However, ECM has a second option. We can choose another random elliptic curve.

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We can consider an analogous mapping

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- Note the quotations. There is a subtly in the difference between $E(\mathbb{Z}/N\mathbb{Z})$ and $\mathbb{Z}/N\mathbb{Z}$.
- Let $P = (0 : 1 : 1) \in E(\mathbb{Z}/1763\mathbb{Z})$
  $P_1 = (0 : 1 : 1) \in E(\mathbb{Z}/41\mathbb{Z})$ and $P_2 = (0 : 1 : 1) \in E(\mathbb{Z}/43\mathbb{Z})$

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i	$i * P_1$	$i * P_2$	$i * P$

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7	$(10 : 18 : 1)$	$(33 : 20 : 1)$	$(420 : 1740 : 1)$
8	$(22 : 19 : 1)$	$(2 : 20 : 1)$	$(432 : 880 : 1)$
9	$(40 : 11 : 1)$	$(13 : 26 : 1)$	$(1475 : 585 : 1)$
10	$(19 : 25 : 1)$	$(8 : 20 : 1)$	$(1126 : 1009 : 1)$
11	$(32 : 19 : 1)$	$(1 : 2 : 1)$	$(1549 : 1249 : 1)$
12	$(13 : 25 : 1)$	$(0 : 42 : 1)$	$\gcd(\text{denom}, N) = 43$
13	$(12 : 21 : 1)$	$(0 : 1 : 0)$	

Implementation

- Generate a random elliptic curve $E \pmod{N}$ and let $P = (0, 1)$.
- Compute $m = lcm(1, 2, ..., B)$.
- Compute mP (don't be naive!).
- If the calculation fails, you have found a non-trivial factor of N.
- Otherwise, just generate a new Elliptic curve and try again.

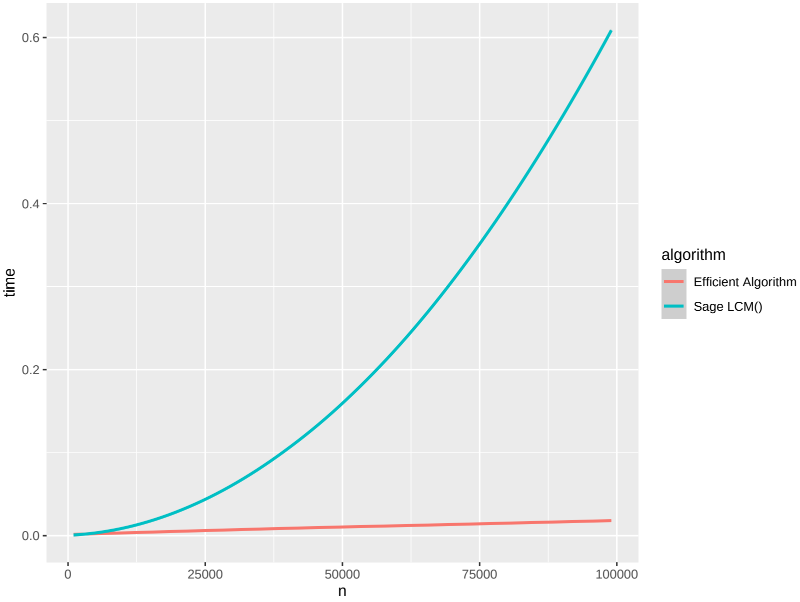
Computing lcm(1,2,...,B)

Recall,

$$lcm(1,2,...B) = \prod_{p \in P} p^r$$

where $r = \max\{r \in \mathbb{Z} \mid p^r \leq B\}$.

$$\begin{aligned} p^r &\leq B \\ r\log(p) &\leq \log(B) \\ r &\leq \log_p(B) \\ r &= \lfloor \log_p(B) \rfloor \end{aligned}$$



0.6
0.4
0.2
0.0
time
0
25000
50000
75000
100000
n
algorithm
Efficient Algorithm
Sage LCM()

Computing mP

$$mP = \overbrace{P + P + P \dots P}^{m \text{ times}}$$

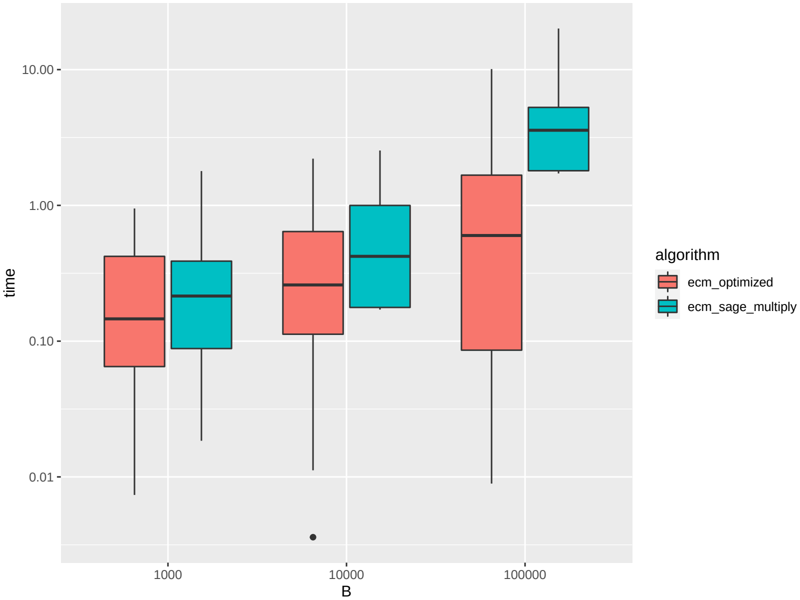
A very bad way to compute mP.

There are many algorithms for computing general elliptic curve point multiplication efficiently, and given the known make-up of m, we can save time by being thoughtful here.
Consider,

$$m_n = q_1^{r_1} \cdot q_2^{r_2} \dots q_n^{r_n}$$

then

$$m_n P = q_n^{r_n} \cdot m_{n-1} P$$


10.00
1.00
0.10
0.01
time
1000
10000
100000
B
algorithm
ecm_optimized
ecm_sage_multiply

Coded Example

```
def ecm(n, B=10^4, trials=100):
    R = Zmod(n)
    primes = list(prime_range(B+1))

    for _ in range(trials):
        while True:
            a = R.random_element()
            if gcd(4 * Integer(a)^3 + 27, n) == 1:
                break

        E = EllipticCurve([a, 1])
        P = E([0,1])

        try:
            for p in primes:
                P = P * p^floor(math.log(B,p))

        except ZeroDivisionError as e:
            return gcd(Integer(str(e).split()[2]), n)

    return -1
```

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- $LCM(1, 2, 3, ..., B) \approx e^B$.
- This implies computing mP given m is $O\left(\frac{B}{\log 2}\right)$.
- Therefore, computing mP should take roughly $O(B \log \log B)$

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- Recall Hasse-Weil bound lets us limit the group by $2 \cdot \sqrt{p}$
- $O\left(e^{\sqrt{2\log(p)\log(\log(p))}}\right)$